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# NECESSARY CONDITIONS FOR CONSTRAINED OPTIMAL CONTROL PROBLEMS

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**NECESSARY CONDITIONS  
FOR  
CONSTRAINED  
OPTIMAL CONTROL PROBLEMS**

Maria do Rosário de Pinho



0. The UP Team

1. Constrained Optimal Control Problems

2. Challenges on Necessary Conditions

 3. New developments on Mixed Constrained Problems



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$$\text{Min } l(a, x(a), b, x(b))$$

s.t.

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.}$$

$$(\mathbf{x}(t), \mathbf{u}(t)) \in \mathbf{S}(t) \quad \text{a.e.}$$

$$(a, x(a), b, x(b)) \in E$$

$$u(t) \in U(t)$$

$$\begin{cases} x(t) \in X(t) \\ u(t) \in U(t) \end{cases}$$

If  $l(x(a), x(b))$  and  $(x(a), x(b)) \in E$ , FIXED TIME PROBLEM.



## **NONSMOOTH PROBLEMS**

1. Nondegenerated Necessary Conditions for State Constrained Problems
2. Refining Necessary Conditions
  - weak assumptions and/or stronger conclusions
  - easily translated in computational schemes
  - special interest on state constrained problems
3. Necessary Conditions for various non standard problems
  - Hybrid optimal control, Free time problems,
  - infinite horizon problems, nonregular mixed constrained problems
5. Relations between sets of Necessary Conditions
6. Applications vs Theory

## Necessary conditions under “minimal assumptions”

Based on Clarke’s monograph 2005.

Multifunction  $\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}^m$  multifunction.

Graph of  $\Gamma$  :  $G \subset \mathbb{R}^n \times \mathbb{R}^m$ .

**Definition:**  $\Gamma$  satisfies the **bounded slope condition** near  $(\bar{x}, \bar{v})$  (with parameters  $\varepsilon, R, K$ ), if  $x \in B(\bar{x}, \varepsilon), v \in B(\bar{v}, R), (\alpha, \beta) \in N_G^P(x, v) \implies |\alpha| \leq K|\beta|$ .

If  $\Gamma$  is **pseudo Lipschitz**, then it satisfies the **bounded slope condition**.



$$\left\{ \begin{array}{l} \text{Min } l(x(a), x(b)) \\ \text{s.t.} \\ \dot{x}(t) \in F(t, x(t)) \quad \text{a.e.} \\ (x(a), x(b)) \in E \end{array} \right.$$

•  $R : [a, b] \rightarrow ]0, +\infty]$  a measurable function.

$\bar{x}$  **local minimum of radius  $R$**  if

$$\exists \epsilon > 0 : |\dot{x}(t) - \dot{\bar{x}}(t)| \leq R(t) \text{ a.e. ,}$$

$$\int_a^b |\dot{x}(t) - \dot{\bar{x}}(t)| dt \leq \epsilon, \quad \|x - \bar{x}\|_\infty \leq \epsilon$$

we have  $l(x(a), x(b)) \geq l(\bar{x}(a), \bar{x}(b))$

## Necessary Conditions

$$\left\{ \begin{array}{l} (\lambda, p(t)) \neq 0 \\ \dot{p}(t) \in \text{co} \left\{ w : (w, p(t)) \in N_{G(t)}^L(\bar{x}(t), \dot{\bar{x}}(t)) \right\} \text{ a.e.} \\ \langle p(t), v \rangle \leq \langle p(t), \dot{\bar{x}}(t) \rangle \quad \forall v \in F(t, \bar{x}(t)) \cap \bar{B}(\dot{\bar{x}}(t), R(t)) \text{ a.e.} \\ (p(a), -p(b)) \in \partial^L \lambda l(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)) \end{array} \right.$$



This result is a main tool in the derivation of refinements of necessary conditions for various problems.

For example:

(New) Nonsmooth Maximum Principle

- ✓ For standard optimal control problems—for convex-linear case it is a sufficient and necessary condition
- ✓ For general mixed constrained problems including problems with DAE
  - ❑ For problems with state constraints ?



Necessary conditions for general free time problems



$$\left\{ \begin{array}{l} \text{Min } l(x(a), x(b)) \\ \text{s.t.} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.} \\ u(t) \in U(t) \quad \text{a.e.} \\ (x(a), x(b)) \in E \end{array} \right.$$

Assuming Lipschitz continuity  
of  $f$  wrt  $(x,u)$  jointly

**NEW NONSMOOTH MAXIMUM PRINCIPLE**

$\exists p, \lambda_0 :$

$$(\lambda_0, p(t)) \neq 0,$$

$$(-\dot{p}(t), 0) \in \partial_{x,u}^C \langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle + \{0\} \times N_{U(t)}^C(\bar{u}(t))$$

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle = \max_{u \in U(t)} \langle p(t), f(t, \bar{x}(t), u) \rangle$$

$$(p(a), -p(b)) \in \lambda_0 \partial^L l(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)).$$



## MIXED CONTRAINED O.C. P.

Min  $l(x(a), x(b))$   
s.t.

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.}$$

$$(x(a), x(b)) \in E$$

$$F(t, x) = \{v : v = f(t, x, u), (x, u) \in S(t)\}$$

Necessary conditions for such problems called for some **regularity assumptions** on the mixed constraints

**For Example:**

If  $S(t) = \{(x, u) : b(t, x, u) = 0\}$ ,

then assume that  $\det B(t)B^T(t) > K$ , where  $B(t) = \nabla_u b(t, x(t), u(t))$

called full rank conditions

## MIXED CONSTRAINED O.C. P.

Min  $l(x(a), x(b))$

s.t.

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.}$$

$$(x(a), x(b)) \in E$$

$$F(t, x) = \{v : v = f(t, x, u), (x, u) \in S(t)\}$$

$$S_*^\varepsilon(t) = \{(x, u) \in S(t) : |x - \bar{x}(t)| \leq \varepsilon, |u - \bar{u}| \leq R(t)\}$$

### MAJOR HYPOTHESES

(i)  $S(t)$  locally closed at  $(x, u) \in S_*^\varepsilon(t)$ .

(ii)  $\exists k_x^f, k_u^f \in L^1 :$

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq k_x^f(t)|x_1 - x_2| + k_u^f(t)|u_1 - u_2|,$$

$$\forall (x_i, u_i) \in S_*^\varepsilon(t)$$

(iii)  $\exists K_S \in L^1 : (x, u) \in S_*^\varepsilon(t), (\alpha, \beta) \in N_{S(t)}^P(x, u) \Rightarrow |\alpha| \leq K_S(t)|\beta|.$



$$\text{Min } l(x(a), x(b))$$

s.t.

$$\dot{x}(t) \in F(t, x(t)) \quad \text{a.e.}$$

$$(x(a), x(b)) \in E$$

$$F(t, x) = \{v : v = f(t, x, u), (x, u) \in S(t)\}$$

### NEW NECESSARY CONDITIONS

$$\exists p, \lambda_0 :$$

$$(i) (\lambda_0, p(t)) \neq 0,$$

$$(ii) (-\dot{p}(t), 0) \in \partial_{x,u}^C \langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle + N_{S(t)}^C(\bar{x}(t), \bar{u}(t)),$$

$$(iii) \quad \forall (\bar{x}(t), u) \in S(t), |u - \bar{u}(t)| \leq R(t)$$

$$\langle p(t), f(t, \bar{x}(t), u) \rangle \leq \langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle,$$

$$(iv) (p(a), -p(b)) \in \lambda_0 \partial_L l(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)).$$



IMPORTANCE OF SUCH RESULT:

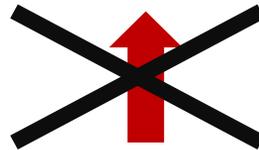
Necessary conditions in the form of a

maximum principle

under somewhat

minimal assumptions

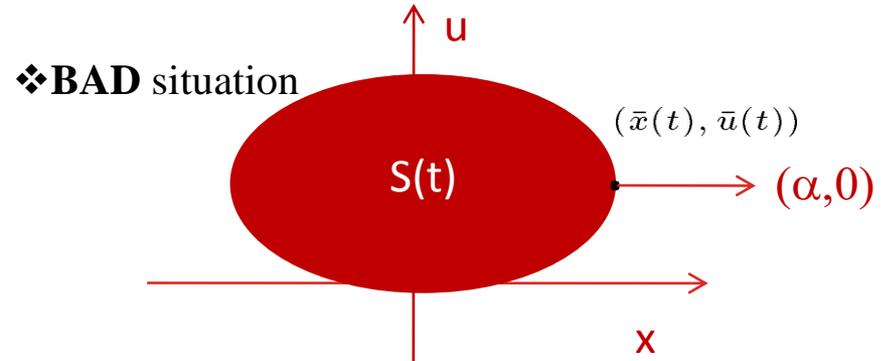
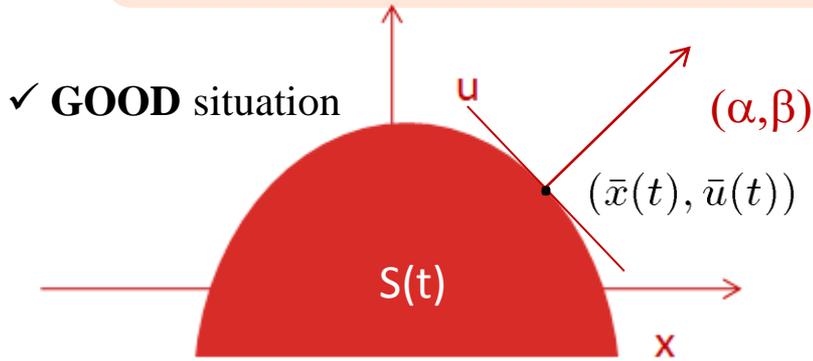
KNOWN REGULARITY ASSUMPTIONS ON  $S(t)$



BOUNDED SLOPE CONDITION ON  $S(t)$

**THE BOUNDED SLOPE CONDITION**

$$\exists K_S \in L^1 : (x, u) \in S_*^\varepsilon(t), (\alpha, \beta) \in N_{S(t)}^P(x, u) \Rightarrow |\alpha| \leq K_S(t)|\beta|.$$

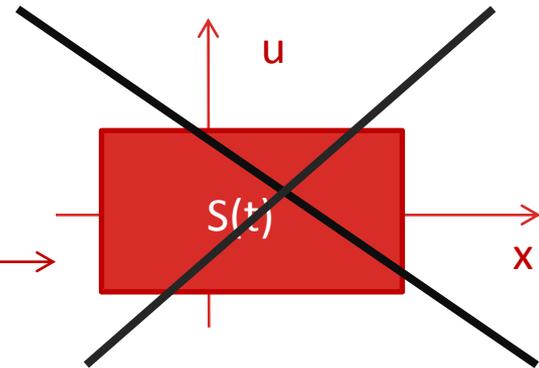


**Some sets S(t)**

$$S(t) = \{(x, u) : \varphi(t, x, u) \in \Phi(t)\}$$

$$S(t) = \{(x, u) : b(t, x, u) = 0, g(t, x, u) \leq 0, u \in U(t)\}$$

~~$$S(t) = \{(x, u) : u(t) \in U(t), x(t) \in X(t)\}$$~~





**SPECIAL CASEs**

In many problems we have state-control constraints with "two" controls:

$$u(t) = (v(t), w(t)), \quad (x(t), v(t)) \in S(t), \quad w(t) \text{ unconstrained.}$$

Then the **bounded slope condition** is imposed only on  $S(t)$ .

**PROBLEMS WITH DIFFERENTIAL AND ALGEBRAIC EQUATIONS**

**The Problem**

$$\text{Min } l(x(a), x(b))$$

s.t.

$$h(t, x(t), u(t), \dot{x}(t)) = 0 \quad \text{a.e.}$$

$$u(t) \in U(t) \quad \text{a.e.}$$

$$(x(a), x(b)) \in E$$

**"Bounded Slope" condition**

$$(x, u, v) \in S_*^\varepsilon(t), \quad \lambda \in \mathbb{R}^N, \quad \mu \in N_{U(t)}L(\bar{u}(t),$$

$$(\alpha, \beta - \mu, \gamma) \in \partial^L \langle h(t, x, u, v), \lambda \rangle$$

$$\Rightarrow |\alpha| \leq M(t)|(\beta, \gamma)|.$$

## Euler Inclusion:

$$(-\dot{p}(t), 0) \in \partial_{x,u}^C \langle p(t), \bar{f}(t, \bar{x}(t), \bar{u}(t)) \rangle + N_{S(t)}^C(\bar{x}(t), \bar{u}(t))$$

With

$$S(t) = \{(x, u) : u \in U, g(t, x, u) \leq 0, b(t, x, u) = 0\}$$

one would hope to have some **MULTIPLIERS** in the Euler inclusion instead of the Normal cone.

This is possible if one assumes  $g$  and  $b$  to be continuously differentiable, for example.

# Assuming some smoothness...

$$\exists K_S \in L^1 : (x, u) \in S_*^\varepsilon(t), (\alpha, \beta) \in N_{S(t)}^P(x, u) \Rightarrow |\alpha| \leq K_S(t)|\beta|.$$

**Regularity Assumption**

$$(x, u) \in S_*^\varepsilon(t) : \text{all } \lambda, \gamma \text{ such that } \gamma \geq 0, \langle \gamma, g(t, x, u) \rangle = 0,$$

$$D_u \langle (\lambda, \gamma), (b, g) \rangle (t, x, u) \in N_U^L(u) \implies \lambda = 0, \gamma = 0$$



## Euler Inclusion: Nonsmooth to Smooth

$$S(t) = \{(x, u) : u \in U(t), g(t, x, u) \leq 0, b(t, x, u) = 0\}$$

**Nonsmooth  
b and g:**



$$(-\dot{p}(t), 0) \in \partial_{x,u}^C \langle p(t), \bar{f}(t) \rangle + N_{S(t)}^C(\bar{x}(t)(t), \bar{u}(t))$$

$$(-\dot{p}(t), 0) \in D_{x,u} \langle (p, \lambda, \gamma), (\bar{f}, -\bar{b}, -\bar{g}) \rangle + \{0\} \times N_{U(t)}^C(\bar{u}(t))$$

**Smooth b and g:**





**Drawback:**  $|f(t, x_1, u_1) - f(t, x_2, u_2)| \leq k_x^f(t)|x_1 - x_2| + k_u^f(t)|u_1 - u_2|$

**Question:** Any alternative regularity assumption for the bounded slope condition and w/o Lipschitz continuity wrt  $u$ ?

**Answer:** in some cases, YES. Roughly speaking we get NC assuming, for

$$S(t) = \{(x, u) : \varphi(t, x, u) \in \Phi(t)\}, \quad u(t) \in U(t)$$

that the following holds

$$B(\varphi(t, \bar{x}(t), \bar{u}(t)), \delta) \subset \varphi(t, \bar{x}(t), U(t))$$

**COVERING  
CONDITION**

but we also need to assume that

$$\{(f(x, u), \varphi(t, x, u)) : u \in U(t)\} \text{ is convex.}$$



## Open Problems

- ❖ For mixed (state-control) constraints  
Identify other situations where multipliers can be stated in terms of multipliers and not the normal cone to the constrained set.
- ❖ Extensions to state constrained problems and problems with both state and mixed constraints.
- ❖ Determination of NC for general Free Time problems



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