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# Model Predictive Control of Nonholonomic Vehicle Formations

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ITN-SADCO 1st Industrial workshop and kick-off meeting  
Paris, March 3, 2011

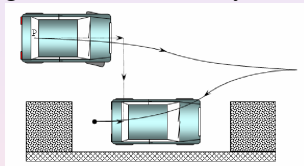
- 1 The Problem
  - Challenges
  - Control of Nonholonomic Systems
- 2 The MPC framework
- 3 Control of vehicle formations
- 4 Path-following
- 5 Mission objective





## Which are the main challenges of this problem?

**nonholonomy** simple controllers (with continuous feedback) cannot, in general, stabilize the system.



**constraints** input saturation, collision avoidance (between vehicles and with obstacles).

**distributed control** stability of the formation, communication delays or failure, computational burden in each agent.

## Why is MPC a good tool for this problem?

- **MPC can overcome nonholonomy challenges**
  - It involves *planning*, not just reactive control.
  - Can generate required nonlinear, discontinuous feedback.
- **MPC can overcome constraints challenges**
  - It is known to be a (if not the main) technique to deal appropriately with constraints.
  - Simply dealt with within the optimization
- **Appropriate for distributed control**
  - Computational burden can be distributed among agents, while kept at feasible levels
  - Overall stability can be guaranteed
  - In case of communication delays or failures, we have the computed open-loop control as backup plan

### Moreover

- It has desirable performance characteristics (optimization-based method)
- It has intrinsic robustness

# Nonholonomic Systems

Are completely controllable but instantaneously cannot move in certain directions (Ex: wheeled vehicles, robot manipulators, rolling sphere, ...)

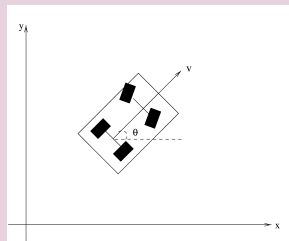
Example: A car-like vehicle

$$\begin{cases} \dot{x} = v \cdot \cos \theta \\ \dot{y} = v \cdot \sin \theta \\ \dot{\theta} = v \cdot c \end{cases}$$

with control inputs  $v$  and  $c$  satisfying

$$v \in [0, v_{max}] \text{ and } c \in [-c_{max}, c_{max}].$$

(Minimum turning radius  $R_{min} = c_{max}^{-1}$ )



- Nonholonomic constraint: velocity vector is orthogonal to the wheel axle  
 $(\dot{x}(t), \dot{y}(t))^T (\sin \theta(t), -\cos \theta(t)) = 0$
- Linearizing around any point  $(x_0, y_0, \theta_0, v = 0)$  results in an uncontrollable system.
- Cannot be stabilized by a continuous (time-invariant) feedback.



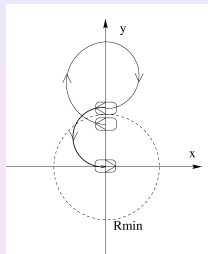
Cannot be stabilized by a continuous feedback.

Suppose  $x = 0$  and  $\theta = \pi$

If  $y \geq 2R_{min} \Rightarrow$  turn left.

If  $y < 2R_{min} \Rightarrow$  turn right.

which is a discontinuous feedback.



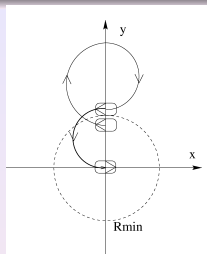
Cannot be stabilized by a continuous feedback.

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which is a discontinuous feedback.



More precisely ...

**Theorem (Brockett 83)**

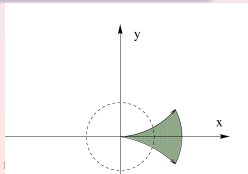
*If the system  $\dot{x} = f(x, u)$ , with  $u \in U$ , admits a continuous stabilizing feedback, then for any neighbourhood  $\mathcal{X}$  of zero, the set  $f(\mathcal{X}, U)$  contains a neighbourhood of zero.*

In the car example, if

$\mathcal{X} := \{(x, y, \theta) : |x| < 1, |y| < 1, |\theta| < \pi/6\}$

then

$f(\mathcal{X}, U)$  has no points of the form  $(\dot{x}, \dot{y}, \dot{\theta})$  with  $\dot{x} < 0$ .



## Discontinuous Feedback: Difficulties

- The trajectory  $x$ , solution to

$$\dot{x}(t) = f(x(t), k(x(t))), \quad x(0) = x_0$$

is only defined, in classical (Caratheodory) sense, if (among other requirements) if the right-hand side is a locally Lipschitz continuous function of  $x$ .

- Using Filippov solution-concept some controllable systems cannot be stabilized. (Ryan 94, Coron&Rosier 94).
- Problem solved if we use “sampling-feedback” solution concept (Clarke,Ledyaev,Sontag,Subbotin 97)

## Discontinuous Feedback: “sampling-feedback” solution

(Clarke, Ledyev, Sontag, Subbotin IEEE TAC97)

- Take a sequence of sampling instants  $\pi := \{t_i\}_{i \geq 0}$  in  $[0, +\infty)$  with  $t_0 < t_1 < t_2 < \dots$

$$\dot{x}(t) = f(x(t), k(x(\lfloor t \rfloor_\pi))), \quad x(0) = x_0$$

where  $\lfloor t \rfloor_\pi := \max_i \{t_i \in \pi : t_i \leq t\}$

i.e. The feedback is not a function of the state at every instant of time, rather it is a function of the state at the last sampling instant.

This solution concept combines naturally with our MPC sampled-data framework.

# Outline

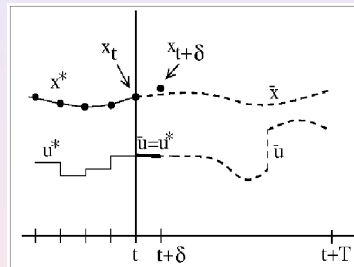
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## Our MPC framework

Consider a sequence  $\{t_i\}_{i \geq 0}$  s.t.  $t_{i+1} = t_i + \delta$ ,  $\delta > 0$ :

1. Measure state of the plant  $x_{t_i}$
2. Get  $\bar{u} : [t_i, t_i + T] \mapsto \mathbb{R}^m$  solution to the OCP:

$$\begin{aligned} &\text{Minimize} && \int_{t_i}^{t_i+T} L(t, x(t), u(t)) dt + W(x(t_i + T)) \\ &\text{subject to} && \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [t_i, t_i + T] \\ & && x(t_i) = x_{t_i} \\ & && u(t) \in U(t) \quad \text{a.e. } t \in [t_i, t_i + T] \\ & && x(t) \in X \quad \text{a.e. } t \in [t_i, t_i + T] \\ & && x(t_i + T) \in S \end{aligned}$$



$\{t_i\}$  Sequence of sampling instants

$\delta$  Sampling interval

$x_{t_i}$  Plant state measured at time  $t_i$ .

$(\bar{x}, \bar{u})$  OL state/control solving the OCP

$(x^*, u^*)$  CL state/control of the MPC

$T, L, W, S$  Design parameters

3. Apply to the plant the control  $u^*(t) := \bar{u}(t)$  in the interval  $[t_i, t_i + \delta]$ . (the remaining control  $\bar{u}(t)$ ,  $t > t_i + \delta$  is discarded)
4. Repeat for the next sampling time  $t_i = t_i + \delta$

## A Sufficient Condition for Stability

Choose the design parameters: time horizon  $T$ , objective functions  $L$  and  $W$ , and terminal constraint set  $S$ , to satisfy:

SC1–SC4 ... $S$  is closed, contains the origin is reachable in time  $T$ ...  $W$  is locally Lipschitz continuous and positive definite ...

SC5  $\forall x_t \in S, \exists \tilde{u} : [t, t + \delta] \rightarrow U$  [or  $\tilde{u}_t \in U$ ] such that

$$W(x(t + \delta; t, x_t, \tilde{u})) - W(x_t) \leq - \int_t^{t+\delta} L(x(s), \tilde{u}(s)) ds, \quad (\text{SC5a})$$

$$\text{[or } \langle \xi, f(x_t, \tilde{u}_t) \rangle \leq -L(x_t, \tilde{u}_t), \quad \forall \xi \in \partial_P W(x_t)]$$

and

$$x(s; t, x_t, \tilde{u}) \in S, \quad s \in [t, t + \delta]. \quad (\text{SC5b})$$

$$\text{[or } \langle \xi, f(x_t, \tilde{u}_t) \rangle \leq 0, \quad \forall \xi \in N_S^P(x_t)]$$

Then, the MPC strategy is stabilizing (i.e. for a sufficiently small inter-sample time  $\delta$  we have  $\|x^*(t)\| \rightarrow 0$  as  $t \rightarrow +\infty$ ).

Consider the system

$$\begin{cases} \dot{x}(t) = (u_1(t) + u_2(t)) \cdot \cos \theta(t) \\ \dot{y}(t) = (u_1(t) + u_2(t)) \cdot \sin \theta(t) \\ \dot{\theta} = (u_1(t) - u_2(t)). \end{cases}$$

with  $\theta(t) \in [-\pi, \pi]$ , and the controls  
 $u_1, u_2(t) \in [-1, 1]$ .

Stability achieved choosing:

$S$  to be the set of points s.t. the heading angle  $\theta$   
 is pointing towards the origin of the plane  $(x, y) = (0, 0)$ .

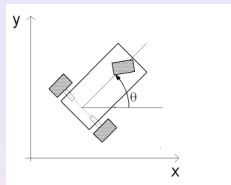
$$T = \pi/2.$$

$$L(x, y, \theta) := x^2 + y^2 + \theta^2,$$

$$W(x_0, y_0, \theta_0) := \int_0^{\bar{t}} L(x(t), y(t), \theta(t)) dt$$

where  $\bar{t}$  is the time to reach the origin.

SC is satisfied. MPC is stable.





## Example: A car-like vehicle

Consider the system

$$\begin{cases} \dot{x} = v \cdot \cos \theta \\ \dot{y} = v \cdot \sin \theta \\ \dot{\theta} = v \cdot c \end{cases}$$

with control inputs  $v$  and  $c$  satisfying

$$v \in [0, v_{max}] \text{ and } c \in [-c_{max}, c_{max}].$$

Stability achieved choosing

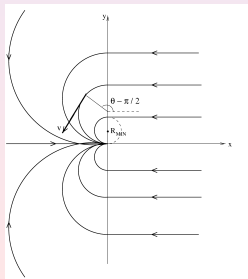
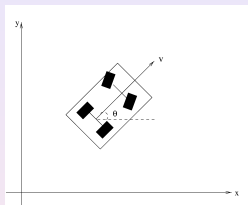
$S$  as in picture

$$T = 2\pi R_{min}/v_{max}.$$

$$L(x, y, \theta) := x^2 + y^2 + \theta^2,$$

$$W(x_0, y_0, \theta_0) := \int_0^{\bar{t}} L(x(t), y(t), \theta(t)) dt$$

where  $\bar{t}$  is the time to reach the origin.

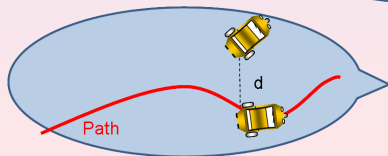
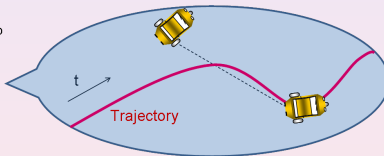


## Problem Objective

### What do we mean by “to control”?

- point-to-point motion (what we've seen so far)
- trajectory tracking (much easier)
- path-following (one more degree of freedom)
- accomplish mission objective (why not explore further degrees of freedom?)

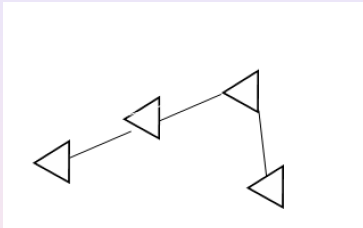
**Trajectory tracking:** The aim is to follow, as close as possible, a trajectory in state space (i.e., a geometric path in the cartesian space together with an associated timing law) starting from a given initial configuration .



**Path following:** The robot must reach and follow a geometric path in the cartesian space starting from a given initial configuration, but no associated timing law is specified



## Control of formations



**Method:** A two-layer model predictive control (MPC) scheme

- The top layer controls the trajectory of the whole group of vehicles.  
*Based on Nonlinear MPC allowing discontinuous feedbacks.*
- The bottom layer controls the relative position between vehicles.  
*Based on MPC with a linearized model (solves constrained LQR).*

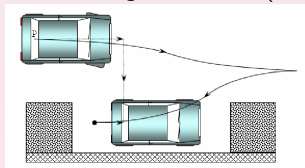
## Why two layers?

Because there are two intrinsically different control problems

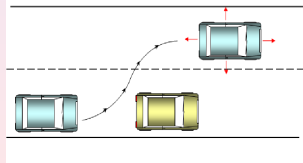
- The nonholonomic vehicles require a nonlinear controller allowing discontinuous feedbacks (point-to-point motion)
- However, while in motion, the relative position between the vehicles in a formation can be changed in all directions (as if they were holonomic). Can be controlled by a linear controller. (trajectory tracking)

In some robotics literature called leader-follower formation control.

**Example:** Consider a car in a parking manoeuvre (nonholonomic), or in a overtaking manoeuvre (can move in all directions relative to the other car)



Parking manoeuvre



Overtaking manoeuvre

## The two layer MPC scheme

- The top layer, **the trajectory controller**, applied to the group of vehicles as a whole.
  - Is a nonlinear MPC controller allowing discontinuous feedbacks. The control is computed centrally and a unique control is broadcasted to all vehicles.
- The bottom layer, **the formation controller**, applied to each vehicle individually.
  - Aims to compensate for small changes around a nominal trajectory maintaining the relative positions between vehicles
  - can be adequately carried out by a linear controller accommodating input and state constraints
  - This has the advantage that the control laws for each vehicle are much simpler to compute (are piecewise affine feedback laws and can even be pre-computed off-line [Bemporad, Morari, Dua, Pistikopoulos 2002]) They are computed and implemented in a distributed way in each vehicle.

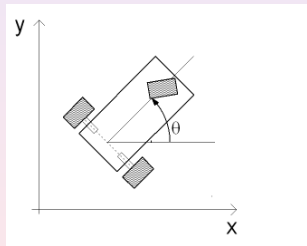
In the end the two controls are added and applied to the actuators in each vehicle.

## Nonholonomic vehicle model for the Trajectory controller

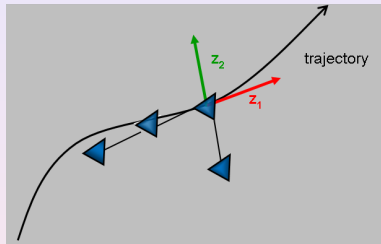
Our vehicle: A differential-drive mobile robot

$$\begin{cases} \dot{x}(t) = (u_1(t) + u_2(t)) \cdot \cos \theta(t) \\ \dot{y}(t) = (u_1(t) + u_2(t)) \cdot \sin \theta(t) \\ \dot{\theta} = (u_1(t) - u_2(t)). \end{cases}$$

with constrained controls  $u_1, u_2(t) \in [-1, 1]$ .



## Linearized vehicle model for the Formation Controller



We consider a linearized model for each of the differential drive mobile robots:

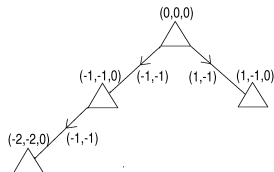
- with the  $z_1$  axis aligned with the velocity of the *reference vehicle*
- for the nominal velocity  $v_n = (u_{1n} + u_{2n})/2$ , with  $u_{1n} = u_{2n}$  being the nominal velocities of the wheels.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & w_n & 0 \\ -w_n & 0 & v_n \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$



## Formation Model

- Consider  $M$  vehicles and a *reference vehicle, leader* (that might not exist) that follows exactly the trajectory predicted by the trajectory controller.
- Consider that the  $M + 1$  vehicles are nodes of a directed graph  $G = (V, E)$ , that is a directed tree rooted at the reference vehicle. That is, all vertices are connected, there are no cycles, and all edges are directed in a path away from the root.
- To each vertex  $i$  associate a triple  $z^i = (z_1^i, z_2^i, z_3^i)$  with the relative position w.r.t. the reference vehicle.
- To each edge  $(i, j) \in E$  associate the pair  $\tilde{z}^{ij} = (\tilde{z}_1^{ij}, \tilde{z}_2^{ij})$  with the desired relative position in the plane of vehicle  $i$  with respect to its **parent** node  $j$ .



The performance index for each vehicle  $i$  with parent  $j$  is

$$J^i(z) = \int_{t=0}^T \left( \|(\tilde{z}_1^{ij}, \tilde{z}_2^{ij}, 0) - (z^j(t) - z^i(t))\|_Q^2 + \|(v(t), w(t))\|_R^2 \right) dt + \|(\tilde{z}_1^{ij}, \tilde{z}_2^{ij}, 0) - (z^j(N) - z^i(N))\|_P^2$$

## Formation Controller (cont.)

### Stability of the formation

- If we choose the design parameter  $L, W, T$ , and  $S$  for the trajectory controller as seen previously, and
- if we choose the matrices  $Q$  and  $R$  as p.d. and  $P$  solving the Riccati equation  $A'P + PA - PBR^{-1}B'P + Q = 0$ , in the objective function of the formation controller.

Then,

- The trajectory of the reference vehicle is stable.
- The trajectory of each vehicle is stable with respect to the desired relative position to its parent vehicle.
- Since there is exactly one path from each vehicle to the reference vehicle, stability of any vehicle follows recursively

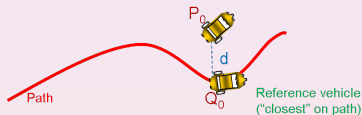


## Path-following

Given a path parameterized by  $\tau \mapsto (x_d(\tau), y_d(\tau))$  for  $\tau \in [0, T_d]$ :

- 1 Find an initial point  $Q_0$  in the path to start the trajectory of the virtual reference vehicle. Given the initial (current) position of our vehicle  $(x_0, y_0)$ , find the path parameter value that corresponds to a point in the path that is the closest to  $(x_0, y_0)$ .

$$\tau_0 = \operatorname{argmin}_{\tau \in [0, T_d]} \|(x_d(\tau), y_d(\tau)) - (x_0, y_0)\|$$



- 2 Select a speed profile at which the path is to be followed

$$\nu : [0, T] \mapsto \mathbb{R}_+$$

- 3 Find a feedback control to track the trajectory of the virtual reference vehicle  $t \mapsto (x_r(t), y_r(t))$

## Path-following with MPC

Minimize over  $\nu \in \mathbb{R}_+$  and  $u$

$$\int_0^{T_d/\nu} (\|q(t) - q_d(\tau(t))\|_Q^2 + \|u(t)\|_R^2) dt + \|q(T_d/\nu)\|_P^2, \quad (1)$$

subject to:

$$\begin{aligned} \dot{q}(t) &= f(t, q(t), u(t)) && a.e. t \in [0, T_d/\nu] \\ q(0) &= q_{t_i} \\ \tau_0 &= \operatorname{argmin}_{\tau \in [0, T_d]} \|q_d(\tau) - q_{t_i}\| && (2) \\ \tau(t) &= \tau_0 + \nu t \\ u(t) &\in U && \forall t \in [0, T_d/\nu]. \end{aligned}$$

where the path is parameterized by

$$t \mapsto q_d(\tau), \quad \tau \in [0, T_d]$$

## Path-following example

Consider a differential-drive robot to follow a eight-shaped path

$$\begin{aligned}x_d(\tau) &= \sin(\tau/10), & \tau \in [0, 38\pi] \\y_d(\tau) &= \sin(\tau/20), & \tau \in [0, 38\pi]\end{aligned}$$

By selecting  $\nu$  and an initial point in the path we define the trajectory for a virtual reference vehicle

$$\begin{aligned}x_r(t) &= \sin(\tau(t)/10) \\y_r(t) &= \sin(\tau(t)/20) \\ \tau(t) &= \tau_0 + \int_0^t \nu(s) ds.\end{aligned}$$

and by inverse kinematics we obtain the control of the reference vehicle

$$\theta_r(t) = \text{ATAN2}(\dot{y}_r, \dot{x}_r)$$

$$v_r(t) = \pm \sqrt{\dot{x}_d^2(t) + \dot{y}_d^2(t)}$$

$$w_r(t) = \frac{\ddot{y}_d(t)\dot{x}_d(t) - \ddot{x}_d(t)\dot{y}_d(t)}{\dot{y}_d^2(t) + \dot{x}_d^2(t)}, \quad \text{a.e.}$$

## Path-following example

Apply the MPC with a linearized model:

Minimize

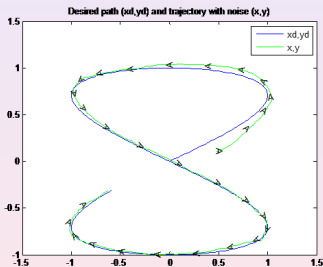
$$\int_0^{T_d/\nu} (\|q(t) - q_d(\tau(t))\|_Q^2 + \|u(t)\|_R^2) dt + \|q(T_d/\nu)\|_P^2, \quad (3)$$

subject to:

$$\begin{aligned} \dot{e}(t) &= Ae(t) + Bu(t) & a.e. t \in [0, T_d/\nu] \\ e(0) &= \mathcal{R}_\theta(q_d(\tau_0) - q(t_i)) \\ \tau_0 &= \operatorname{argmin}_{\tau \in [0, T_d]} \|q_d(\tau) - q_{t_i}\| \\ \tau(t) &= \tau_0 + \nu t & \forall t \in [0, T_d/\nu] \\ |u_1(t)| &\leq U_{max} & \forall t \in [0, T_d/\nu] \\ |u_2(t)| &\leq U_{max} & \forall t \in [0, T_d/\nu]. \end{aligned} \quad (4)$$

where  $Q$  and  $R$  are chosen to be p.d., and  $P$  solves the Riccati equation

$A'P + PA - PBR^{-1}B'P + Q = 0$ , guaranteeing stability.



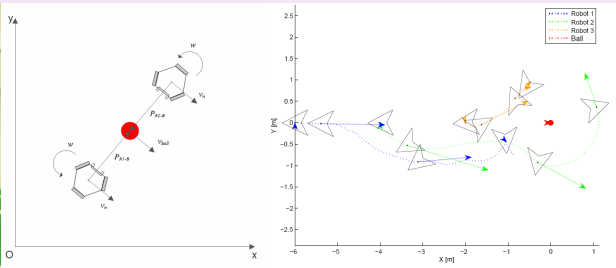
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## Current and Future Research

Mission Objective: Minimize the estimation error of a target (ball) position and velocity.





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