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Optimal control under probability constraint

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SADCO Workshop

March 3, 2011



Presentation outline

- 1 Problem formulation
- 2 Modeling improvement
- 3 Stochastic Arrow-Hurwicz algorithm
- 4 Numerical results

- 1 Problem formulation
 - Satellite model and deterministic optimization problem
 - Engine failure
 - Stochastic formulation
- 2 Modeling improvement
- 3 Stochastic Arrow-Hurwicz algorithm
- 4 Numerical results

Satellite model

$$\frac{dr}{dt} = v, \quad \frac{dv}{dt} = -\mu \frac{r}{\|r\|^3} + \frac{F}{m} \kappa, \quad (1a)$$

$$\frac{dm}{dt} = -\frac{T}{g_0 I_{sp}} \delta. \quad (1b)$$

(1a) : 6-dimensional state vector (position r and velocity v).

(1b) : 1-dimensional state vector (mass m including fuel).

κ involves the direction cosines of the thrust and the on-off switch δ of the engine (3 controls), and μ, F, T, g_0, I_{sp} are constants.

The deterministic control problem is to drive the satellite from the initial condition at t_i to a known **final position r_f and velocity v_f** at t_f (given) while **minimizing fuel consumption $m(t_i) - m(t_f)$** .

Deterministic optimization problem

Using **equinoctial coordinates** for the position and velocity

↷ **state vector** $x \in \mathbb{R}^7$,

and **cartesian coordinates** for the thrust of the engine

↷ **control vector** $u \in \mathbb{R}^3$,

the deterministic optimization problem is written as follows:

$$\min_{u(\cdot)} K(x(t_f)) \quad (2a)$$

subject to:

$$x(t_i) = x_i, \quad \dot{x}(t) = f(x(t), u(t)), \quad (2b)$$

$$\|u(t)\| \leq 1 \quad \forall t \in [t_i, t_f], \quad (2c)$$

$$C(x(t_f)) = 0. \quad (2d)$$

Engine failure

- Sometimes, the engine may **fail to work** when needed: the satellite **drift away** from the deterministic optimal trajectory. After the engine control is recovered, it is not always possible to **drive the satellite to the final target** at t_f .
- By **anticipating** such possible failures and by **modifying** the trajectory followed **before** any such failure occurs, one may **increase** the possibility of eventually reaching the target. But such a deviation from the deterministic optimal trajectory results in a **deterioration of the economic performance**.
- The problem is thus to **balance** the **increased probability** of eventually reaching the target despite possible failures against the **expected economic performance**, that is, to **quantify** the price of safety one is ready to pay for.

Stochastic formulation (1)

A failure is modeled using two random variables:

- \mathbf{t}_p : random initial time of the failure,
- \mathbf{t}_d : random duration of the failure.

For every realization (t_p^ξ, t_d^ξ) :

- 1 $u(\cdot)$ denotes the control used prior to any failure
 $\rightsquigarrow u$ is defined over $[t_i, t_f]$ but implemented over $[t_i, t_p^\xi]$
 and corresponds to an **open-loop control**,
- 2 the control is 0 in $[t_p^\xi, t_p^\xi + t_d^\xi]$,
- 3 $v^\xi(\cdot)$ denotes the control used after the end of the failure
 $\rightsquigarrow v^\xi$ is defined over $[t_p^\xi + t_d^\xi, t_f]$ (if nonempty)
 and corresponds to a **closed-loop strategy v**.

The **satellite dynamics** in the stochastic formulation writes:

$$x^\xi(t_i) = x_i, \quad \dot{x}^\xi(t) = f^\xi(x^\xi(t), u(t), v^\xi(t)).$$

Stochastic formulation (2)

The problem is to minimize the **expected cost** (fuel consumption)

- w.r.t. the open-loop control u and the closed-loop strategy \mathbf{v} ,
- the **probability to hit the target** at time t_f being at least p .

$$\min_{u(\cdot)} \mathbb{E} \left(\min_{\mathbf{v}^\xi(\cdot)} K(x^\xi(t_f)) \right) \quad (3a)$$

subject to:

$$x^\xi(t_i) = x_i, \quad \dot{x}^\xi(t) = f^\xi(x^\xi(t), u(t), \mathbf{v}^\xi(t)), \quad (3b)$$

$$\|u(t)\| \leq 1 \quad \forall t \in [t_i, t_f], \quad \|\mathbf{v}^\xi(t)\| \leq 1 \quad \forall t \in [t_p^\xi + t_d^\xi, t_f], \quad (3c)$$

$$\mathbb{P} \left(C(x^\xi(t_f)) = 0 \right) \geq p. \quad (3d)$$

- 1 Problem formulation
- 2 **Modeling improvement**
 - Probability as an expectation
 - Cost refinement
 - Final formulation
- 3 Stochastic Arrow-Hurwicz algorithm
- 4 Numerical results

Probability as an expectation

$$\text{Let } I(y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathbb{P}\left(C(x^\xi(t_f)) = 0\right) = \mathbb{E}\left(I(\|C(x^\xi(t_f))\|)\right).$$

Thus, Problem (3) can (shortly) be written:

$$\min_{u(\cdot)} \mathbb{E}\left(\min_{v^\xi(\cdot)} K(x^\xi(t_f))\right) \quad (4a)$$

$$\text{s.t. } \mathbb{E}\left(I(\|C(x^\xi(t_f))\|)\right) \geq p. \quad (4b)$$

Formulation (4) opens the possibility to make use of a **stochastic Arrow-Hurwicz algorithm** in order to obtain the solution $u^\sharp(\cdot)$.

Missing the target should not contribute to the cost

Whenever a failure occurred, setting $v^\xi \equiv 0$ over $[t_p^\xi + t_d^\xi, t_f]$

- generally makes the satellite miss the target and **does not contribute to the probability constraint**,
- but it forces fuel consumption to its minimum and **contributes to minimize the cost function**.

This yields an **artificially** good expected cost; however, one is not interested in what happens whenever the whole mission has failed.

Therefore, a better formulation is to register only **scenarios** when the target is **effectively hit**; instead of the original expected cost in (4), we thus prefer to deal with a **conditional expected cost**:

$$\mathbb{E} \left(K(x^\xi(t_f)) \mid C(x^\xi(t_f)) = 0 \right) = \frac{\mathbb{E} \left(K(x^\xi(t_f)) \times I(\|C(x^\xi(t_f))\|) \right)}{\mathbb{E} \left(I(\|C(x^\xi(t_f))\|) \right)}$$

Conditional expected cost

The problem is reformulated as

$$\min_{u(\cdot), v(\cdot)} \frac{\mathbb{E} \left(K(x^\xi(t_f)) \times \mathbb{I}(\|C(x^\xi(t_f))\|) \right)}{\mathbb{E} \left(\mathbb{I}(\|C(x^\xi(t_f))\|) \right)} \quad (5a)$$

$$\text{s.t. } \mathbb{E} \left(\mathbb{I}(\|C(x^\xi(t_f))\|) \right) \geq p. \quad (5b)$$

Such a formulation is however not well-suited for the stochastic Arrow-Hurwicz algorithm:

a ratio of expectations is not an expectation!

An useful lemma

Using compact notation, Problem (5) is:

$$\min_{\mathbf{u}} \frac{J(\mathbf{u})}{\Theta(\mathbf{u})} \quad \text{s.t.} \quad \Theta(\mathbf{u}) \geq p, \quad (6)$$

in which J and Θ assume positive values.

- ① If $\mathbf{u}^\#$ is a solution of (6) and if $\Theta(\mathbf{u}^\#) = p$, then $\mathbf{u}^\#$ is also a solution of

$$\min_{\mathbf{u}} J(\mathbf{u}) \quad \text{s.t.} \quad \Theta(\mathbf{u}) \geq p. \quad (7)$$

- ② Conversely, if $\mathbf{u}^\#$ is a solution of (7), and if an optimal Kuhn-Tucker multiplier $\beta^\#$ satisfies the condition

$$\beta^\# \geq \frac{J(\mathbf{u}^\#)}{\Theta(\mathbf{u}^\#)},$$

then $\mathbf{u}^\#$ is also a solution of (6).

Back to a cost in expectation

Finally, instead of (5) we aim at solving

$$\min_{u(\cdot), \mathbf{v}(\cdot)} \mathbb{E} \left(K(x^\xi(t_f)) \times \mathbb{I}(\|C(x^\xi(t_f))\|) \right) \quad (8a)$$

$$\text{s.t. } \mathbb{E} \left(\mathbb{I}(\|C(x^\xi(t_f))\|) \right) \geq p. \quad (8b)$$

The cost function again corresponds to a standard expectation!

Final formulation and associated Lagrangian

$$\min_{u(\cdot)} \left(\mathbb{E} \left(\min_{v^\xi(\cdot)} K(x^\xi(t_f)) \times I(\|C(x^\xi(t_f))\|) \right) \right)$$

$$\text{s.t. } \mu \rightsquigarrow p - \mathbb{E} \left(I(\|C(x^\xi(t_f))\|) \right) \leq 0.$$

Assuming there exists a saddle point for the associated Lagrangian, we have finally to solve

$$\max_{\mu \geq 0} \min_{u(\cdot)} \left\{ \mu p + \mathbb{E} \left(\min_{v^\xi(\cdot)} (K(x^\xi(t_f)) - \mu) \times I(\|C(x^\xi(t_f))\|) \right) \right\}.$$

Remark: it is convenient to put apart the **no-failure event**, namely $\{t_p \geq t_f\}$ in the problem formulation. For the sake of simplicity, we do not present this last problem improvement. We will denote by π_f the probability of this event.

- 1 Problem formulation
- 2 Modeling improvement
- 3 **Stochastic Arrow-Hurwicz algorithm**
 - Algorithm overview
 - Downstream closed-loop problem
 - Smoothing function I
 - Solving the resulting approximated problem
- 4 Numerical results

Algorithm overview (1)

In order to solve

$$\max_{\mu \geq 0} \min_{u(\cdot)} \left\{ \mu p + \underbrace{\mathbb{E} \left(\min_{v^\xi(\cdot)} (K(x^\xi(t_f)) - \mu) \times I(\|C(x^\xi(t_f))\|) \right)}_{W(u, \mu, \xi)} \right\}.$$

that is,

$$\max_{\mu \geq 0} \min_{u(\cdot)} \left\{ \mu p + \mathbb{E}(W(u, \mu, \xi)) \right\},$$

we use a **stochastic Arrow-Hurwicz algorithm** (see [2] and [3]).

Algorithm overview (2)

Arrow-Hurwicz algorithm

At iteration k ,

- 1 draw a failure $\xi^k = (t_p^{\xi^k}, t_d^{\xi^k})$ according to its probability law,
- 2 update $u(\cdot)$:

$$u^{k+1} = \Pi_{\mathcal{B}} \left(u^k - \varepsilon^k \nabla_u W(u^k, \mu^k, \xi^k) \right),$$

- 3 update μ :

$$\mu^{k+1} = \max \left(0, \mu^k + \rho^k (\rho + \nabla_\mu W(u^{k+1}, \mu^k, \xi^k)) \right).$$

Downstream closed-loop problem

Consider the inner optimization problem:

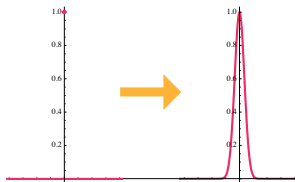
$$W(u, \xi, \mu) = \min_{v^{\xi}(\cdot)} \left\{ (K(x^{\xi}(t_f)) - \mu) \times \mathbf{I}(\|C(x^{\xi}(t_f))\|) \right\},$$

- If reaching the target is **impossible** ($C(x^{\xi}(t_f)) \neq 0 \forall v^{\xi}$), the best thing to do is to **stop immediately**: $W \equiv 0$.
- If reaching the target is **possible** but **too expensive** (that is $K(x^{\xi}(t_f)) \geq \mu$), the best thing to do is again to **stop immediately**: $W \equiv 0$.
- Therefore, it must be checked that one can reach the target while maintaining fuel consumption below a threshold, and the corresponding optimal control $v_*^{\xi}(\cdot)$ must be computed.

At every iteration k , we must **evaluate** function W as well as **its derivatives** w.r.t. $u(\cdot)$ and μ . But W is not differentiable!

First difficulty: I is not a smooth function

$$I(y) = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{otherwise,} \end{cases} \rightsquigarrow I_r(y) = \begin{cases} \left(1 - \frac{y^2}{r^2}\right)^2 & \text{if } y \in [-r, r], \\ 0 & \text{otherwise.} \end{cases}$$



There are specific rules to drive r^k to 0 as the iteration number k goes to infinity in order to obtain the best asymptotic **Mean Quadratic Error** of the gradient estimates (see [1]).

Second difficulty: solving the approximated problem

The approximated closed-loop problem to solve at each iteration is:

$$W_r(u^k, \xi^k, \mu^k) = \min_{v^\xi(\cdot)} \left\{ (K(x^\xi(t_f)) - \mu^k) \times \mathbf{I}_r(\|C(x^\xi(t_f))\|) \right\}.$$

In this setting, we have to check if the target is reached **up to r** .

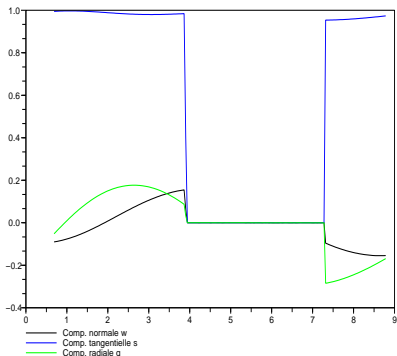
In practice, the solution of the approximated problem is derived from the resolution of two standard optimal control problems.

- 1 Problem formulation
- 2 Modeling improvement
- 3 Stochastic Arrow-Hurwicz algorithm
- 4 Numerical results**
 - Brief description
 - Results for different values of p
 - The price of safety

Mission description

Interplanetary mission (Earth-Mars trajectory):

- duration of the mission: 450 days,
- \mathbf{t}_p : exponential distribution s.t. $\mathbb{P}(\mathbf{t}_p \geq t_f) = \pi_f \approx 0.58$,
- \mathbf{t}_d : exponential distribution s.t. $\mathbb{P}(2 \leq \mathbf{t}_d \leq 7) \approx 0.80$.



Using normalized units:

- $t_i = 0.69$ and $t_f = 8.73$.

The **deterministic optimal control** has a “bang–off–bang” shape. Along the **deterministic optimal path**, the probability to recover a failure is:

$$p^{\text{det}} \approx 0.94.$$

Parameters tuning

Gradient step length:

$$\varepsilon^k = \frac{a}{b+k} \quad , \quad \rho^k = \frac{c}{d+k} \quad ,$$

↪ usual for a stochastic gradient algorithm.

Smoothing parameter:

$$r^k = \frac{\alpha}{\beta + k^{\frac{1}{3}}} \quad ,$$

↪ MQE reduced by a factor 2000 in about 100.000 iterations.

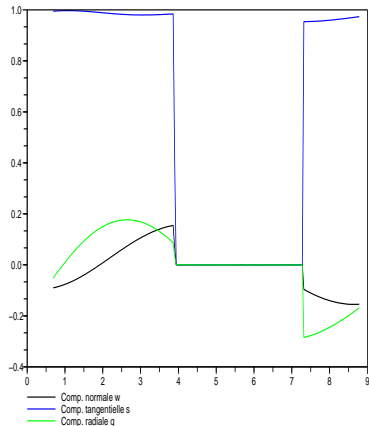
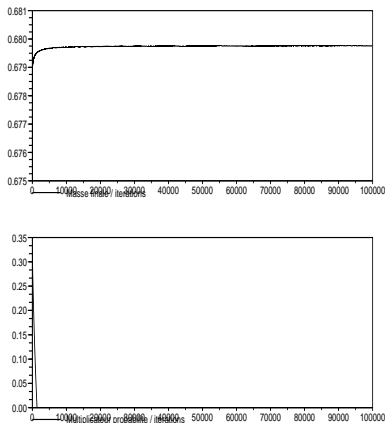


Figure: Probability level $p < \pi_f$

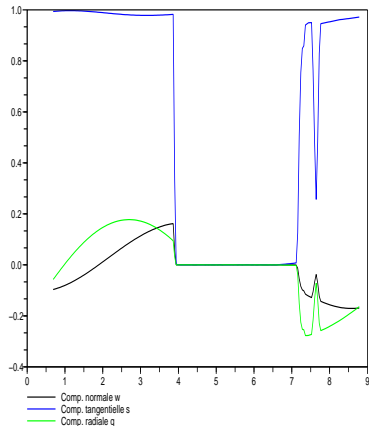
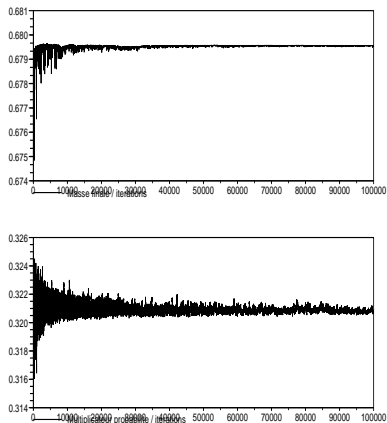


Figure: Probability level $\rho = 0.750$

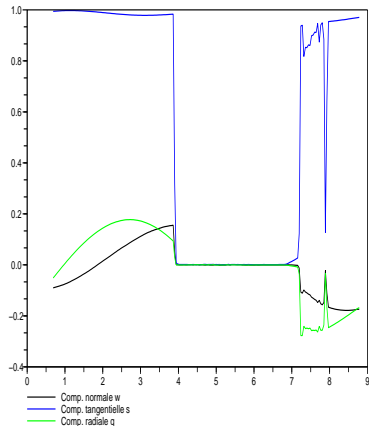
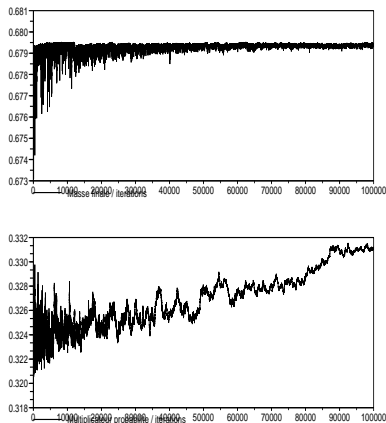


Figure: Probability level $\rho = 0.960$

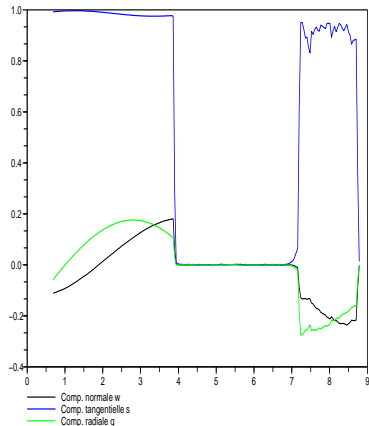
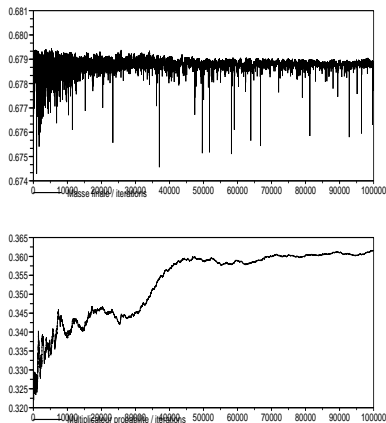
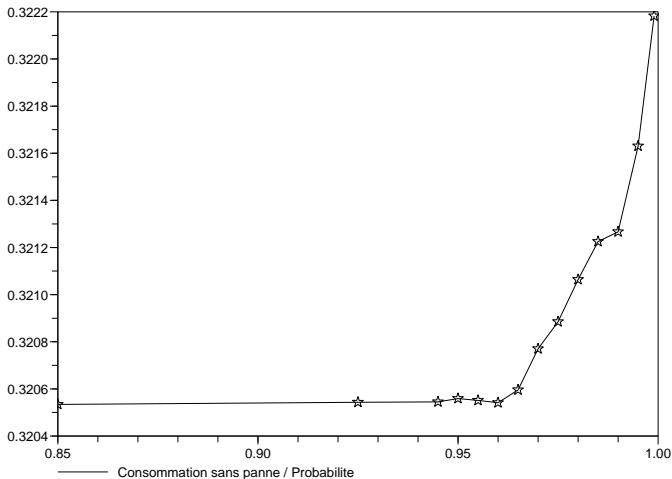


Figure: Probability level $\rho = 0.990$

Fuel consumption versus probability level



Conclusion

Main conclusion

We are able to deal with probability constraints in the optimal control framework.

Future works

- *From the theoretical point of view:*
 - existence of a saddle point for the constrained problem,
 - smoothing process (results available only for inequality constraints).
- *From the numerical point of view:*
 - efficient solver for the downstream problem,
 - computer parallelization.



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Thank you for your attention. Any question?

