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New Semiconcavity Results for Optimal Control Problems

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New Semiconcavity Results for Optimal Control Problems

Piermarco Cannarsa

University of Rome "Tor Vergata" (Italy)

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ITN SADCO – ENSTA PARISTECH

March 3-4, 2011



Outline

Semiconcave functions

Definitions and general properties

Generalized gradients

Singularities of semiconcave functions

Semiconcave functions and optimal control

Mayer problem

Differential inclusions

Semiconcave functions and Hamilton-Jacobi equations

Smoothness of extremal trajectories

Semiconcavity of solutions

Conclusions



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definition

 $\Omega \subset \mathbb{R}^N$ domain

Definition

 $u : \Omega \rightarrow \mathbb{R}$ (linearly) semiconcave

- ▶ u continuous in Ω
- ▶ $\exists C \geq 0$ (semiconcavity constant) such that

$$u(x+h) + u(x-h) - 2u(x) \leq C|h|^2$$

for all $x, h \in \mathbb{R}^N$ such that $[x-h, x+h] \subset \Omega$

more general notion

$$u(x+h) + u(x-h) - 2u(x) \leq |h|\omega(|h|)$$

where $\omega : [0, \infty) \rightarrow [0, \infty)$ (semiconcavity modulus) upper semicontinuous, nondecreasing, $\omega(r) \rightarrow 0$ ($r \rightarrow 0$)



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equivalent properties

$u : \Omega \rightarrow \mathbb{R}$ semiconcave with constant $C \geq 0$

iff

any of the following holds:

- ▶ for all x, y such that $[x, y] \subset \Omega$ and all $\lambda \in [0, 1]$

$$\lambda u(x) + (1 - \lambda)u(y) - u(\lambda x + (1 - \lambda)y) \leq \frac{C}{2} \lambda(1 - \lambda) |x - y|^2$$

- ▶ the function $x \rightarrow u(x) - \frac{C}{2}|x|^2$ is concave in Ω
- ▶ for any $\nu \in \mathbb{R}^N$ such that $|\nu| = 1$

$$\frac{\partial^2 u}{\partial \nu^2} \leq C \quad \text{in (distributional sense on) } \Omega$$

- ▶ $u(x) = \inf_{i \in \mathcal{I}} u_i(x)$ where $\|D^2 u_i\|_\infty \leq C$ for all $i \in \mathcal{I}$



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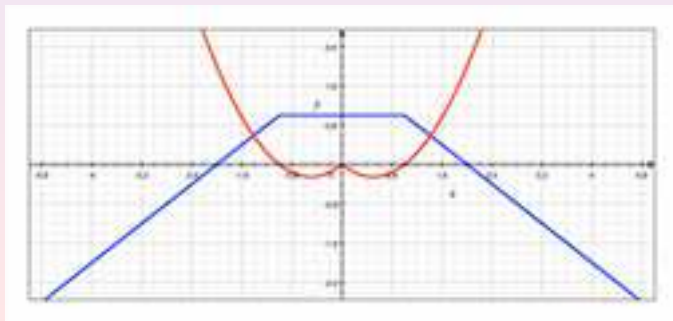
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why study semiconcave functions?

$$u(x) = \inf_{i \in \mathcal{I}} u_i(x) \quad x \in \mathbb{R}^N$$

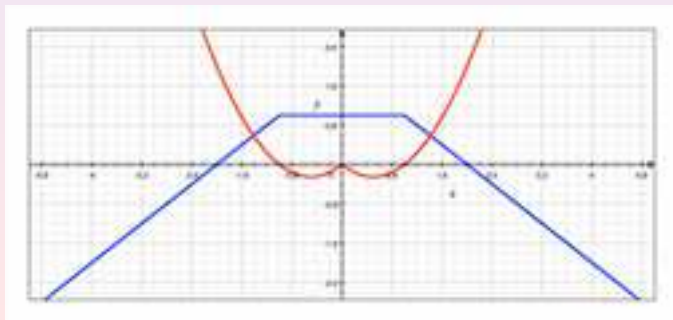
- ▶ u_i affine $\Rightarrow u$ concave
- ▶ $\|D^2 u_i\|_\infty \leq C \Rightarrow u$ semiconcave



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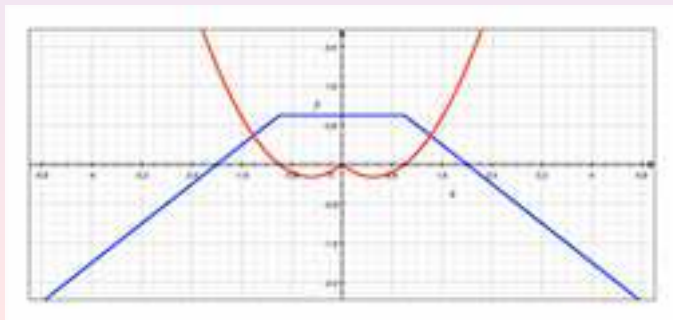
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references

- ▶ uniqueness for pdes Kruzhkov (1960), Douglis (1961)
- ▶ control theory
 - ▶ Hrustalev (1978)
 - ▶ C – Soner (1987) power-like modulus
 - ▶ C – Frankowska (1991) general modulus
 - ▶ Fleming – McEneaney (2000) nonlinear filtering
 - ▶ Rifford (2000, 2002) feedback stabilization
 - ▶ ...
- ▶ nonsmooth analysis Rockafellar (1982)
- ▶ comprehensive exposition
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generalized differentials

$\Omega \subset \mathbb{R}^N$ domain $u : \Omega \rightarrow \mathbb{R}$

u semiconcave \implies u Locally Lipschitz
 \implies u differentiable a.e.

$x \in \Omega$

▶ limiting gradients

$$D^* u(x) = \left\{ \lim_{i \rightarrow \infty} Du(x_i) \mid x_i \rightarrow x \right\}$$

compact $\neq \emptyset$

▶ $D^+ u(x) = \text{co}(D^* u(x))$ superdifferential

($\implies D^+ u(x) = \partial u(x)$ Clarke's generalized differential)

$$D^+ u(x) = \left\{ p \in \mathbb{R}^N \mid \limsup_{y \rightarrow x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}$$



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example 1

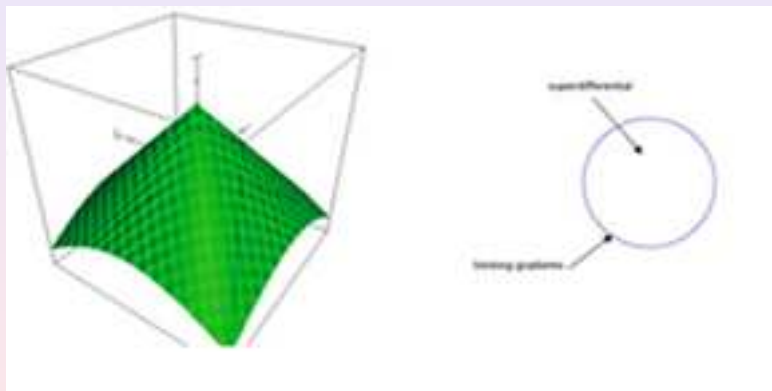


Figure: $u(x, y) = 3 - \sqrt{x^2 + y^2}$ and $D^+ u(0, 0)$



example 2

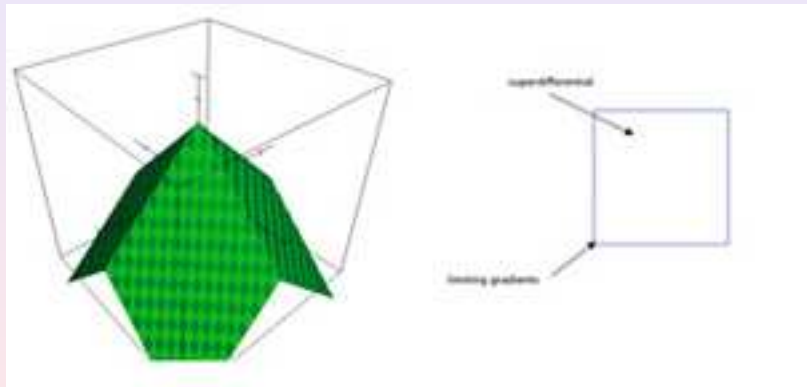


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singular point = nondifferentiability point

$u : \Omega \rightarrow \mathbb{R}$ semiconcave

Definition (singular set)

$$\begin{aligned}\Sigma(u) &= \{x \in \Omega \mid \nexists Du(x)\} \\ &= \{x \in \Omega \mid \dim D^+ u(x) \geq 1\}\end{aligned}$$

- ▶ $u \in \text{Lip}_{loc}(\Omega) \implies |\Sigma(u)| = 0$ (Lebesgue)
- ▶ $Du \in BV_{loc}(\Omega; \mathbb{R}^N) \implies \Sigma(u)$ countably \mathcal{H}^{N-1} -rectifiable
- ▶ magnitude of a singular point x

$$\kappa(x) = \dim D^+ u(x)$$



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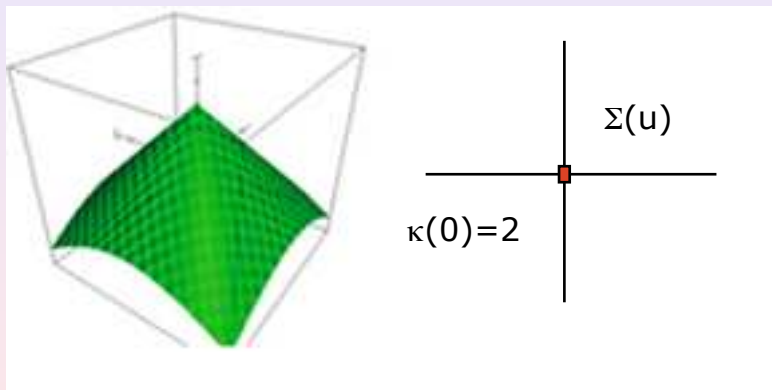


Figure: singular set of $u(x, y) = 3 - \sqrt{x^2 + y^2}$



example 2

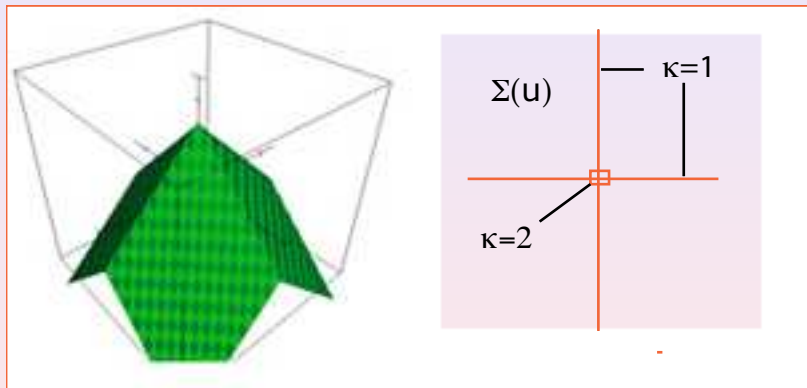


Figure: singular set of $u(x, y) = 3 - |x| - |y|$



sharp rectifiability

$u : \Omega \rightarrow \mathbb{R}$ semiconcave

singular sets of prescribed magnitude $k = 1, \dots, N$

$$\Sigma^k(u) := \{x \in \Omega \mid \dim D^+ u(x) = k\}$$

Theorem

$\Sigma^k(u)$ countably $(N - k)$ -rectifiable

▶ $\Sigma(u) = \bigcup_{k=1}^N \Sigma^k(u)$ countably $(N - 1)$ -rectifiable

▶ references

- ▶ Zajíček (1978), Veselý (1979)
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propagation of singularities

$u : \Omega \rightarrow \mathbb{R}$ semiconcave $x_0 \in \Sigma(u)$

Problem

$$??? \Rightarrow \Sigma(u) \neq \{x_0\}$$

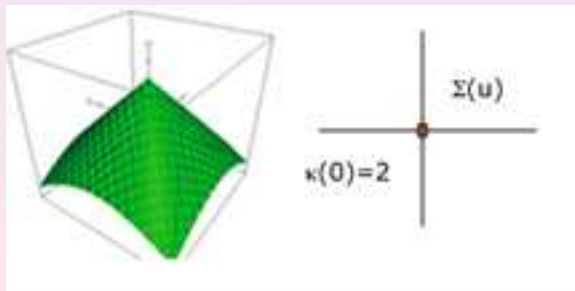


Figure: no propagation for $u(x, y) = 3 - \sqrt{x^2 + y^2}$



do lower magnitude singularities propagate?

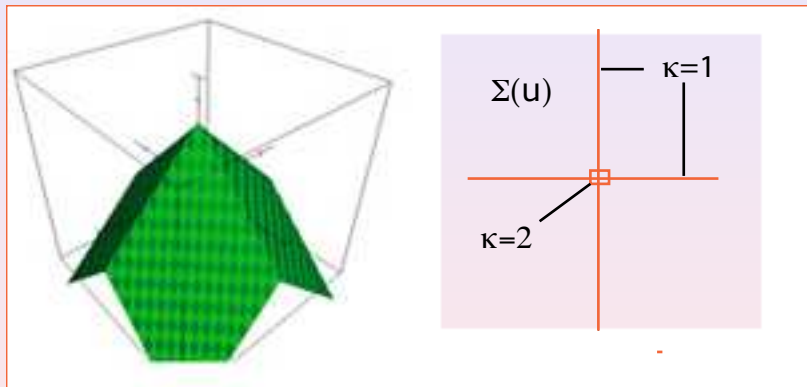


Figure: **magnitude 1** singularities of $u(x, y) = 3 - |x| - |y|$ do propagate along straight lines



semiconcavity results for optimal control

- └ semiconcave functions
- └ singularities

NO

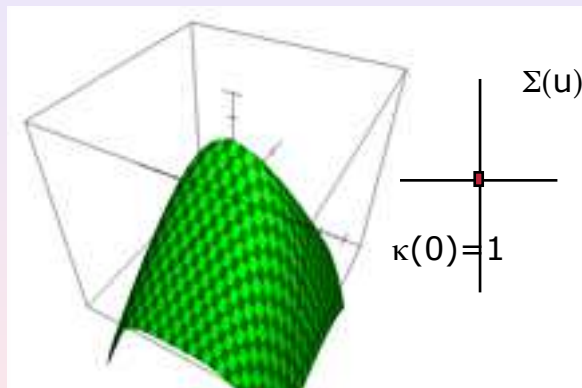


Figure: an **isolated** singularity of **magnitude 1** at the origin

$$u(x, y) = 3 - \sqrt{\left(\frac{3x}{2}\right)^2 + \left(\frac{2y}{3}\right)^4}$$



- └ semiconcave functions
- └ singularities

“pattern recognition”

a closer look at D^+u and D^*u

Example

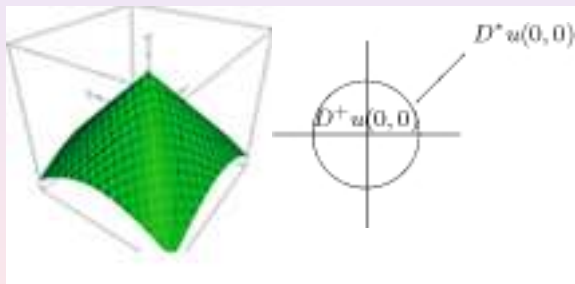


Figure: in example 1

$$D^*u(0,0) = \partial D^+u(0,0)$$



- └ semiconcave functions
- └ singularities

example 2

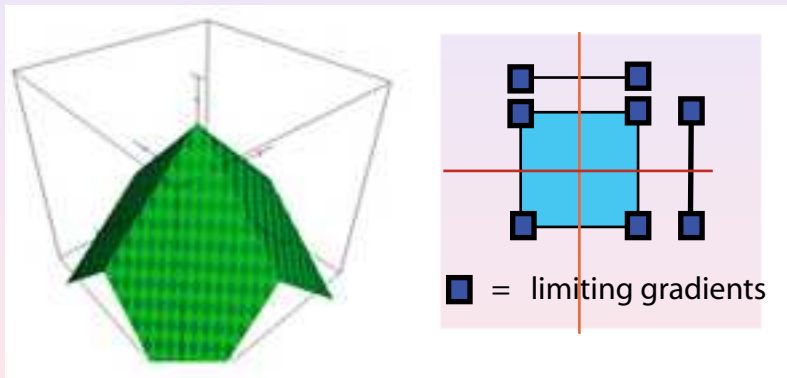


Figure: here

$$D^*u(0,0) \subsetneq \partial D^+u(0,0)$$



example 3

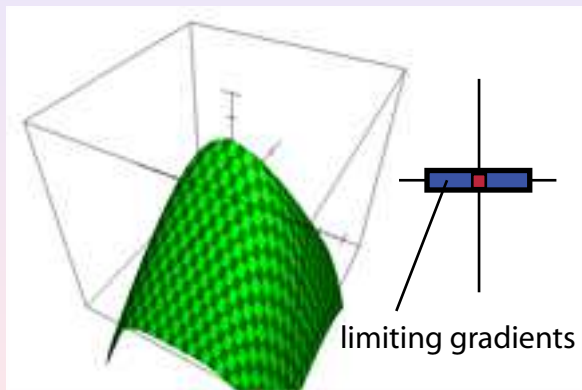


Figure: here

$$D^*u(0,0) = D^+u(0,0)$$



propagation principle

$u : \Omega \rightarrow \mathbb{R}$ semiconcave $x_0 \in \Sigma(u)$

Theorem (Albano – C 1999; C – Yu 2007)

Let

- ▶ $\emptyset \neq \partial D^+ u(x_0) \setminus D^* u(x_0) \ni p_0$
- ▶ $q \in \mathbb{R}^N \setminus \{0\}$ such that $q \cdot (p - p_0) \geq 0 \forall p \in D^+ u(x_0)$

Then $\exists \tau > 0$ and $x(\cdot) : [0, \tau] \rightarrow \Sigma(u)$ Lipschitz

1. $\dot{x}(s) \in q - p_0 + D^+ u(x(s))$ a.e. $s \in [0, \tau]$
2. $\dot{x}^+(0) = q$ & $\dot{x}^+(\cdot)$ continuous from the right
3. $\inf_{s \in [0, \tau]} \text{diam } D^+ u(x(s)) > 0$



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semiconcavity results for optimal control

- └ semiconcave functions
- └ singularities

back to example 2

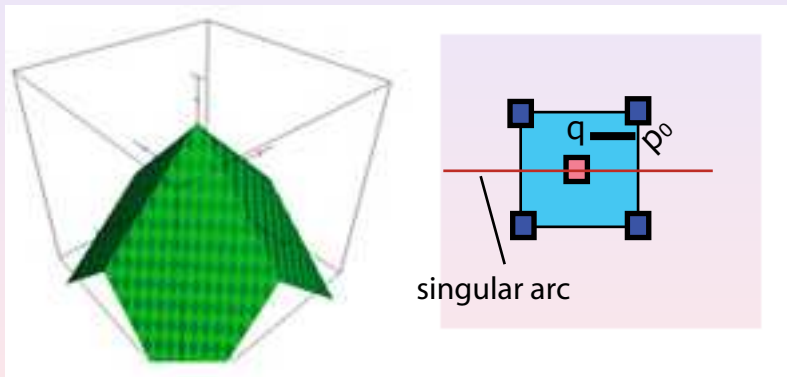


Figure: the propagation principle at work



Outline

Semiconcave functions

Definitions and general properties

Generalized gradients

Singularities of semiconcave functions

Semiconcave functions and optimal control

Mayer problem

Differential inclusions

Semiconcave functions and Hamilton-Jacobi equations

Smoothness of extremal trajectories

Semiconcavity of solutions

Conclusions



value function of a Mayer problem

$t \leq T$ $x \in \mathbb{R}^N$ $y_{t,x,\alpha}(\cdot)$ solution

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in (t, T) \\ y(t) = x \end{cases}$$

where

- A compact
- $\alpha : [t, T] \rightarrow A$ measurable
- $f \in C(\mathbb{R}^N \times A; \mathbb{R}^N)$ continuous
 - ▶ $|f(x, a)| \leq C_0(1 + |x|)$
 - ▶ $|f(x, a) - f(y, a)| \leq C_1|x - y|$

given $\phi \in C(\mathbb{R}^N)$ define value function

$$V(t, x) = \inf_{\alpha} \phi(y_{t,x,\alpha}(T))$$



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an 'old' semiconcavity result

Theorem

Assume

- ϕ *semiconcave*
- $\|f_x(x, a) - f_x(y, a)\| \leq C_2|x - y|$

Then V **(linearly) semiconcave** $(-\infty, T] \times \mathbb{R}^N$

references

- ▶ C – Frankowska (1991)
- ▶ C – Sinestrari (2004)



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proof of semiconcavity of $V(t, \cdot)$

$$x, z \in \mathbb{R}^N$$

- ▶ α optimal at x
- ▶ $y(\cdot) = y_{t,x,\alpha}(\cdot)$, $y_{\pm}(\cdot) = y_{t,x \pm z, \alpha}(\cdot)$

by semiconcavity of ϕ and smoothness of flow

$$\begin{aligned} & V(t, x+z) + V(t, x-z) - 2V(t, x) \\ & \leq \phi(y_+(T)) + \phi(y_-(T)) - 2\phi(y(T)) \\ & = \underbrace{\phi(y_+(T)) + \phi(y_-(T)) - 2\phi\left(\frac{y_+(T) + y_-(T)}{2}\right)}_{\leq c|y_+(T) - y_-(T)|^2 \leq c|z|^2} \\ & \quad + 2 \left[\underbrace{\phi\left(\frac{y_+(T) + y_-(T)}{2}\right) - \phi(y(T))}_{\leq c|y_+(T) + y_-(T) - 2y(T)| \leq c|z|^2} \right] \end{aligned}$$



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parameterizations

- ▶ control system

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) & s \in (t, T) \\ y(t) = x \end{cases}$$

can be recast as differential inclusion

$$\begin{cases} \dot{y}(s) \in F(y(s)) & \text{a.e in } (t, T) \\ y(t) = x \end{cases}$$

defining

$$F(x) = \{f(x, a) : a \in A\} \quad (*)$$



- ▶ F can be parametrized as in $(*)$



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difficulty: smoothness of $f(\cdot, a)$

Example

$$F(x) = [-|x|, |x|] \quad (N = 1)$$

has the C^{1+} parametrization

$$F(x) = \{xu : |u| \leq 1\}$$

as well as the nonsmooth parametrization

$$F(x) = \{|x|u : |u| \leq 1\}$$

trajectories of the two systems coincide but known theorems asserting semiconcavity only apply to former parametrization



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Hamiltonian

$H : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ associated to F is defined by

$$H(x, p) = \sup_{v \in F(x)} v \cdot p$$

one-to-one correspondence between H and F

$$v \in F(x) \iff v \cdot p \leq H(x, p) \quad \forall p \in \mathbb{R}^N$$

Assumptions

- (H1) for all compact convex $K \subseteq \mathbb{R}^N$ there is a constant $c_K \geq 0$ so that $x \mapsto H(x, p)$ semiconvex on K with constant $c_K |p|$
- (H2) $\nabla_p H(x, p)$ locally Lipschitz in x uniformly for $|p| \neq 0$



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a 'new' semiconcavity result

given terminal cost $\varphi \in C(\mathbb{R}^N)$

define value function

$$V(t, x) = \inf \{ \varphi(y(T)) : y \in Y_T(t, x) \}$$

Theorem (C – Wolenski, 2011)

Assume

- ▶ F and H as above
- ▶ φ locally semiconcave

Then V locally semiconcave on $(-\infty, T] \times \mathbb{R}^N$

time optimal control

- ▶ C—, Marino and Wolenski
- ▶ C— and Nguyen



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proof of semiconcavity of $V(t, \cdot)$

- ▶ fix $x, z \in \mathbb{R}^N$ and let $y(\cdot)$ be **optimal** at x : want to show $V(t, x+z) + V(t, x-z) - 2V(t, x) \leq c|z|^2$
- ▶ invoke **maximum principle** to obtain adjoint arc

$$\begin{cases} -\dot{p}(s) \in \partial_x H(y(s), p(s)), & \dot{y}(s) \in \partial_p H(y(s), p(s)) \\ -p(T) \in \partial^P \varphi(y(T)) \end{cases}$$

- ▶ supposing $p(s) \neq 0$ define $y_{\pm}(\cdot)$ by

$$\begin{cases} \dot{y}_{\pm}(s) = \nabla_p H(y_{\pm}(s), p(s)) \\ y_{\pm}(t) = x \pm z \end{cases}$$

observe

- ▶ $|y_+(s) - y_-(s)| \leq c|z|$ and $|y_{\pm}(s) - y(s)| \leq c|z|$
- ▶ $p \cdot \dot{y} = H(y, p)$ and $p \cdot \dot{y}_{\pm} = H(y_{\pm}, p)$



► $V(t, x + z) + V(t, x - z) - 2V(t, x)$

$$\begin{aligned} &\leq \varphi(y_+(T)) + \varphi(y_-(T)) - 2\varphi(y(T)) \\ &= \underbrace{\varphi(y_+(T)) + \varphi(y_-(T)) - 2\varphi\left(\frac{y_+(T) + y_-(T)}{2}\right)}_{\leq c|y_+(T) - y_-(T)|^2 \leq c|z|^2} \\ &\quad + 2\left[\varphi\left(\frac{y_+(T) + y_-(T)}{2}\right) - \varphi(y(T))\right] \end{aligned}$$

► since $-p(T) \in \partial^P \varphi(y(T))$

$$\begin{aligned} &\varphi\left(\frac{y_+(T) + y_-(T)}{2}\right) - \varphi(y(T)) \\ &\leq -p(T) \cdot \left[\frac{y_+(T) + y_-(T)}{2} - y(T)\right] + c \underbrace{\left|\frac{y_+(T) + y_-(T)}{2} - y(T)\right|^2}_{\leq c|z|^2} \end{aligned}$$



- ▶ $-p(T) \cdot \left[\frac{y_+(T)+y_-(T)}{2} - y(T) \right]$
 $= \frac{1}{2} \int_t^T \left[-\dot{p} \cdot (y_+ + y_- - 2y) + p \cdot (2\dot{y} - \dot{y}_+ - \dot{y}_-) \right] ds$
- ▶ $\frac{1}{2} \int_t^T p \cdot (2\dot{y} - \dot{y}_+ - \dot{y}_-) ds$
 $= \int_t^T \left[H(y, p) - \frac{H(y_+, p) + H(y_-, p)}{2} \right] ds$
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 $+ \int_t^T \underbrace{\left[H\left(\frac{y_+ + y_-}{2}, p\right) - \frac{H(y_+, p) + H(y_-, p)}{2} \right]}_{\leq c|p||y_+ - y_-|^2 \leq c|z|^2} ds$
- ▶ $\int_t^T \left[H(y, p) - H\left(\frac{y_+ + y_-}{2}, p\right) \right] ds \quad (-\dot{p} \in \partial_x H(y, p)!)$
 $\leq \int_t^T \left[\dot{p} \cdot \left(\frac{y_+ + y_-}{2} - y \right) + \underbrace{c|p| \left| \frac{y_+ + y_-}{2} - y \right|^2}_{\leq c|z|^2} \right] ds$



nonsmooth HJ equations

$$\begin{cases} H(x, -Du(x)) = 1 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

Example

$$|A(x)Du(x)| = 1 \quad \text{in } \Omega$$

where

- ▶ $A : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times N}$ is α -Hölder continuous on \mathbb{R}^N
- ▶ $A(x)$ is invertible for all $x \in \mathbb{R}^N$, and

$$|A(x)| \leq C \quad \text{and} \quad |A(x)^{-1}| \leq C \quad \forall x \in \mathbb{R}^N$$



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assumptions on H

$\exists C_0, r, R$, and $\alpha \in (0, 1/2)$ such that

- ▶ $|H(x, p) - H(y, p)| \leq C_0 |x - y|^{2\alpha} |p| \quad \forall x, y, p \in \mathbb{R}^N$
- ▶ $H(x, \cdot)$ is convex, 1-homogeneous, and

$$r|p| \leq H(x, p) \leq R|p| \quad \forall x, p \in \mathbb{R}^N$$

- ▶ $H(x, \cdot)$ is continuously differentiable on $\mathbb{R}^N \setminus \{0\}$, and

$$\begin{aligned} & -\frac{1}{2r} |D_p H(x, q) - D_p H(x, p)|^2 \\ & \leq \langle D_p H(x, q) - D_p H(x, p), \frac{p}{|p|} \rangle \\ & \leq -\frac{1}{2R} |D_p H(x, q) - D_p H(x, p)|^2 \end{aligned}$$



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associated control problem

define

$$F(x) = \text{co} \{ D_p H(x, p) : p \in \mathbb{R}^N \setminus \{0\} \} \quad \forall x \in \mathbb{R}^N$$

then

$$\begin{cases} H(x, -Du(x)) = 1 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

if and only if

$$u(x) = \inf \left\{ t \geq 0 : \exists \begin{cases} y'(s) \in F(y(s)) & s \geq 0 \text{ a.e.} \\ y(0) = x, & y(t) \in \partial\Omega \end{cases} \right\}$$



associated control problem

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$$F(x) = \text{co} \{ D_p H(x, p) : p \in \mathbb{R}^N \setminus \{0\} \} \quad \forall x \in \mathbb{R}^N$$

then

$$\begin{cases} H(x, -Du(x)) = 1 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

if and only if

$$u(x) = \inf \left\{ t \geq 0 : \exists \begin{cases} y'(s) \in F(y(s)) & s \geq 0 \text{ a.e.} \\ y(0) = x, & y(t) \in \partial\Omega \end{cases} \right\}$$



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properties of $F(\cdot)$

- ▶ $F(x) \subset F(y) + C_0|x - y|^{2\alpha}B \quad \forall x, y \in \mathbb{R}^N$
- ▶ *curvature* estimates

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Mayer problem

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Semiconcave functions and Hamilton-Jacobi equations

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Semiconcavity of solutions

Conclusions



$C^{1,\alpha/2}$ regularity of extremal trajectories

$K \subset \mathbb{R}^N$ closed

$$\begin{cases} x'(t) \in F(x(t)) & t \geq 0 \text{ a.e.} \\ x(0) \in K \end{cases} \quad (DI)$$

reachable set (from K) in time t

$$\mathcal{R}(t) = \{x(t) : x(\cdot) \text{ is a trajectory of } (DI)\}$$

$x(\cdot)$ **extremal trajectory** on $[0, t]$ if $x(t) \in \partial \mathcal{R}(t)$

Theorem (C–Cardaliaguet)

$x(\cdot)$ extremal on $[0, T] \Rightarrow |x'(t) - x'(s)| \leq C|t - s|^{\alpha/2} \forall t, s$

$F(x) = c(x)B$

- ▶ Su and Berger (2006, 2007) $N = 2$
- ▶ Cardaliaguet, Ley and Monteillet



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'new' semiconcavity for HJ

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$$u(x+h) + u(x-h) - 2u(x) \leq C|h|^{1+\theta}$$



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conclusions

- ▶ **semiconcavity** is a useful property to study **nonsmooth** solutions of
 - ▶ optimal control problems
 - ▶ partial differential equations

not only for smooth data
- ▶ **extensions** of such a property (**φ -concavity, exterior ball property of hypograph...**) have been developed
- ▶ modelling control systems by **differential inclusions** provides intrinsic description which is independent of smoothness of parameterization
- ▶ **maximum principle** and **geometric properties of dynamics** (rather than smoothness of flow) should play an essential role in such generalized contexts



Thank you for your attention

