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Optimal control techniques for reachable set computations

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Optimal control techniques for reachable set computations

Matthias Gerds
joint work with Robert Baier



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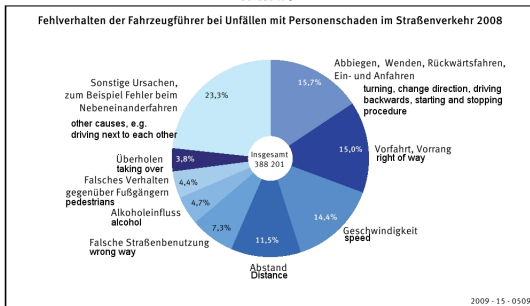
- 1 Motivation: Driver Assistance Systems
- 2 Nonlinear Control Problems and Reachable Sets
- 3 Discrete Approximations of Reachable Sets using Optimal Control
- 4 Numerical Examples

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Safety by Driver Assistance Systems

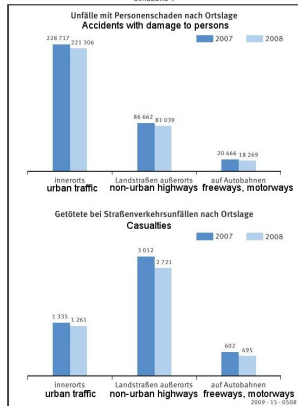
Statistics (source: Statistisches Bundesamt, www.destatis.de)

Schaubild 5



Goal: development of driver assistance systems that help to reduce severeness of accidents

Schaubild 4



Safety by Driver Assistance Systems

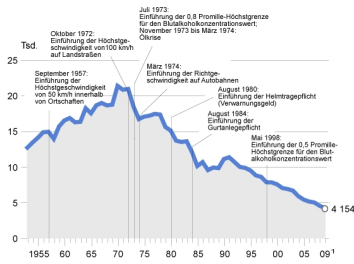
Passive safety systems:

chassis, airbags, seat belts, seat belt tightener, ...

Driver assistance systems in use:

- anti-blocking system (ABS), braking assistant (BAS), active brake assist in trucks (ABA)
- anti-slip regulation (ASR)
- electronic stability control (ESC,ESP,DSC,...)
- adaptive cruise control (ACC)
- lane departure warning (LDW), blind spot intervention (BSI)
- ...

Entwicklung der Zahl der im Straßenverkehr Getöteten 1953 bis 2009



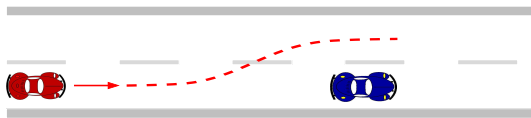
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Future driver assistance systems:

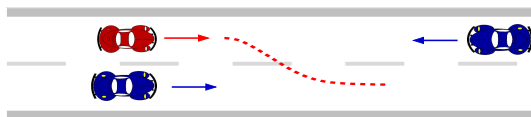
collision avoidance,
active steering,
car-to-car
communication,...

Scenarios

Scenario 1: avoiding an obstacle (time to collision 0.5...2 s)



Scenario 2: overtaking maneuver



Questions: Can a collision be avoided at all? If yes, how?

Decision Making based on Reachable Sets

Once an obstacle has been detected by suitable sensors (e.g. radar, lidar), **can a collision be avoided?** Approaches:

- compute an (optimal) trajectory to a secure target state
- compute (projected) reachable set from initial position
- compute backward oriented (projected) reachable set starting from a secure target state

Decision Making based on Reachable Sets

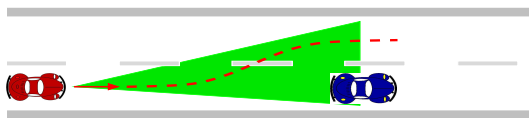
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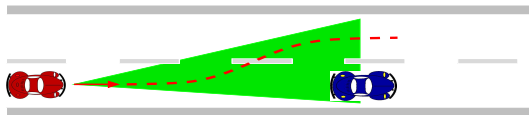


- compute backward oriented (projected) reachable set starting from a secure target state

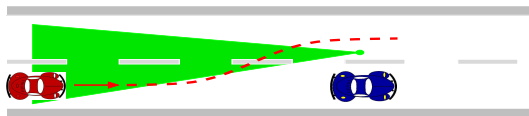
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Reachable Sets

Let $t_0 < t_f$, $\mathcal{U} \neq \emptyset$ convex, compact, and $X_0 \subseteq \mathbb{R}^n$ be given.

Control problem

For a given $u \in L^\infty([t_0, t_f], \mathbb{R}^m)$ find $x \in W^{1,\infty}([t_0, t_f], \mathbb{R}^n)$ with

$$\begin{aligned}x'(t) &= f(t, x(t), u(t)) && \text{a.e. in } [t_0, t_f] \\x(t_0) &\in X_0 \\ \psi(x(t_f)) &= 0 \\ s(t, x(t)) &\leq 0 && \text{in } [t_0, t_f] \\ u(t) &\in \mathcal{U} && \text{a.e. in } [t_0, t_f]\end{aligned}$$

Reachable set at t :

$$\mathcal{R}(t, t_0, X_0) := \{y \in \mathbb{R}^n \mid \exists u(\cdot) \text{ control function and } \exists x(\cdot) \text{ corresponding solution of control problem with } x(t) = y\}$$

Approximation of Reachable Sets

Approaches:

- set-valued integration schemes [Baier'95]
- **optimal control techniques** [Varaiya'00, Baier et al.'07]
- external and inner ellipsoidal techniques [Kurzhanski and Varaiya'00,'01,'02]
- estimation methods [Gajek'86]
- discretization methods for nonlinear problems with state constraints [Chahma'03, Beyn an Rieger'07]
- level set methods using Hamilton-Jacobi equations [Mitchell'07,'08]
- ...

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Discrete Reachable Sets (Euler Discretization)

Approximation on grid with stepsize $h = (t_f - t_0)/N$:

grid functions $u_h(t_i) = u_i$, $x_h(t_i) = x_i$, $i = 0, \dots, N$

Discrete Control Problem (Euler)

For a discretized control function $u_h(\cdot)$ find a solution $x_h(\cdot)$ with

$$x_h(t_{i+1}) = x_h(t_i) + hf(t_i, x_h(t_i), u_h(t_i)), \quad i = 0, 1, \dots, N-1$$

$$x_h(t_0) \in X_0$$

$$\psi(x_h(t_N)) = 0$$

$$s(t_i, x_h(t_i)) \leq 0, \quad i = 0, 1, \dots, N$$

$$u_h(\cdot) \in \mathcal{U}_h$$

Discrete reachable set at t_i :

$$\mathcal{R}_h(t_i, t_0, X_0) := \{y \in \mathbb{R}^n \mid \exists u_h(\cdot) \text{ discretized control function and} \\ \exists x_h(\cdot) \text{ corresponding solution} \\ \text{with } x_h(t_i) = y\}$$

Convergence

Differential inclusion:

$$x'(t) \in F(t, x(t)), \quad F(t, x) := \bigcup_{u \in \mathcal{U}} \{f(t, x, u)\}$$

Theorem 1 (Dontchev/Farkhi'89)

Let $F : I \times \mathbb{R}^n \Rightarrow \mathbb{R}^n$ be Lipschitz with compact, convex, nonempty images, *no boundary conditions, no state constraints*. Then:

$$d_H(\mathcal{R}(T, t_0, x_0), \mathcal{R}_h(T, t_0, x_0)) \leq C_1 h.$$

Hausdorff distance:

$$d_H(S, \tilde{S}) = \max\{d(S, \tilde{S}), d(\tilde{S}, S)\}, \quad d(S, \tilde{S}) = \sup_{s \in S} \text{dist}(s, \tilde{S})$$

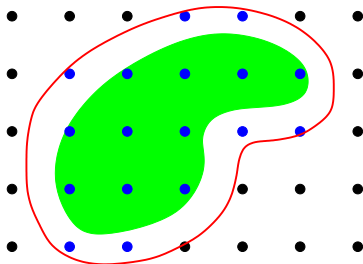
Approximation of Discrete Reachable Set I

Strategy 1: collect all grid points with distance $\frac{\sqrt{n}}{2}\rho$

$$\mathcal{R}_\rho^1 := \bigcup_{g_\rho \in \mathbb{G}_\rho} \{g_\rho\} \\ \text{dist}(g_\rho, \mathcal{R}_h) \leq \frac{\sqrt{n}}{2} \cdot \rho$$

To be computed:

$$\text{dist}(g_\rho, \mathcal{R}_h) = \inf_{r \in \mathcal{R}_h} \|g_\rho - r\|$$



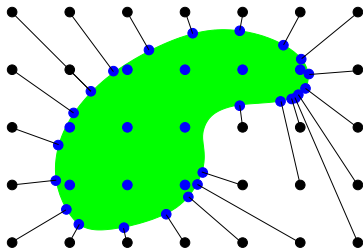
Approximation of Discrete Reachable Set II

Strategy 2: collect all best approximations

$$\mathcal{R}_\rho^2 := \bigcup_{\substack{g_\rho \in \mathbb{G}_\rho \\ \hat{s}_\rho \in \Pi_{\mathcal{R}_h}(g_\rho)}} \{\hat{s}_\rho\}$$

To be computed:

$$\hat{s}_\rho \in \Pi_{\mathcal{R}_h}(g_\rho) = \{r \in \mathcal{R}_h : \|g_\rho - r\| = \text{dist}(g_\rho, \mathcal{R}_h)\}$$



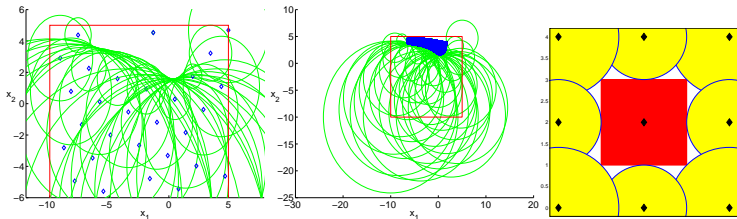
Approximation of Discrete Reachable Set III

Strategy 3: outer approximation by complement of open balls

$$\mathcal{R}_\rho^3 := \mathbb{R}^n \setminus \bigcup_{g_\rho \in \rho \mathbb{Z}^n} \text{int } B_{\text{dist}(g_\rho, \mathcal{R}_h)}(g_\rho)$$

To be computed:

$$\text{dist}(g_\rho, \mathcal{R}_h) = \inf_{r \in \mathcal{R}_h} \|g_\rho - r\|$$



Computing Distance and Best Approximations

Algorithm:

- Choose a region $\mathbb{G} \subseteq \mathbb{R}^n$ and cover \mathbb{G} by a grid \mathbb{G}_h with step-size h (or h^p)
- For every $g_h \in \mathbb{G}_h$ solve (discretized) optimal control problem:

$$\begin{array}{ll} \text{Min} & \frac{1}{2} \|x(t_f) - g_h\|_2^2 \\ \text{s.t.} & x'(t) = f(t, x(t), u(t)) \\ & x(t_0) \in X_0 \\ & \psi(x(t_f)) = 0 \\ & s(t, x(t)) \leq 0 \\ & u(t) \in \mathcal{U} \end{array} \quad (OCP_{g_h})$$

Solution: $x^*(\cdot; g_h)$ and $u^*(\cdot; g_h)$

- Reachable set approximation (relative to \mathbb{G}_h) according to one of the three strategies.

Error Estimates without state constraints and boundary conditions [Baier,G.'09]

Strategy 1 (points with small distance):

$$d_H(\mathcal{R}, \mathcal{R}_\rho^1) \leq (C_1 + \frac{\sqrt{n}}{2} \cdot C_2) \cdot h$$

Strategy 2 (best approximations):

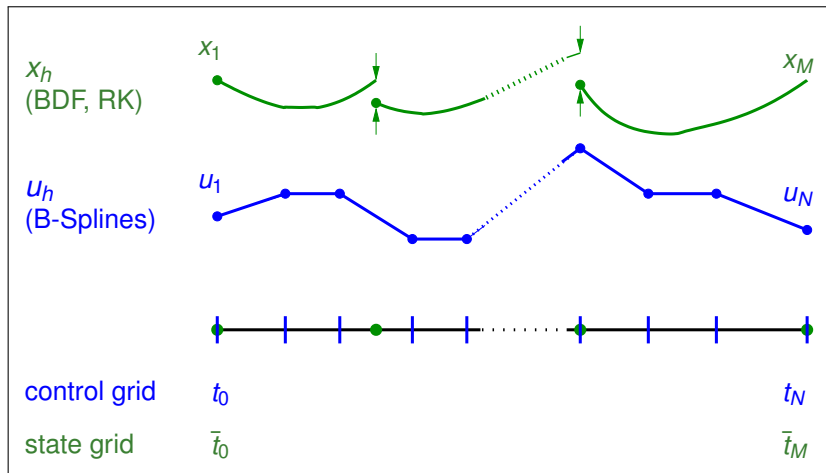
$$d_H(\mathcal{R}, \mathcal{R}_\rho^2) \leq (C_1 + \sqrt{n} \cdot C_2) \cdot h$$

Strategy 3 (complement of open balls):

$$d_H(\mathcal{R}, \mathcal{R}_\rho^3) \leq (C_1 + \sqrt{n} \cdot C_2) \cdot h$$

Assumptions: f lipschitz w.r.t. t , C^1 w.r.t. x , C^0 w.r.t. u , $\emptyset \neq \mathcal{U}$ convex, compact, $\rho = C_2 \cdot h$, Euler discretization, **no boundary conditions, no state constraints**

Direct Shooting



Direct Shooting

DOCP

$$\begin{aligned} \text{Minimize} \quad & \varphi(x_h(0; z), x_h(1; z)) \\ \text{s.t.} \quad & c(x_h(t_i; z), y_h(t_i; z), u_h(t_i; z)) \leq 0, \quad \forall i, \\ & s(x_h(t_i; z)) \leq 0, \quad \forall i, \\ & \psi(x_h(0; z), x_h(1; z)) = 0, \\ & x_h(\bar{t}_j; z) - x_j = 0, \quad \forall j \end{aligned}$$

Structure

$$\begin{aligned} M = 1 \quad (\text{single shooting}) \quad & : \quad \text{small \& dense;} \\ & z = (x_1, u_1, \dots, u_N) \\ M > 1 \quad (\text{multiple shooting}) \quad & : \quad \text{large-scale \& sparse;} \\ & z = (x_1, \dots, x_{M-1}, u_1, \dots, u_N) \end{aligned}$$

Numerical Solution

Software **OC-ODE**, **OC-DAE1**, **SODAS** [G.]:

- **direct multiple shooting** discretization
- **SQP method** (non-monotone linesearch, BFGS update, primal active-set QP solver)
- various **integrators** (Runge-Kutta, BDF methods, linearized Runge-Kutta methods)
- various **control approximations** (B-splines of order k)
- gradients by **sensitivity differential equation**
- **sensitivity analysis** and **adjoint estimation**
- extensions to adjoint gradient computation and **mixed-integer optimal control** problems

Large-scale problems:

www.worhp.de (academic licenses available)

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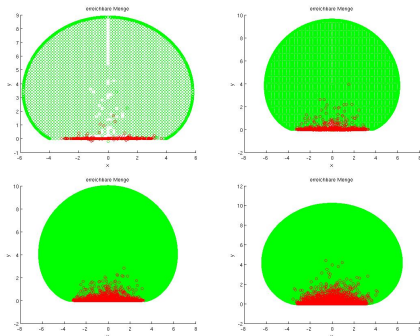
Example 1: Brachistochrone

Control problem: $(t \in [0, 1], u(t) \in [-\pi, \pi])$

$$x'(t) = \sqrt{2gy(t)} \cos(u(t)), \quad x(0) = 0$$

$$y'(t) = \sqrt{2gy(t)} \sin(u(t)), \quad y(0) = 1$$

Reachable sets for $N = 5, 10, 20, 40$:



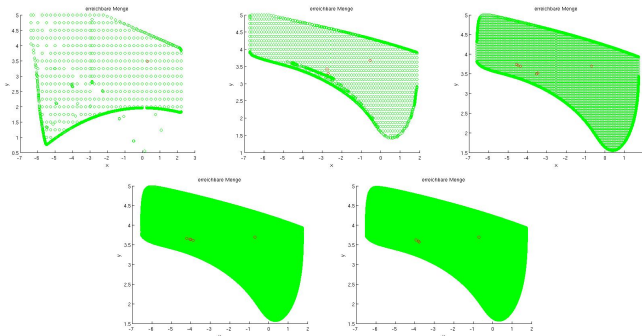
Example 2: Rayleigh problem

Control problem: ($t \in [0, 2.5]$, $u(t) \in [-1, 1]$)

$$x'(t) = y(t), \quad x(0) = -5$$

$$y'(t) = -x(t) + y(t)(1.4 - 0.14y(t)^2) + 4u(t), \quad y(0) = -5$$

Reachable sets for $N = 10, 20, 40, 80, 160$:



Example 3: Kenderov

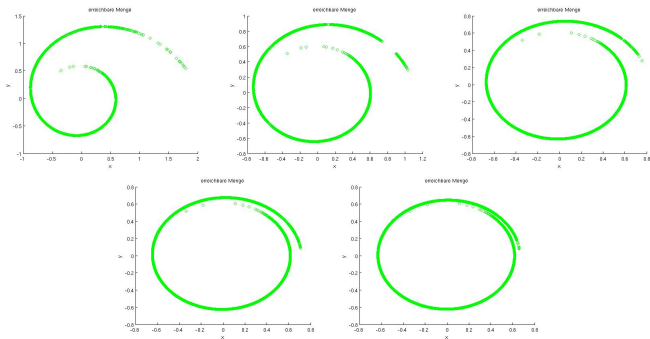
Control problem: ($t \in [0, 1]$, $u(t) \in [-1, 1]$)

$$x'(t) = 8(a_{11}x(t) + a_{12}y(t) - 2a_{12}y(t)u(t)), \quad x(0) = 2$$

$$y'(t) = 8(-a_{12}x(t) + a_{11}y(t) + 2a_{12}x(t)u(t)), \quad y(0) = 2$$

$$(a_{11} = \sigma^2 - 1, a_{12} = \sigma\sqrt{1 - \sigma^2}, \sigma = 0.9)$$

Reachable sets for $N = 20, 40, 80, 160, 320$:



Example 3: Kenderov

Control problem: $(t \in [0, 1], u(t) \in [-1, 1])$

$$x'(t) = 8(a_{11}x(t) + a_{12}y(t) - 2a_{12}y(t)u(t)), \quad x(0) = 2$$

$$y'(t) = 8(-a_{12}x(t) + a_{11}y(t) + 2a_{12}x(t)u(t)), \quad y(0) = 2$$

$(a_{11} = \sigma^2 - 1, a_{12} = \sigma\sqrt{1 - \sigma^2}, \sigma = 0.9)$

CPU times:

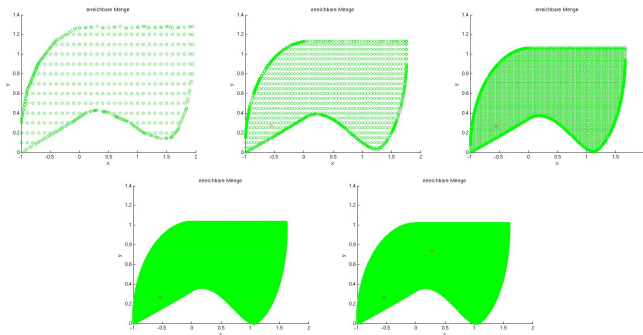
N	CPU User full	CPU User adaptive
20	0m1.296s	0m0.152s
40	0m14.313s	0m0.752s
80	3m54.151s	0m5.980s
160	86m48.758s	1m6.528s
320	2802m35469s	21m23.856s

Example 4: Bilinear

Control problem: ($t \in [0, 1]$, $u(t) \in [0, 1]$)

$$\begin{aligned}x'(t) &= \pi y(t), & x(0) &= -1 \\y'(t) &= -\pi u(t)x(t), & y(0) &= 0\end{aligned}$$

Reachable sets for $N = 10, 20, 40, 80, 160$:



Example 4: Bilinear

Control problem: ($t \in [0, 1]$, $u(t) \in [0, 1]$)

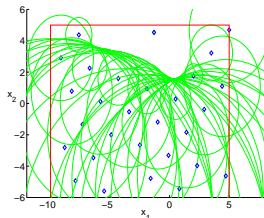
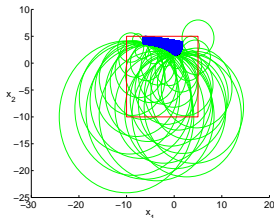
$$\begin{aligned}x'(t) &= \pi y(t), & x(0) &= -1 \\y'(t) &= -\pi u(t)x(t), & y(0) &= 0\end{aligned}$$

CPU times:

N	CPU User full	CPU User adaptive
10	0m0.404s	0m0.268s
20	0m5.016s	0m2.224s
40	1m35.818s	0m34.526s
80	38m34.489s	13m14.846s
160	1204m35.461s	457m28.067s

Idea:

- Let g_h be a grid point and $x^*(t_f; g_h)$ an optimal solution.
- Every grid point within the ball $B_r(g_h)$ and radius $r = \|x^*(t_f; g_h) - g_h\|$ is **not reachable** and thus needs not to be projected.



Potential Advantages and Extensions

Advantages:

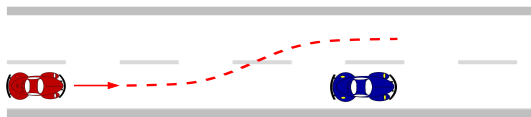
- approximation of reachable sets with higher order methods
- zooming into interesting sub-regions possible
- state-space grid $O(h)$ only once and not $O(h^2)$ in each Euler step as in Chahma'03, Rieger'07
- adaptivity possible
- easy to parallelize
- state and control constraints and terminal conditions can be considered

Drawbacks:

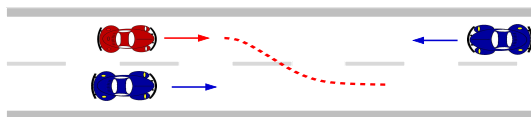
- high computation effort for higher dimensions
- need for global solutions of OCP

Scenarios

Scenario 1: avoiding an obstacle (time to collision 0.5...2 s)



Scenario 2: overtaking maneuver



Questions: Can a collision be avoided at all? If yes, how?

Optimal Trajectory to a Secure Target State

Objective: (t_f = free final time)

$$(y(t_f) - y_{target})^2 \rightarrow \min$$

Single track model:

$$x'' = (F_x \cos(\psi) - F_y \sin(\psi)) / m$$

$$y'' = (F_x \sin(\psi) + F_y \cos(\psi)) / m$$

$$\psi'' = (\ell_v F_{sv} \cos(\delta) - \ell_h F_{sh} + \ell_v F_{uv} \sin(\delta)) / I_z$$

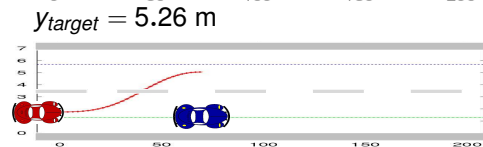
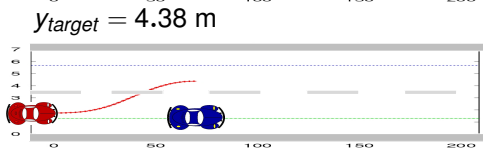
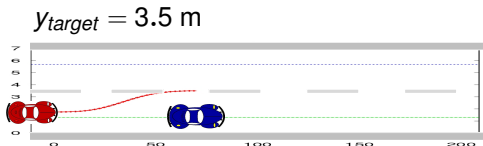
$$\delta' = w_\delta$$



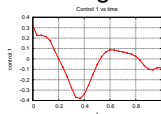
Constraints: initial conditions and

- (a) **state constraints:** $1.3 \leq y(t) \leq 5.7$ (stay on road),
 $\|(F_{sv}, F_{uv})\| \leq F_{max,v}$, $\|(F_{sh}, F_{uh})\| \leq F_{max,h}$ (Kamm's circle)
- (b) **boundary conditions:** $x(t_f) = d$ (d = initial distance to obstacle),
 $y'(t_f) = 0$ (no velocity in y-direction when passing obstacle)
- (c) **control constraints:** $w_{\delta,min} \leq w_\delta \leq w_{\delta,max}$ (steering velocity),
 $F_{B,min} \leq F_B \leq F_{B,max}$ (braking force)

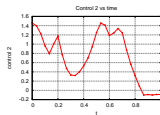
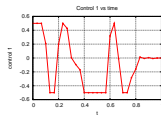
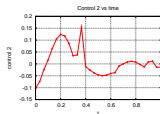
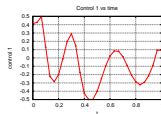
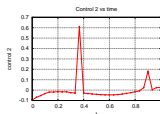
Optimal Trajectory to a Secure Target State



steering vel.



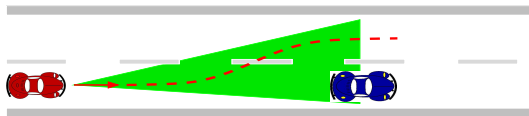
brake force



Data: car width 2.6 m, road width 7 m, initial y-position of car 1.75 m, distance 70 m, velocity 200 km/h, CPU: 0.05 s - 0.07 s

Scenario 1: Avoiding an Obstacle

Scenario 1: avoiding an obstacle (time to collision 0.5...2 s)



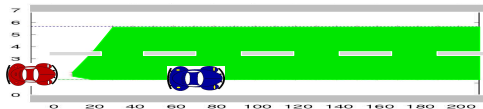
Projected reachable set at distance d :

$$\mathcal{PR}(d) := \left\{ \hat{y} \in \mathbb{R} \mid \exists \text{ final time } t_f > 0, \text{ controls } w_\delta, F_B, \right. \\ \left. \text{and states } x, y, \psi, \delta \text{ such that} \right. \\ \left. \text{dynamics and constraints are satisfied} \right. \\ \left. \text{and } \hat{y} = y(t_f), x(t_f) = d, y'(t_f) = 0 \right\}$$

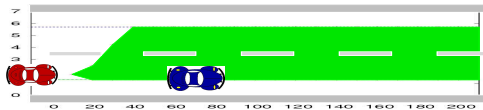
Scenario 1: Avoiding an Obstacle

Projected reachable sets (green) for different initial velocities:

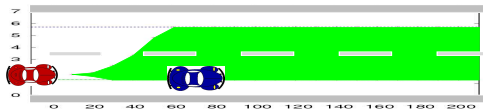
$v(0) = 75 \text{ km/h}$



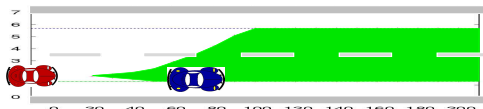
$v(0) = 100 \text{ km/h}$



$v(0) = 150 \text{ km/h}$

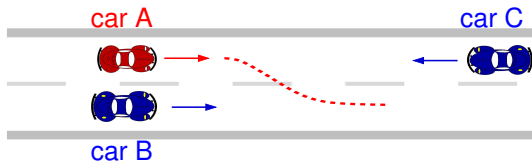


$v(0) = 250 \text{ km/h}$



Data: car width 2.6 m, road width 7 m, initial y-position of car 1.75 m

Scenario 2: Overtaking Maneuver



Difficulty: Additional (potentially infeasible) state constraints

$$(x_A(t) - x_B(t))^2 + (y_A(t) - y_B(t))^2 \geq B^2 \quad (\text{don't hit car B})$$

$$(x_A(t) - x_C(t))^2 + (y_A(t) - y_C(t))^2 \geq B^2 \quad (\text{don't hit car C})$$

(B = car width)

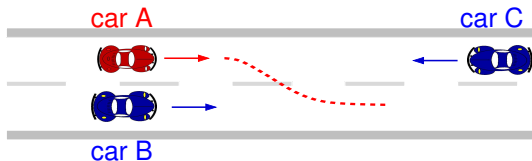
Approach: Minimize constraint violation α !

$$(x_A(t) - x_B(t))^2 + (y_A(t) - y_B(t))^2 + \alpha \geq B^2$$

$$(x_A(t) - x_C(t))^2 + (y_A(t) - y_C(t))^2 + \alpha \geq B^2$$

Collision detection: If $\alpha_{opt} > 0$, collision cannot be avoided!

Scenario 2: Overtaking Maneuver



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(B = car width)

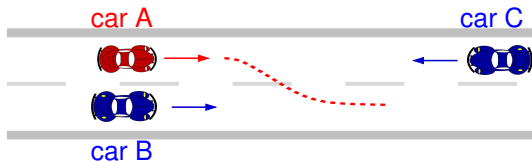
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Collision detection: If $\alpha_{opt} > 0$, collision cannot be avoided!

Scenario 2: Overtaking Maneuver



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$$(x_A(t) - x_C(t))^2 + (y_A(t) - y_C(t))^2 \geq B^2 \quad (\text{don't hit car C})$$

(B = car width)

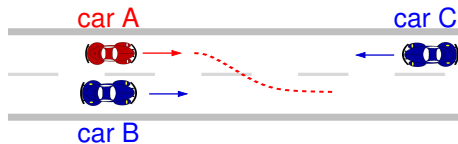
Approach: Minimize constraint violation α !

$$(x_A(t) - x_B(t))^2 + (y_A(t) - y_B(t))^2 + \alpha \geq B^2$$

$$(x_A(t) - x_C(t))^2 + (y_A(t) - y_C(t))^2 + \alpha \geq B^2$$

Collision detection: If $\alpha_{opt} > 0$, collision cannot be avoided!

Scenario 2: Results of feasibility problem



Data:

- car A: 100 km/h, car B: 75 km/h, car C: 100 km/h
- car width 2.6 m, road width 7 m
- initial y-position of car A : 5.25 m
initial y-position of car B : 1.75 m
initial y-position of car C : 5.25 m

CPU: 2.67 s on average (53.43 s for 20 feasibility problems with 81 grid points)

init. dist. [m]	con. violation [m]	collision
10	0.24780E+01	yes
20	0.22789E+01	yes
30	0.21355E+01	yes
40	0.19351E+01	yes
50	0.94517E-01	yes
60	0.74140E-08	no
70	0.73879E-08	no
80	0.82019E-08	no
90	0.74505E-08	no
100	0.74506E-08	no
⋮	⋮	⋮
200	0.74760E-08	no

SADCO: Optimal control approaches to reachability analysis industrial partner: Volkswagen

- computation of **driver friendly** controls for active steering driver assistance systems (change of objective function)
- backward oriented reachable sets and reachable sets for overtaking maneuvers
- more complicated road geometries
- real-time capability
- dependence on (sensor) perturbations: **sensitivity and robustness of methods**
- incorporation of statistical data: **propagation of probabilities**

Thanks for your attention!



Questions?



Further information:

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