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# NECESSARY CONDITIONS FOR CONSTRAINED OPTIMAL CONTROL PROBLEMS

Maria Do Rosário de Pinho

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**NECESSARY CONDITIONS  
FOR  
CONSTRAINED  
OPTIMAL CONTROL PROBLEMS**

Maria do Rosário de Pinho



0. The UP Team

1. Constrained Optimal Control Problems

2. Challenges on Necessary Conditions

 3. New developments on Mixed Constrained Problems



JBSousa



Sofia  
Lopes

Fernando  
Fontes

Margarida  
Ferreira

Fernando  
Lobo  
Pereira

Gueorgiu  
Smirnov



Rosário  
de Pinho

Delfim  
F. M. Torres

Paula  
Malonek



JBSousa



JBSousa





$$\text{Min } l(a, x(a), b, x(b))$$

s.t.

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.}$$

$$(\mathbf{x}(t), \mathbf{u}(t)) \in \mathbf{S}(t) \quad \text{a.e.}$$

$$(a, x(a), b, x(b)) \in E$$

$$u(t) \in U(t)$$

$$\begin{cases} x(t) \in X(t) \\ u(t) \in U(t) \end{cases}$$

If  $l(x(a), x(b))$  and  $(x(a), x(b)) \in E$ , FIXED TIME PROBLEM.



## **NONSMOOTH PROBLEMS**

1. Nondegenerated Necessary Conditions for State Constrained Problems
2. Refining Necessary Conditions
  - weak assumptions and/or stronger conclusions
  - easily translated in computational schemes
  - special interest on state constrained problems
3. Necessary Conditions for various non standard problems
  - Hybrid optimal control, Free time problems,
  - infinite horizon problems, nonregular mixed constrained problems
5. Relations between sets of Necessary Conditions
6. Applications vs Theory

## Necessary conditions under “minimal assumptions”

Based on Clarke’s monograph 2005.

Multifunction  $\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}^m$  multifunction.

Graph of  $\Gamma$  :  $G \subset \mathbb{R}^n \times \mathbb{R}^m$ .

**Definition:**  $\Gamma$  satisfies the **bounded slope condition** near  $(\bar{x}, \bar{v})$  (with parameters  $\varepsilon, R, K$ ), if  $x \in B(\bar{x}, \varepsilon), v \in B(\bar{v}, R), (\alpha, \beta) \in N_G^P(x, v) \implies |\alpha| \leq K|\beta|$ .

If  $\Gamma$  is **pseudo Lipschitz**, then it satisfies the **bounded slope condition**.



$$\left\{ \begin{array}{l} \text{Min } l(x(a), x(b)) \\ \text{s.t.} \\ \dot{x}(t) \in F(t, x(t)) \quad \text{a.e.} \\ (x(a), x(b)) \in E \end{array} \right.$$

•  $R : [a, b] \rightarrow ]0, +\infty]$  a measurable function.

$\bar{x}$  **local minimum of radius  $R$**  if

$$\exists \epsilon > 0 : |\dot{x}(t) - \dot{\bar{x}}(t)| \leq R(t) \text{ a.e. ,}$$

$$\int_a^b |\dot{x}(t) - \dot{\bar{x}}(t)| dt \leq \epsilon, \quad \|x - \bar{x}\|_\infty \leq \epsilon$$

we have  $l(x(a), x(b)) \geq l(\bar{x}(a), \bar{x}(b))$

## Necessary Conditions

$$\left\{ \begin{array}{l} (\lambda, p(t)) \neq 0 \\ \dot{p}(t) \in \text{co} \left\{ w : (w, p(t)) \in N_{G(t)}^L(\bar{x}(t), \dot{\bar{x}}(t)) \right\} \text{ a.e.} \\ \langle p(t), v \rangle \leq \langle p(t), \dot{\bar{x}}(t) \rangle \quad \forall v \in F(t, \bar{x}(t)) \cap \bar{B}(\dot{\bar{x}}(t), R(t)) \text{ a.e.} \\ (p(a), -p(b)) \in \partial^L \lambda l(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)) \end{array} \right.$$





This result is a main tool in the derivation of refinements of necessary conditions for various problems.

For example:

(New) Nonsmooth Maximum Principle

- ✓ For standard optimal control problems—for convex-linear case it is a sufficient and necessary condition
- ✓ For general mixed constrained problems including problems with DAE
  - For problems with state constraints ?



Necessary conditions for general free time problems



$$\left\{ \begin{array}{l} \text{Min } l(x(a), x(b)) \\ \text{s.t.} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.} \\ u(t) \in U(t) \quad \text{a.e.} \\ (x(a), x(b)) \in E \end{array} \right.$$

Assuming Lipschitz continuity  
of  $f$  wrt  $(x,u)$  jointly

**NEW NONSMOOTH MAXIMUM PRINCIPLE**

$\exists p, \lambda_0 :$

$$(\lambda_0, p(t)) \neq 0,$$

$$(-\dot{p}(t), 0) \in \partial_{x,u}^C \langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle + \{0\} \times N_{U(t)}^C(\bar{u}(t))$$

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle = \max_{u \in U(t)} \langle p(t), f(t, \bar{x}(t), u) \rangle$$

$$(p(a), -p(b)) \in \lambda_0 \partial^L l(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)).$$

## MIXED CONTRAINED O.C. P.

Min  $l(x(a), x(b))$

s.t.

$\dot{x}(t) \in F(t, x(t))$  a.e.

$(x(a), x(b)) \in E$

$$F(t, x) = \{v : v = f(t, x, u), (x, u) \in S(t)\}$$

Necessary conditions for such problems called for some **regularity assumptions** on the mixed constraints

**For Example:**

If  $S(t) = \{(x, u) : b(t, x, u) = 0\}$ ,

then assume that  $\det B(t)B^T(t) > K$ , where  $B(t) = \nabla_u b(t, x(t), u(t))$

called full rank conditions

## MIXED CONSTRAINED O.C. P.

Min  $l(x(a), x(b))$

s.t.

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.}$$

$$(x(a), x(b)) \in E$$

$$F(t, x) = \{v : v = f(t, x, u), (x, u) \in S(t)\}$$

$$S_*^\varepsilon(t) = \{(x, u) \in S(t) : |x - \bar{x}(t)| \leq \varepsilon, |u - \bar{u}| \leq R(t)\}$$

### MAJOR HYPOTHESES

(i)  $S(t)$  locally closed at  $(x, u) \in S_*^\varepsilon(t)$ .

(ii)  $\exists k_x^f, k_u^f \in L^1 :$

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq k_x^f(t)|x_1 - x_2| + k_u^f(t)|u_1 - u_2|,$$

$$\forall (x_i, u_i) \in S_*^\varepsilon(t)$$

(iii)  $\exists K_S \in L^1 : (x, u) \in S_*^\varepsilon(t), (\alpha, \beta) \in N_{S(t)}^P(x, u) \Rightarrow |\alpha| \leq K_S(t)|\beta|.$

$$\text{Min } l(x(a), x(b))$$

s.t.

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.}$$

$$(x(a), x(b)) \in E$$

$$F(t, x) = \{v : v = f(t, x, u), (x, u) \in S(t)\}$$

### NEW NECESSARY CONDITIONS

$$\exists p, \lambda_0 :$$

$$(i) (\lambda_0, p(t)) \neq 0,$$

$$(ii) (-\dot{p}(t), 0) \in \partial_{x,u}^C \langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle + N_{S(t)}^C(\bar{x}(t), \bar{u}(t)),$$

$$(iii) \quad \forall (\bar{x}(t), u) \in S(t), |u - \bar{u}(t)| \leq R(t)$$
$$\langle p(t), f(t, \bar{x}(t), u) \rangle \leq \langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle,$$

$$(iv) (p(a), -p(b)) \in \lambda_0 \partial_L l(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)).$$



IMPORTANCE OF SUCH RESULT:

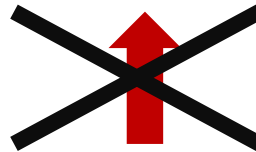
Necessary conditions in the form of a

maximum principle

under somewhat

minimal assumptions

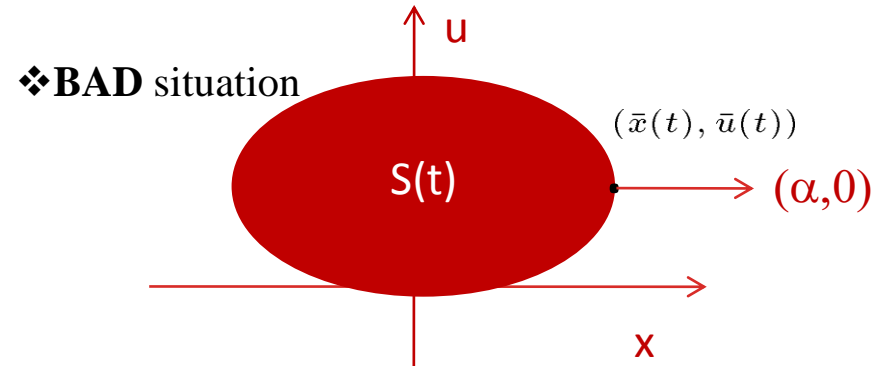
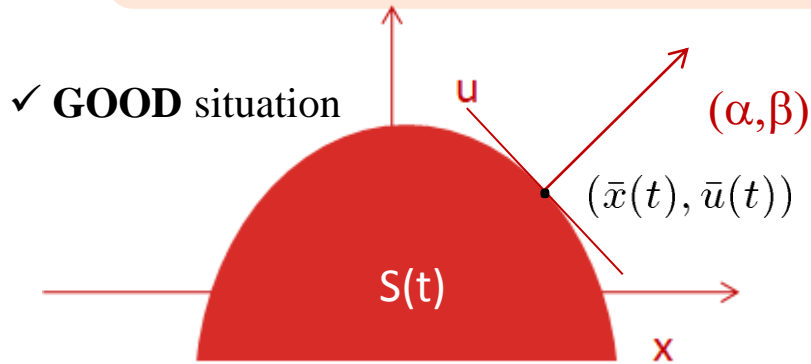
KNOWN REGULARITY ASSUMPTIONS ON  $S(t)$



BOUNDED SLOPE CONDITION ON  $S(t)$

**THE BOUNDED SLOPE CONDITION**

$$\exists K_S \in L^1 : (x, u) \in S_*^\varepsilon(t), (\alpha, \beta) \in N_{S(t)}^P(x, u) \Rightarrow |\alpha| \leq K_S(t)|\beta|.$$

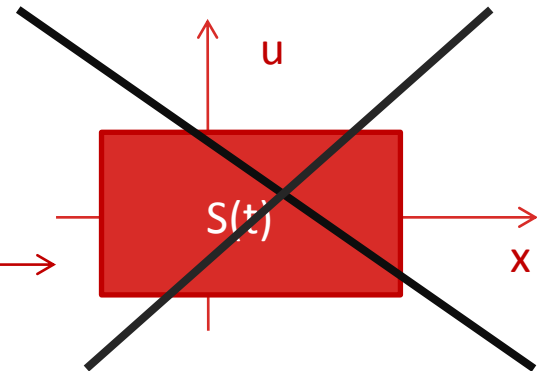


**Some sets S(t)**

$$S(t) = \{(x, u) : \varphi(t, x, u) \in \Phi(t)\}$$

$$S(t) = \{(x, u) : b(t, x, u) = 0, g(t, x, u) \leq 0, u \in U(t)\}$$

~~$$S(t) = \{(x, u) : u(t) \in U(t), x(t) \in X(t)\}$$~~





**SPECIAL CASEs**

In many problems we have state-control constraints with "two" controls:

$$u(t) = (v(t), w(t)), \quad (x(t), v(t)) \in S(t), \quad w(t) \text{ unconstrained.}$$

Then the **bounded slope condition** is imposed only on  $S(t)$ .

**PROBLEMS WITH DIFFERENTIAL AND ALGEBRAIC EQUATIONS**

**The Problem**

$$\text{Min } l(x(a), x(b))$$

s.t.

$$h(t, x(t), u(t), \dot{x}(t)) = 0 \quad \text{a.e.}$$

$$u(t) \in U(t) \quad \text{a.e.}$$

$$(x(a), x(b)) \in E$$

**"Bounded Slope" condition**

$$(x, u, v) \in S_*^\varepsilon(t), \quad \lambda \in \mathbb{R}^N, \quad \mu \in N_{U(t)}L(\bar{u}(t),$$

$$(\alpha, \beta - \mu, \gamma) \in \partial^L \langle h(t, x, u, v), \lambda \rangle$$

$$\Rightarrow |\alpha| \leq M(t)|(\beta, \gamma)|.$$



## Euler Inclusion:

$$(-\dot{p}(t), 0) \in \partial_{x,u}^C \langle p(t), \bar{f}(t, \bar{x}(t), \bar{u}(t)) \rangle + N_{S(t)}^C(\bar{x}(t), \bar{u}(t))$$

With

$$S(t) = \{(x, u) : u \in U, g(t, x, u) \leq 0, b(t, x, u) = 0\}$$

one would hope to have some **MULTIPLIERS** in the Euler inclusion instead of the Normal cone.

This is possible if one assumes  $g$  and  $b$  to be continuously differentiable, for example.

# Assuming some smoothness...

$$\exists K_S \in L^1 : (x, u) \in S_*^\varepsilon(t), (\alpha, \beta) \in N_{S(t)}^P(x, u) \Rightarrow |\alpha| \leq K_S(t)|\beta|.$$

**Regularity Assumption**

$$(x, u) \in S_*^\varepsilon(t) : \text{all } \lambda, \gamma \text{ such that } \gamma \geq 0, \langle \gamma, g(t, x, u) \rangle = 0,$$

$$D_u \langle (\lambda, \gamma), (b, g) \rangle (t, x, u) \in N_U^L(u) \implies \lambda = 0, \gamma = 0$$



## Euler Inclusion: Nonsmooth to Smooth

$$S(t) = \{(x, u) : u \in U(t), g(t, x, u) \leq 0, b(t, x, u) = 0\}$$

**Nonsmooth  
b and g:**



$$(-\dot{p}(t), 0) \in \partial_{x,u}^C \langle p(t), \bar{f}(t) \rangle + N_{S(t)}^C(\bar{x}(t)(t), \bar{u}(t))$$

$$(-\dot{p}(t), 0) \in D_{x,u} \langle (p, \lambda, \gamma), (\bar{f}, -\bar{b}, -\bar{g}) \rangle + \{0\} \times N_{U(t)}^C(\bar{u}(t))$$

**Smooth b and g:**





**Drawback:**  $|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq k_x^f(t)|x_1 - x_2| + k_u^f(t)|u_1 - u_2|$

**Question:** Any alternative regularity assumption for the bounded slope condition and w/o Lipschitz continuity wrt  $u$ ?

**Answer:** in some cases, YES. Roughly speaking we get NC assuming, for

$$S(t) = \{(x, u) : \varphi(t, x, u) \in \Phi(t)\}, \quad u(t) \in U(t)$$

that the following holds

$$B(\varphi(t, \bar{x}(t), \bar{u}(t)), \delta) \subset \varphi(t, \bar{x}(t), U(t))$$

**COVERING  
CONDITION**

but we also need to assume that

$$\{(f(x, u), \varphi(t, x, u)) : u \in U(t)\} \text{ is convex.}$$



## Open Problems

- ❖ For mixed (state-control) constraints  
Identify other situations where multipliers can be stated in terms of multipliers and not the normal cone to the constrained set.
- ❖ Extensions to state constrained problems and problems with both state and mixed constraints.
- ❖ Determination of NC for general Free Time problems



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