Stability and robustness of nonlinear predictive control without stabilizing terminal constraints

Lars Grüne, Anders Rantzer, Nils Altmüller, Thomas Jahn, Jürgen Pannek, Karl Worthmann

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in collaboration with

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Nils Altmüller (Bayreuth), Thomas Jahn (Bayreuth),
Jürgen Pannek (Perth), Karl Worthmann (Bayreuth)

supported by DFG priority research program 1305
"Control theory for digitally networked dynamical systems"

SADCO Kick-Off Meeting, Paris, March 2011

Setup

We consider nonlinear discrete time control systems

$$x(n+1) = f(x(n), u(n))$$

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For a running cost $\ell : X \times U \to \mathbb{R}_0^+$ penalizing the distance to the desired equilibrium solve

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possibly subject to state/control constraints

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Direct solution of the problem is numerically hard

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We obtain a feedback law F_N by a moving horizon technique

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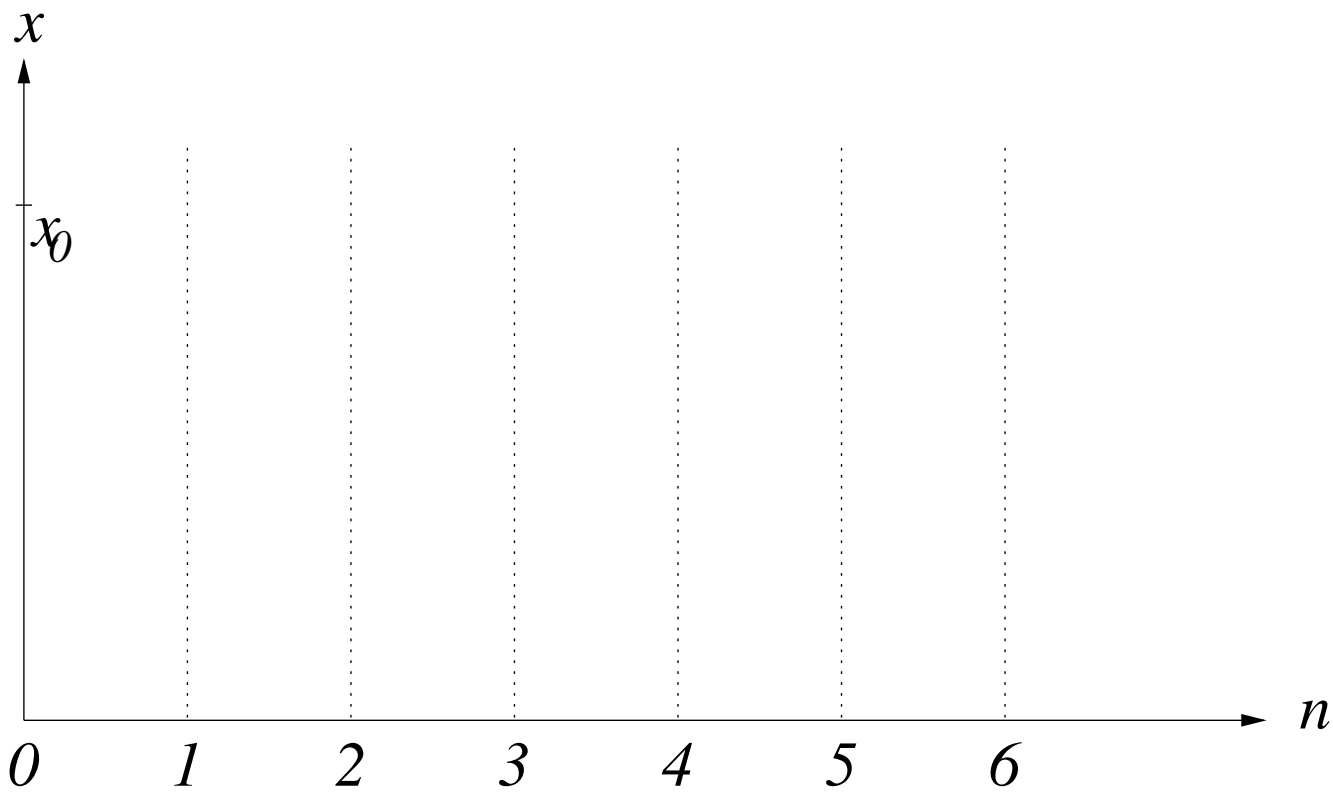
$\rightsquigarrow$ feedback controlled system ("closed loop")

$$x(n+1) = f(x(n), F_N(x(n))) = f(x^{opt}(0), u^{opt}(0)) = x^{opt}(1)$$

MPC from the trajectory point of view



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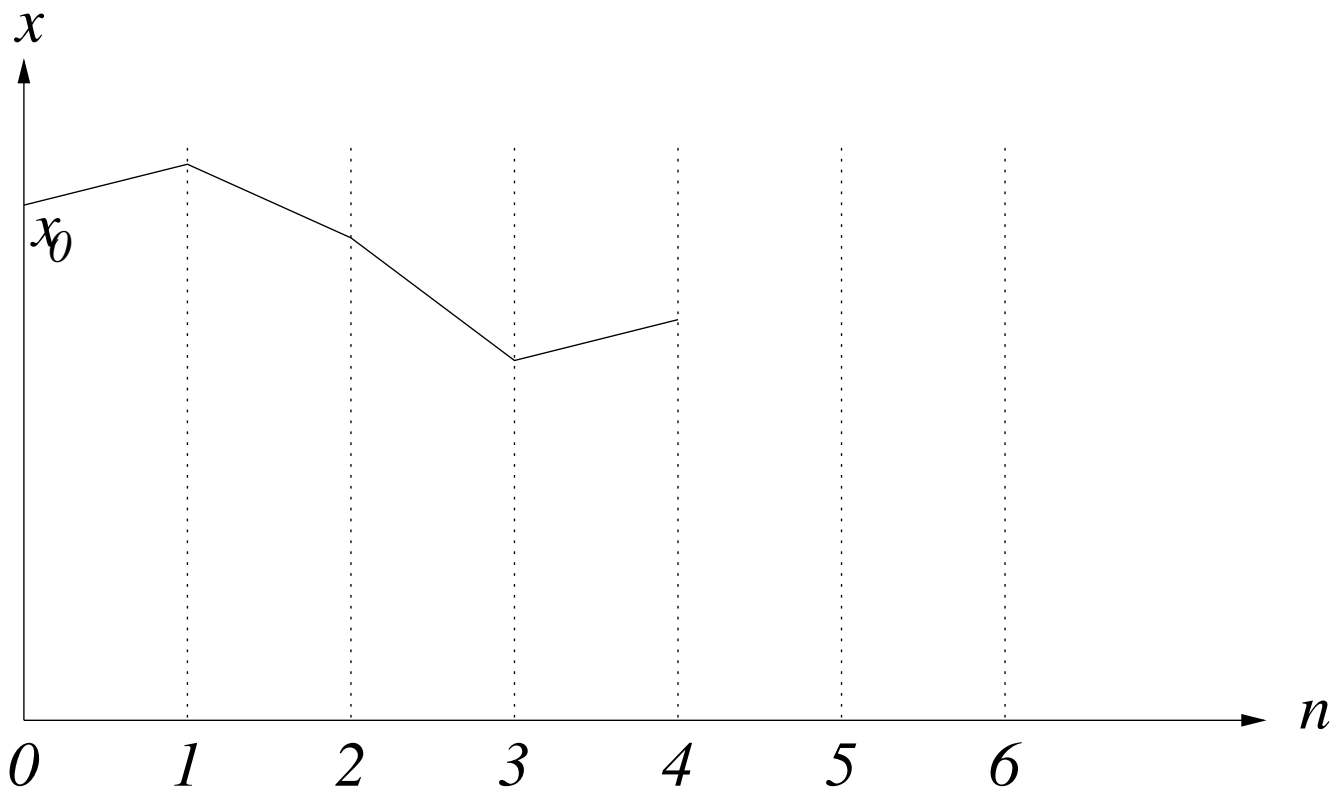

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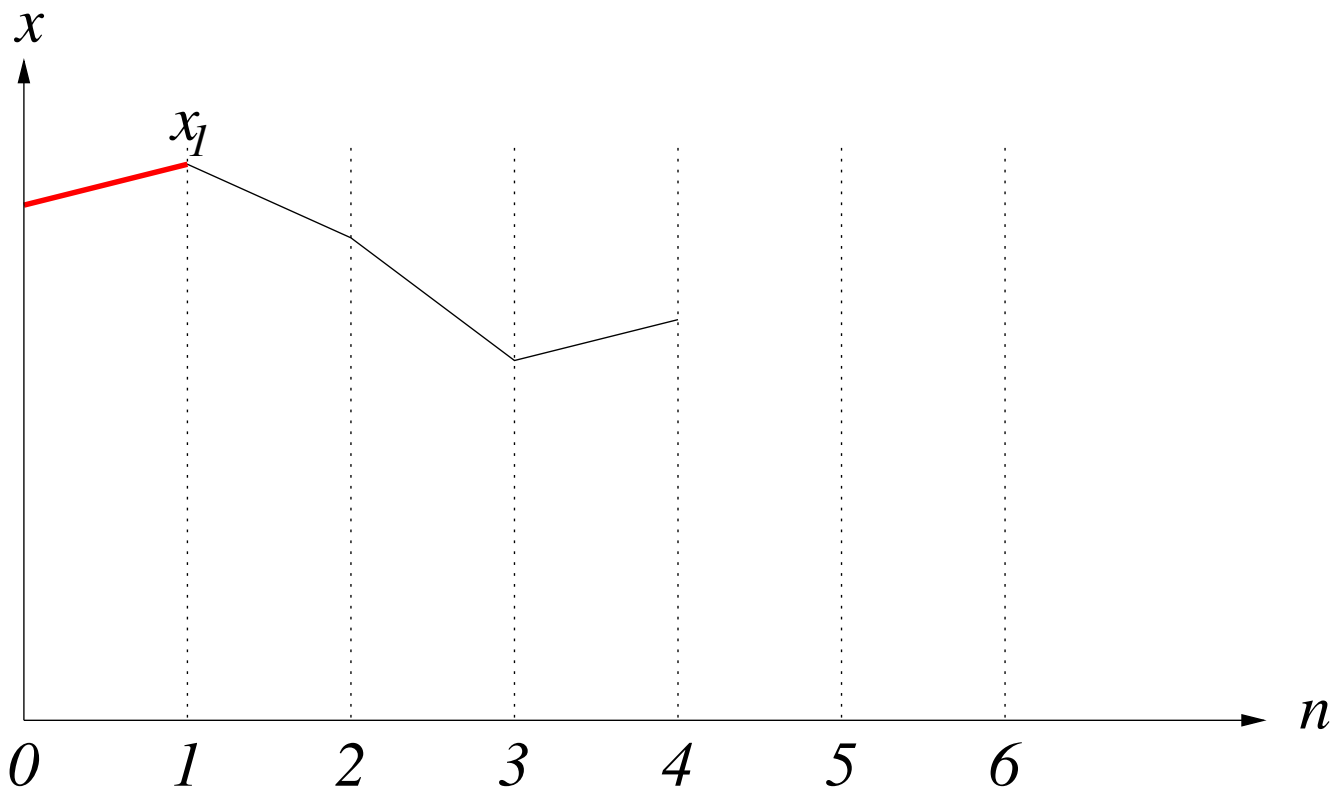

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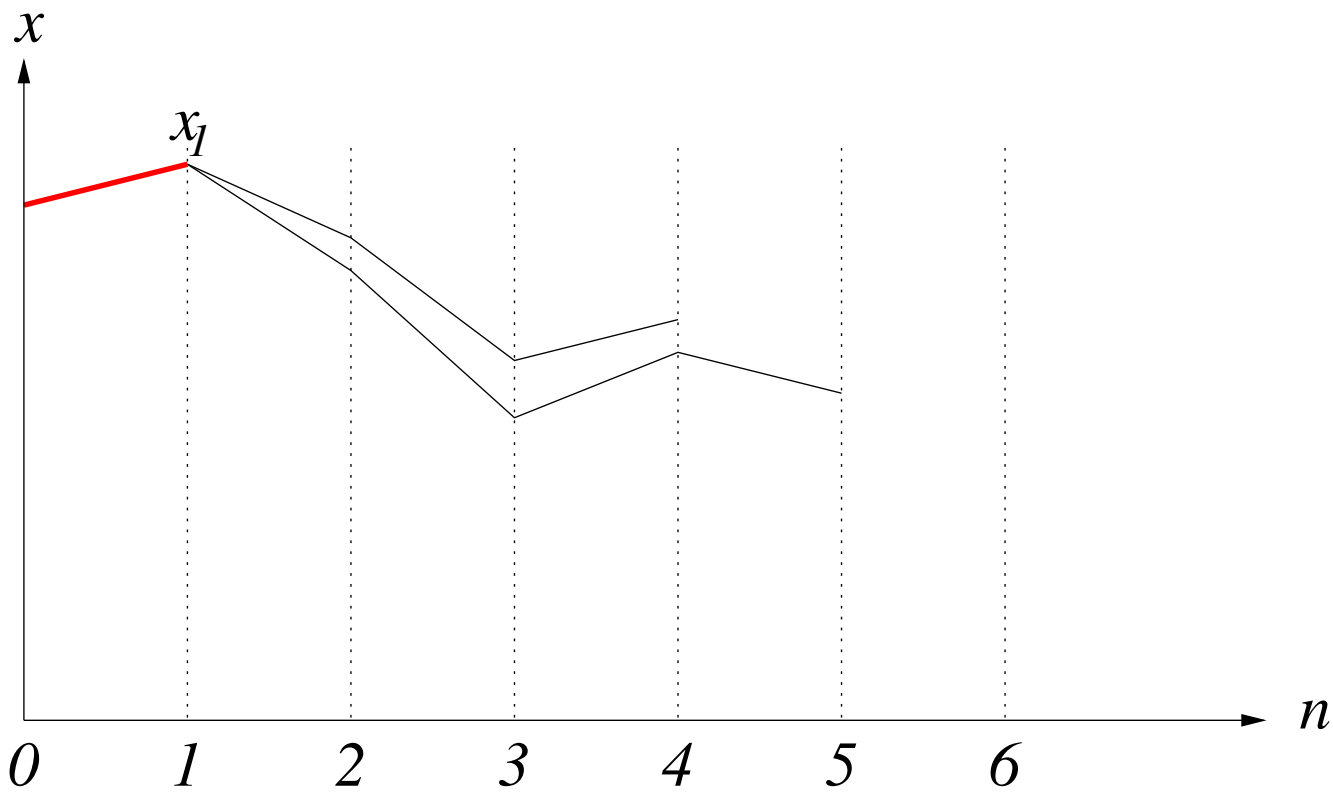

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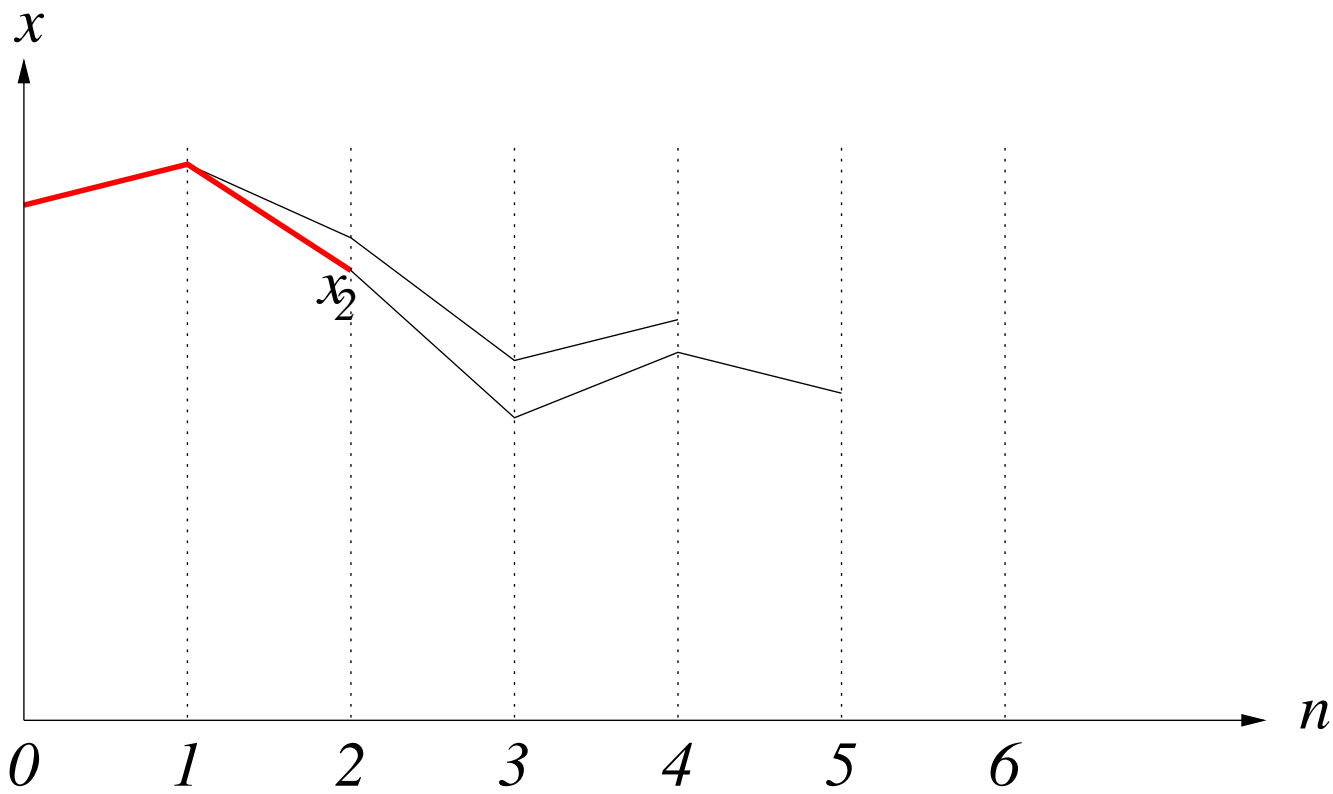

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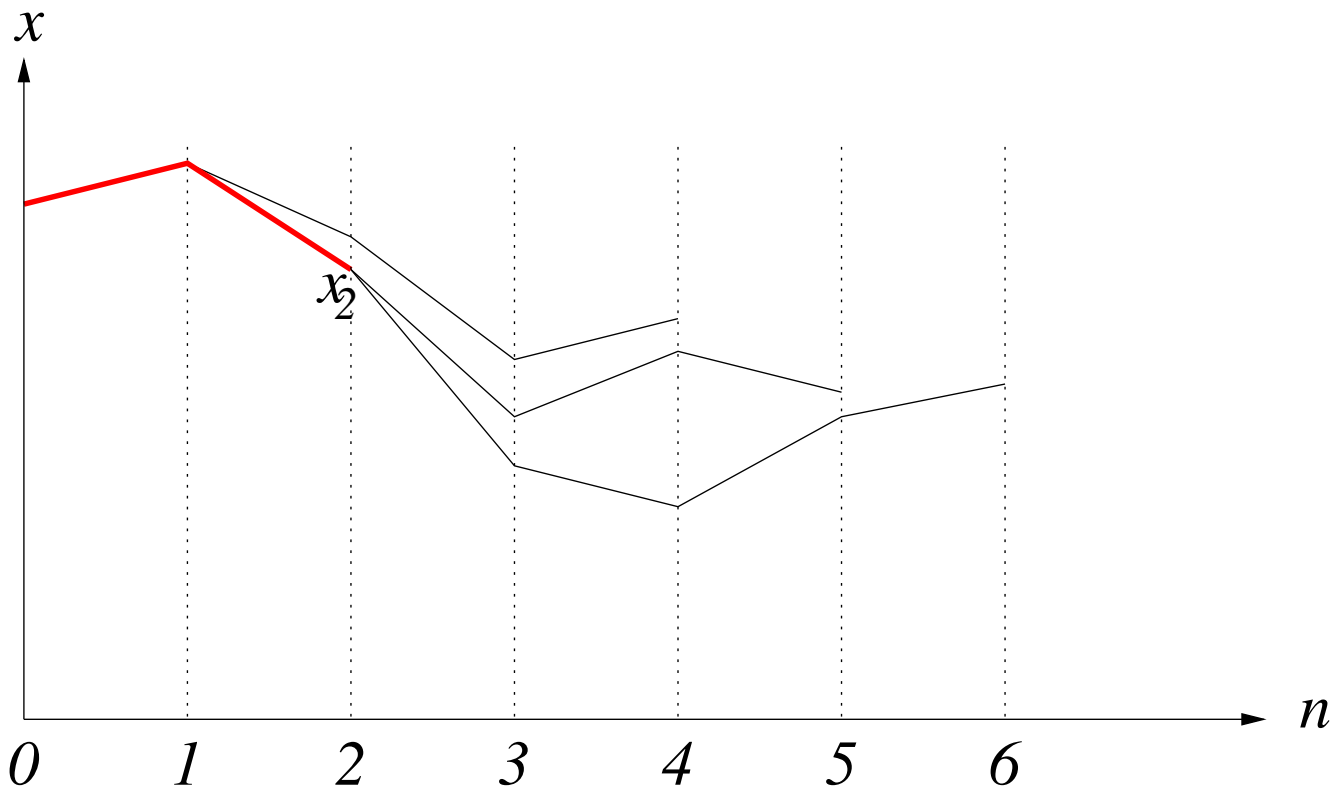

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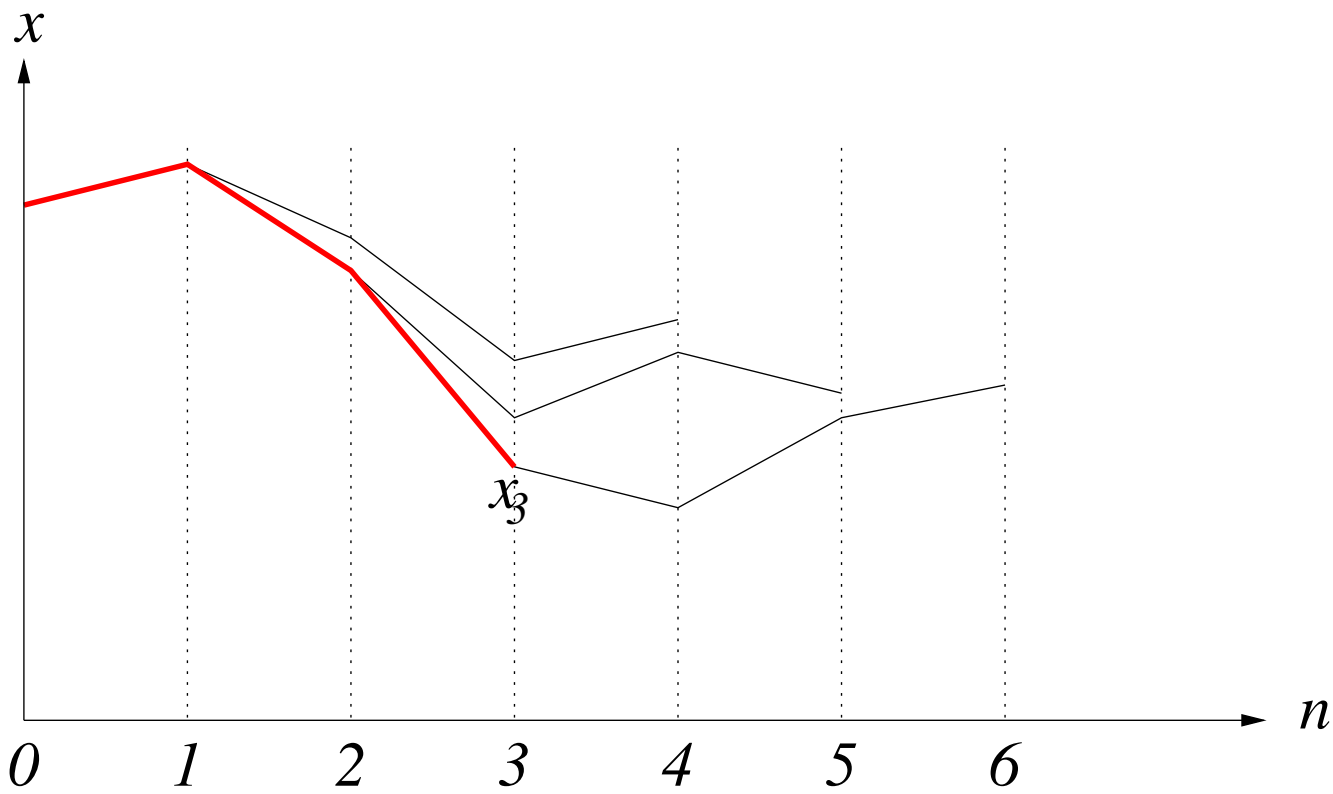

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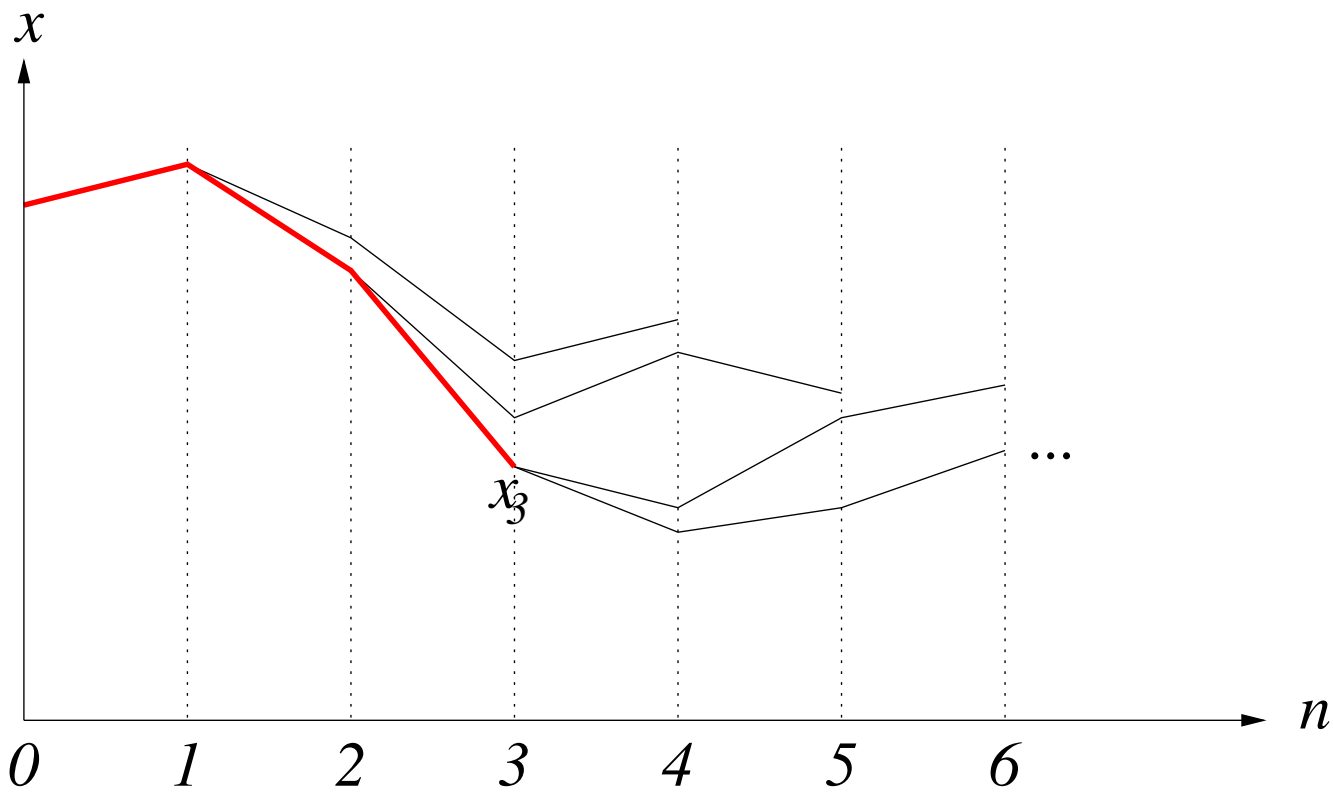

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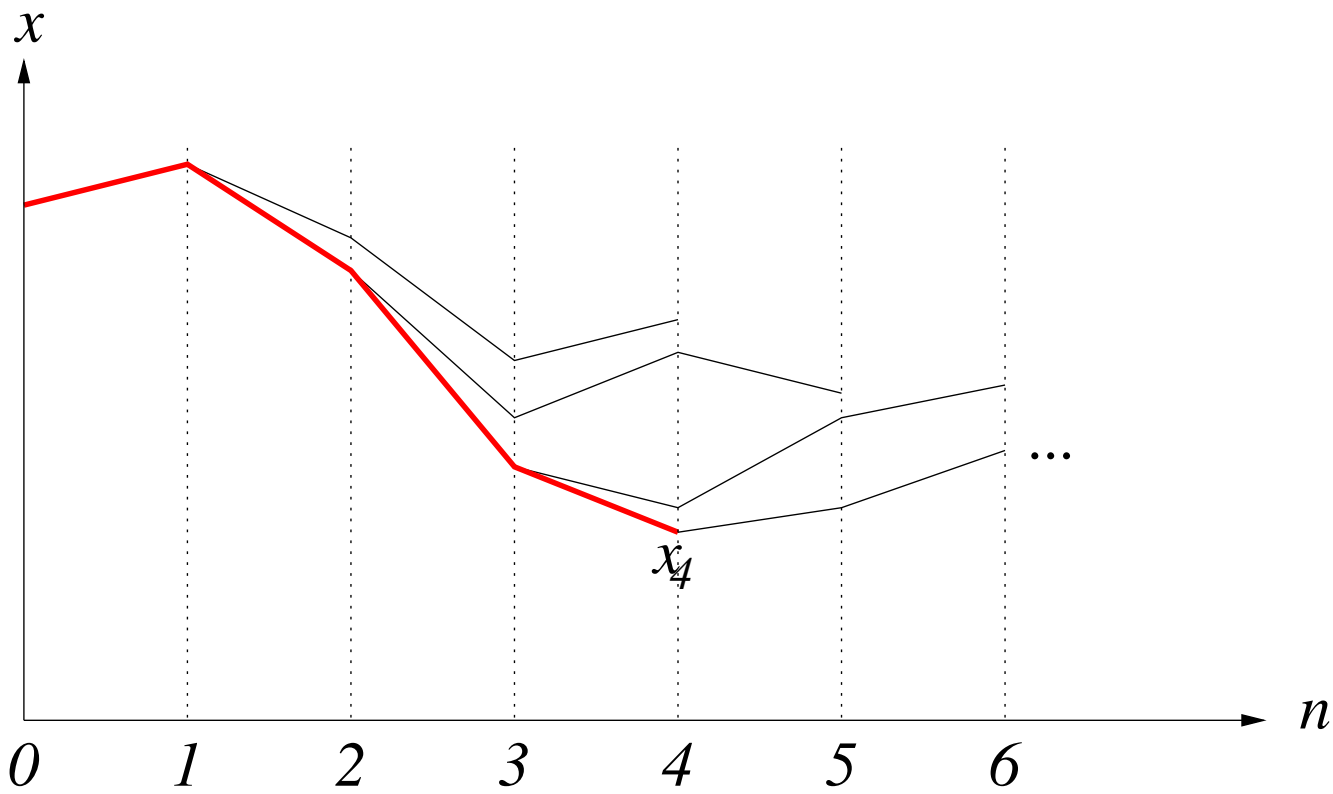

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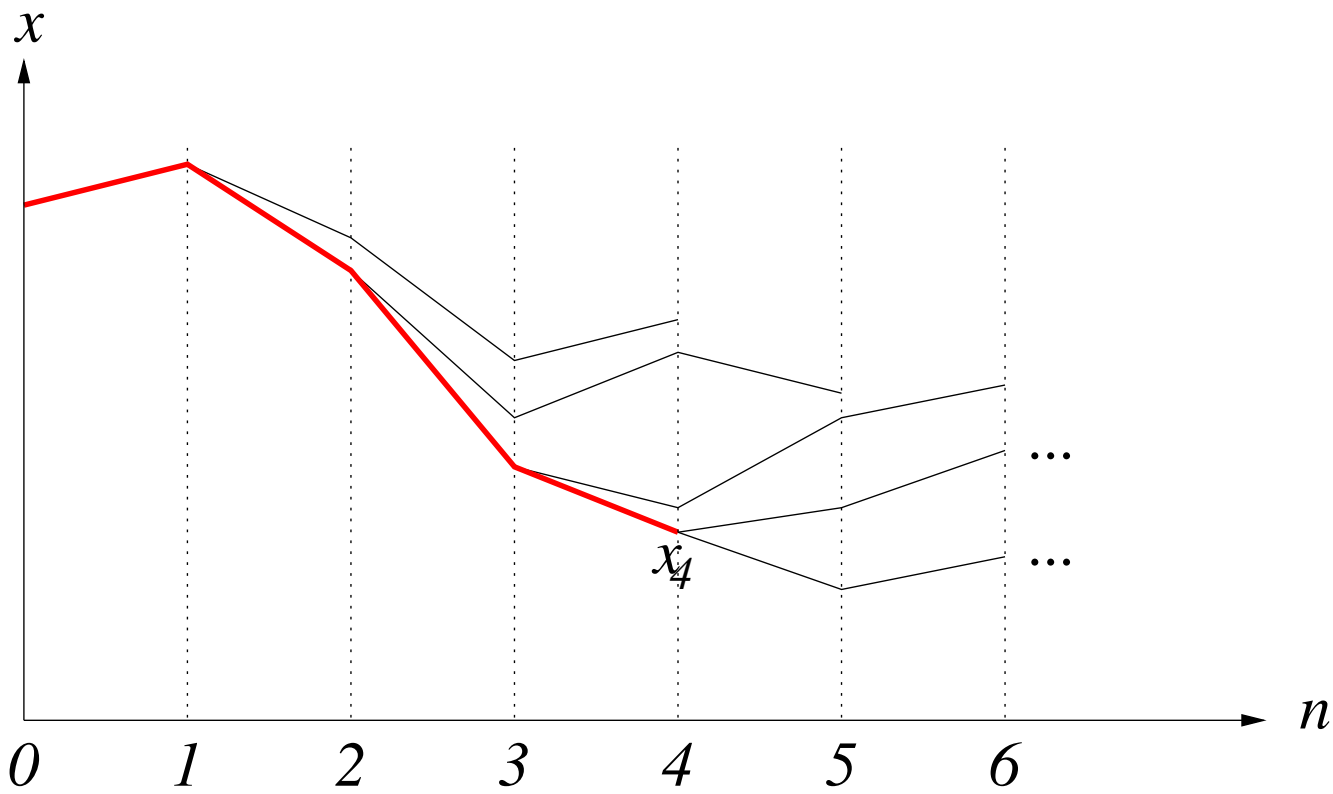

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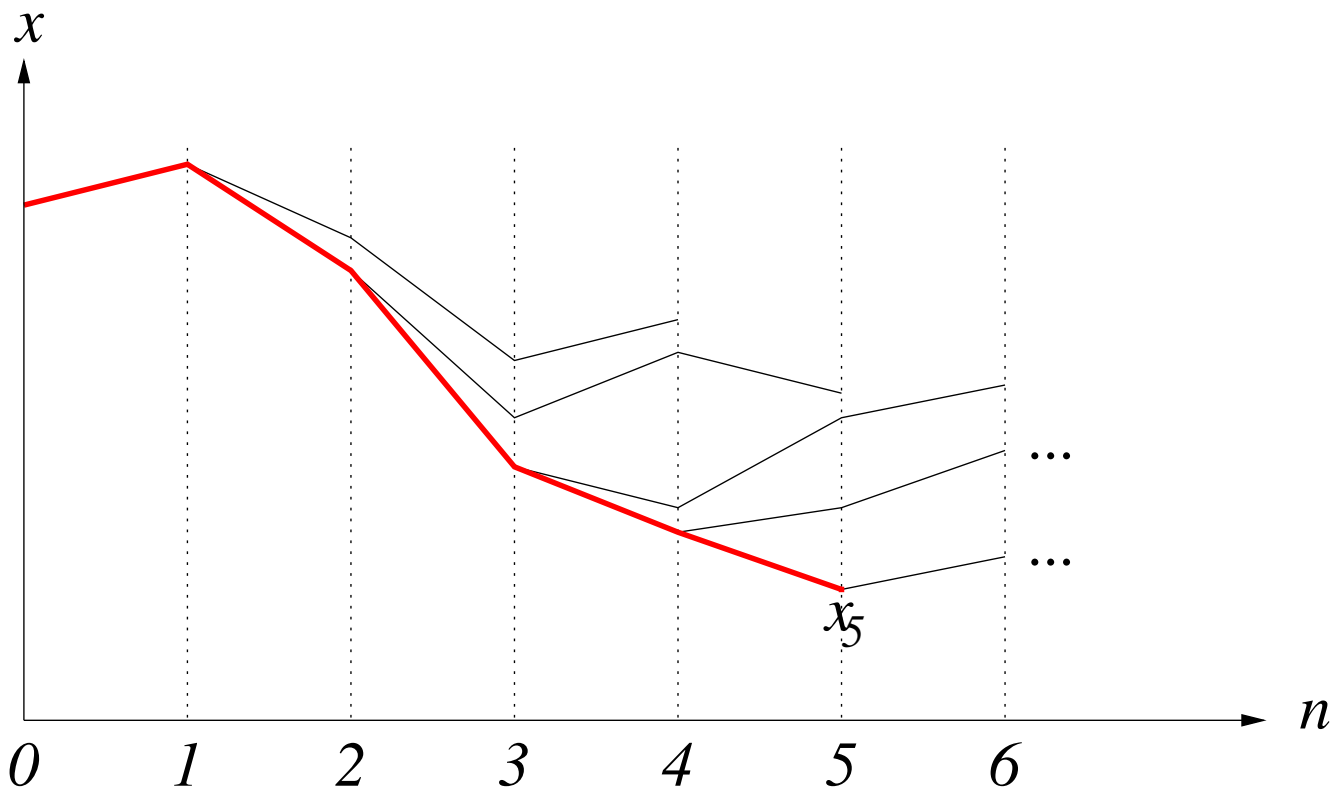

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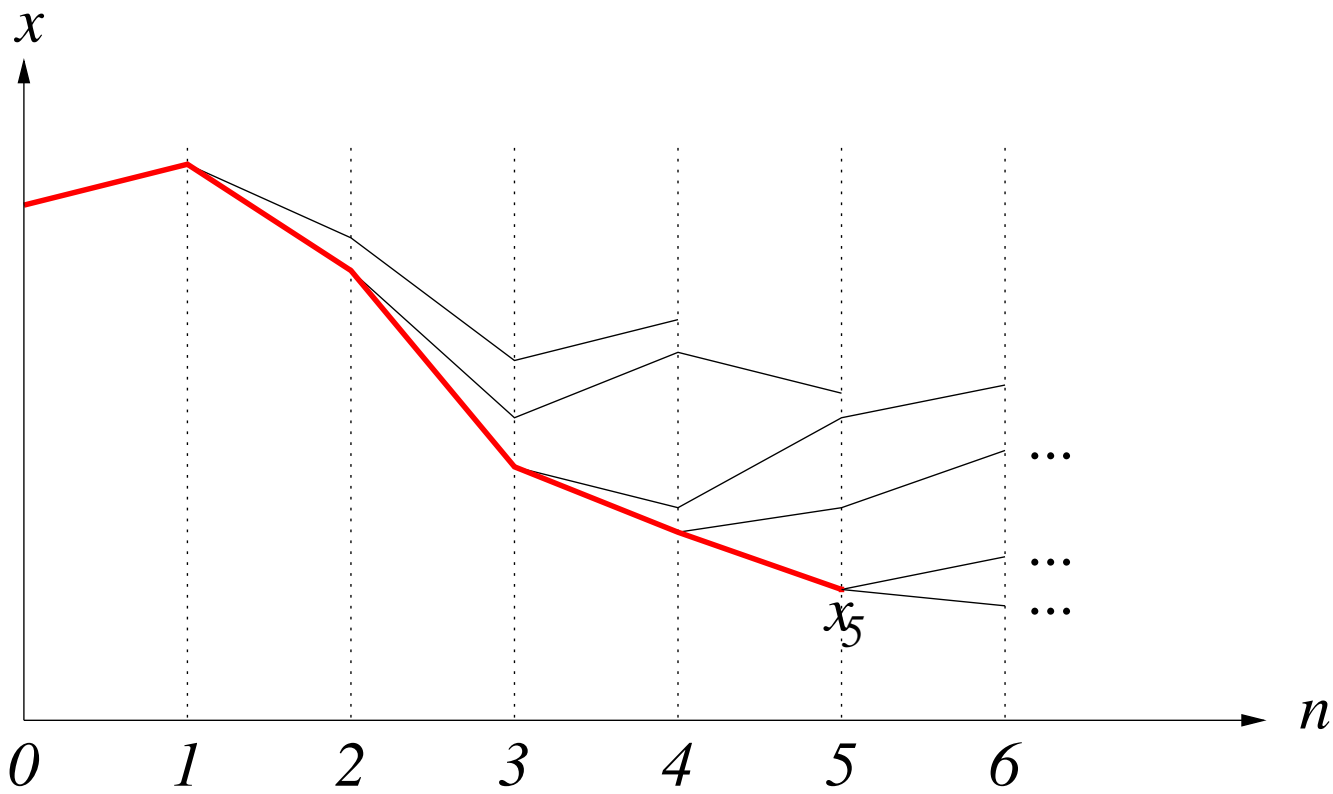

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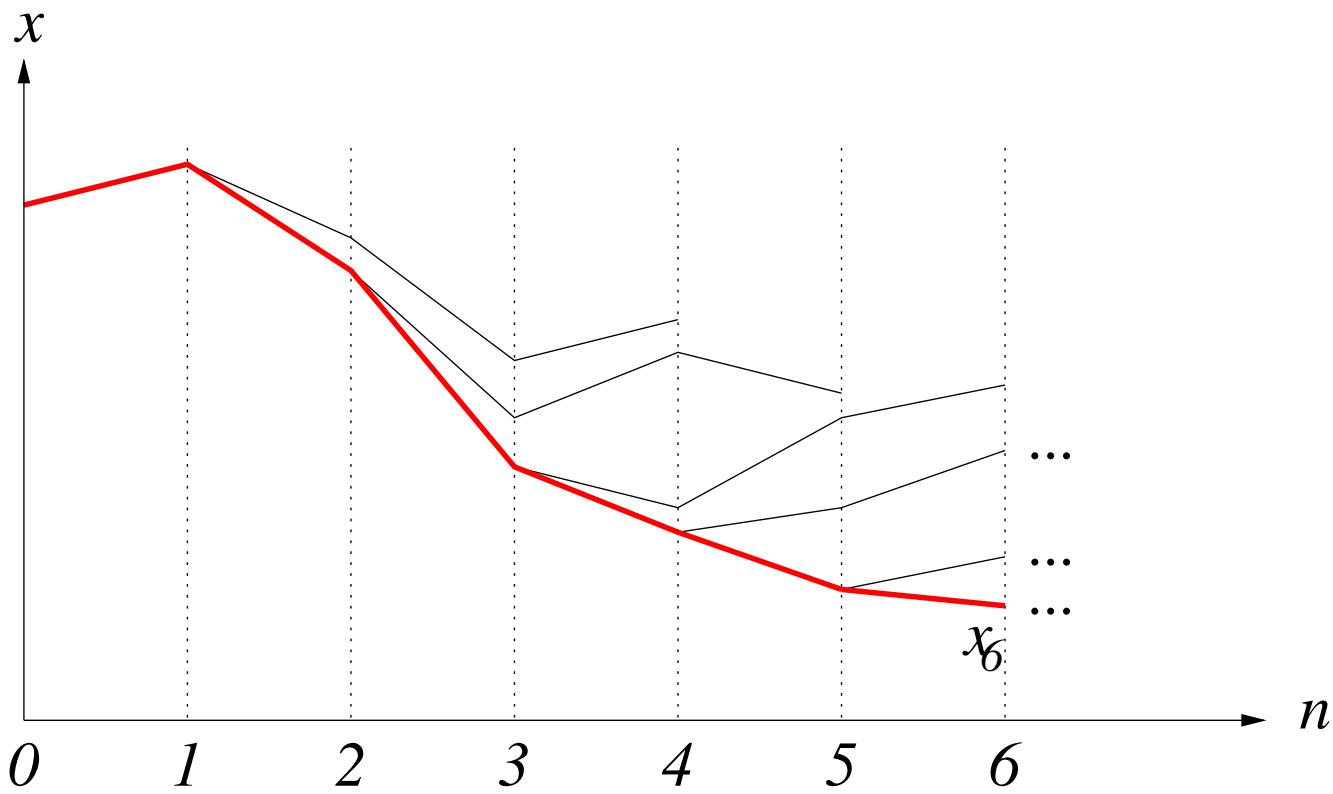

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How large is "sufficiently large"?

Estimating N

For obtaining a quantitative estimate we need quantitative information.

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A suitable condition is "exponential controllability through ℓ":

> there exist real numbers $C > 0$, $\sigma \in (0,1)$ such that for each $x(0) \in X$ there is $u(\cdot)$ with
>
> $$\ell(x(n), u(n)) \leq C\sigma^n \ell^*(x(0))$$
>
> with $\ell^*(x) = \min_u \ell(x,u)$

Stability conditions

C, σ-exponential controllability: $\ell(x(n), u(n)) \leq C\sigma^n \ell^*(x(0))$

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Define $\alpha := 1 - \dfrac{(\gamma_N - 1)\prod_{i=2}^{N}(\gamma_i - 1)}{\prod_{i=2}^{N}\gamma_i - \prod_{i=2}^{N}(\gamma_i - 1)}$ with $\gamma_i = \sum_{k=0}^{i-1} C\sigma^k$

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Theorem: If $\alpha > 0$, then the MPC feedback F_N stabilizes all C, σ-exponentially controllable systems and we get $J_\infty(x, F_N) \leq \inf_{u \in U^\infty} J_\infty(x, u)/\alpha$

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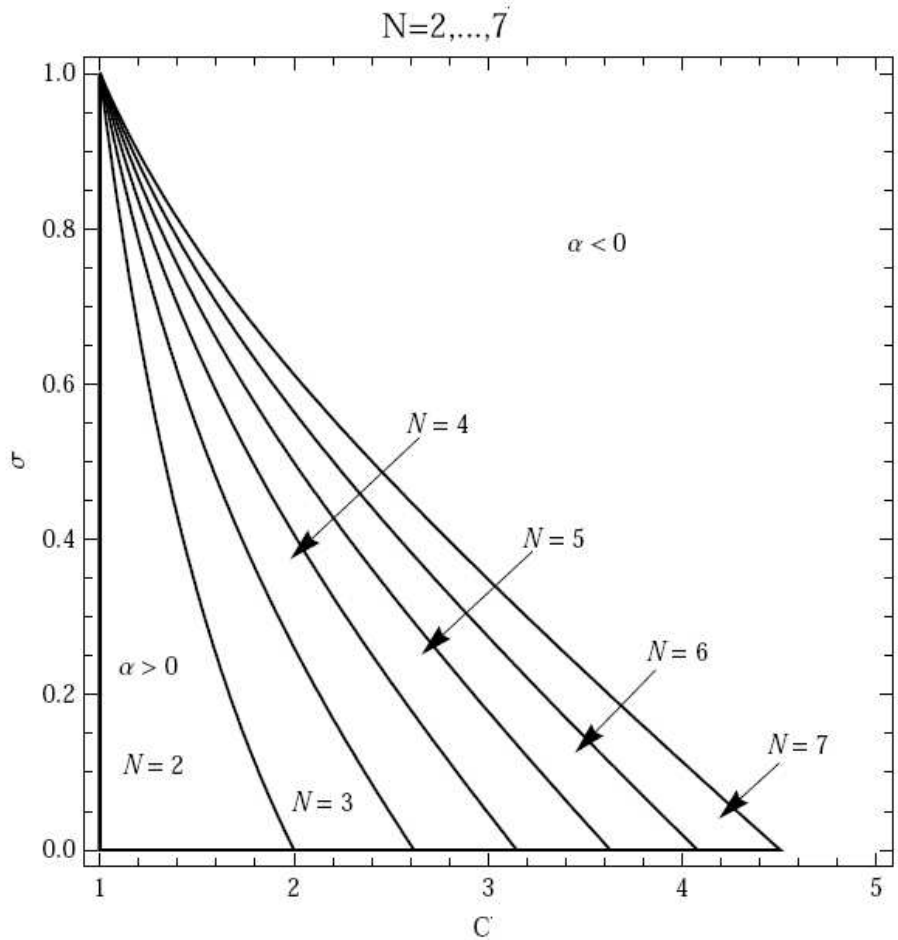
Moreover, $\alpha \to 1$ as $N \to \infty$

Stability chart for C and σ



(Figure: Harald Voit)

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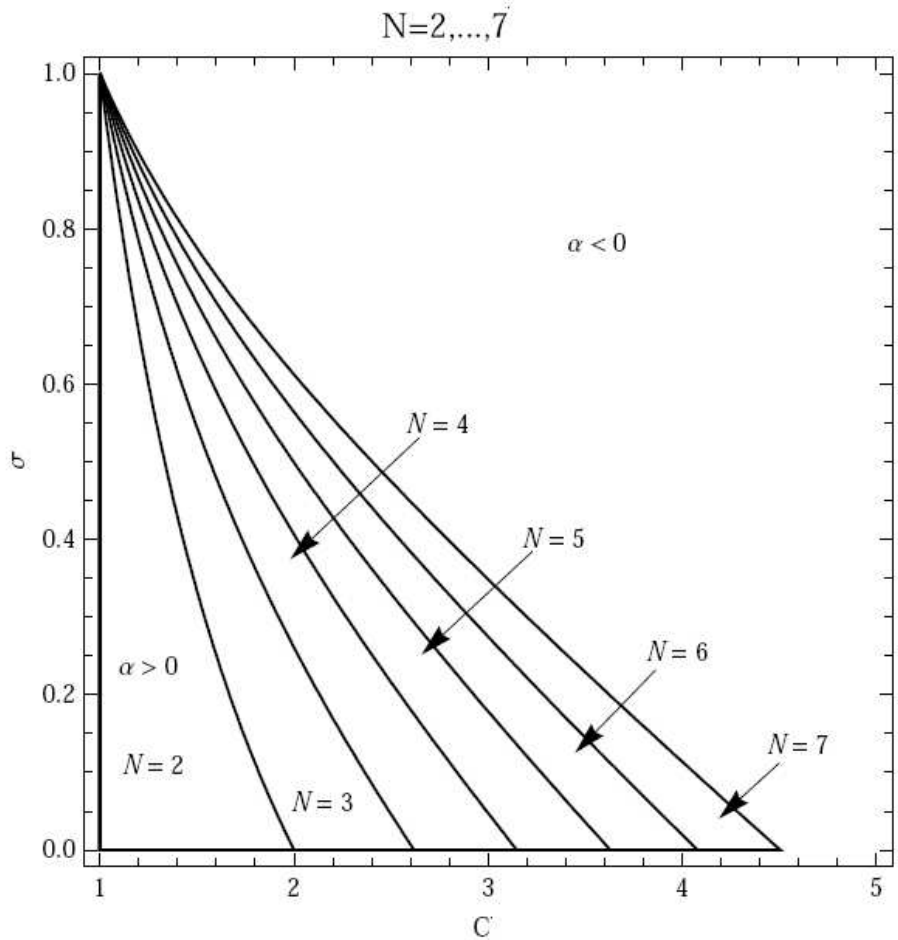

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Conclusion: try to reduce C, e.g., by choosing ℓ appropriately

A PDE example

We illustrate this with the 1d controlled PDE

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y) + u$$

with

domain $\Omega = [0,1]$

solution $y = y(t,x)$

boundary conditions $y(t,0) = y(t,1) = 0$

parameters $\nu = 0.1$ and $\mu = 10$

and distributed control $u : \mathbb{R} \times \Omega \to \mathbb{R}$

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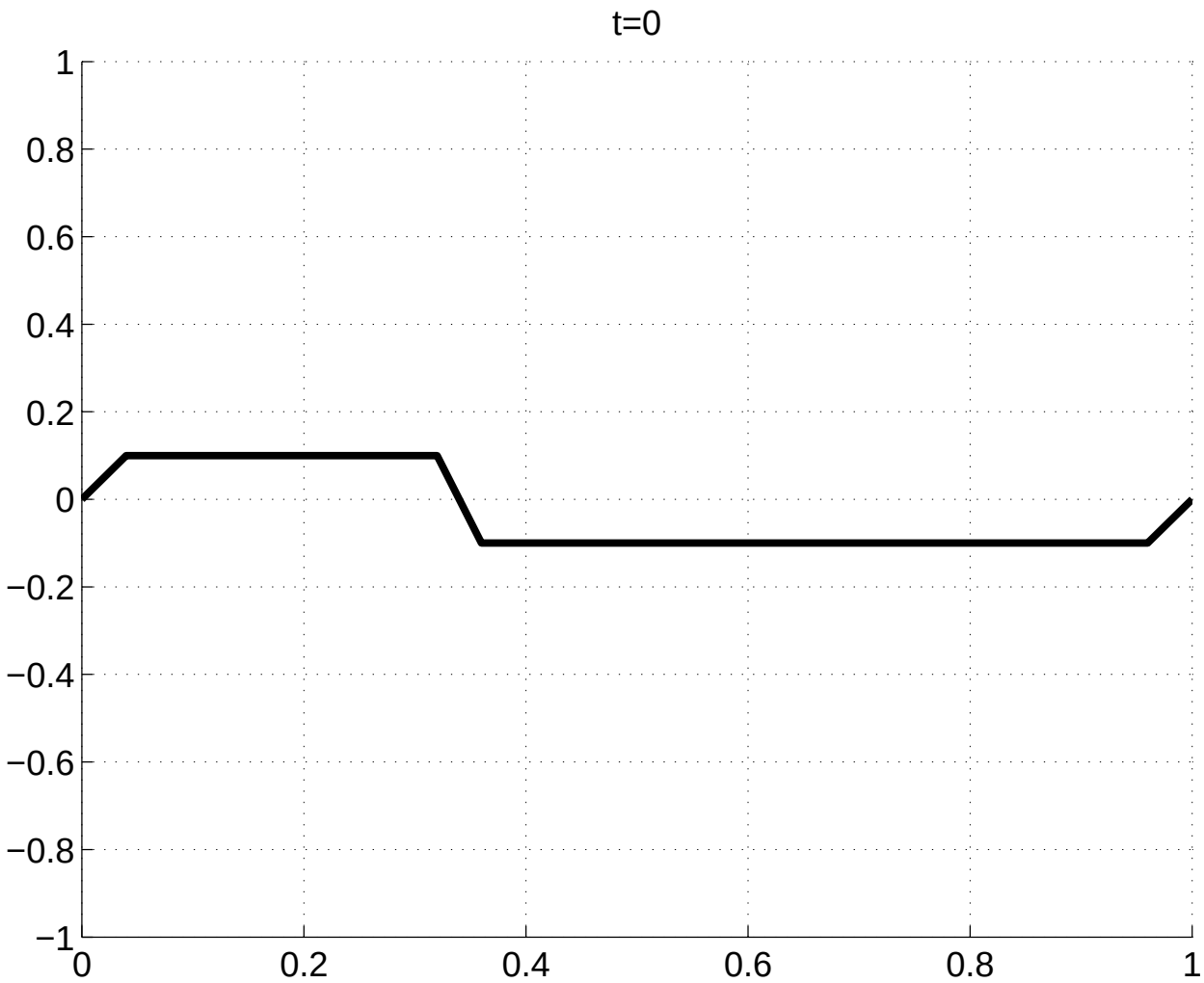
Discrete time system: $y(n) = y(nT, \cdot)$ for some $T > 0$

("sampled data system with sampling time T")

The uncontrolled PDE



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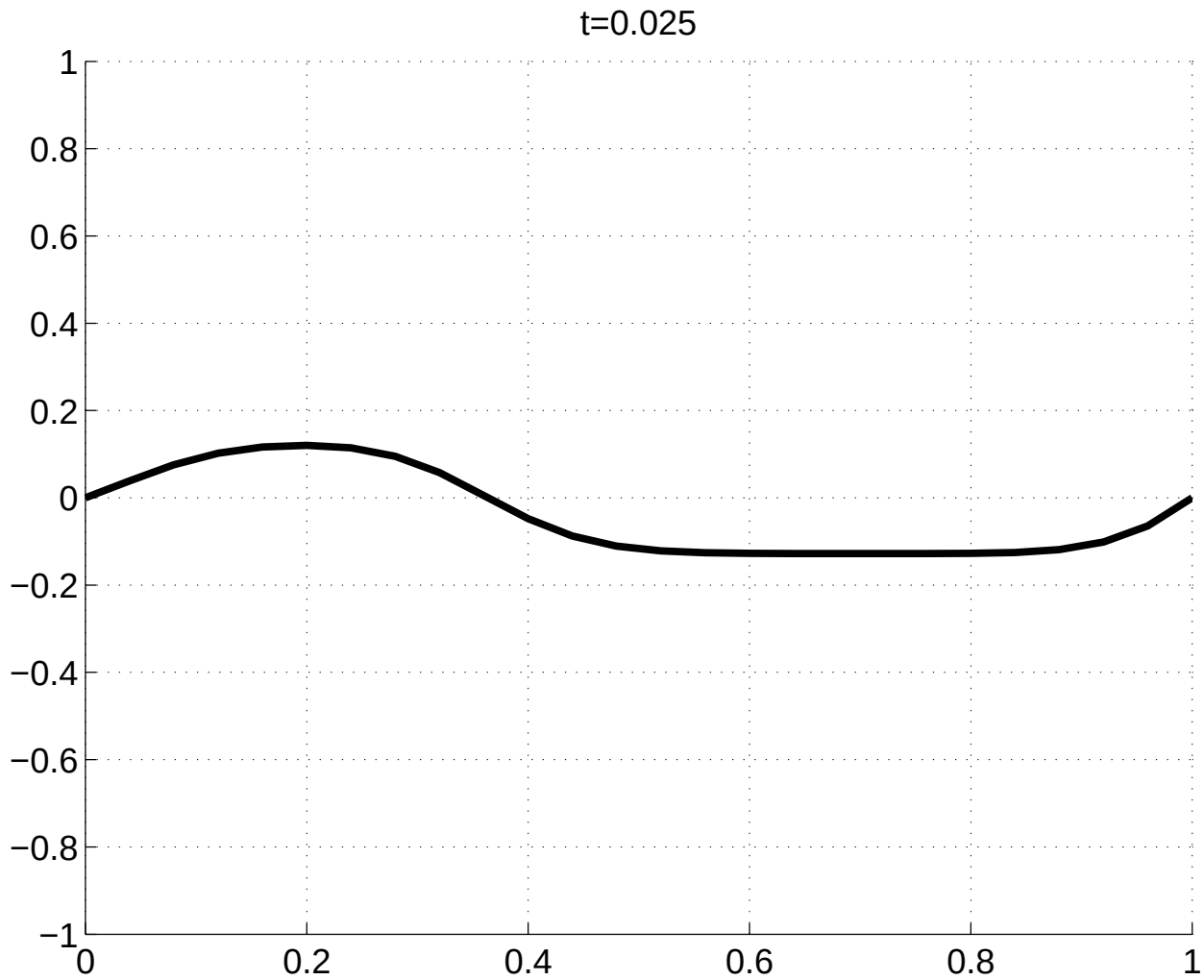

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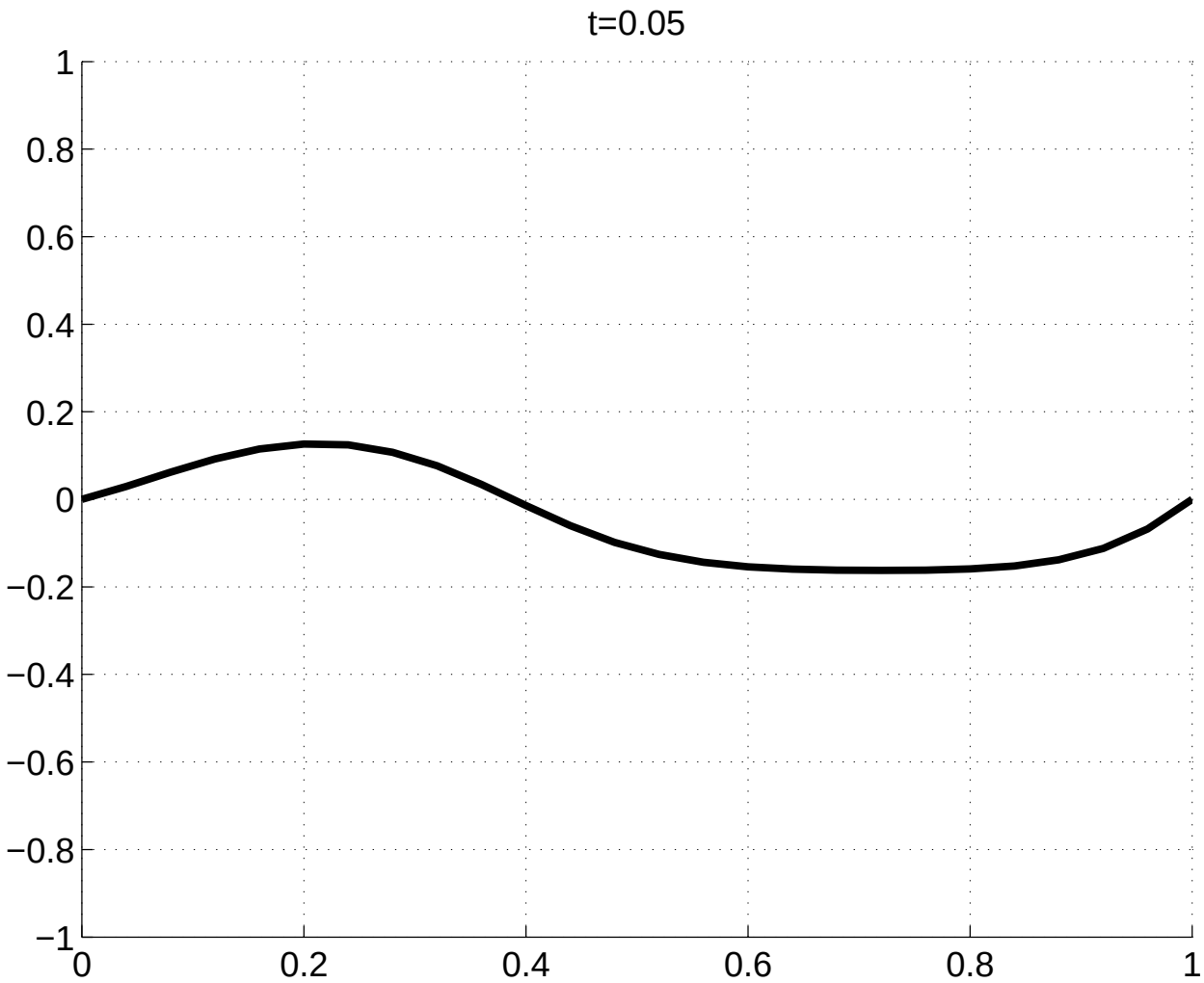

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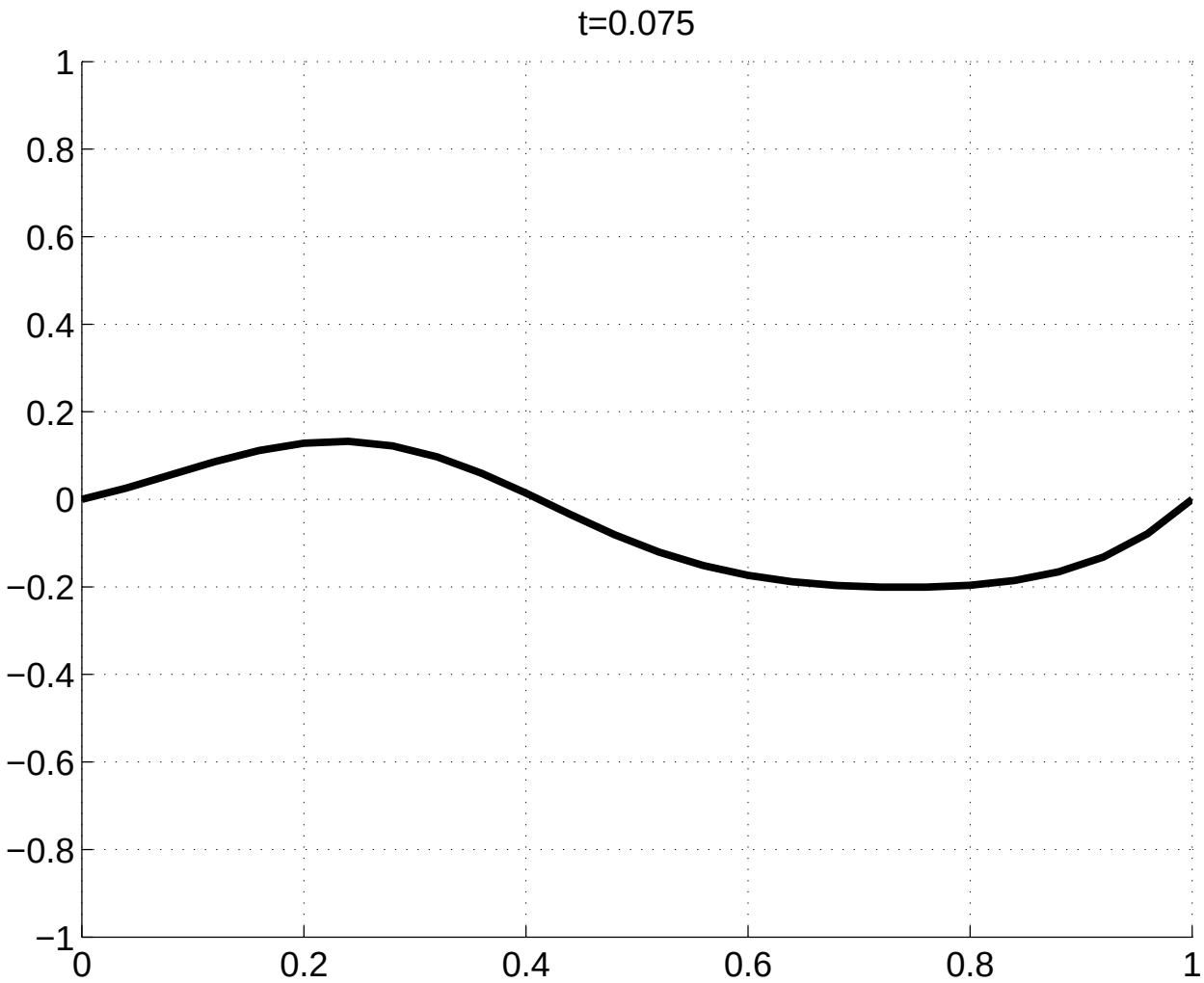

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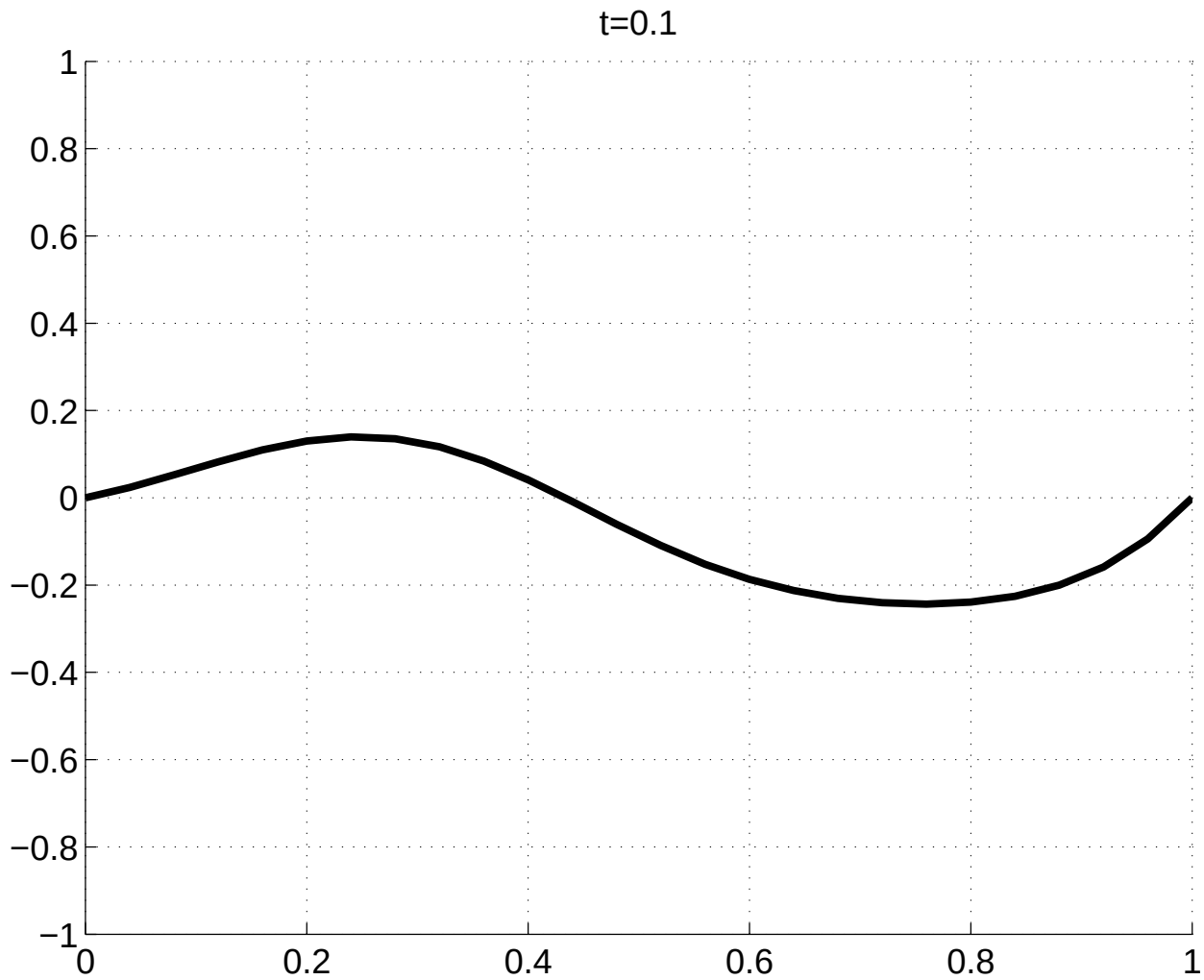

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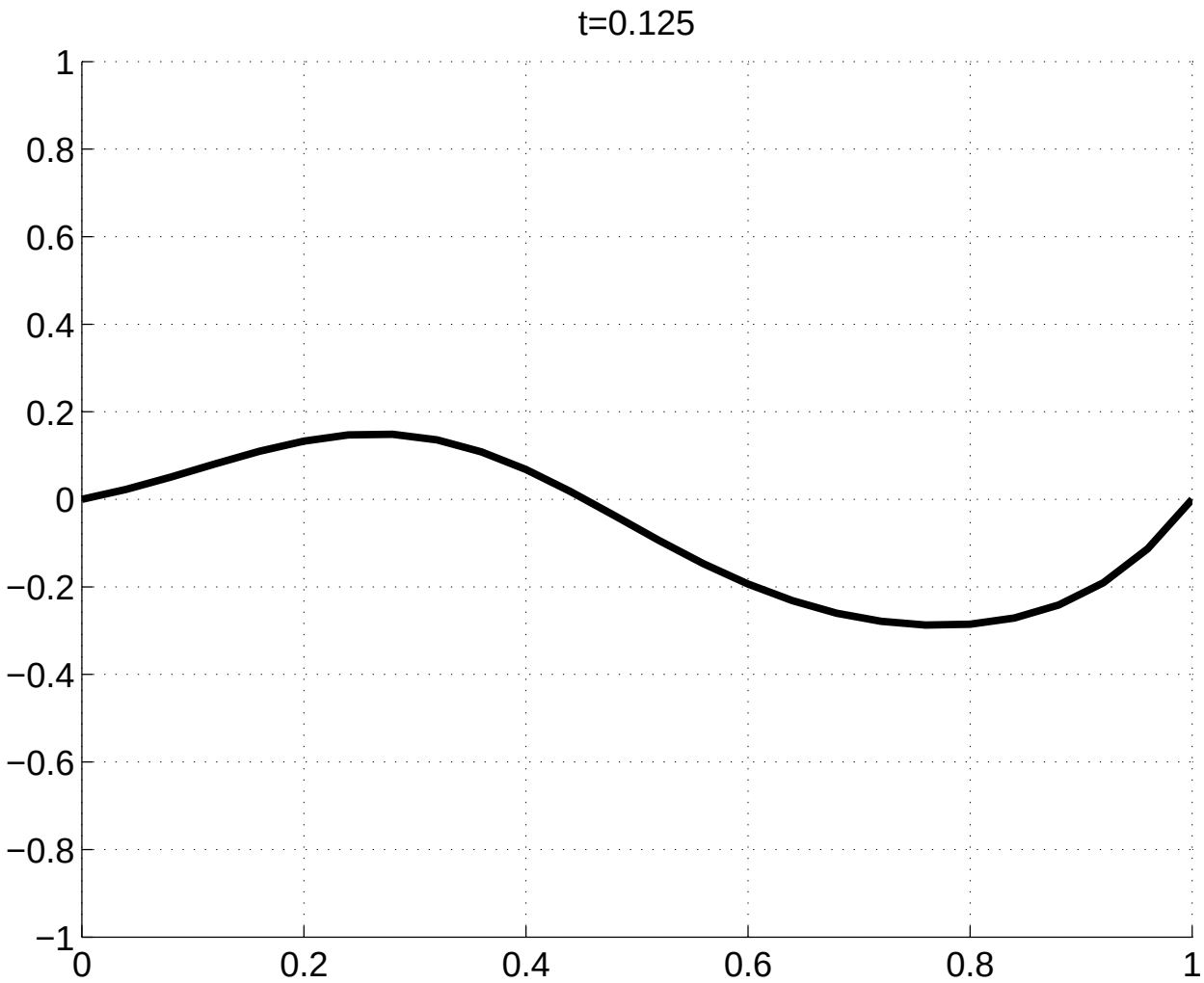

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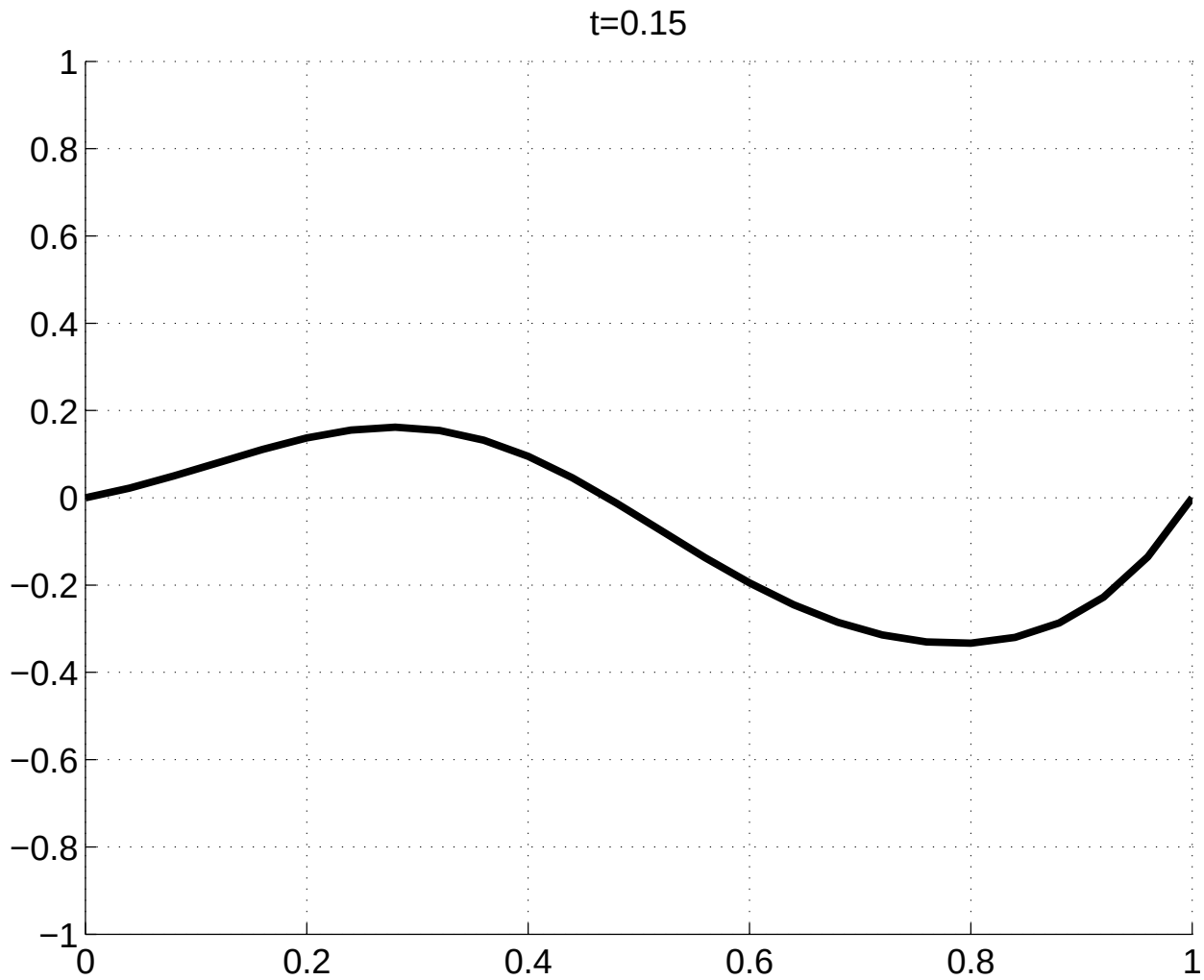

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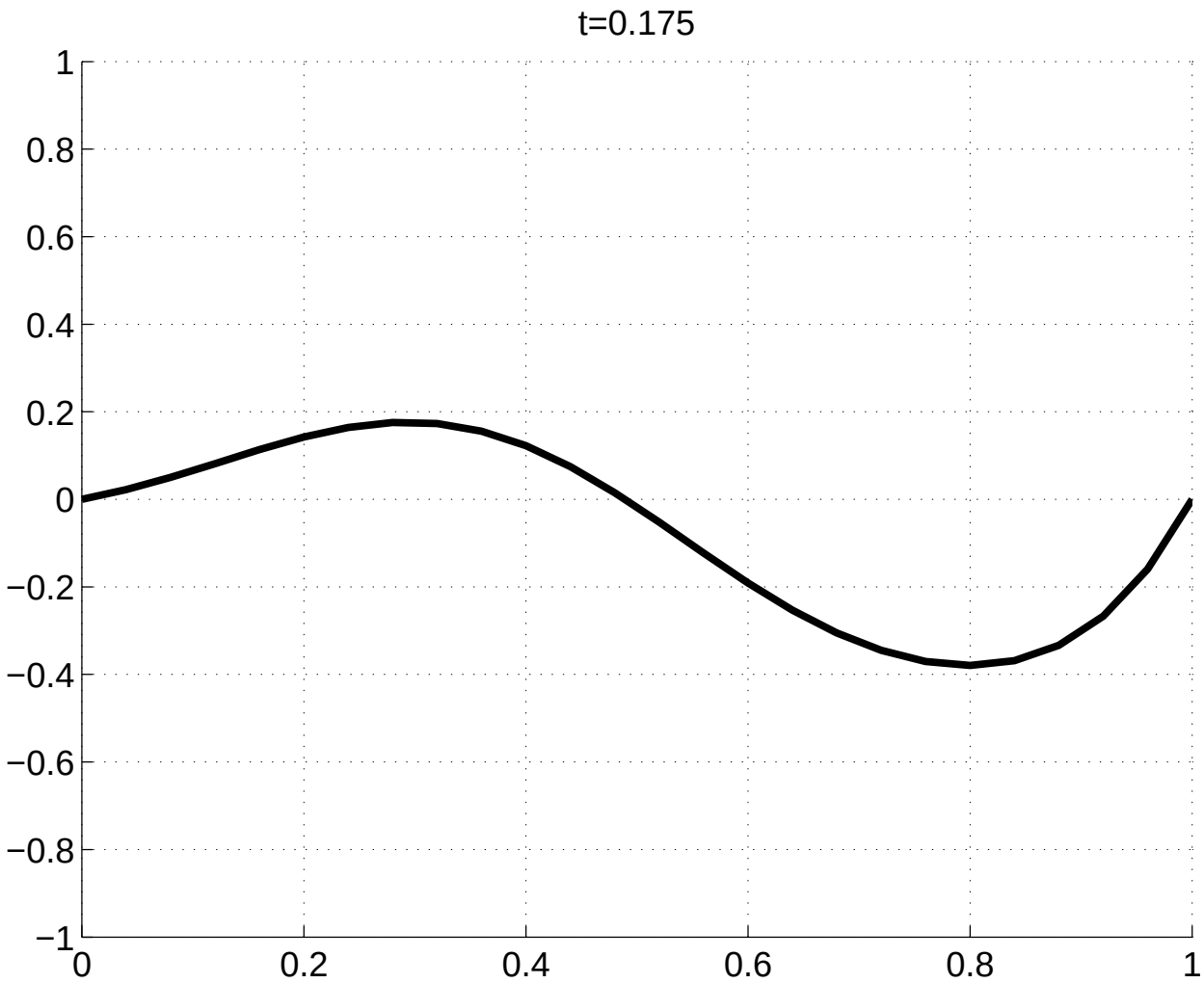

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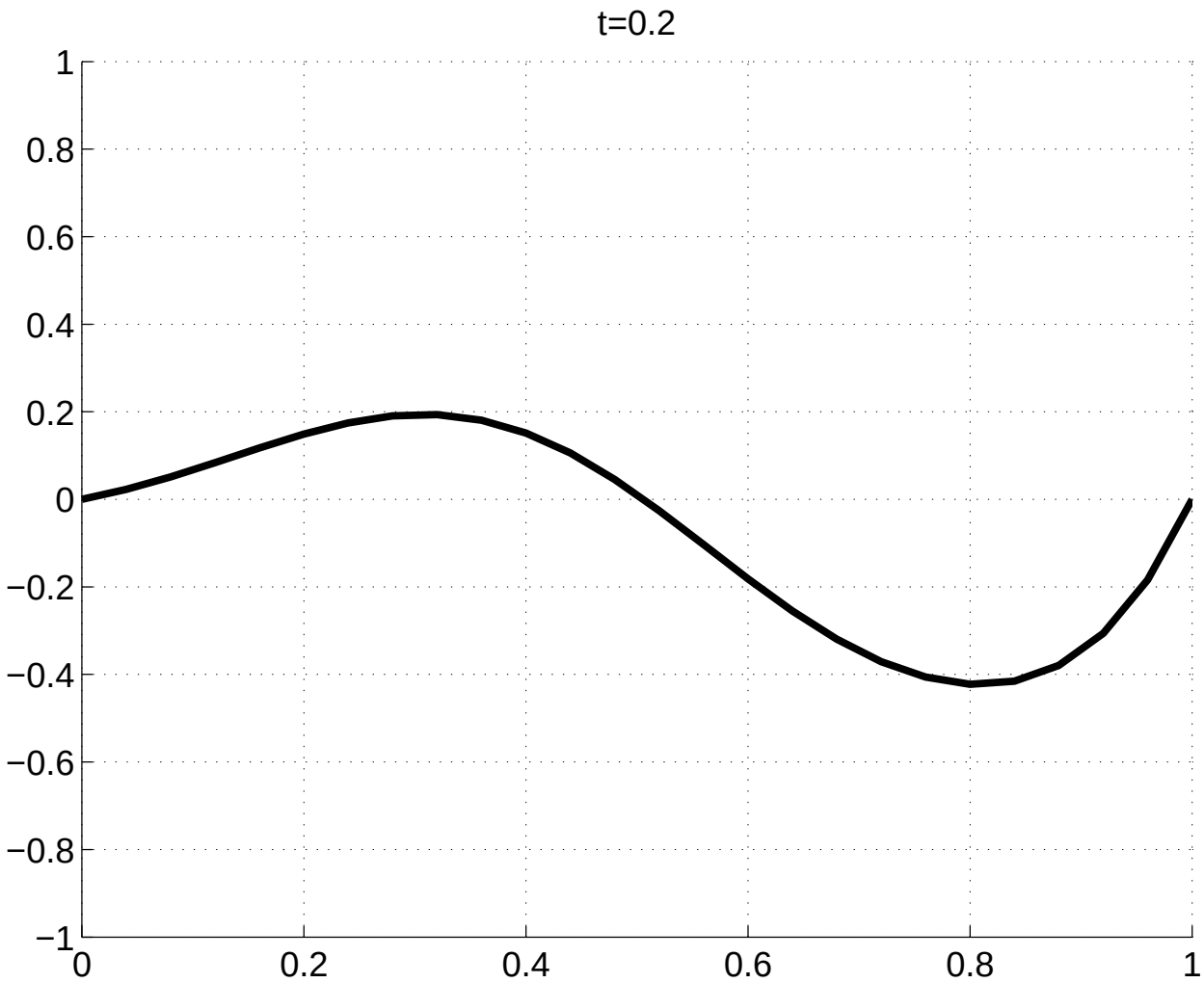

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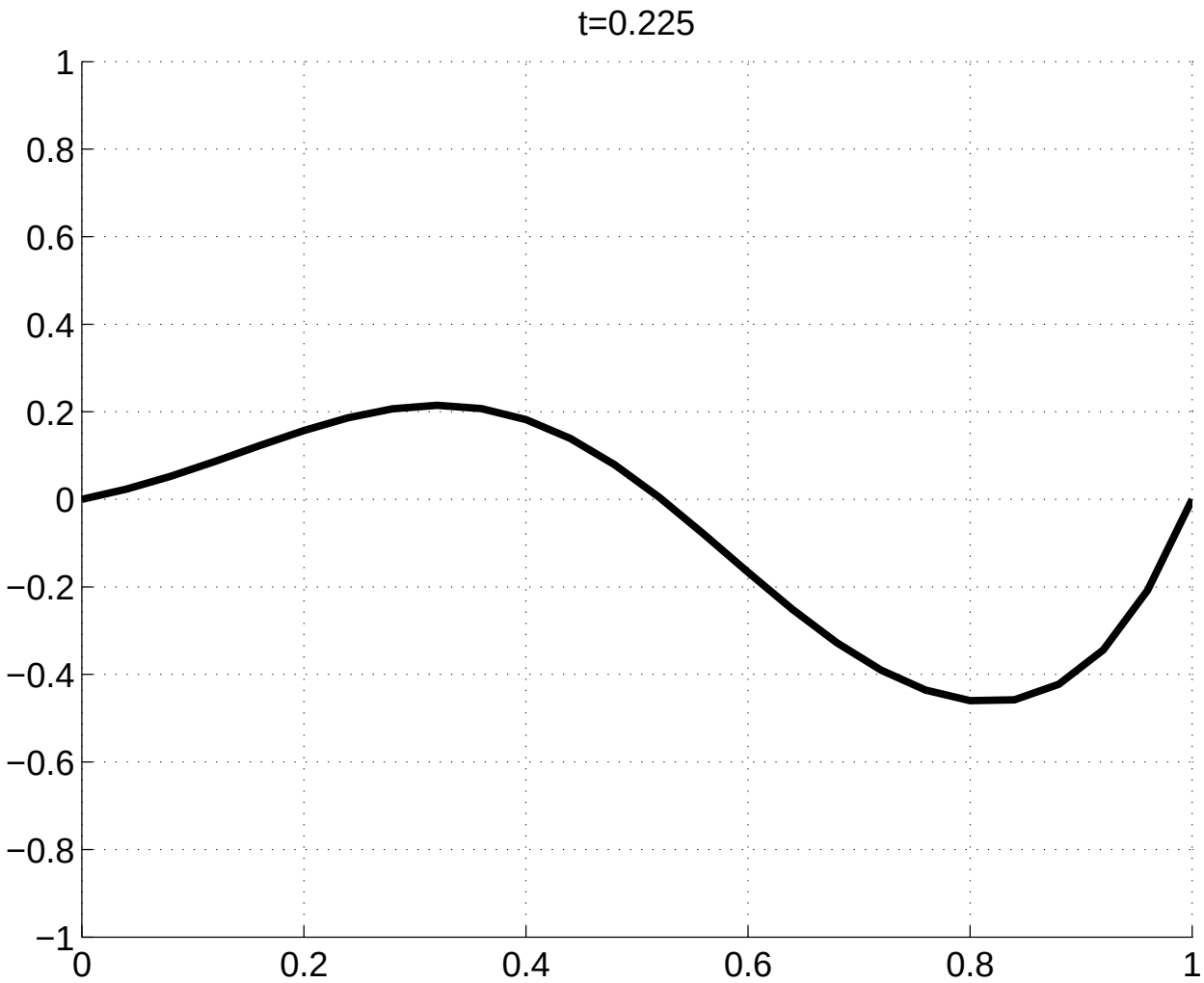

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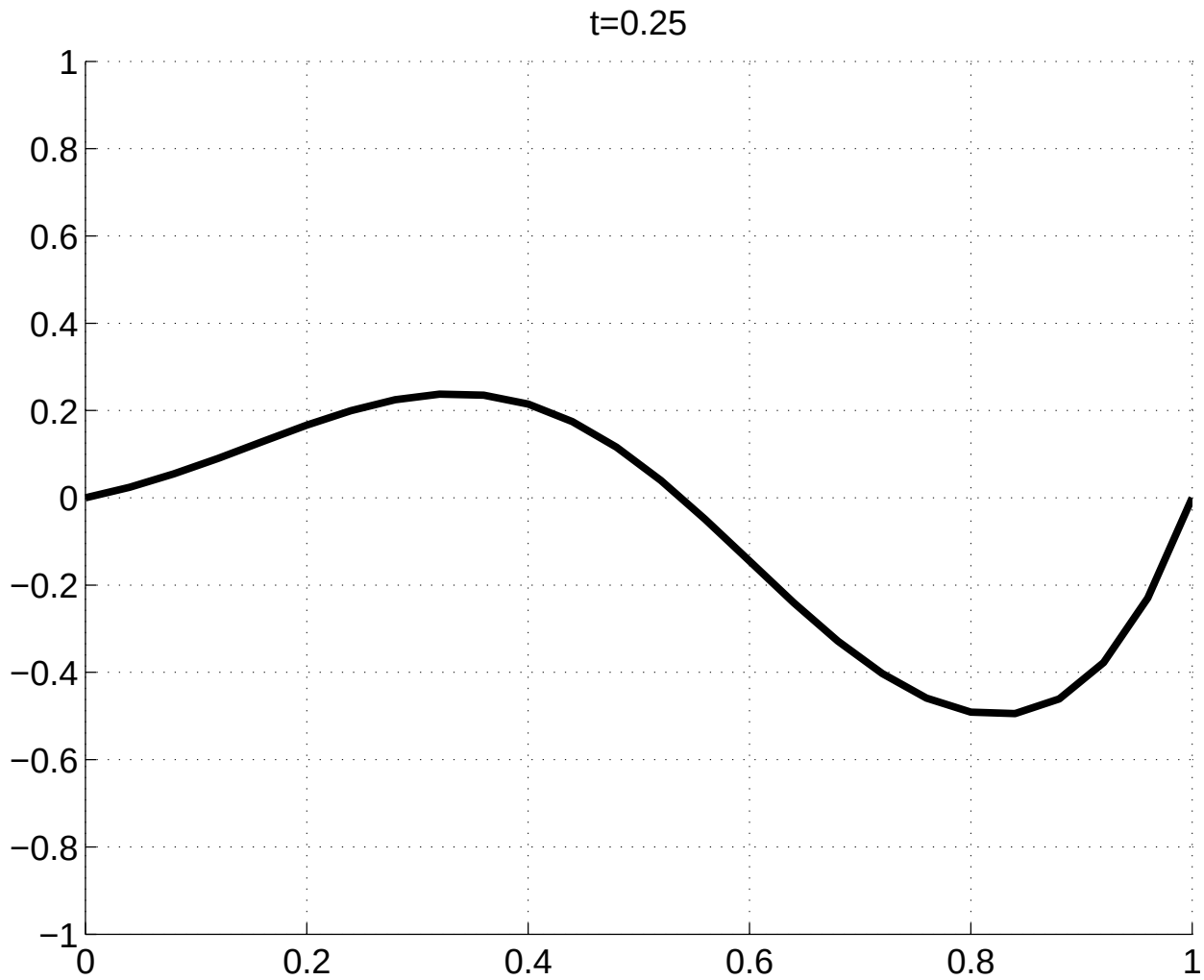

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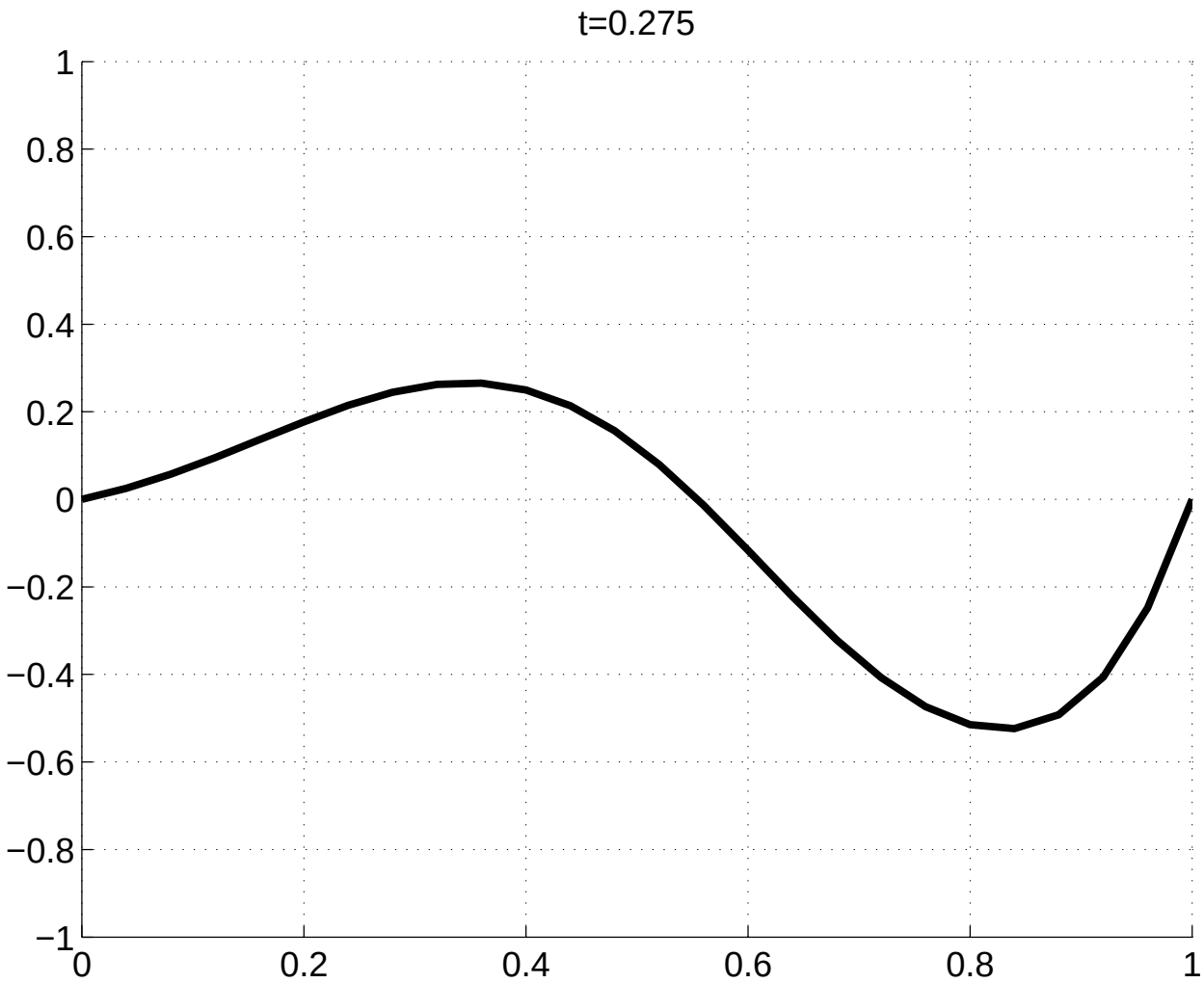

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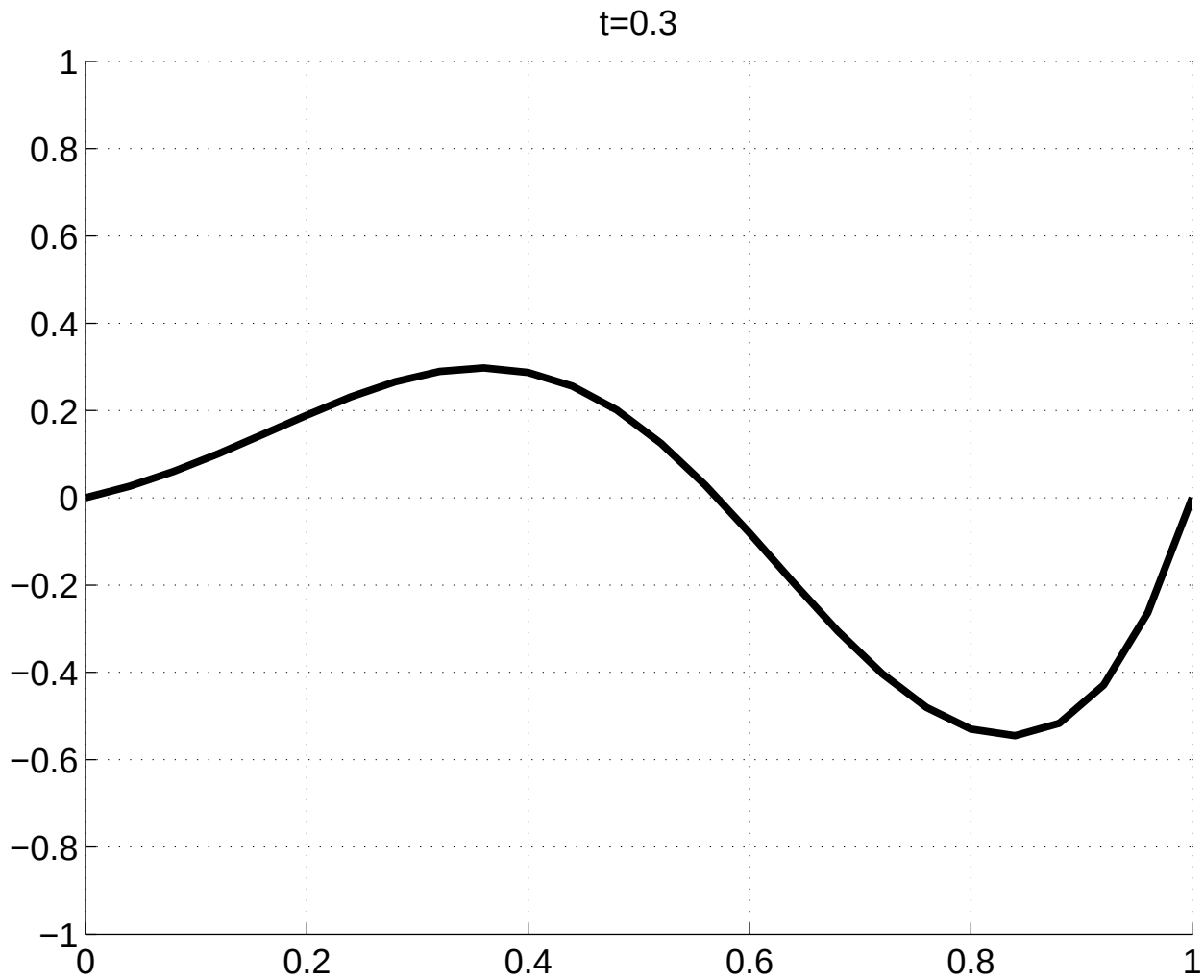

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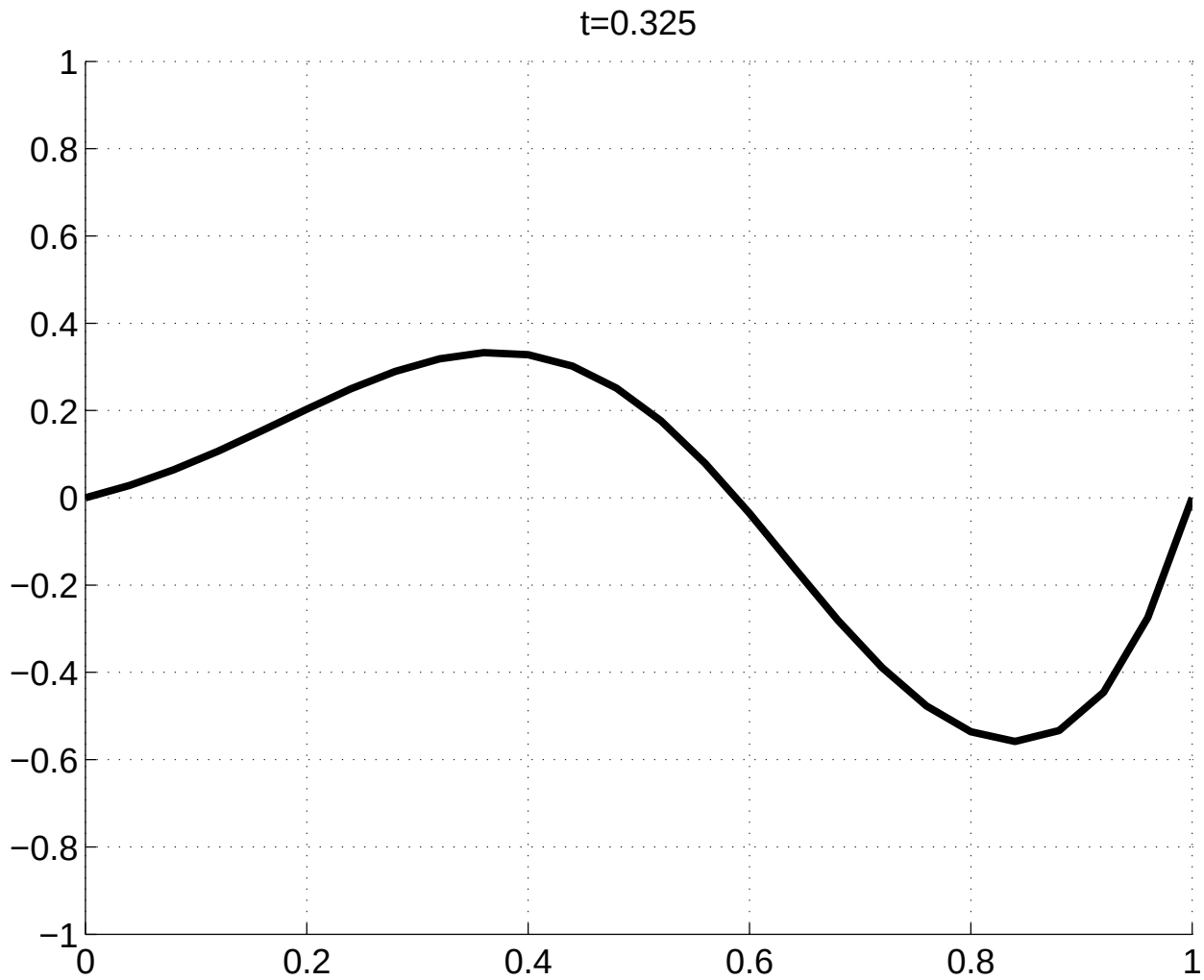

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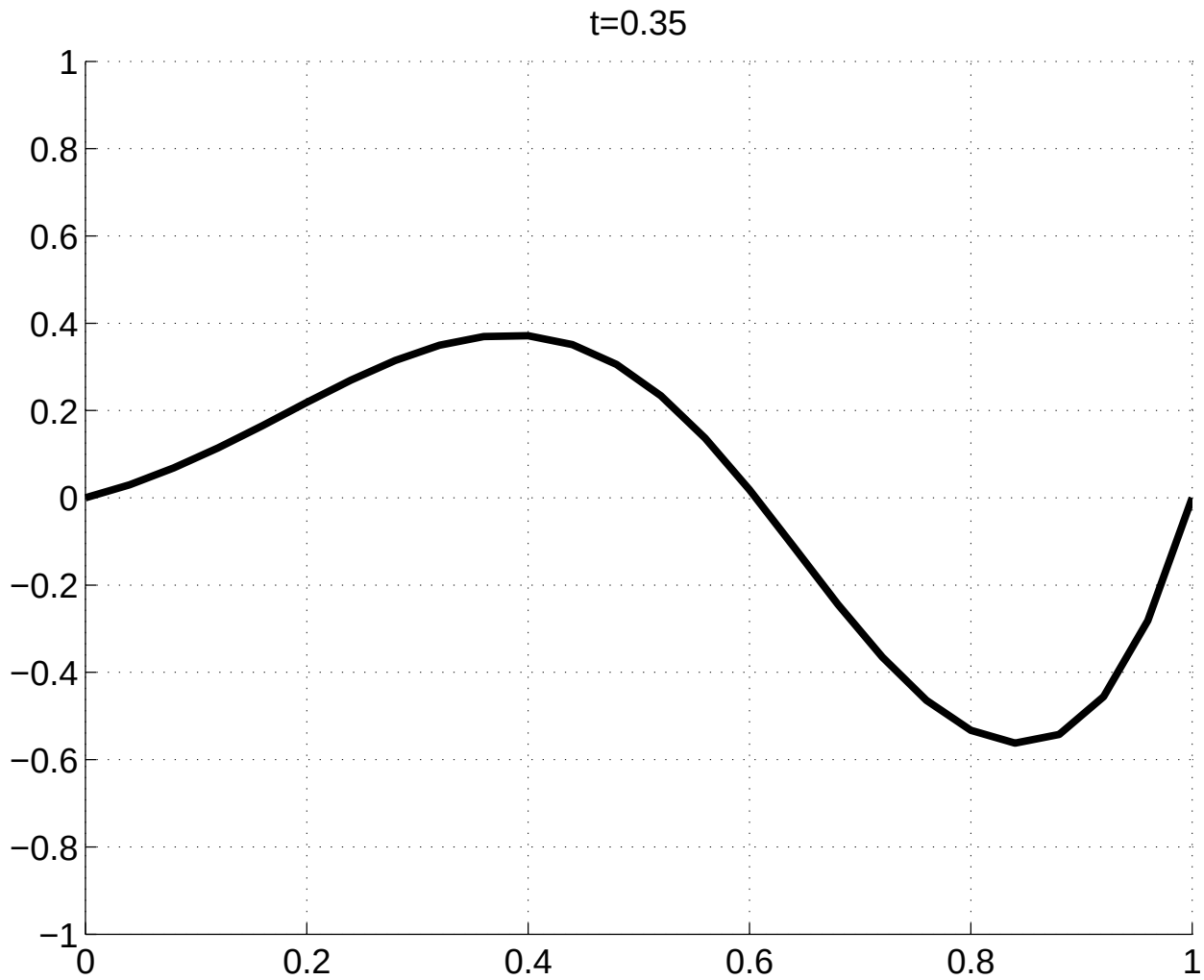

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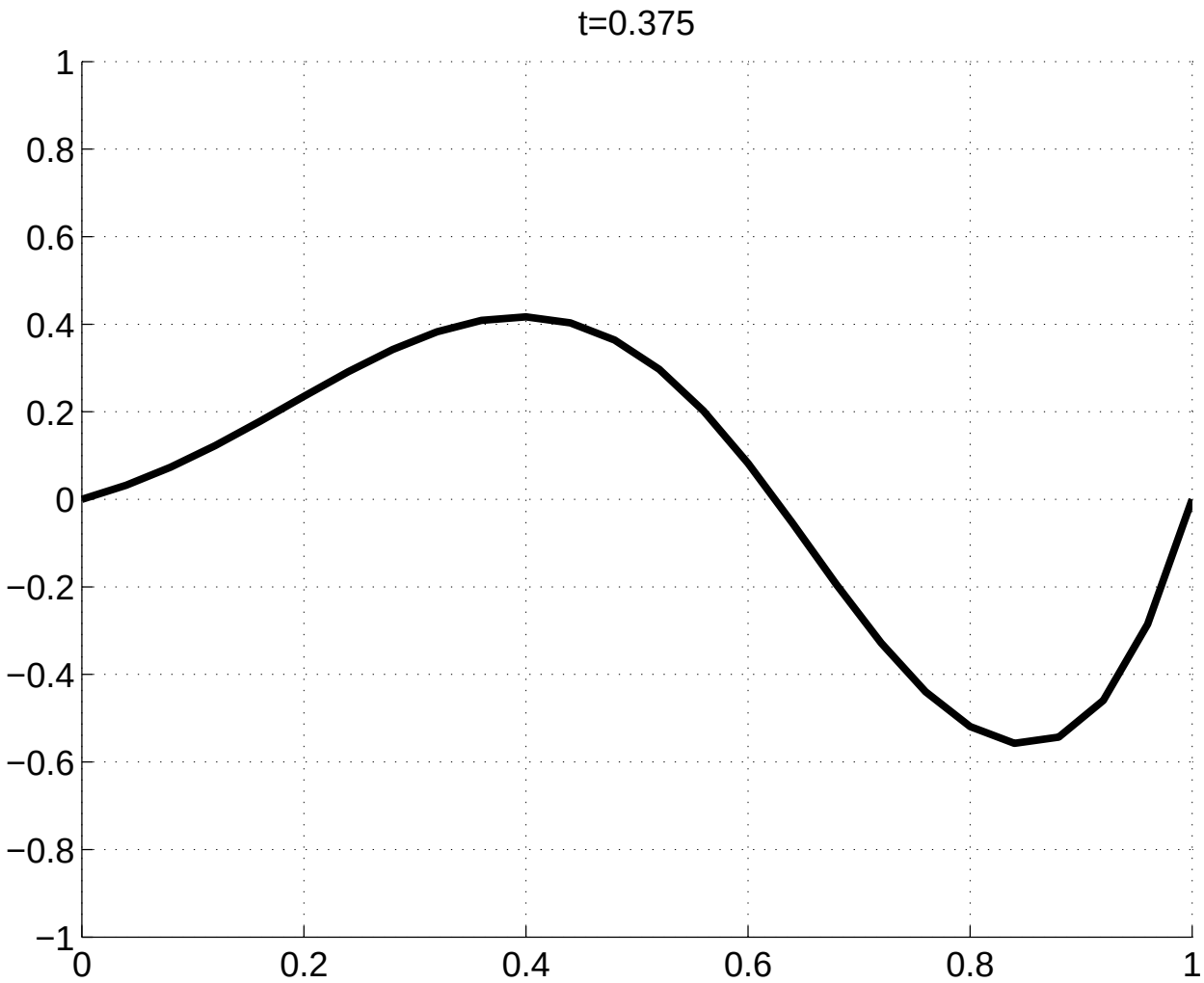

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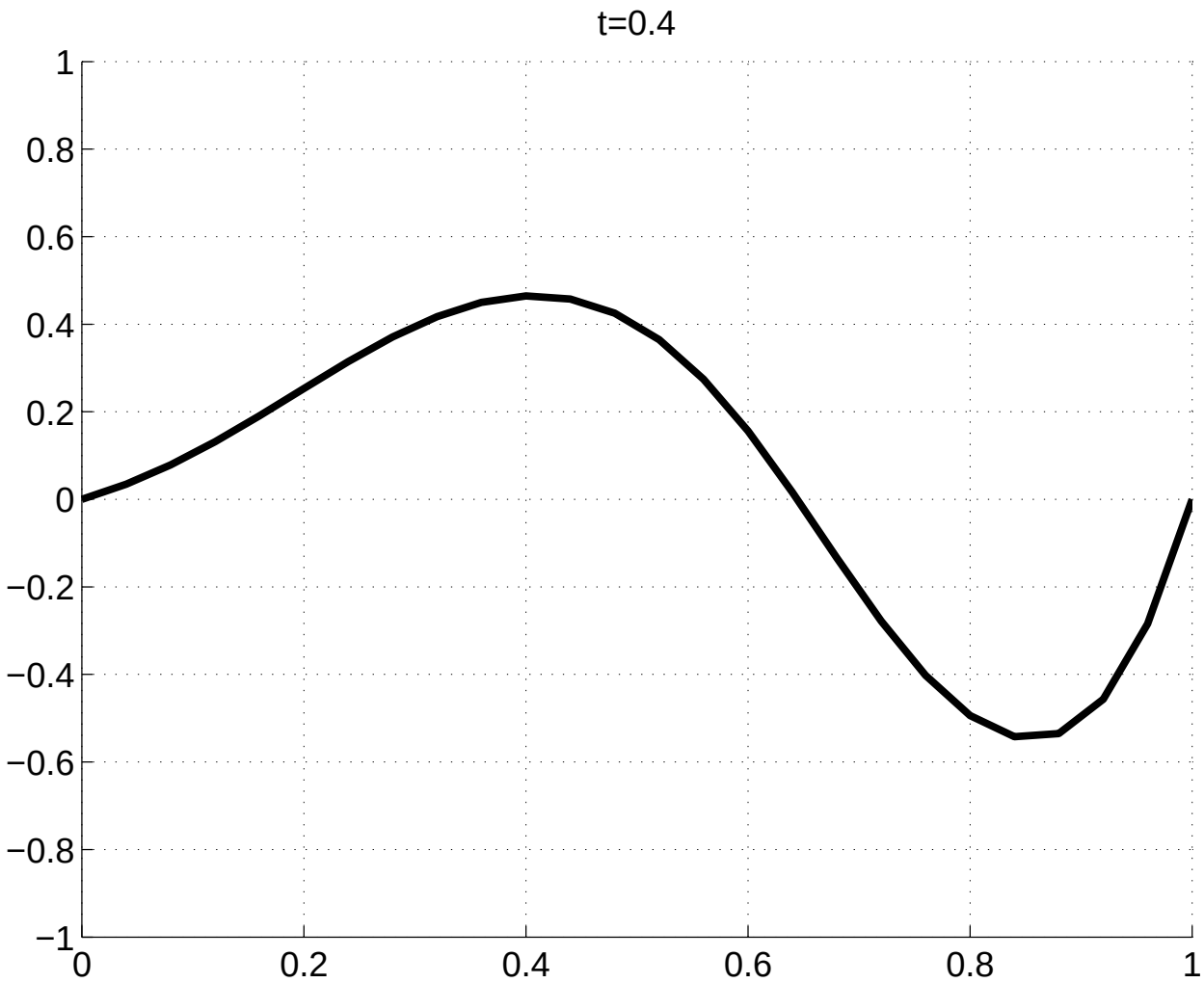

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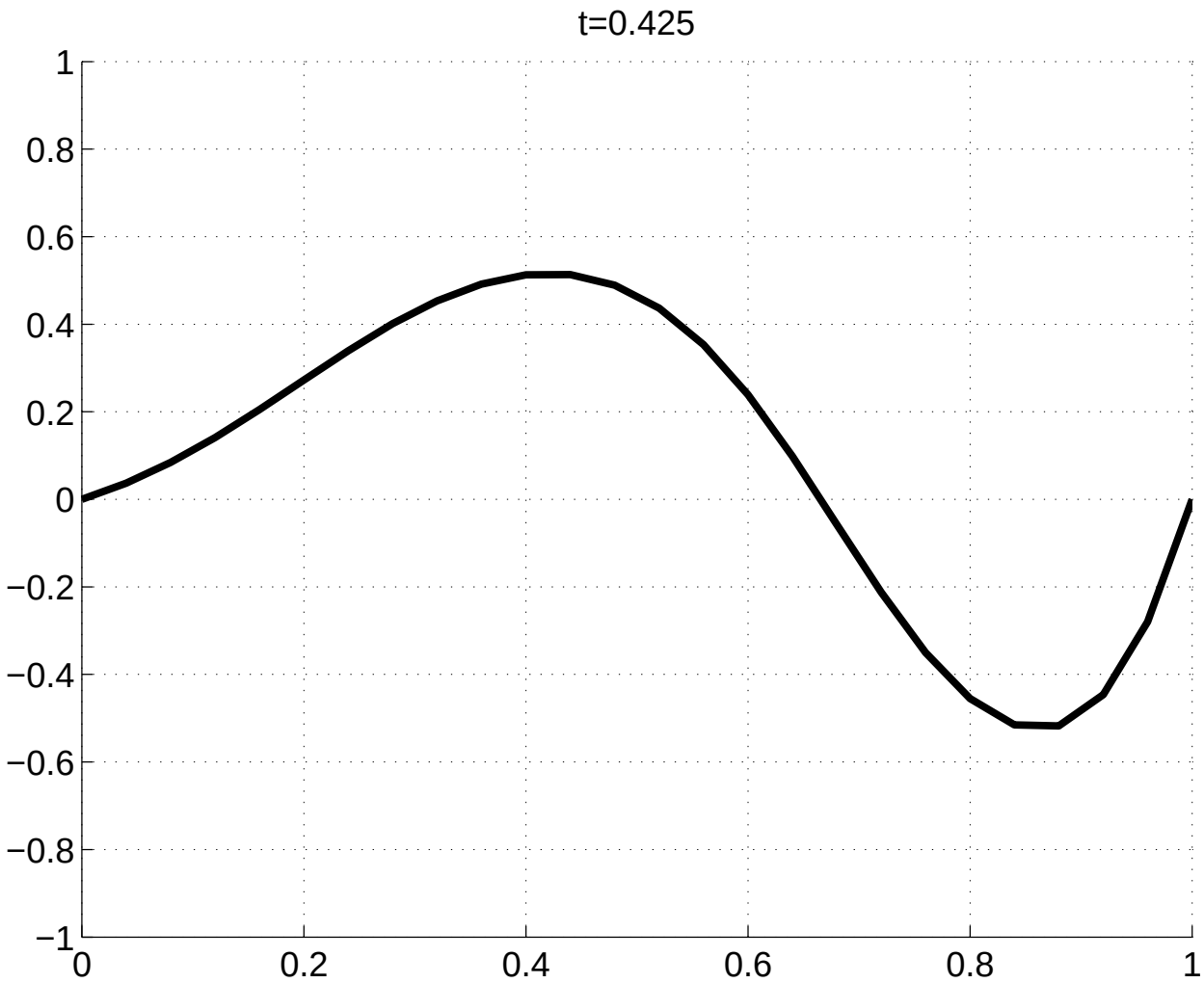

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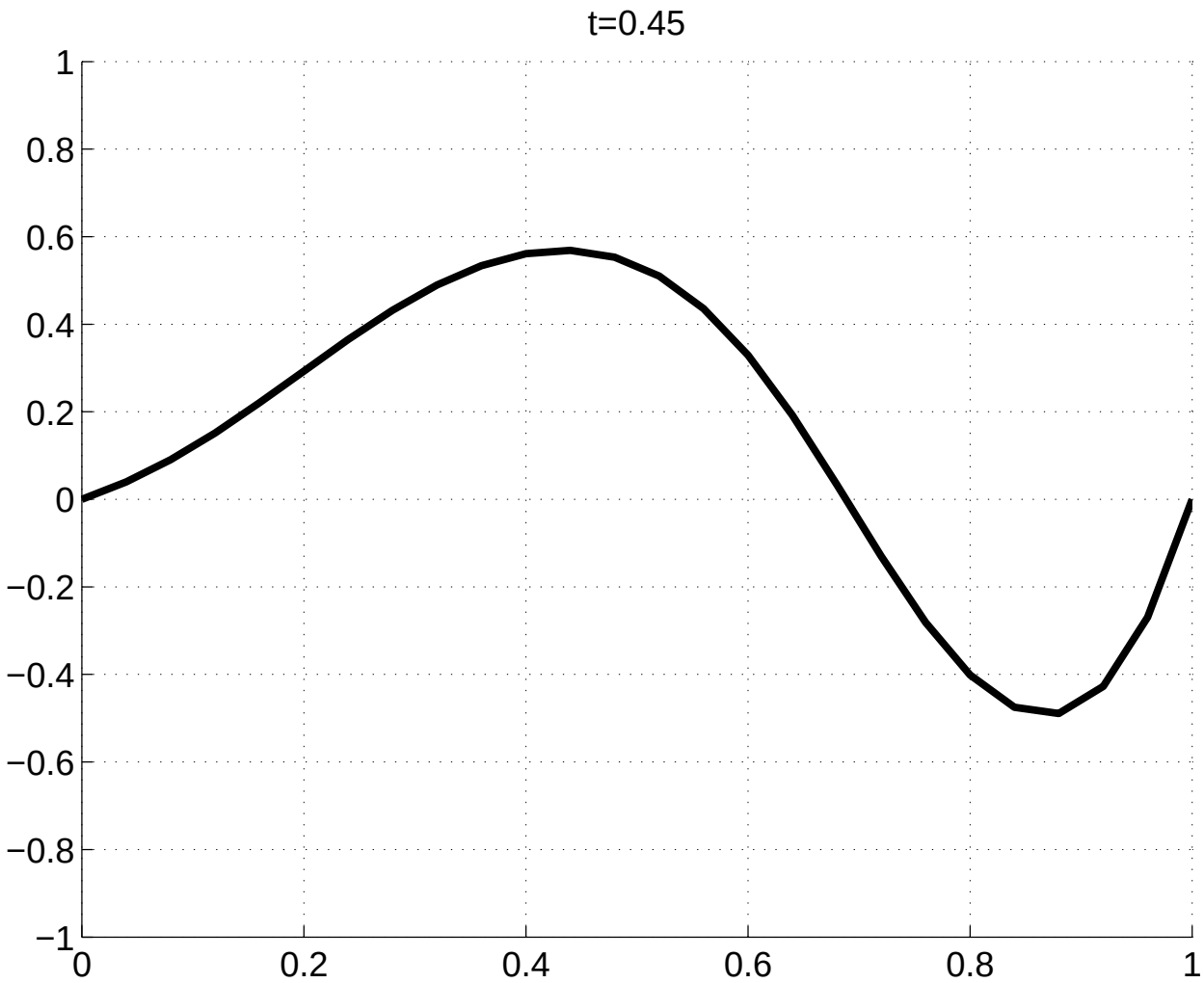

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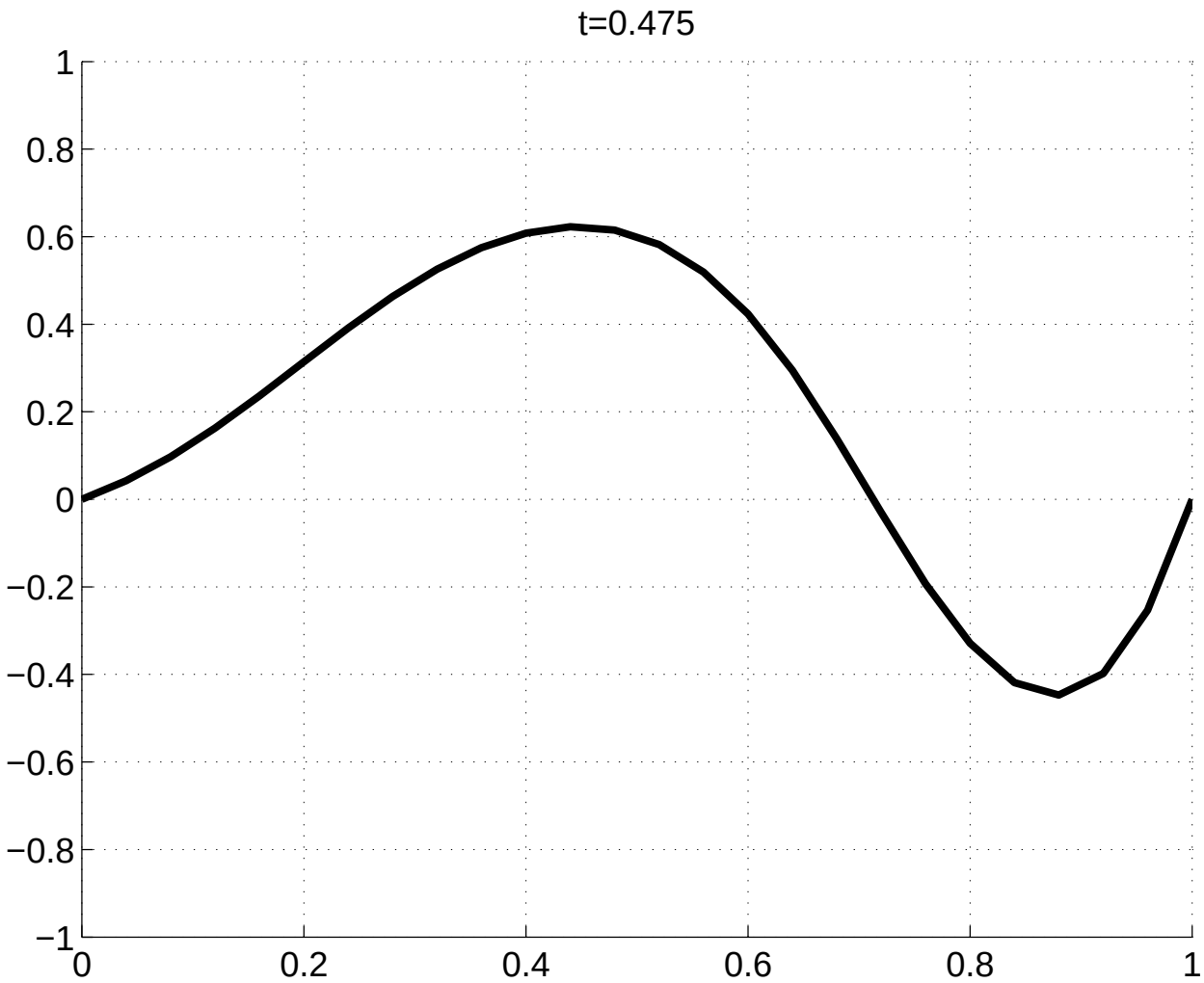

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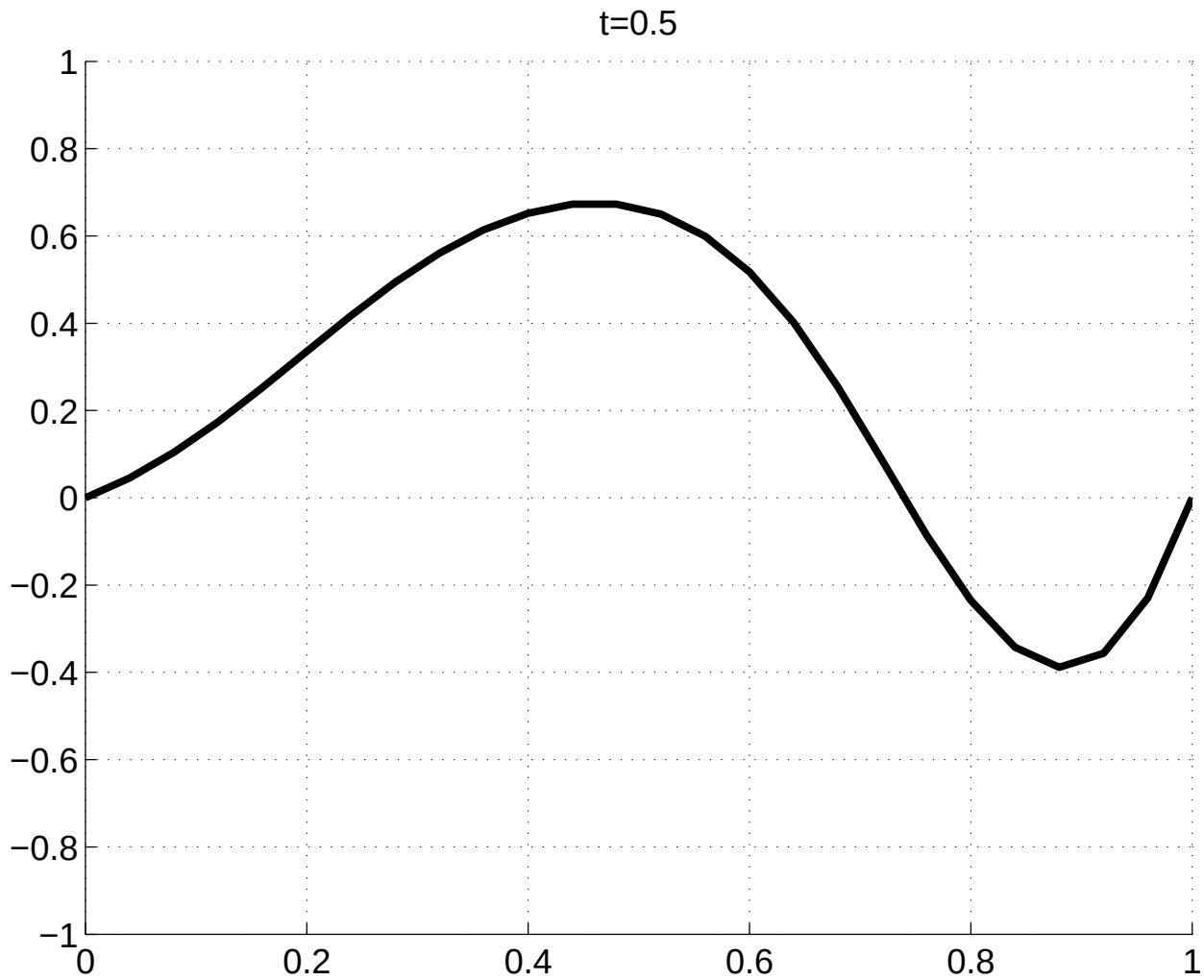

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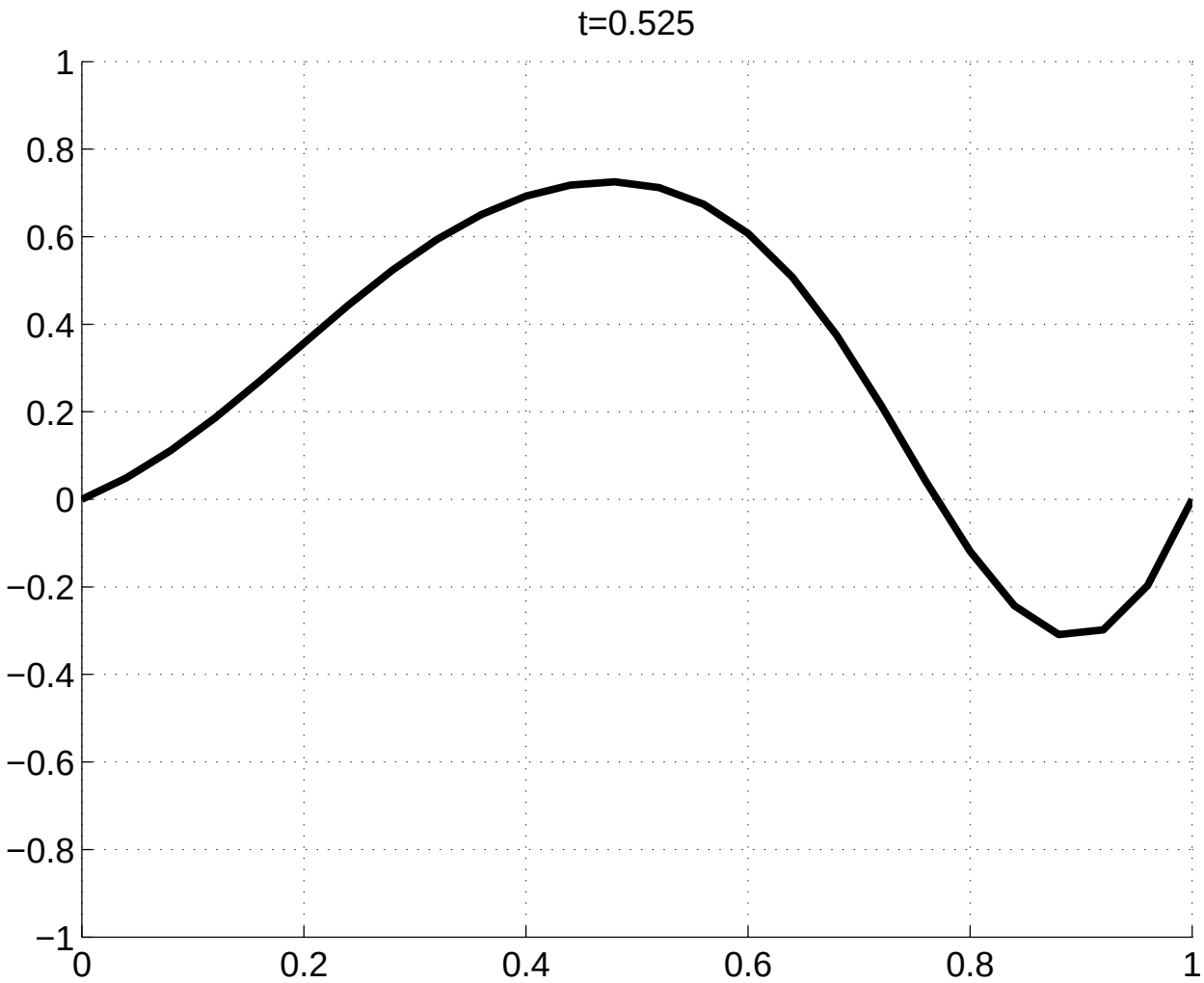

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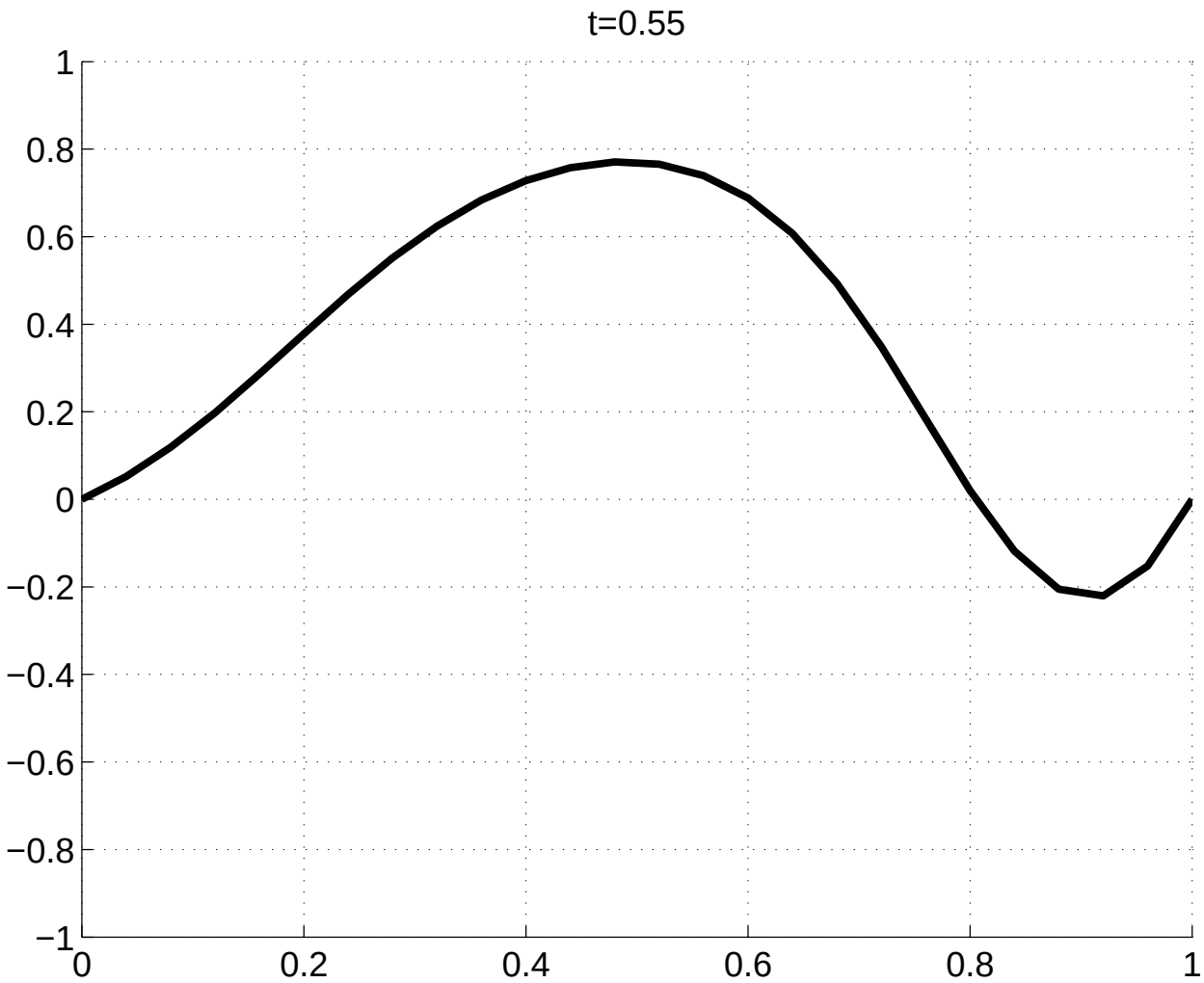

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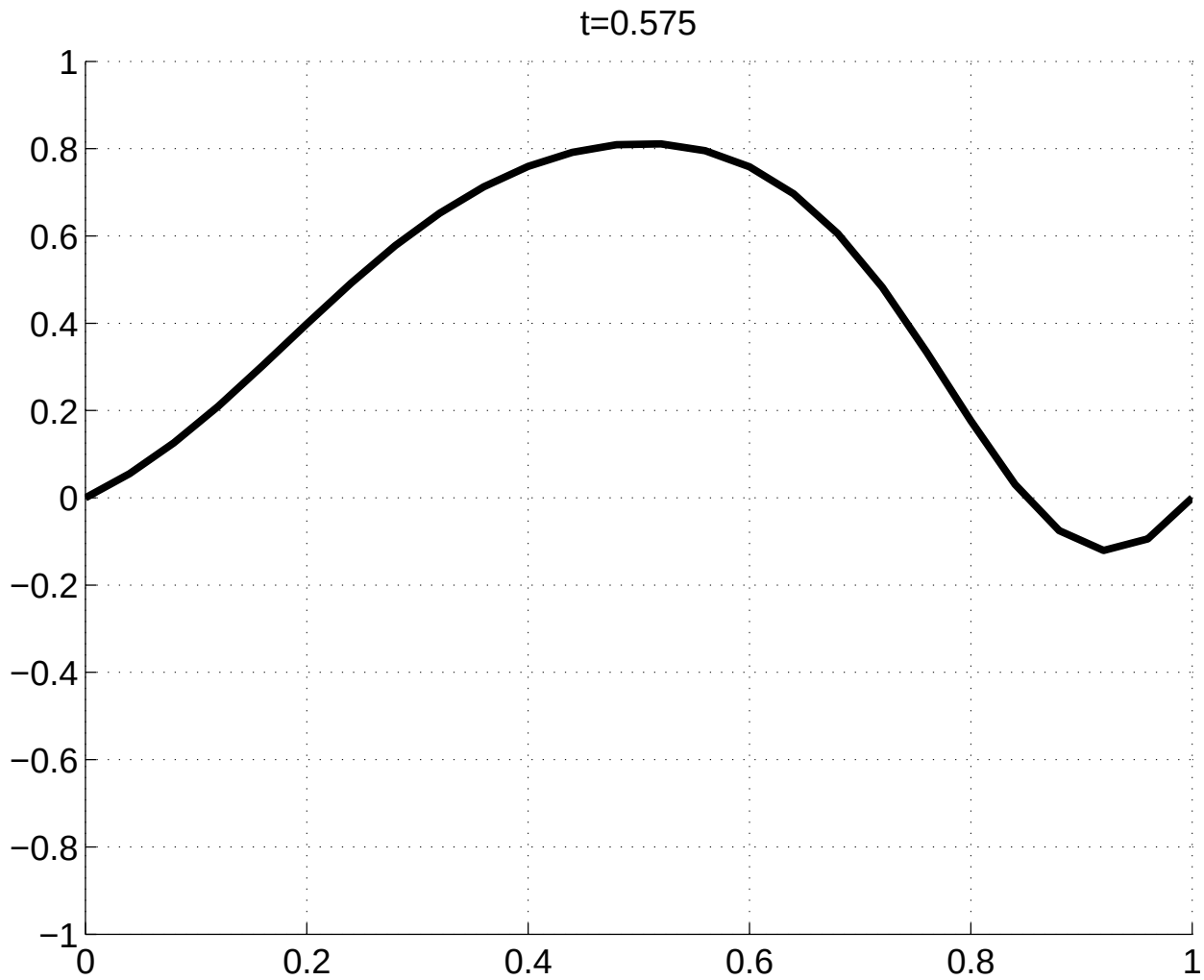

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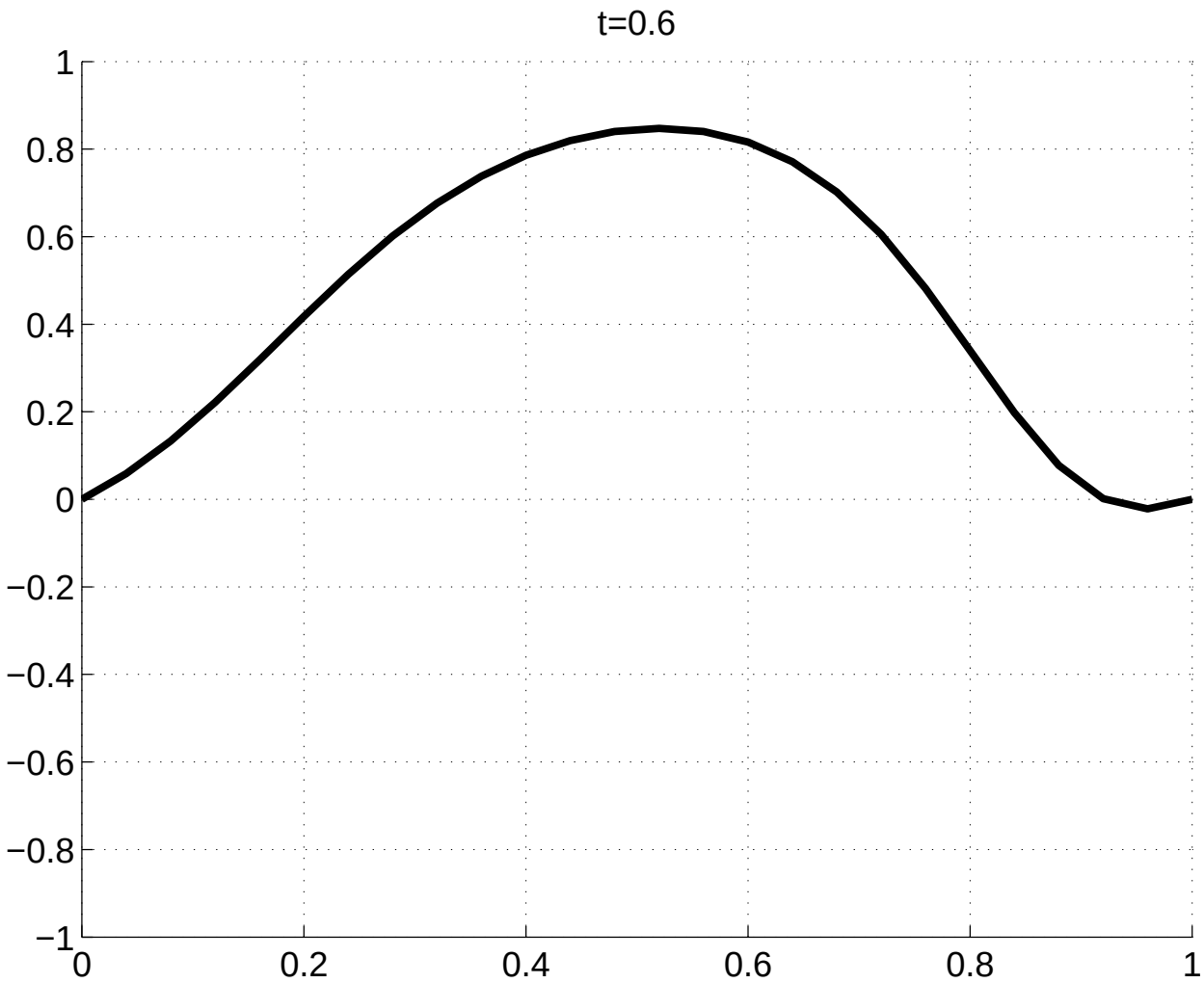

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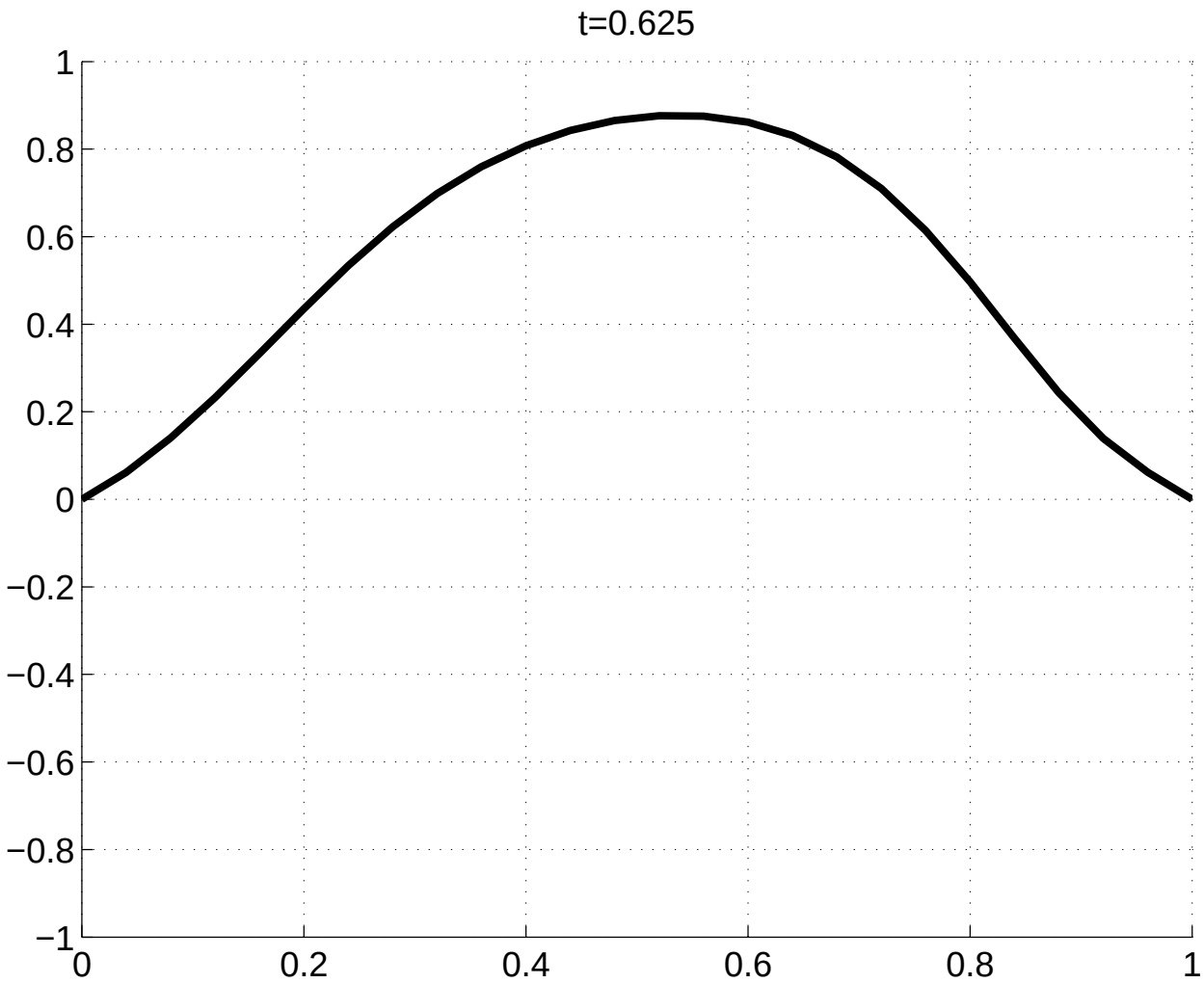

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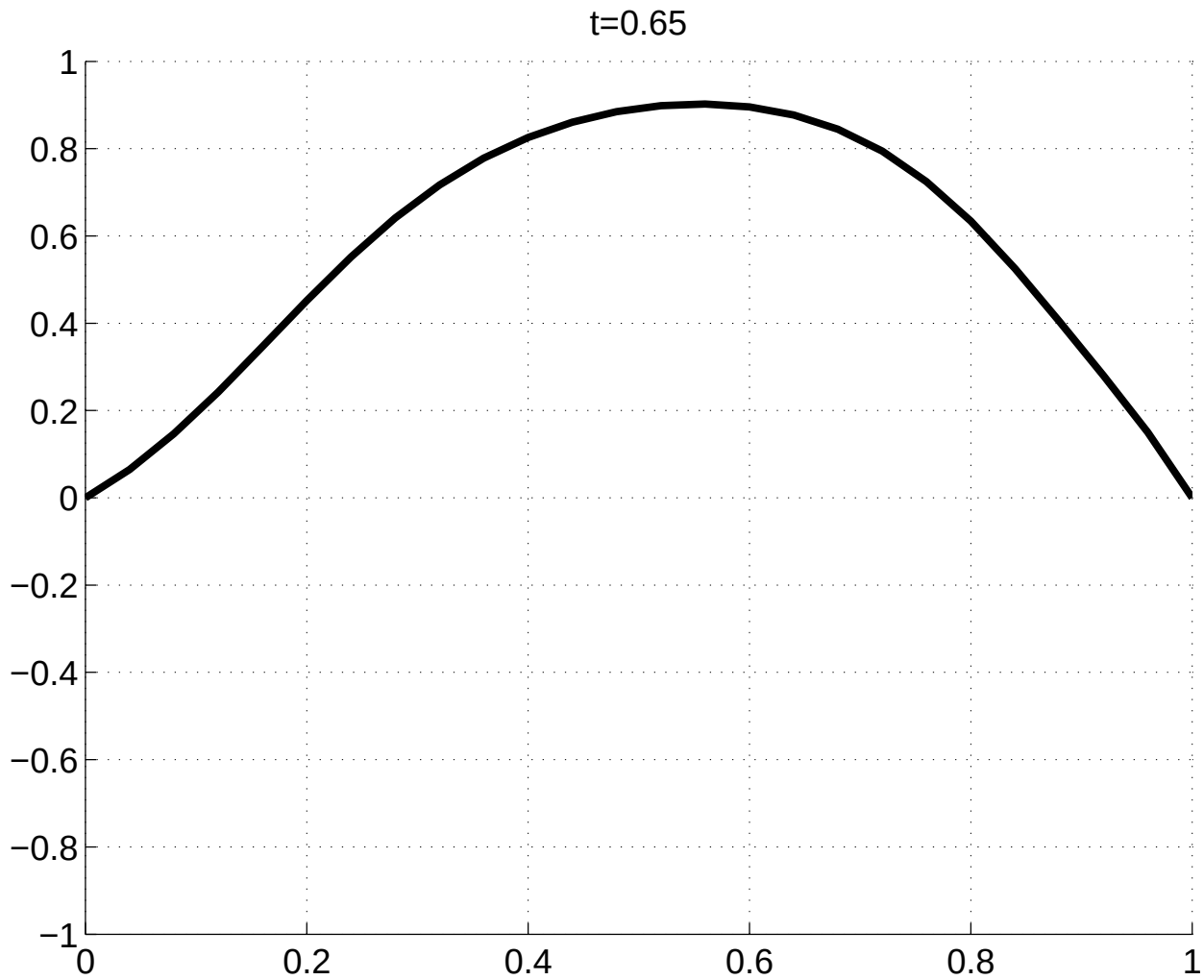

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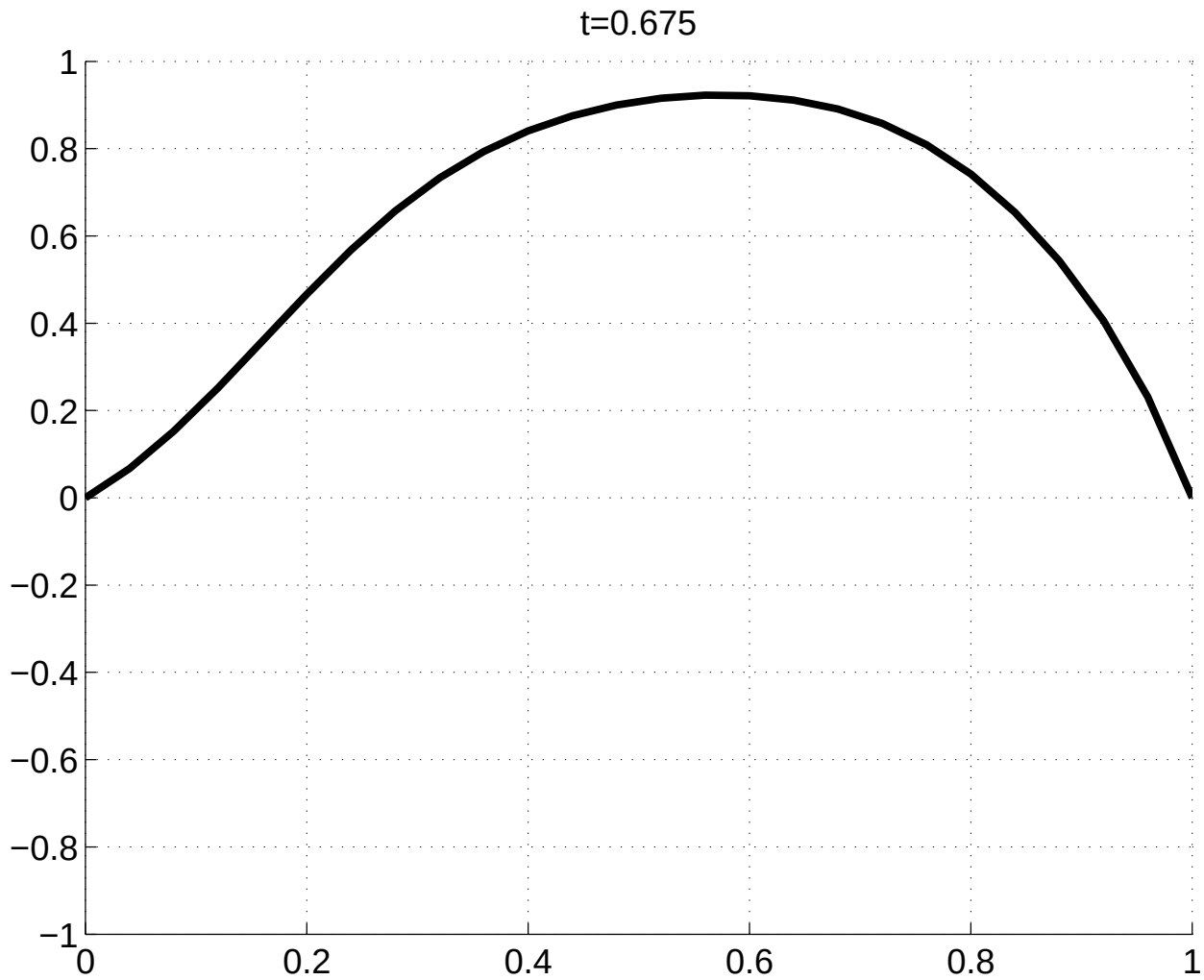

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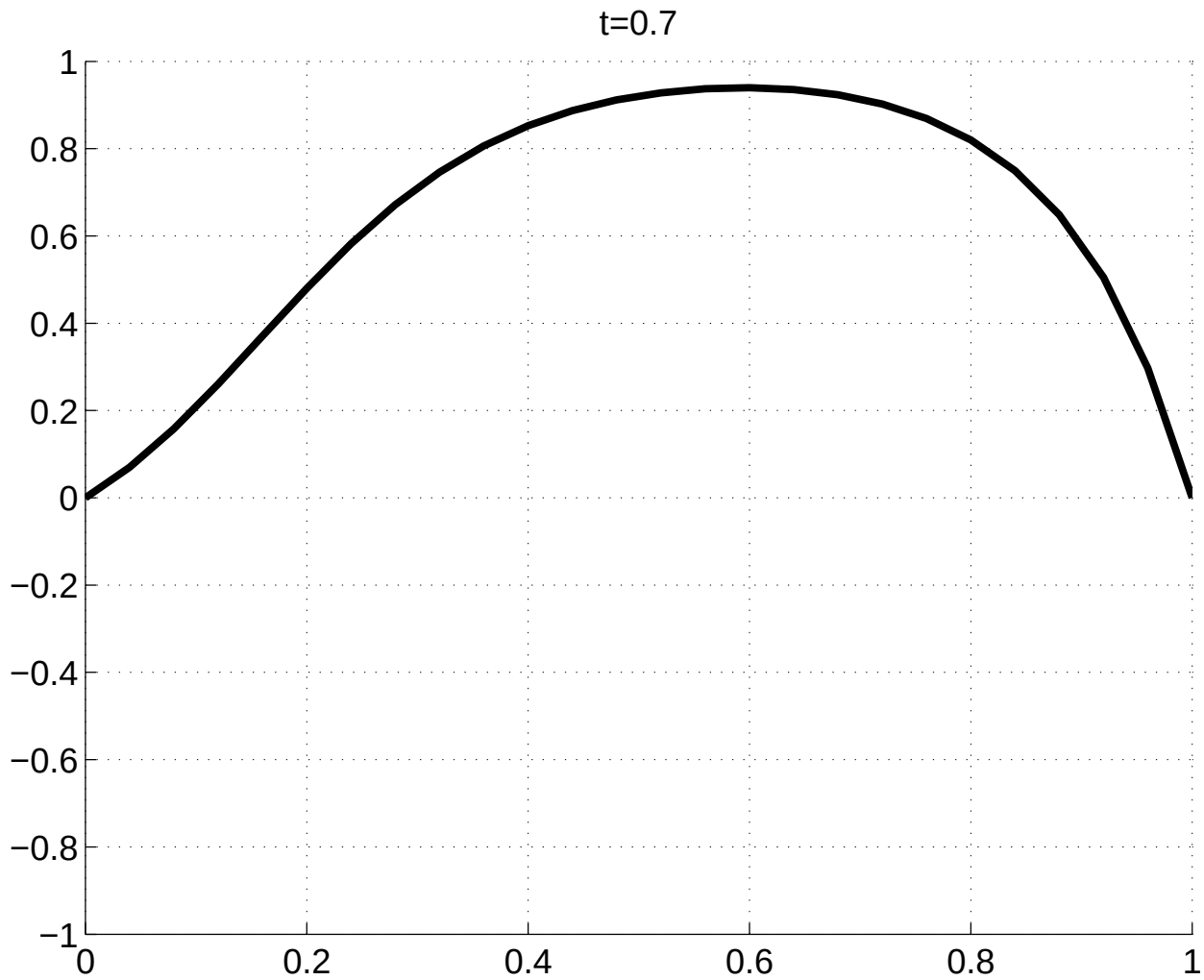

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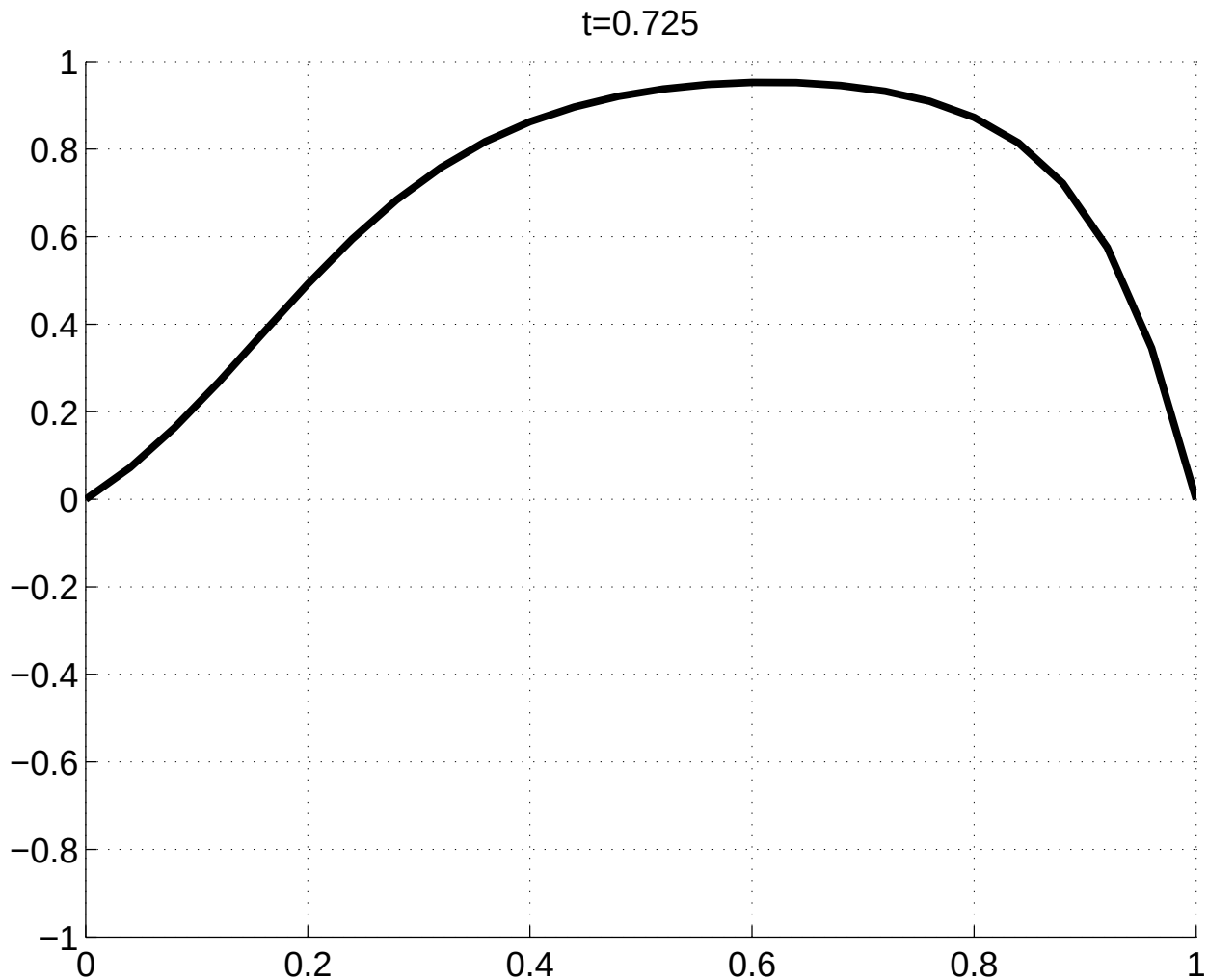

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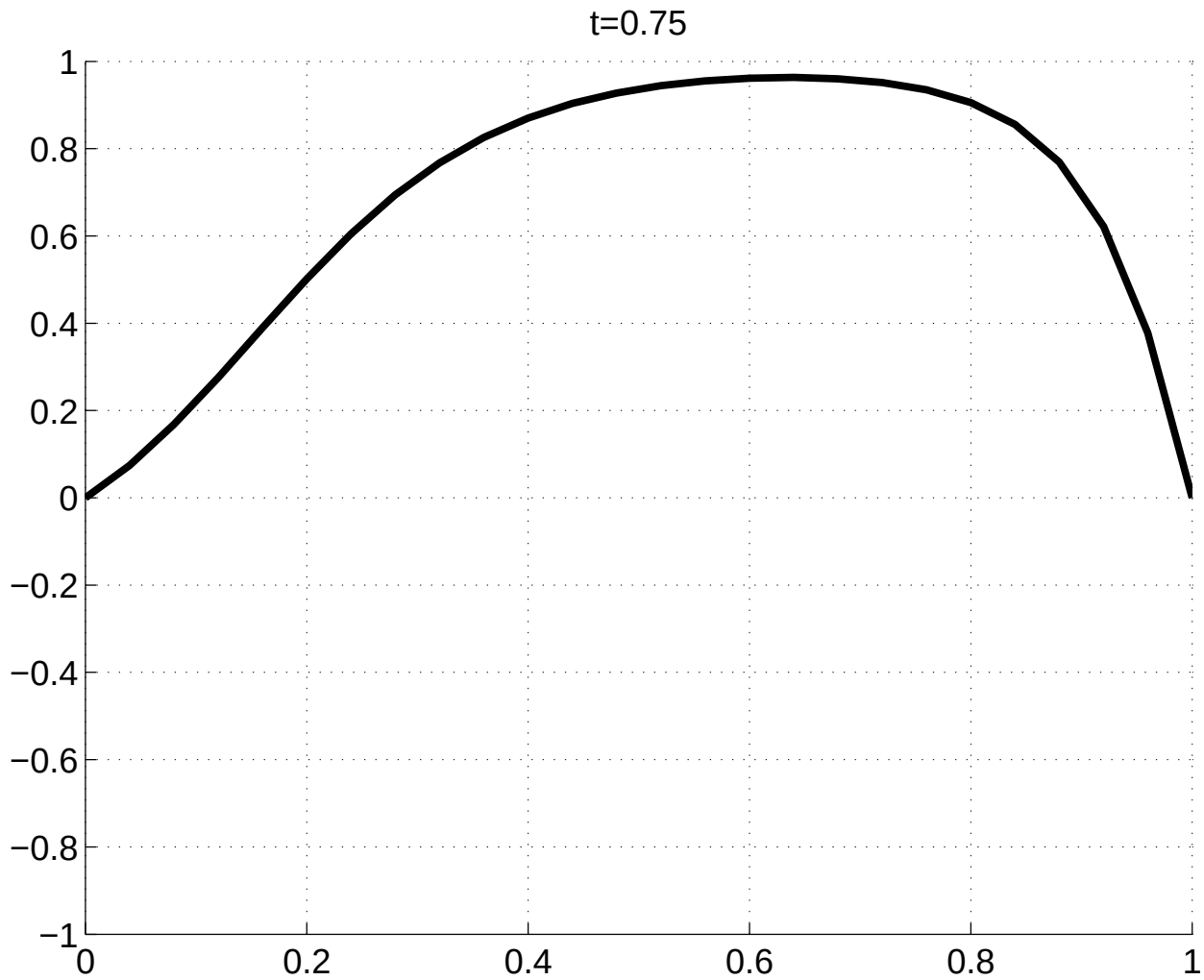

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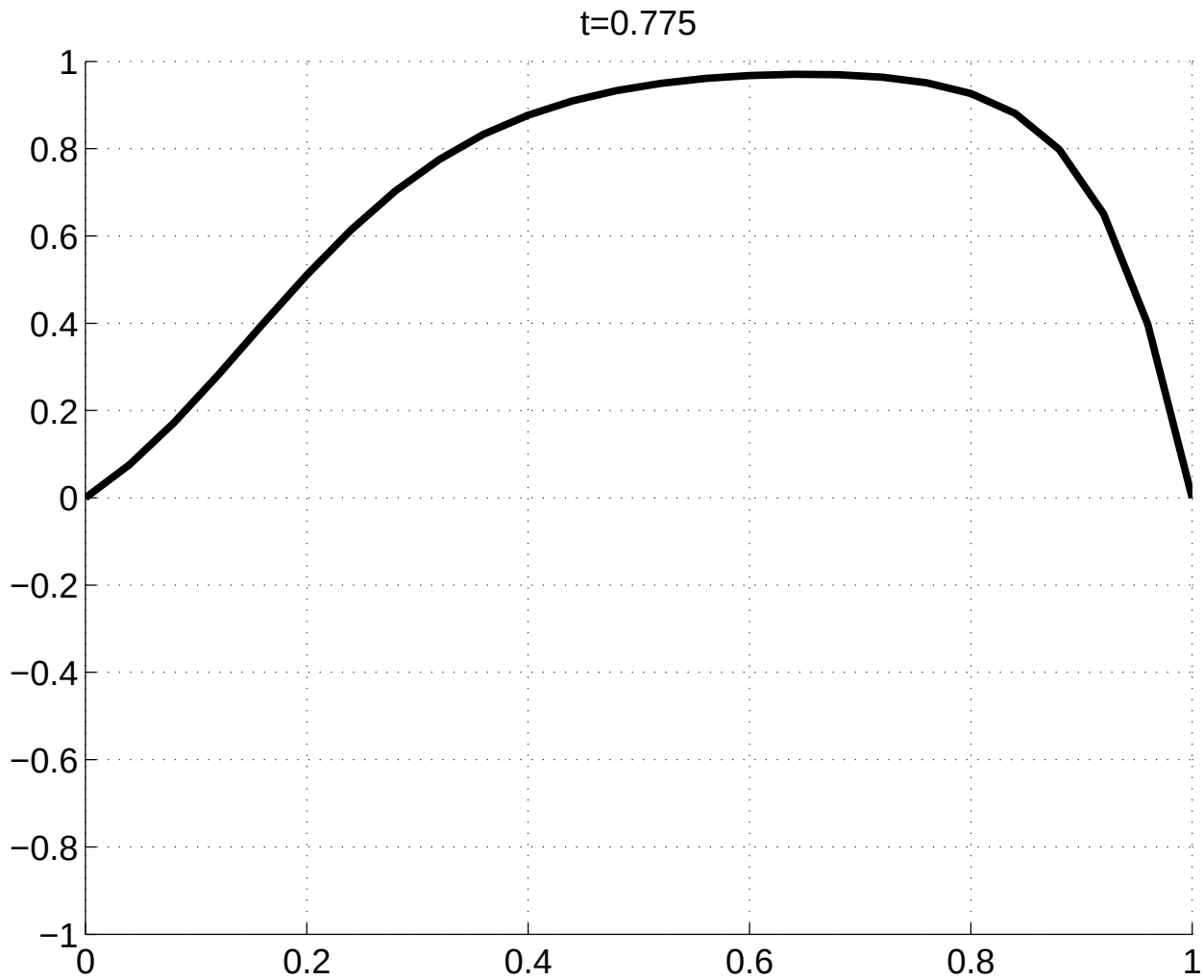

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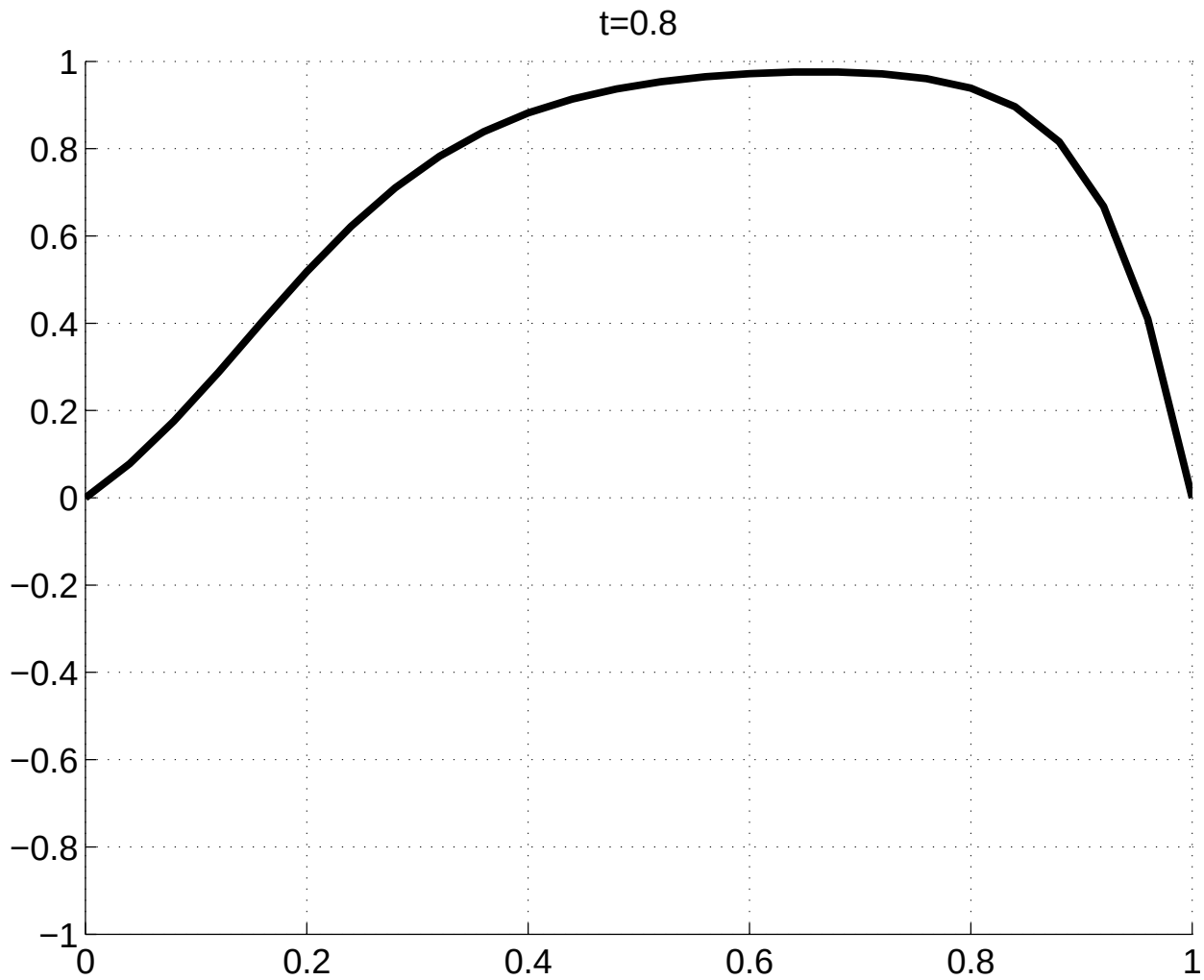

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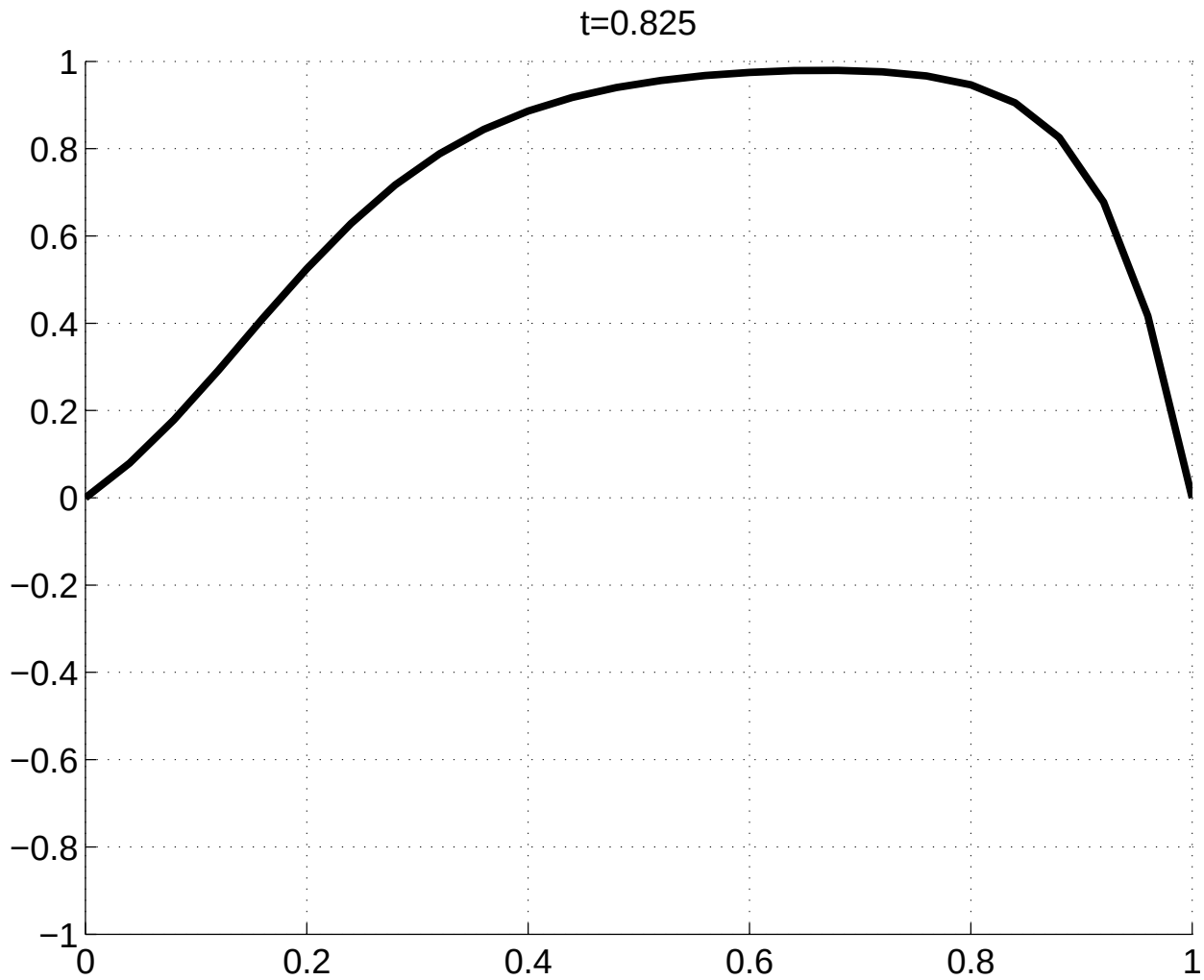

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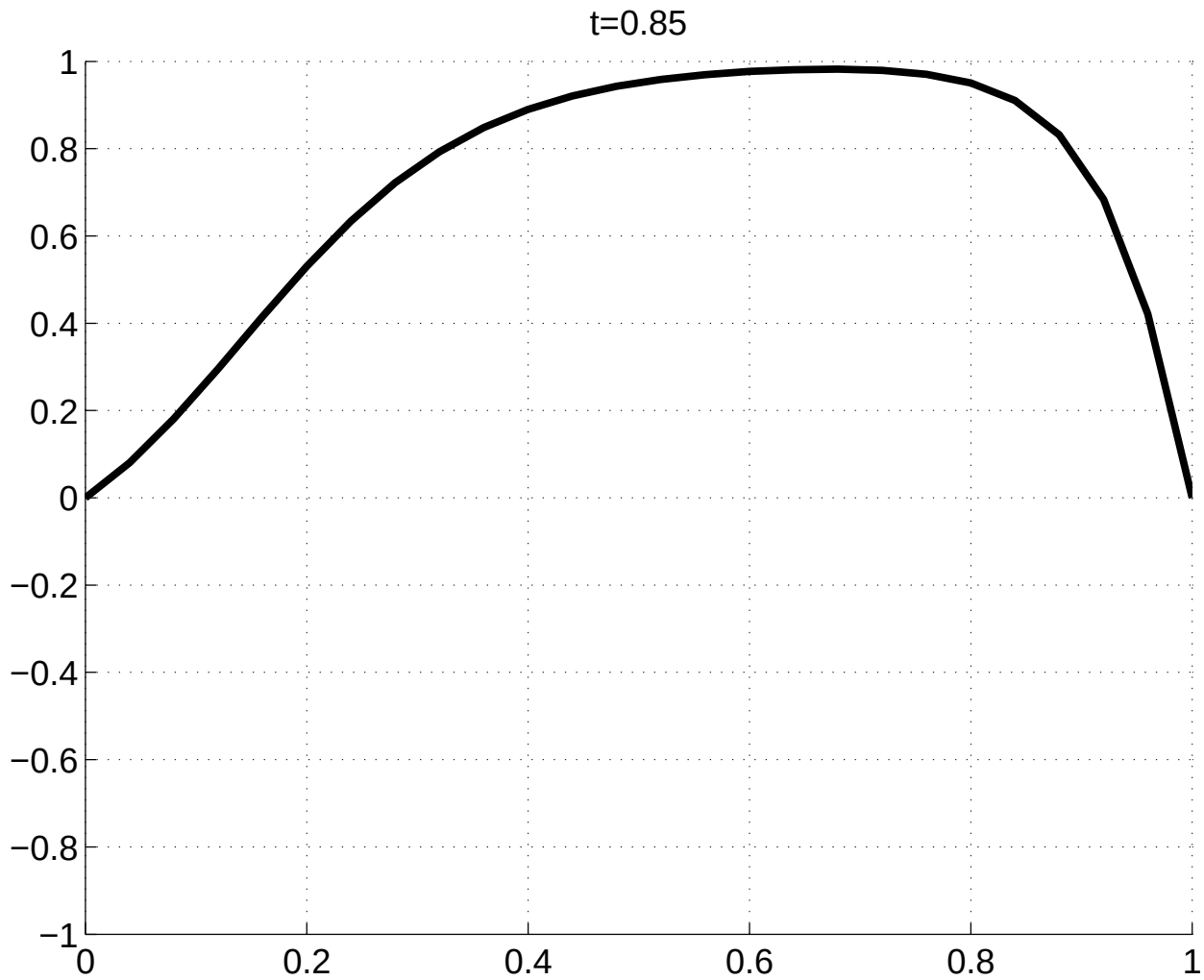

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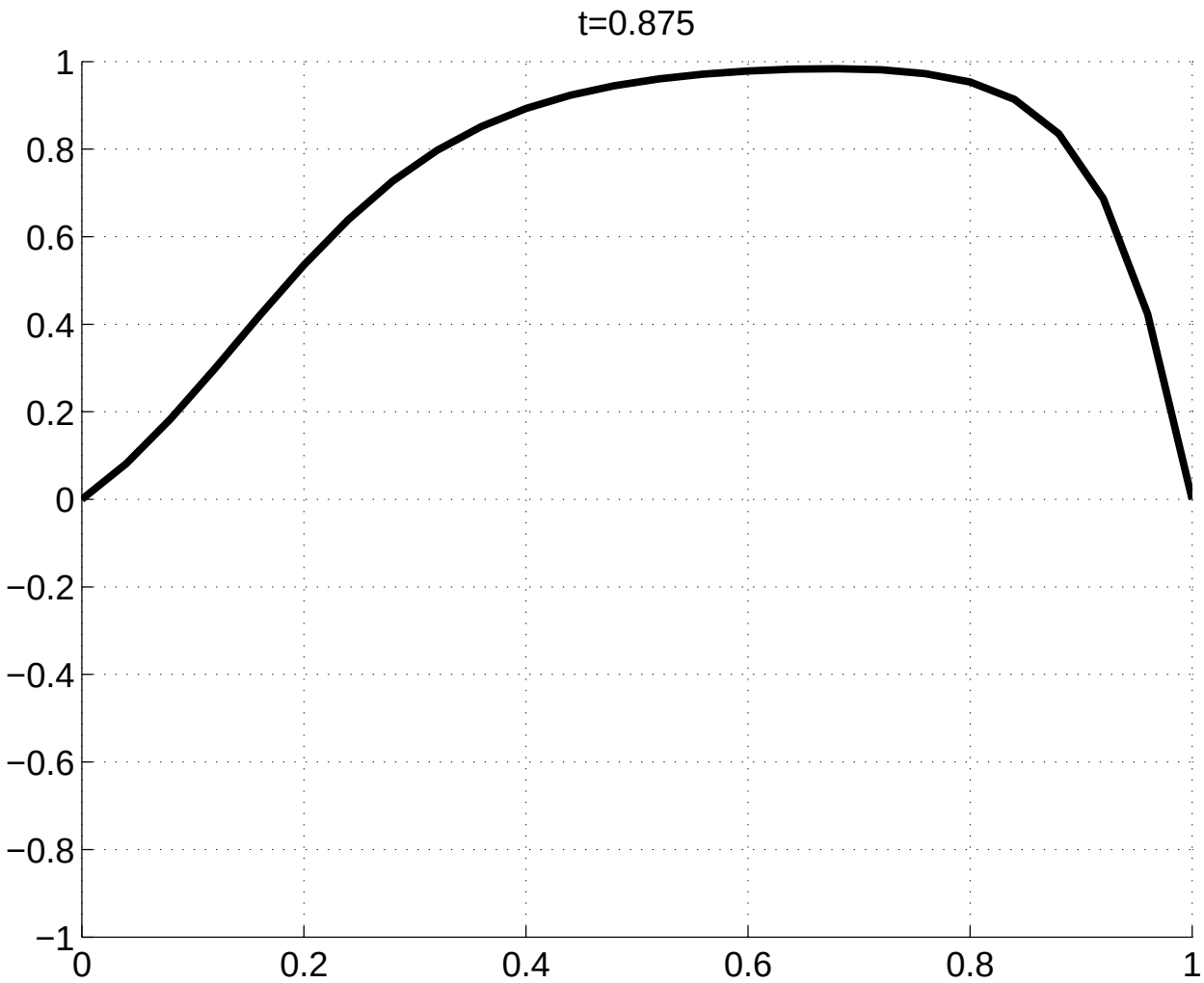

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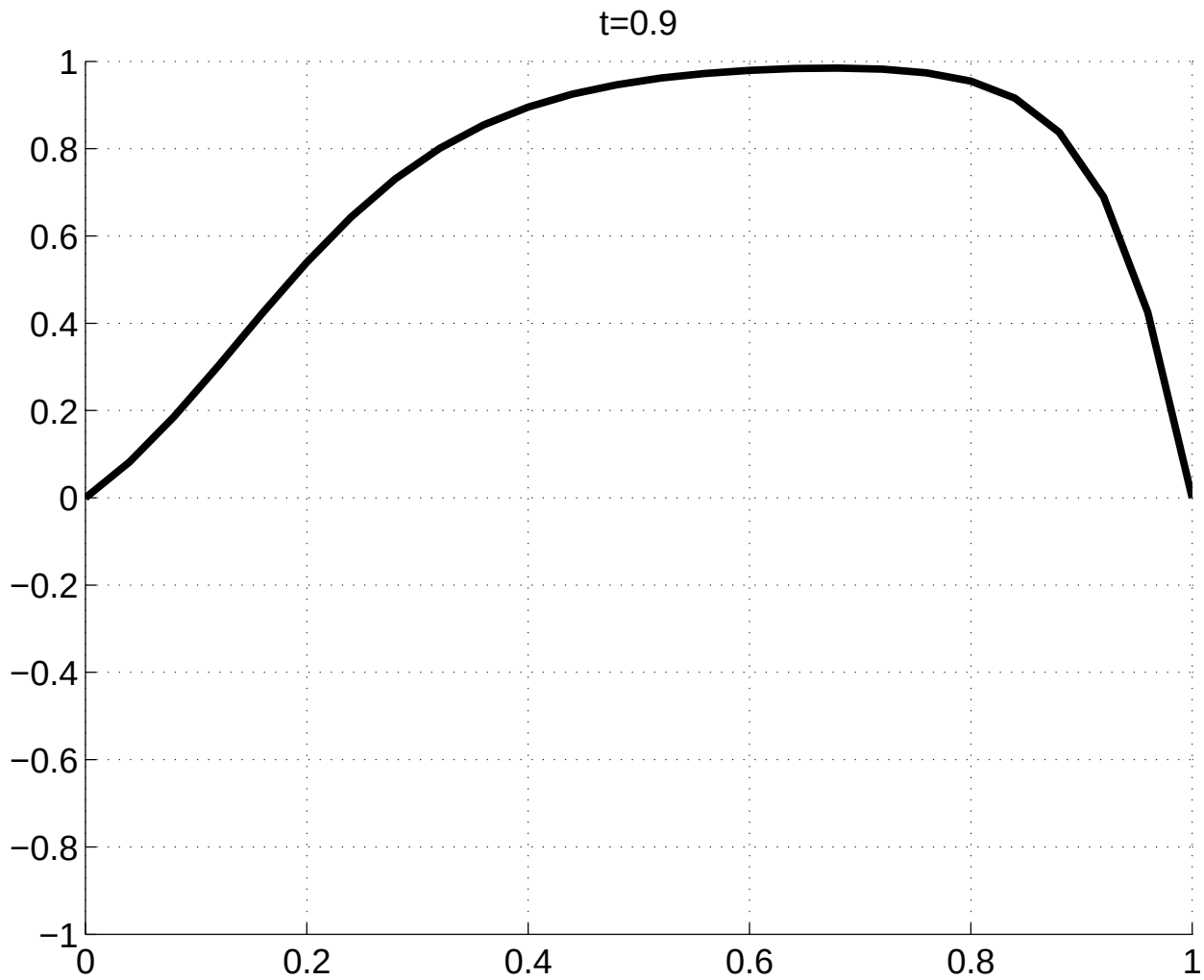

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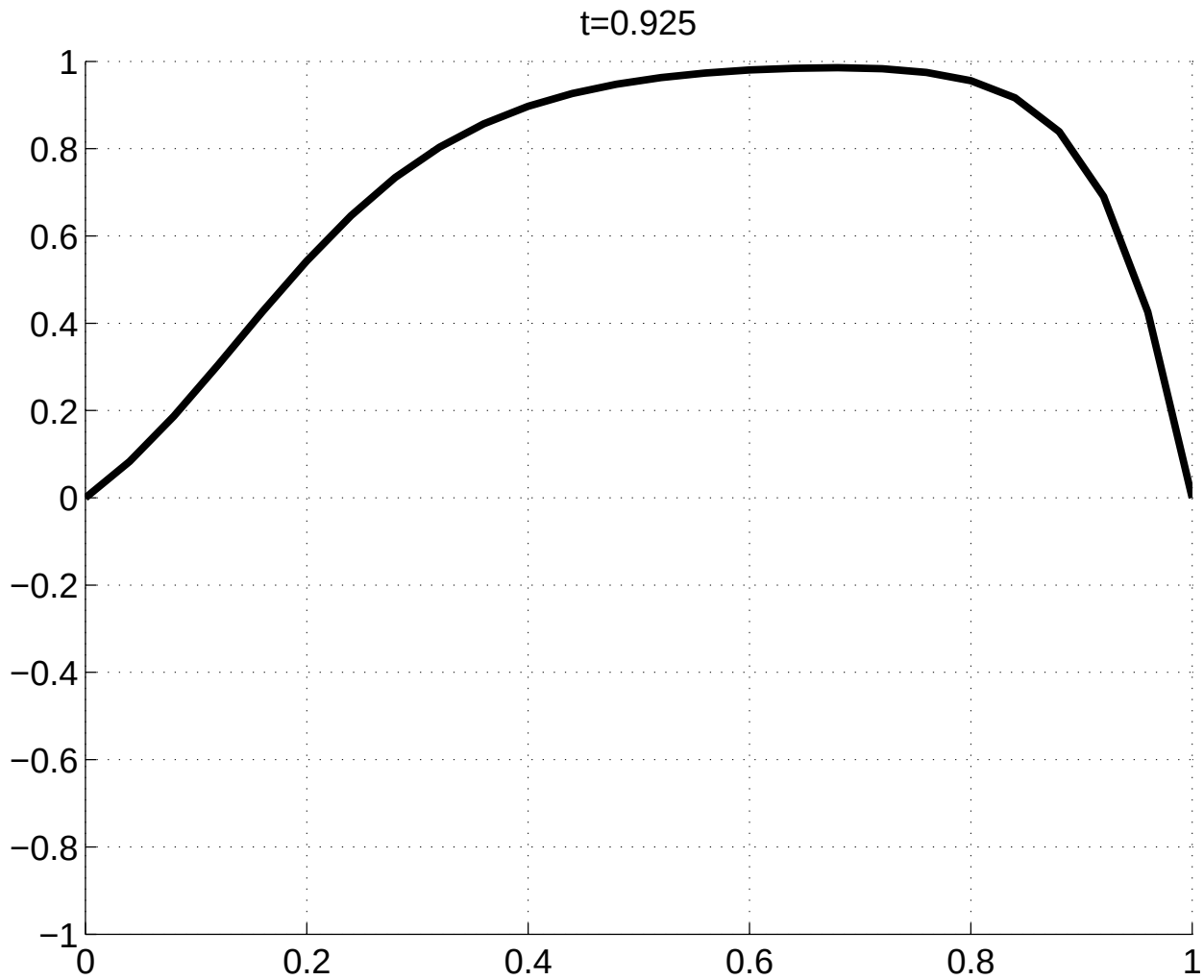

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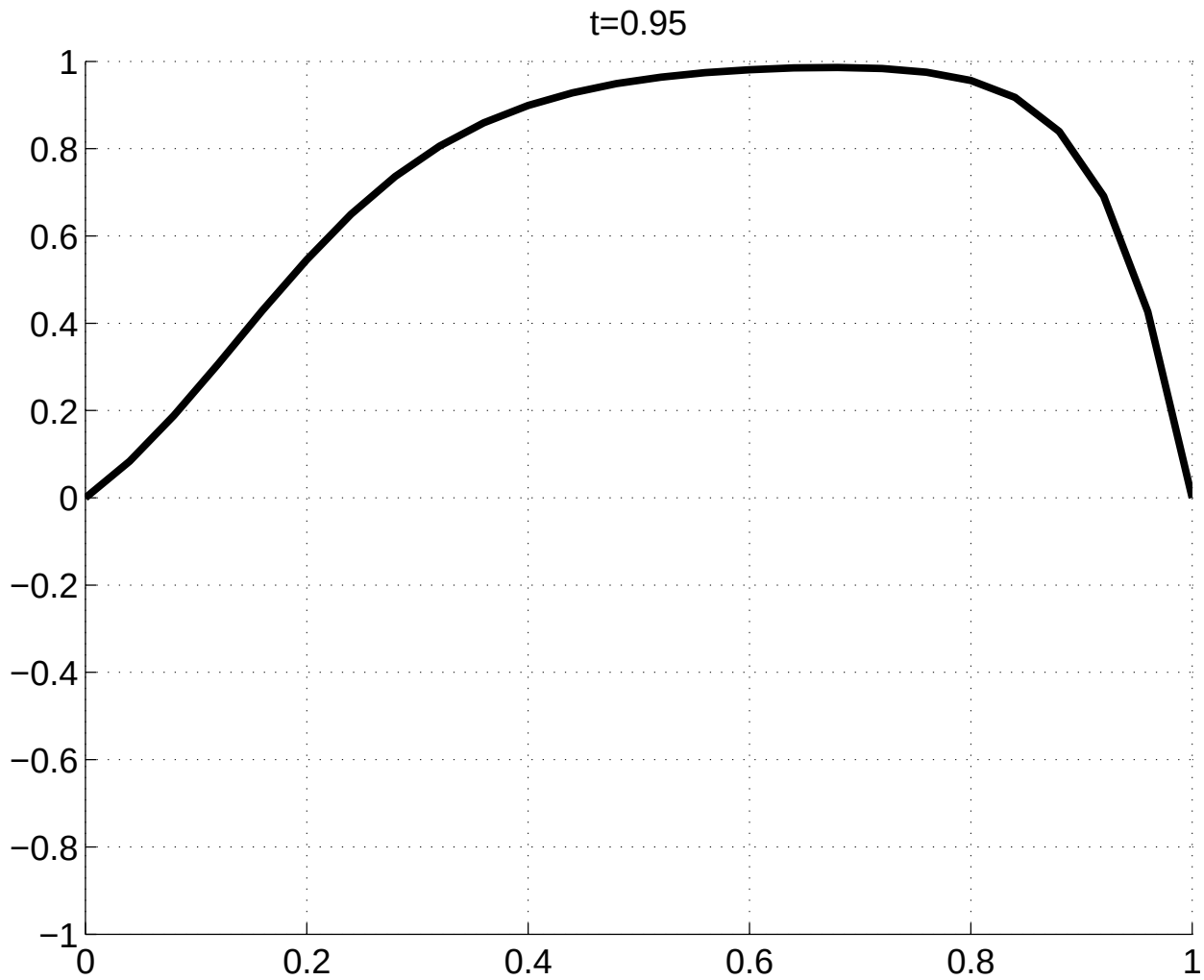

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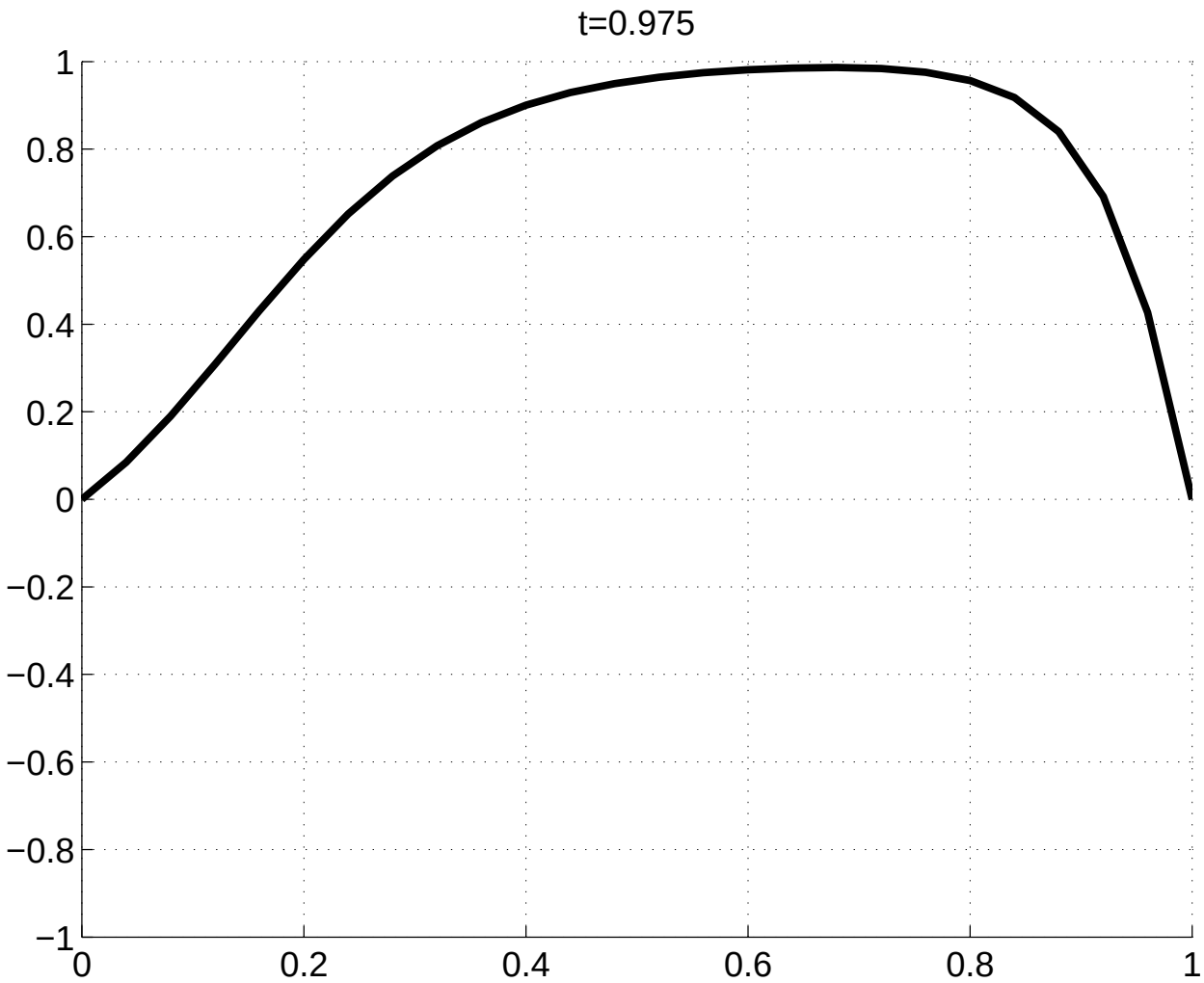

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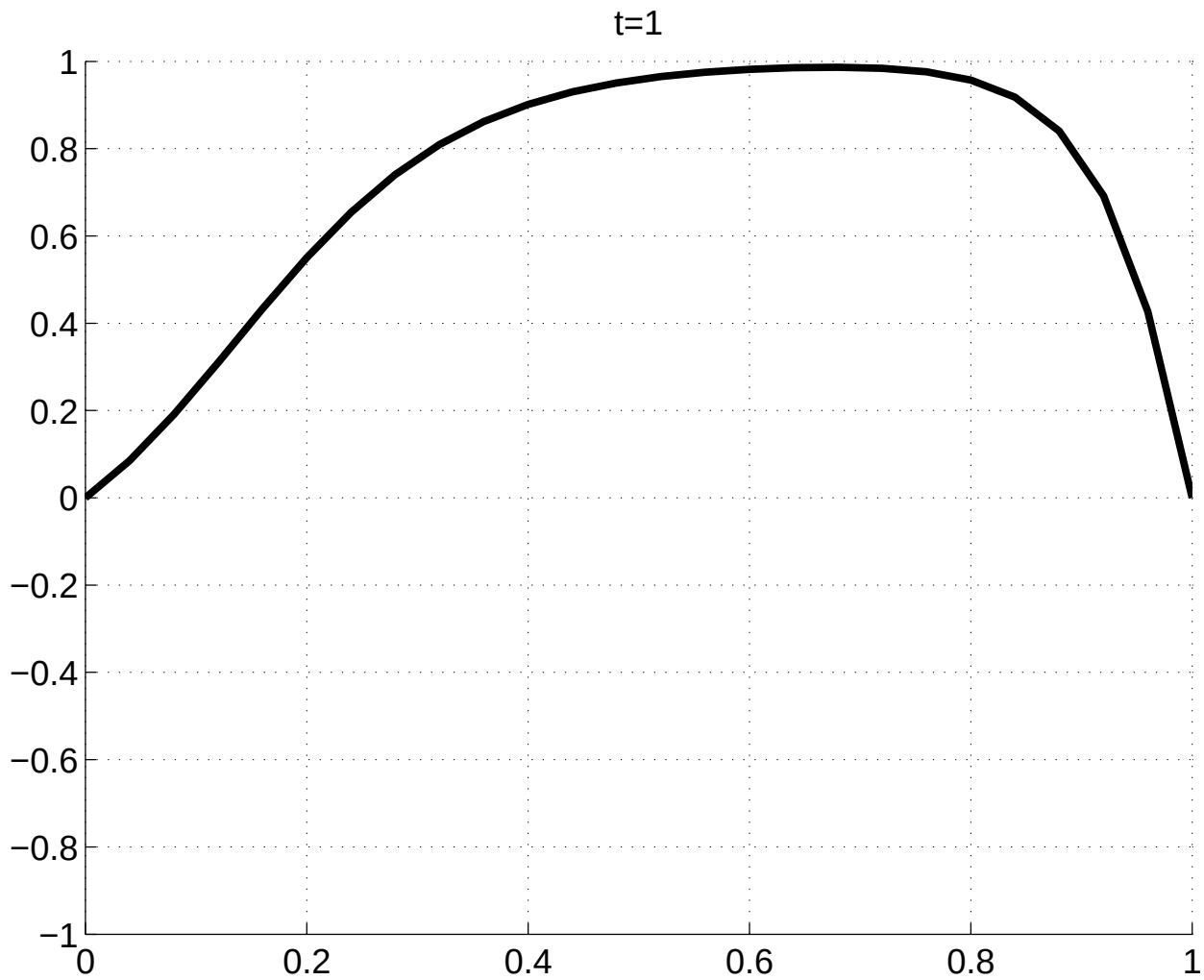

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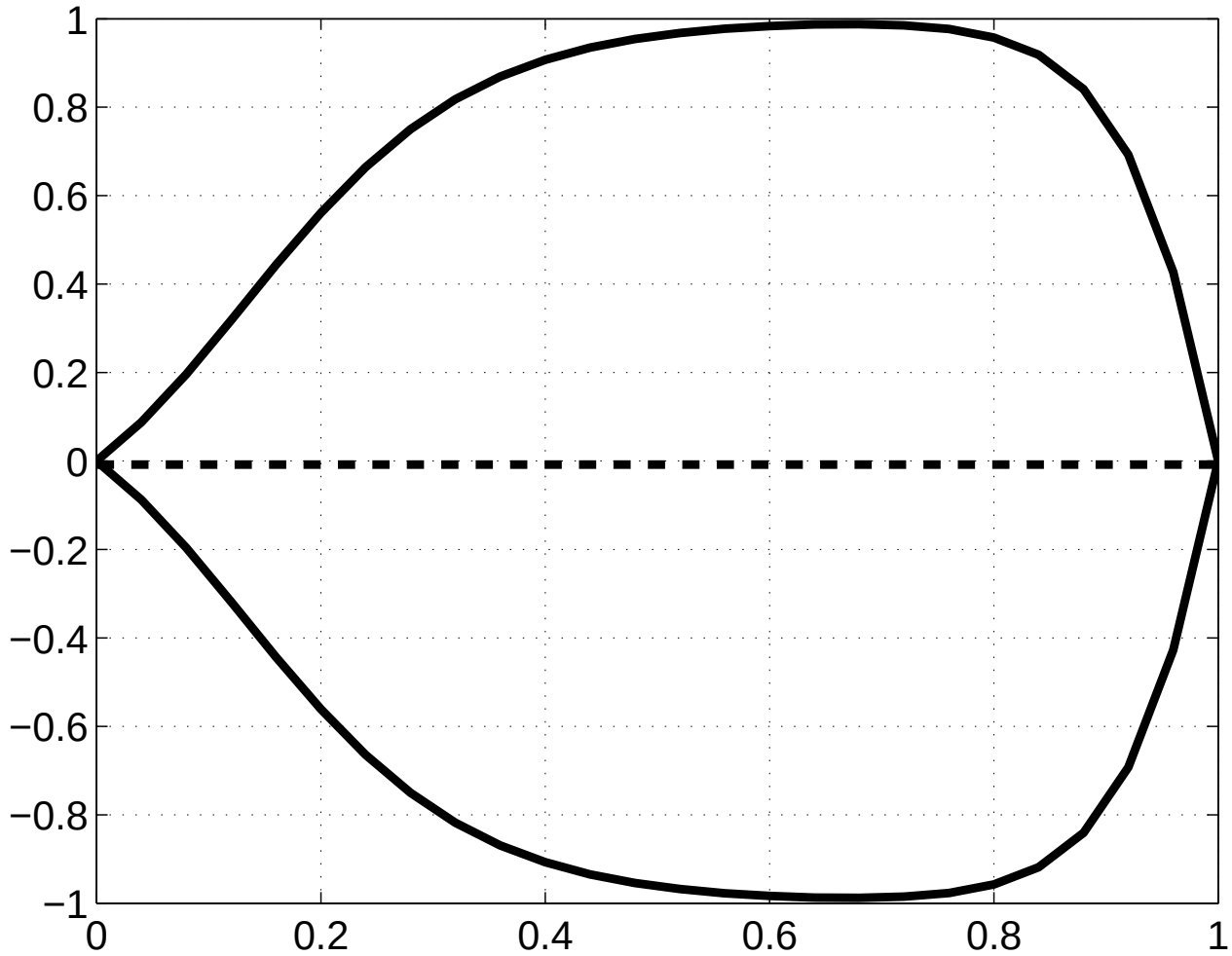

all equilibrium solutions

MPC for the PDE example

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This observation and a little computation reveals:

For the (usual) quadratic L^2 cost

$$\ell(y(n), u(n)) = \|y(n)\|_{L^2}^2 + \lambda \|u(n)\|_{L^2}^2$$

the constant C is much larger than for the quadratic H^1 cost

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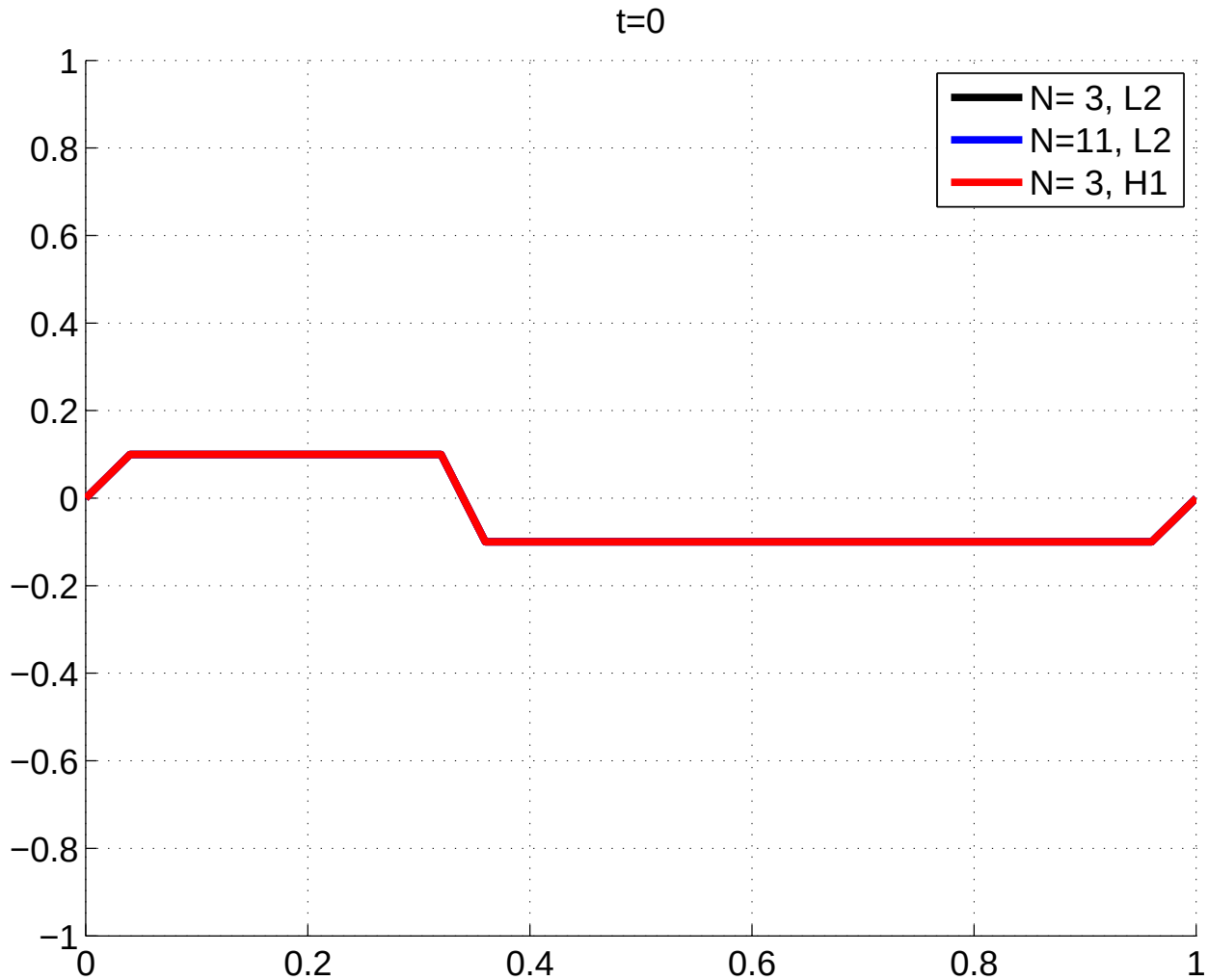
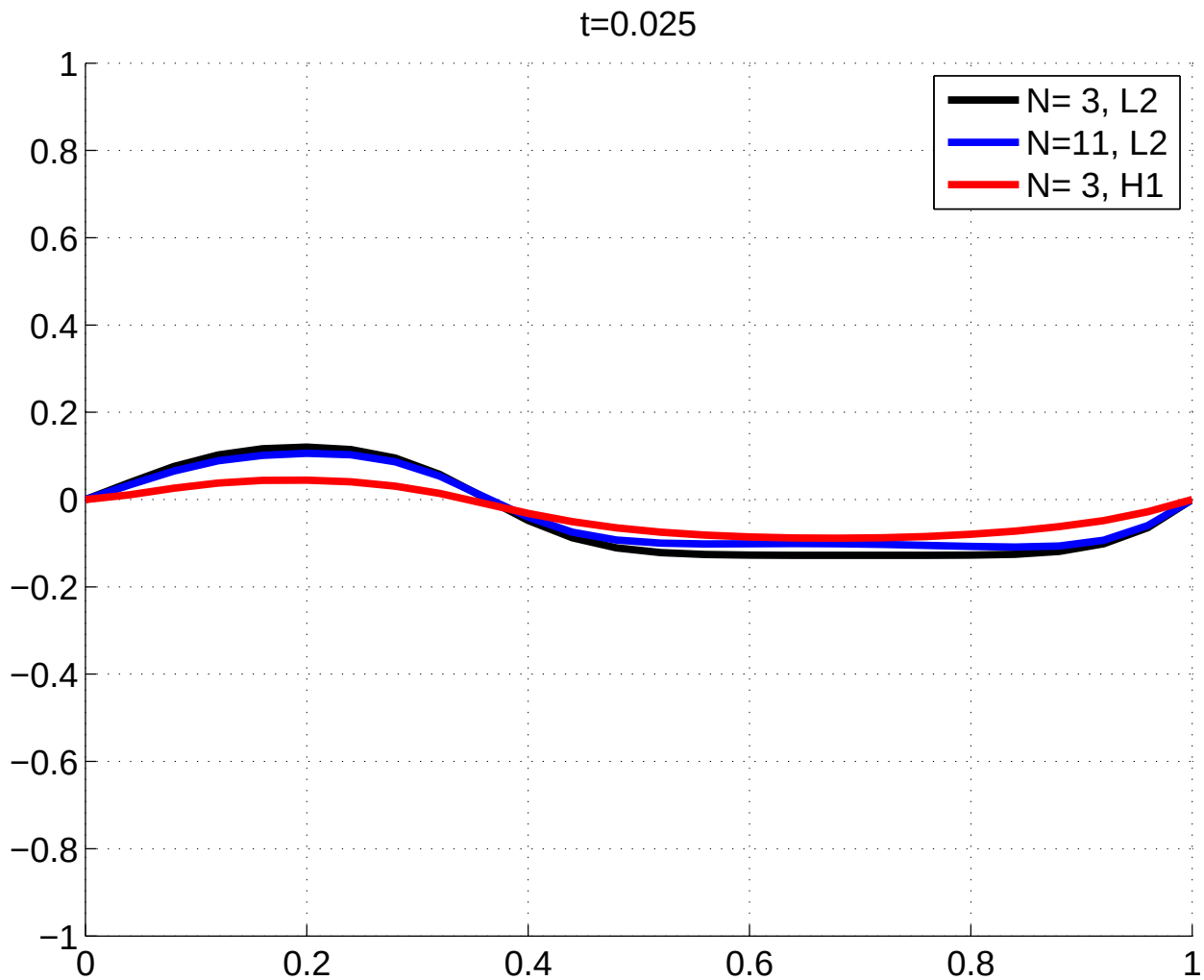
MPC with L_2 vs. H_1 cost



MPC with L_2 and H_1 cost, $\lambda = 0.1$, sampling time $T = 0.025$

MPC with L_2 vs. H_1 cost



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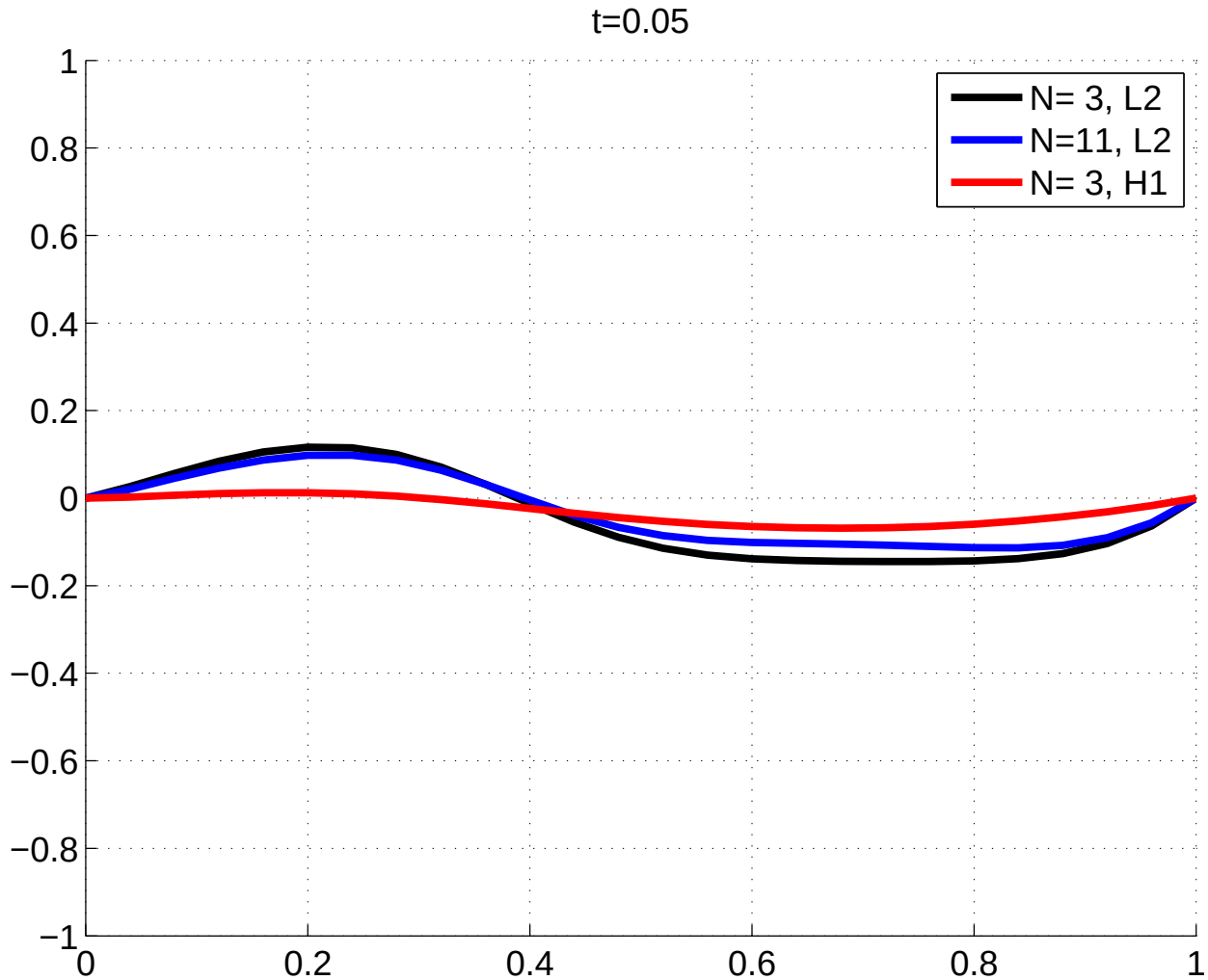

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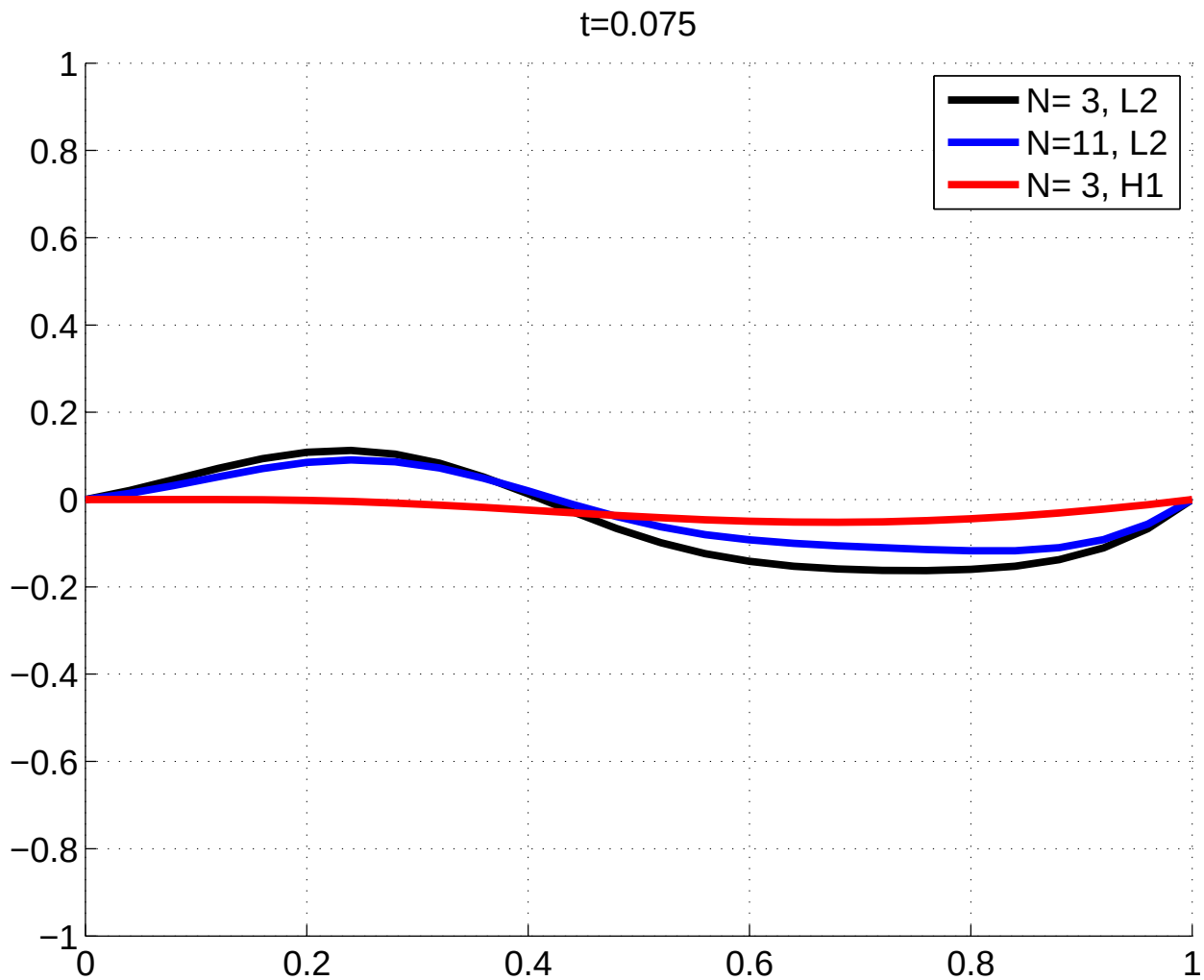

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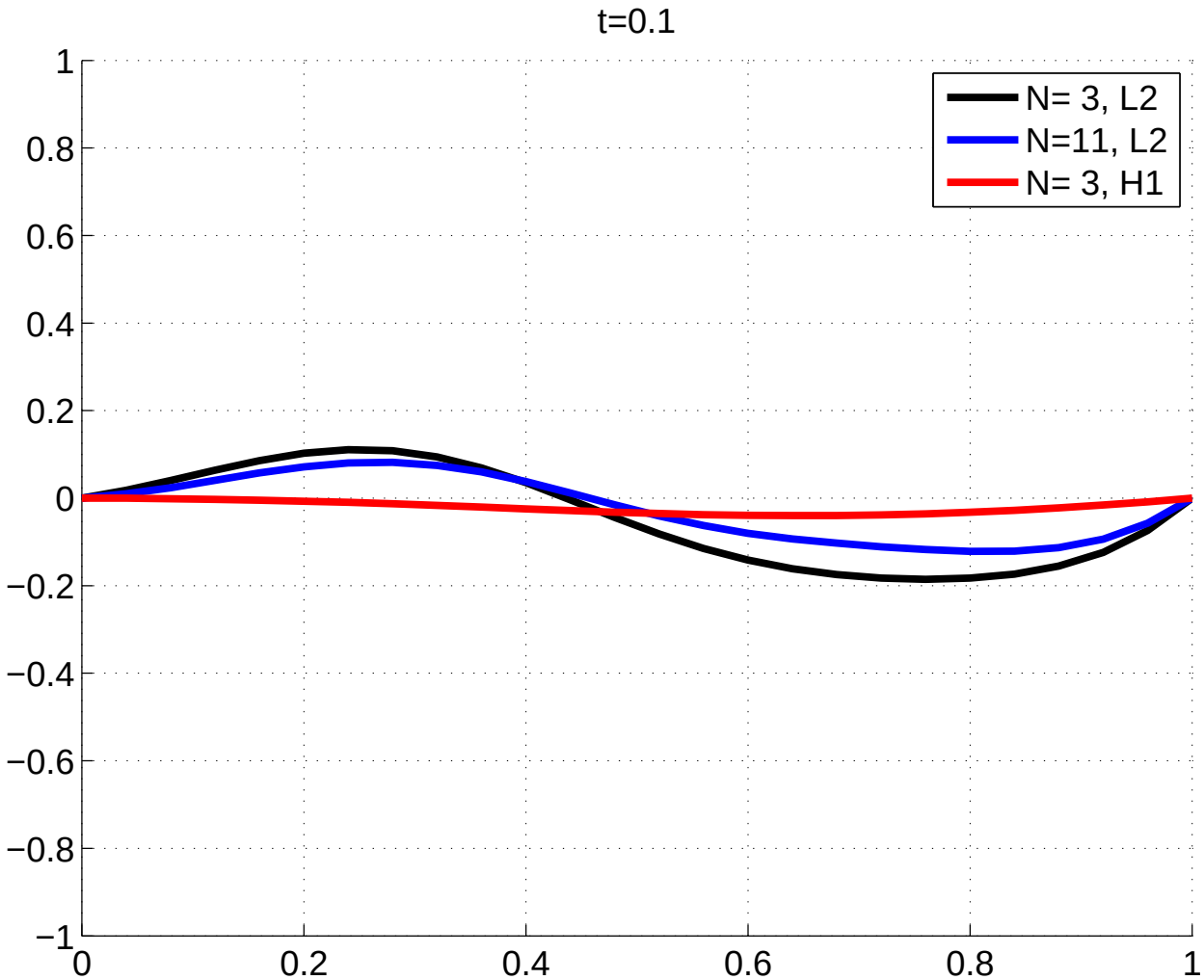

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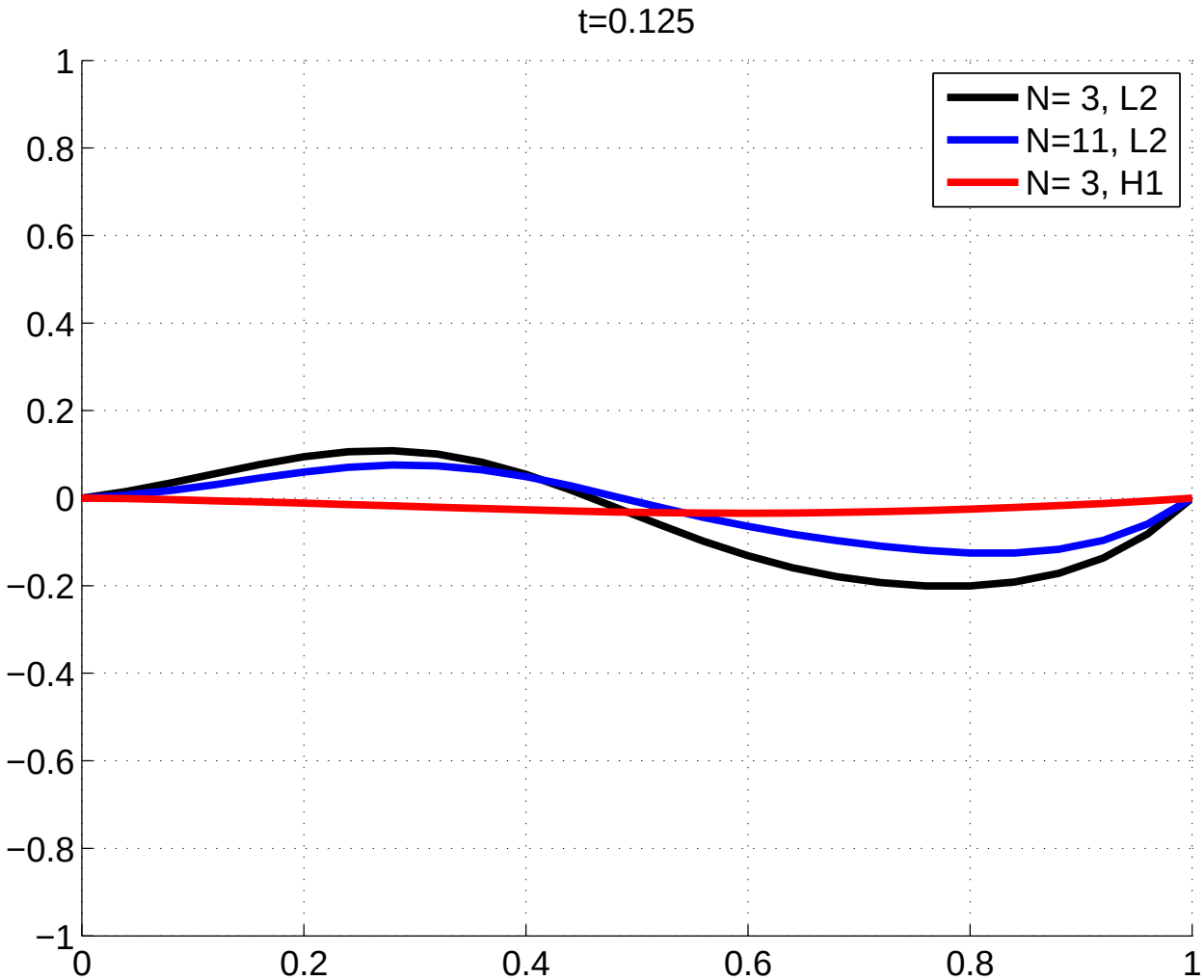

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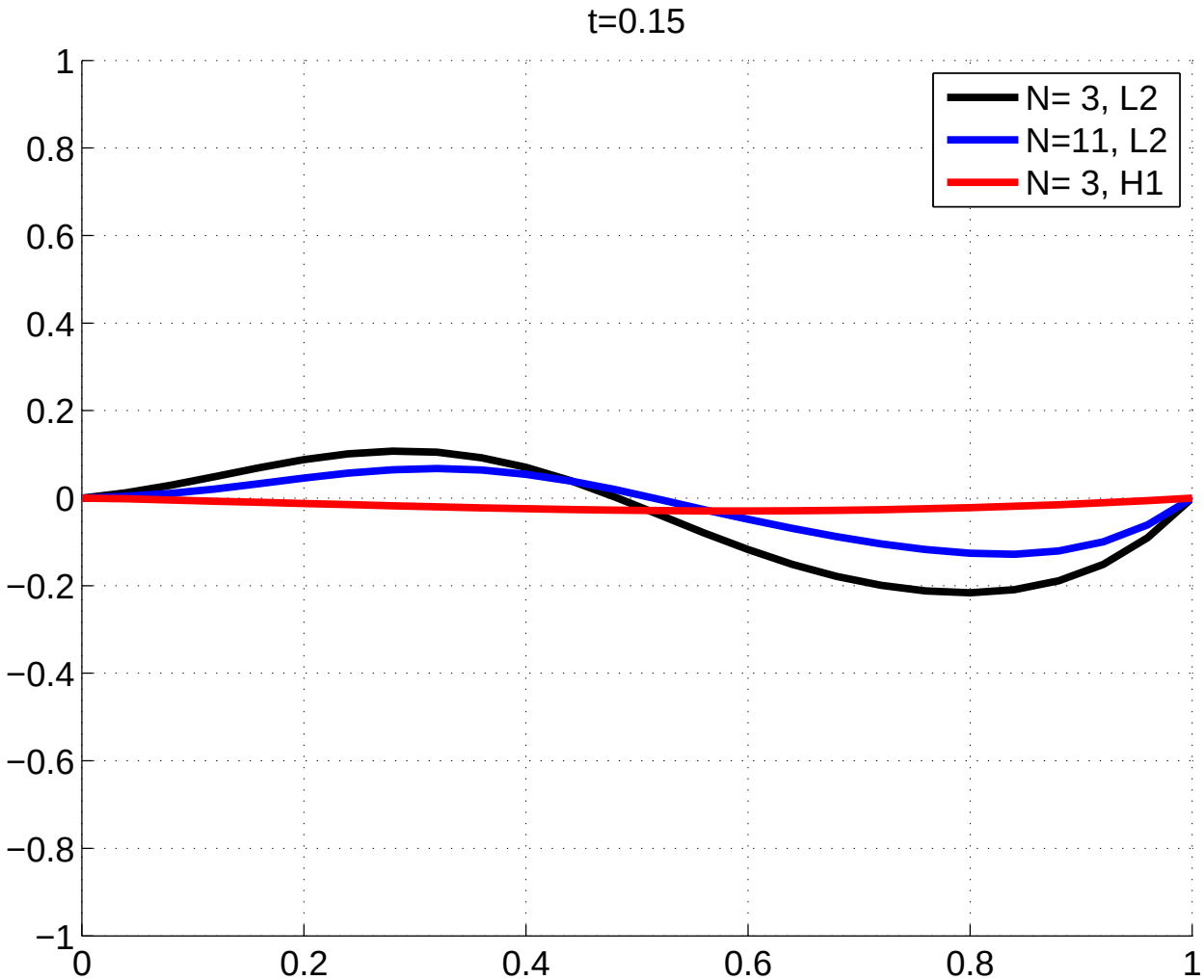

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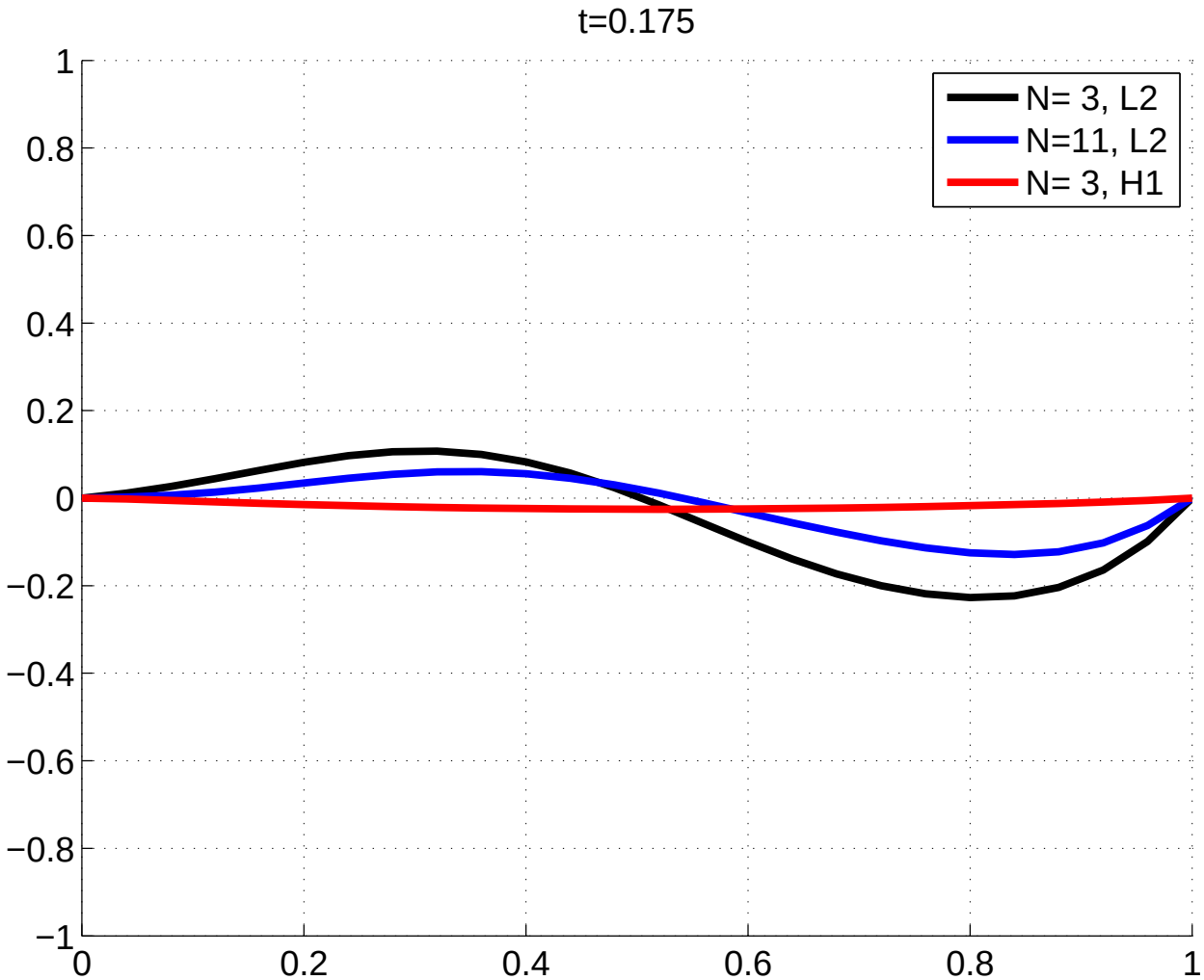

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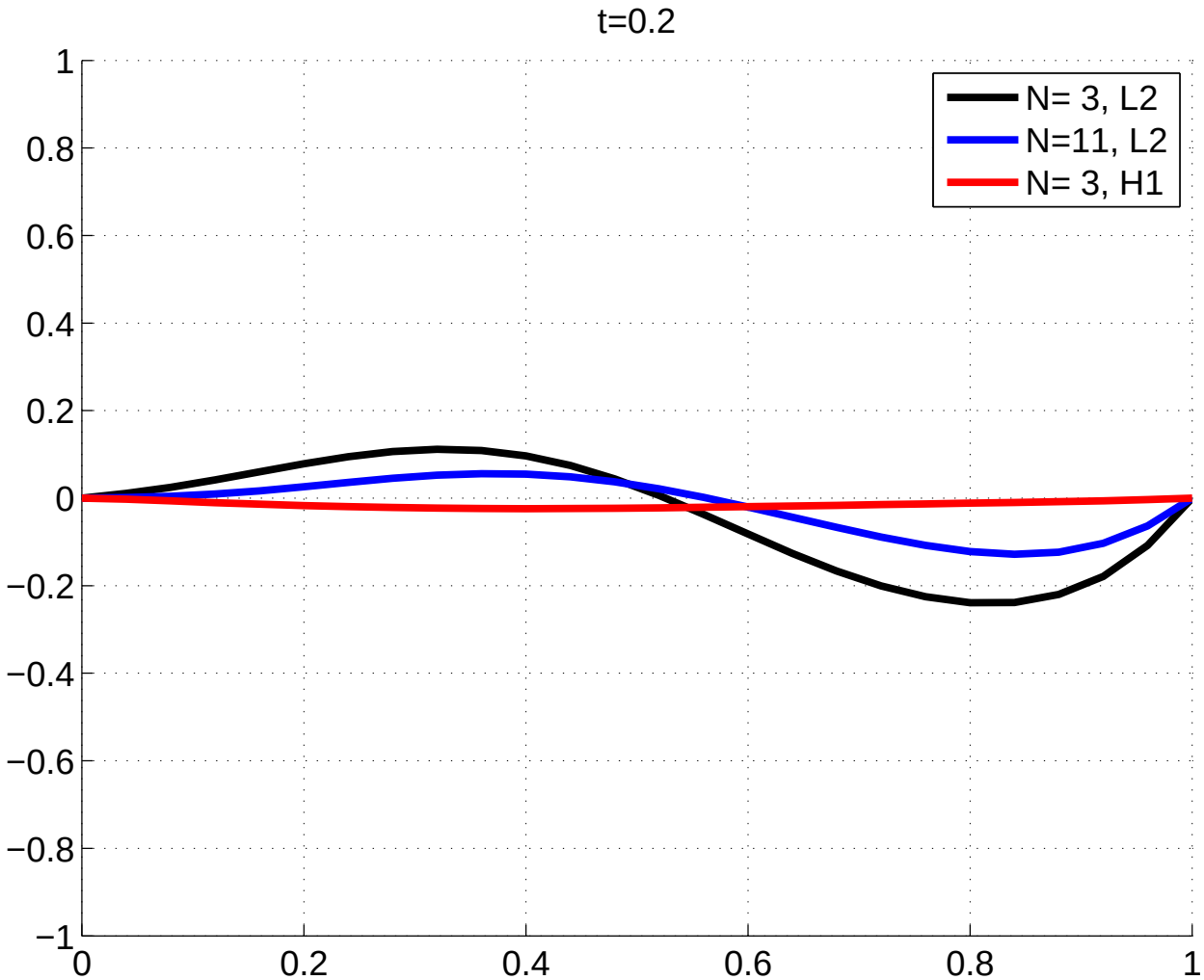

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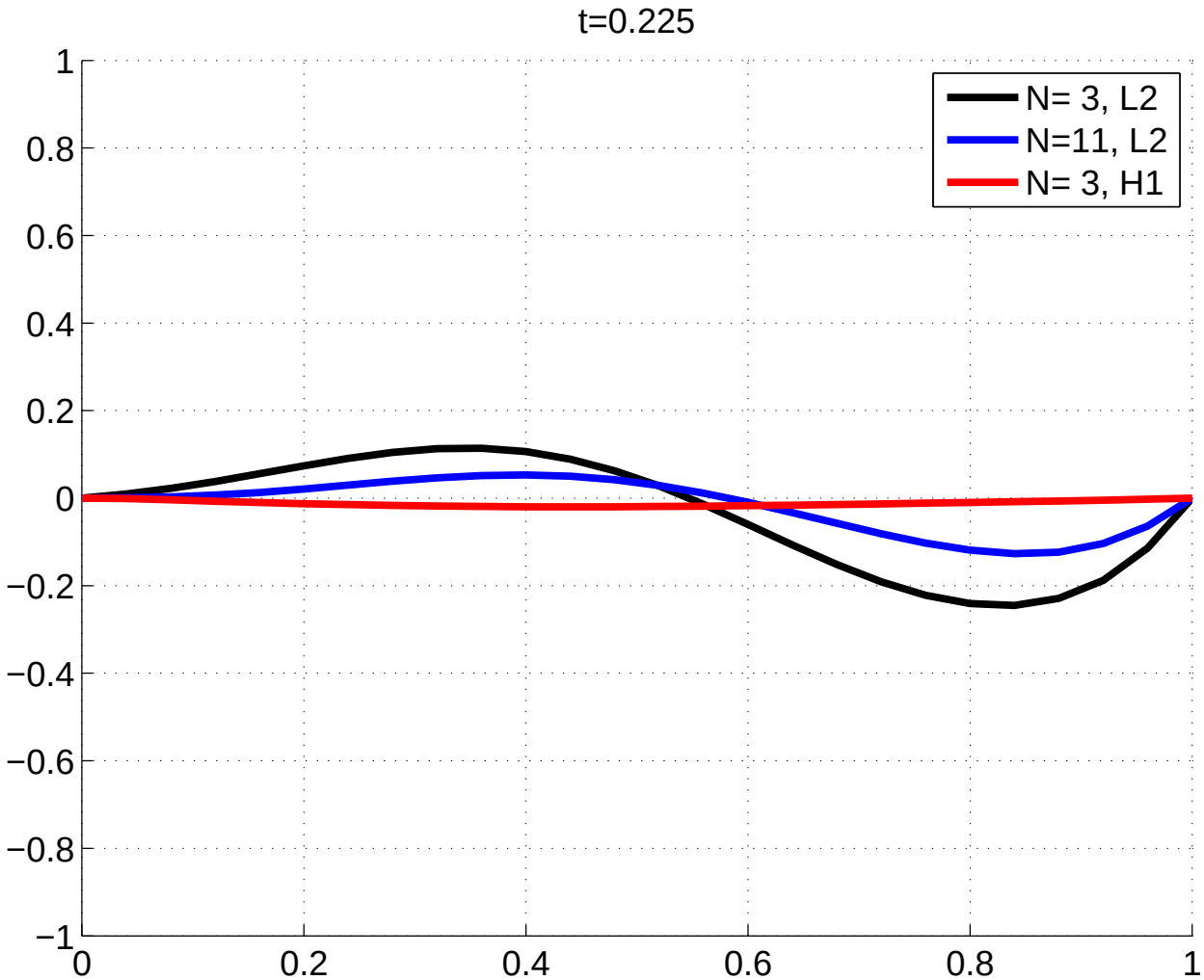

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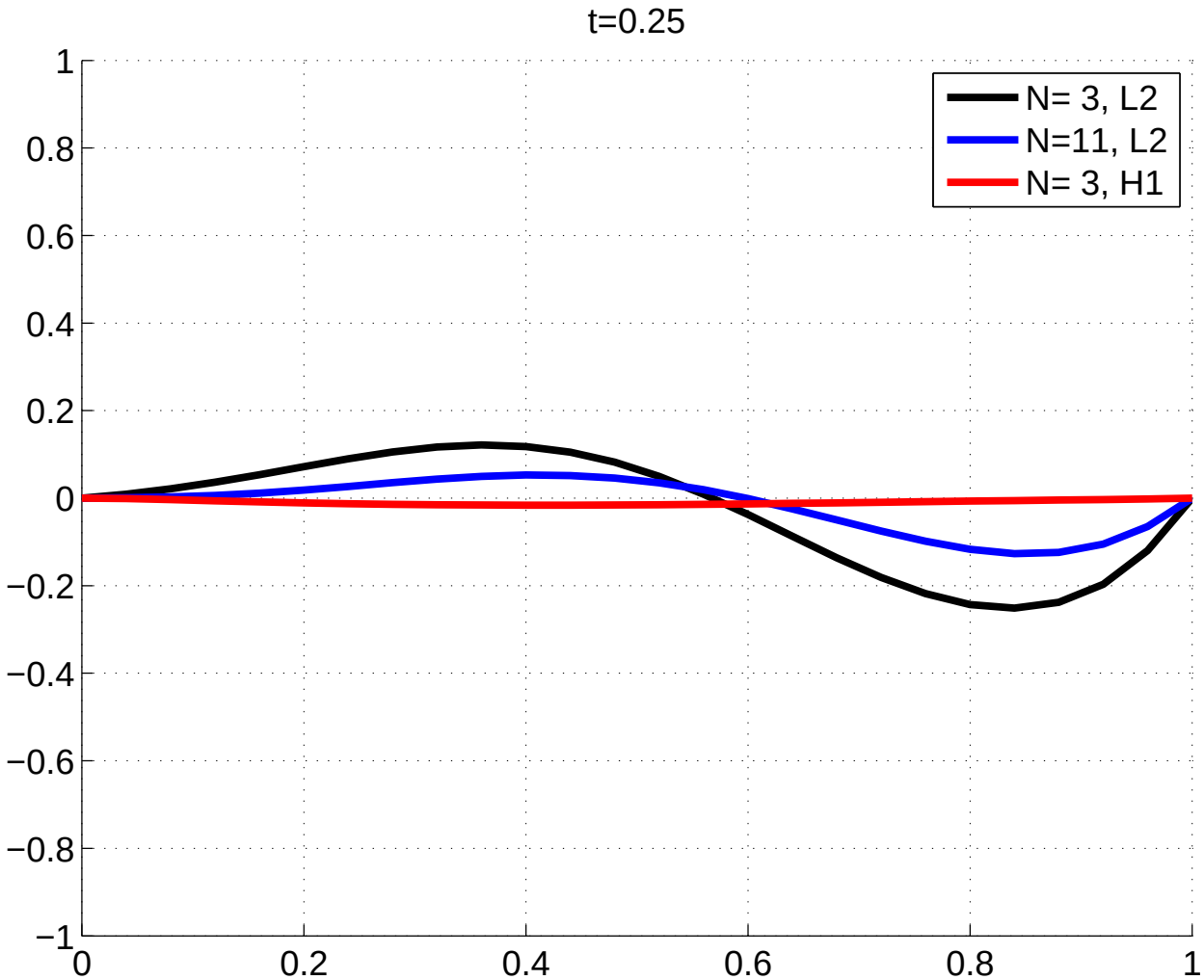

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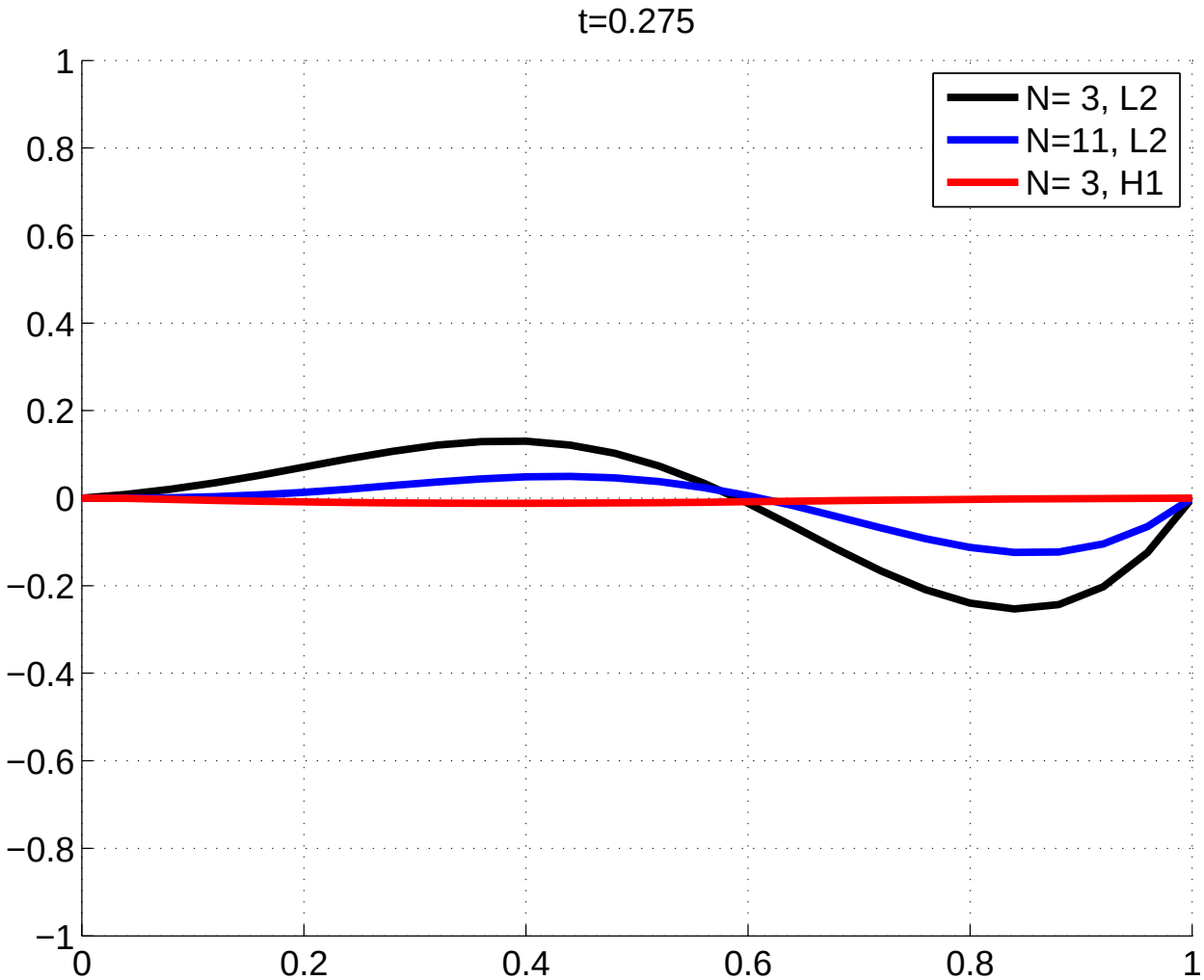

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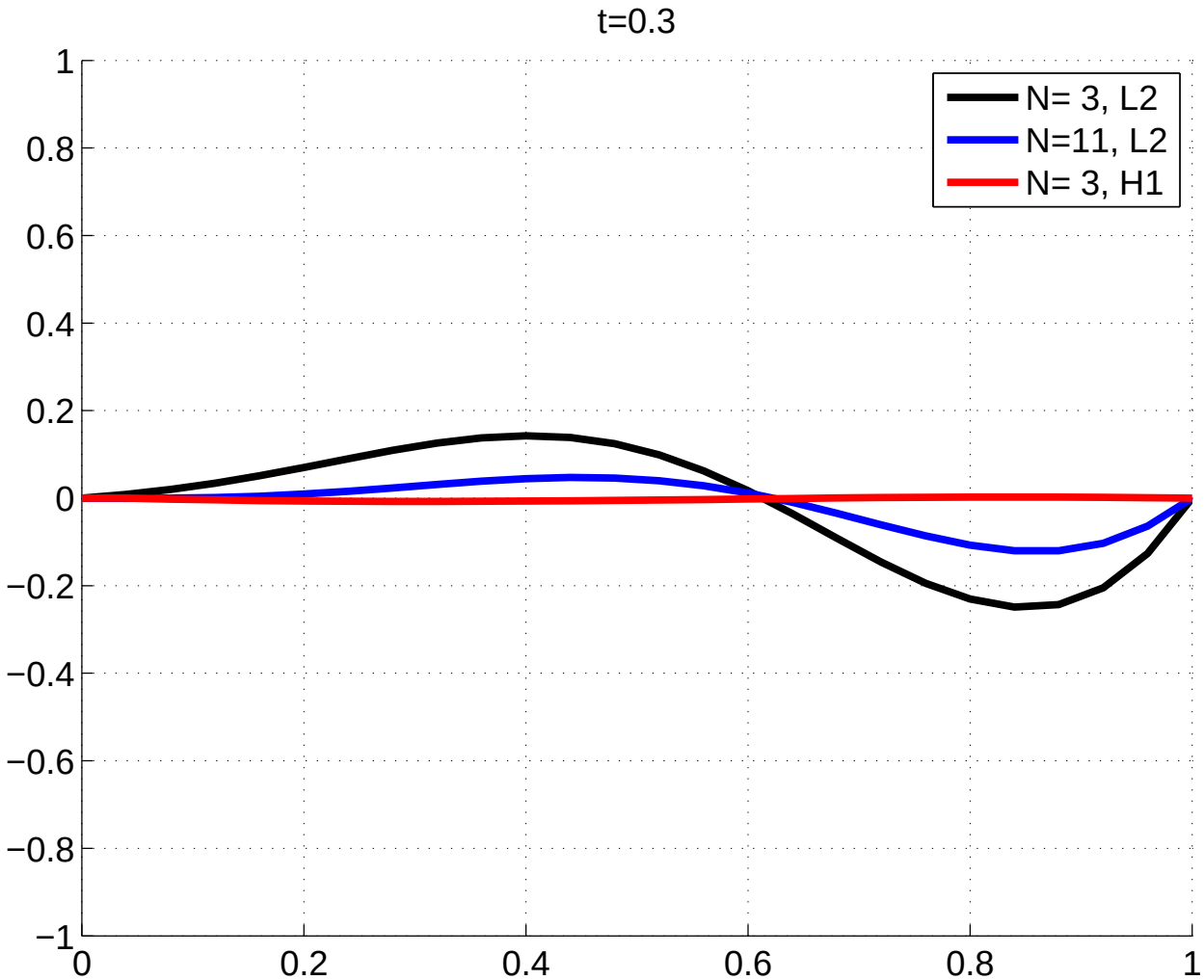

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Boundary Control

Now we change our PDE from distributed to (Dirichlet-) boundary control, i.e.

$$y_t = y_x + \nu y_{xx} + \mu y(y+1)(1-y)$$

with

domain $\Omega = [0,1]$

solution $y = y(t,x)$

boundary conditions $y(t,0) = u_0(t)$, $y(t,1) = u_1(t)$

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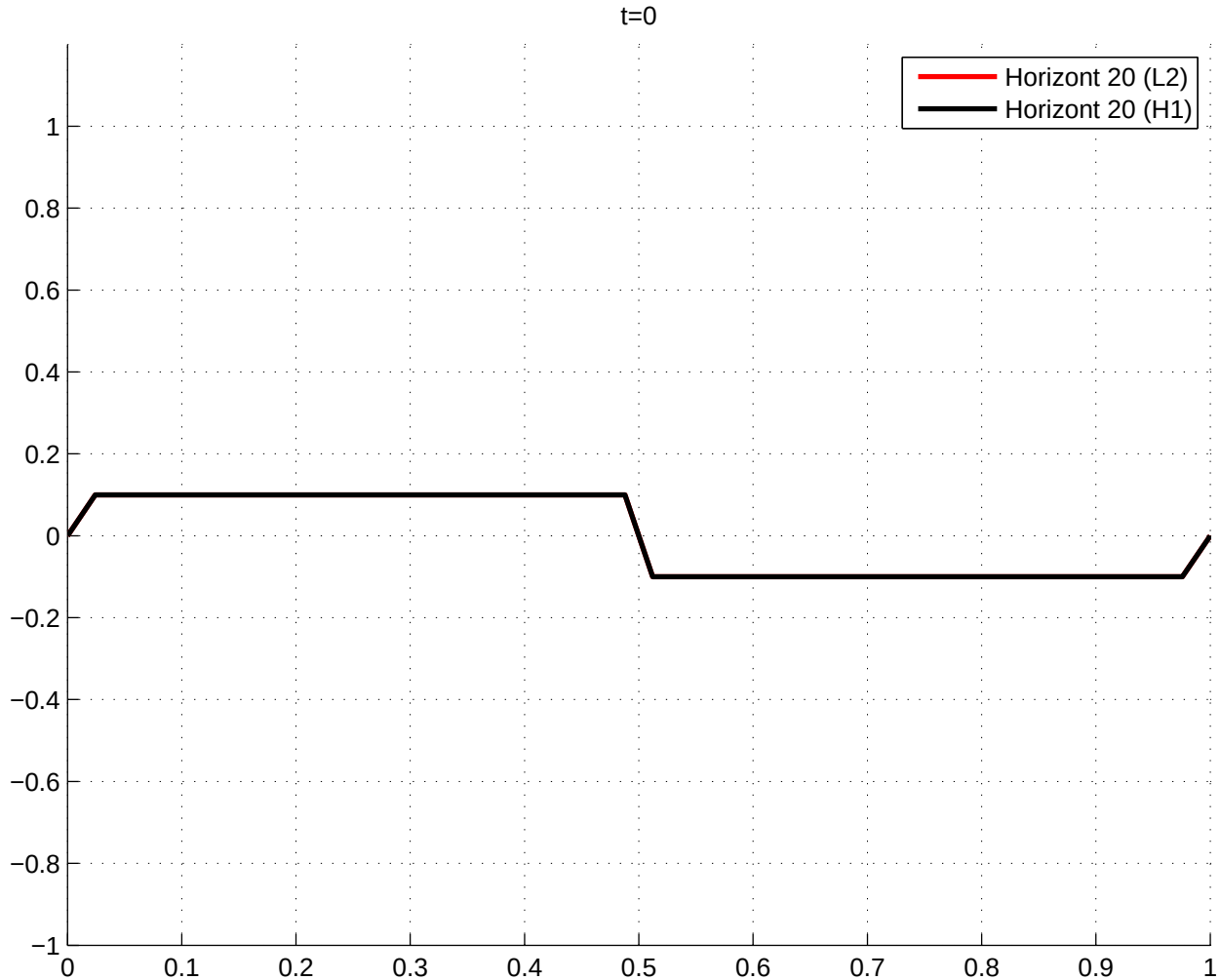
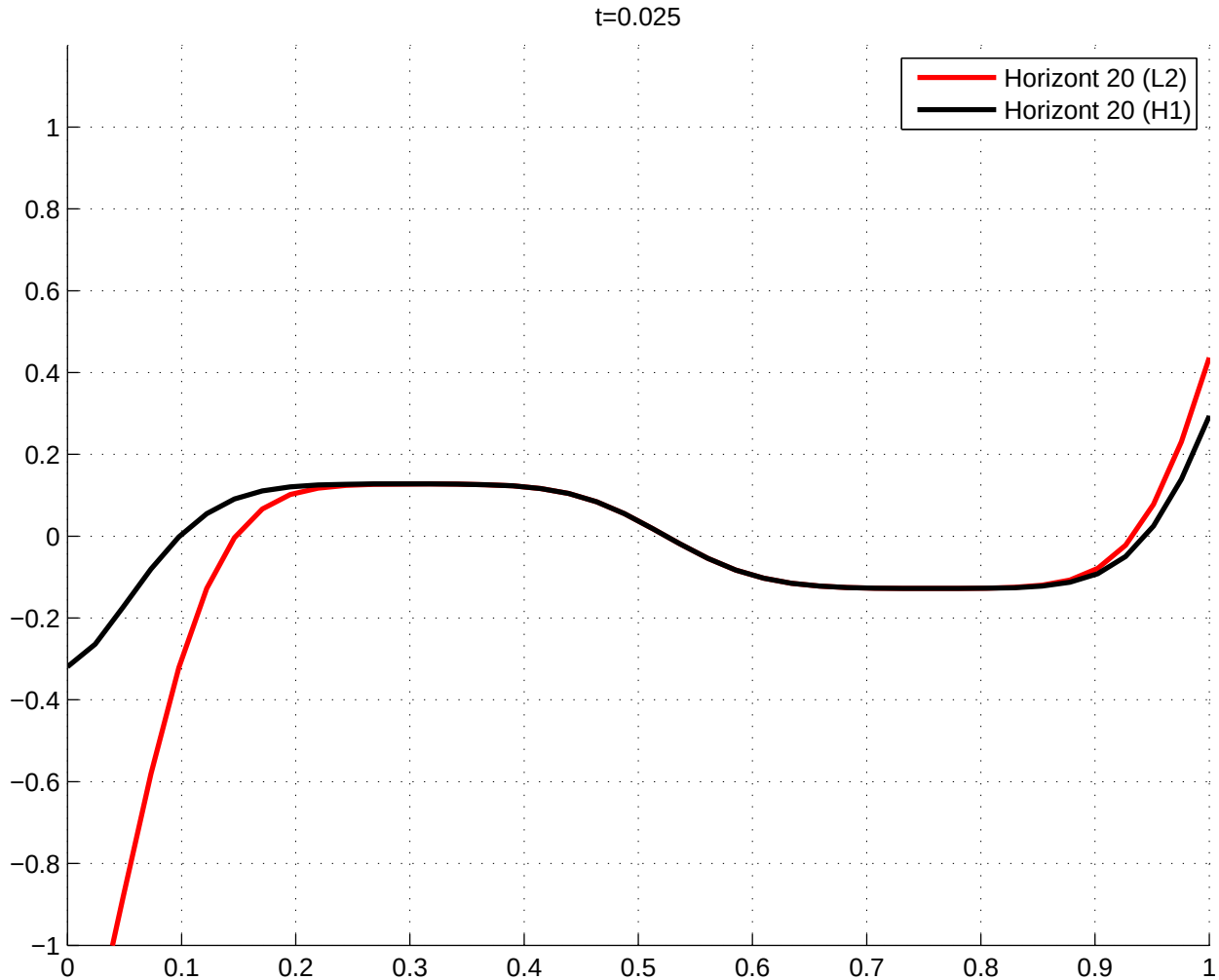
Boundary control, L_2 vs. H_1, $N = 20$



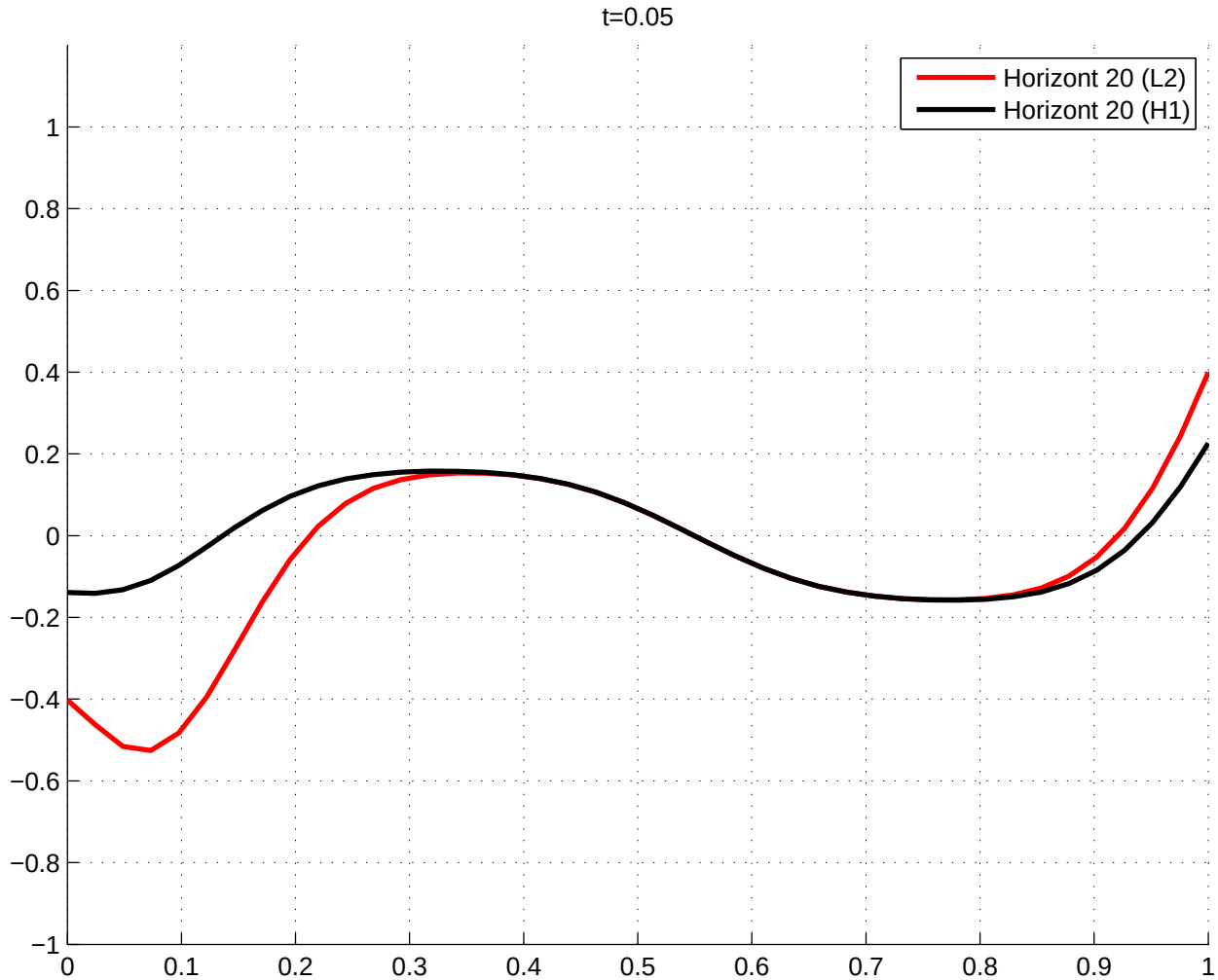
Boundary control, $\lambda = 0.001$, sampling time $T = 0.025$

Boundary control, L_2 vs. H_1, $N = 20$



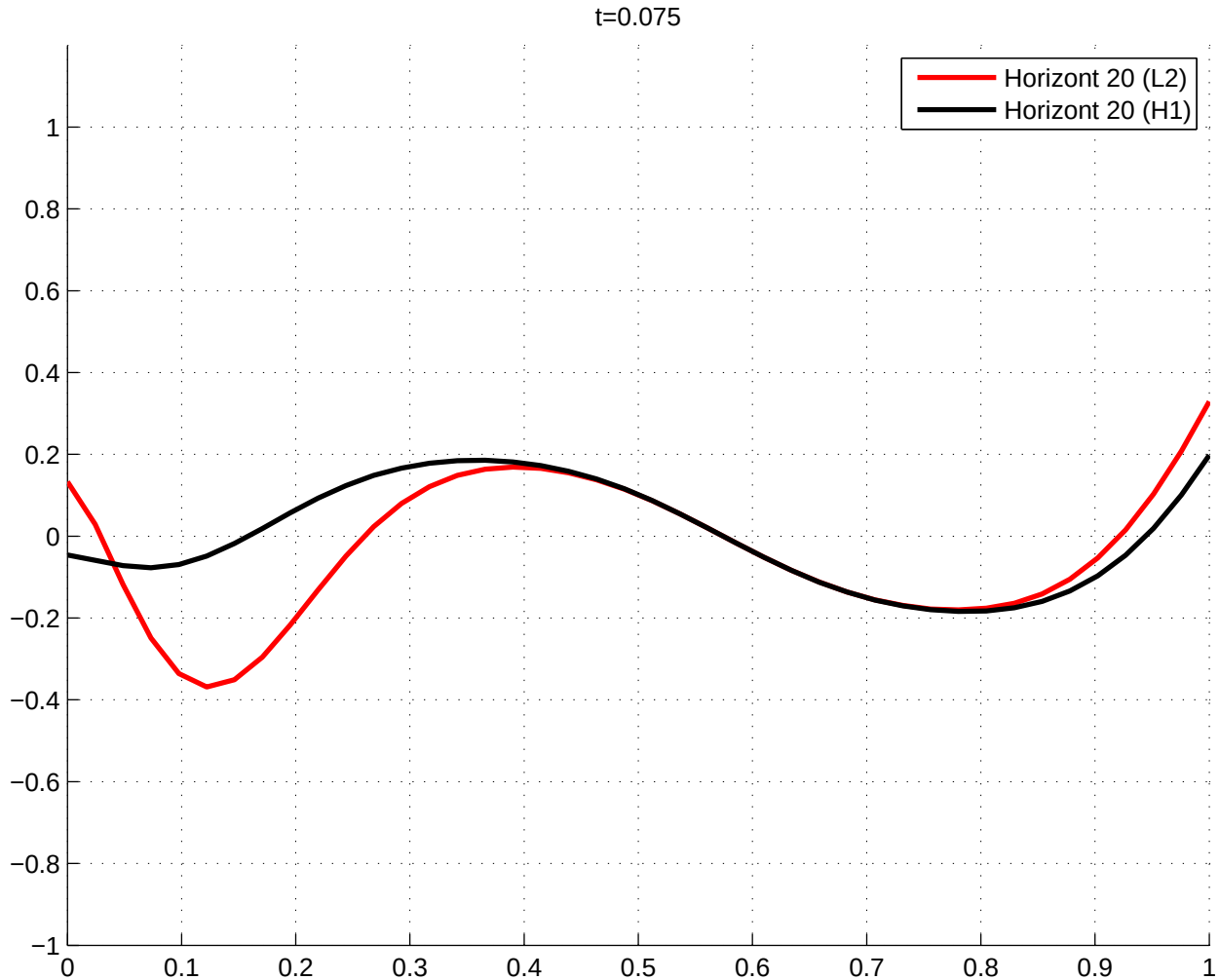
Boundary control, $\lambda = 0.001$, sampling time $T = 0.025$

Boundary control, L_2 vs. H_1, $N = 20$



Boundary control, $\lambda = 0.001$, sampling time $T = 0.025$

Boundary control, L_2 vs. H_1, $N = 20$



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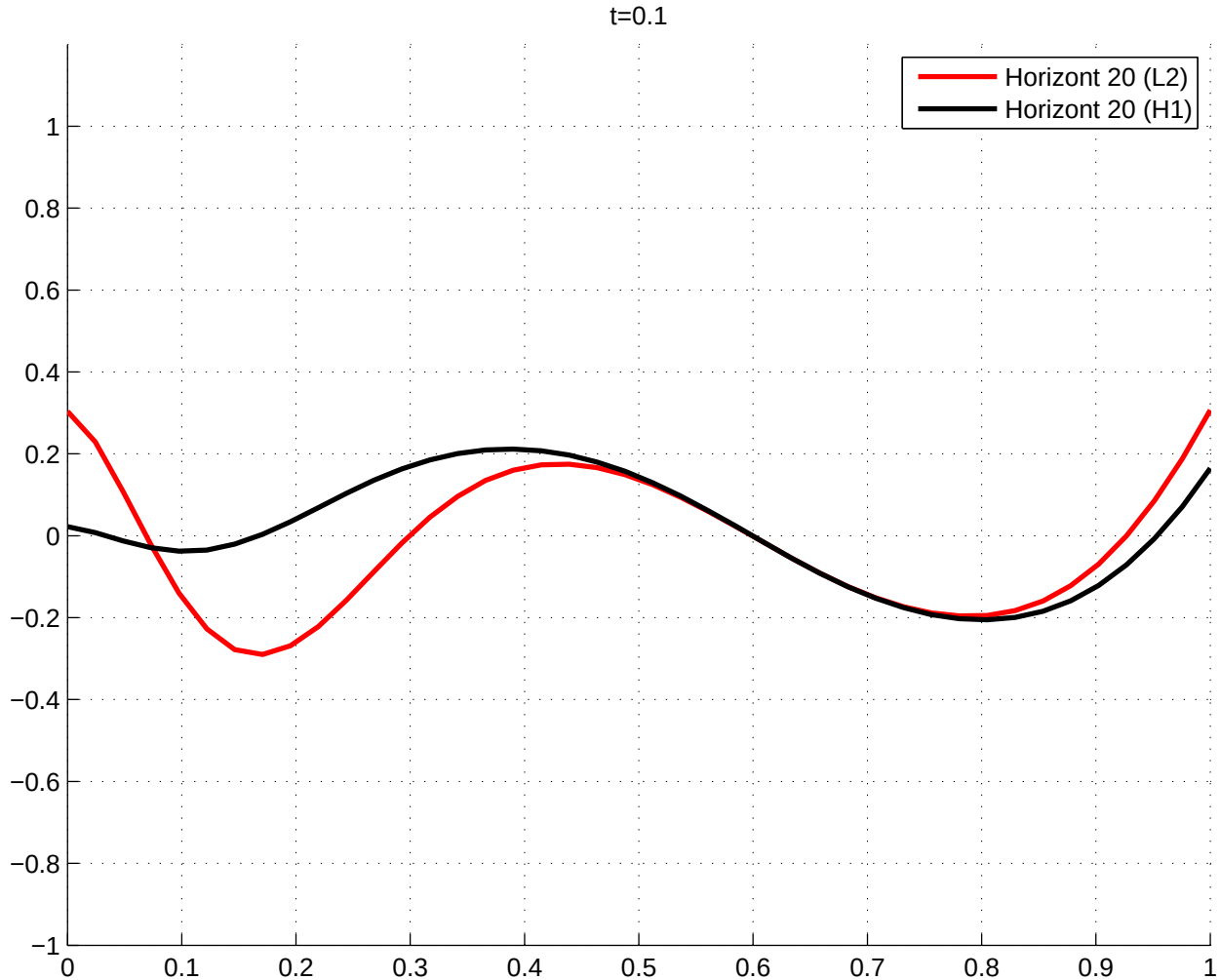

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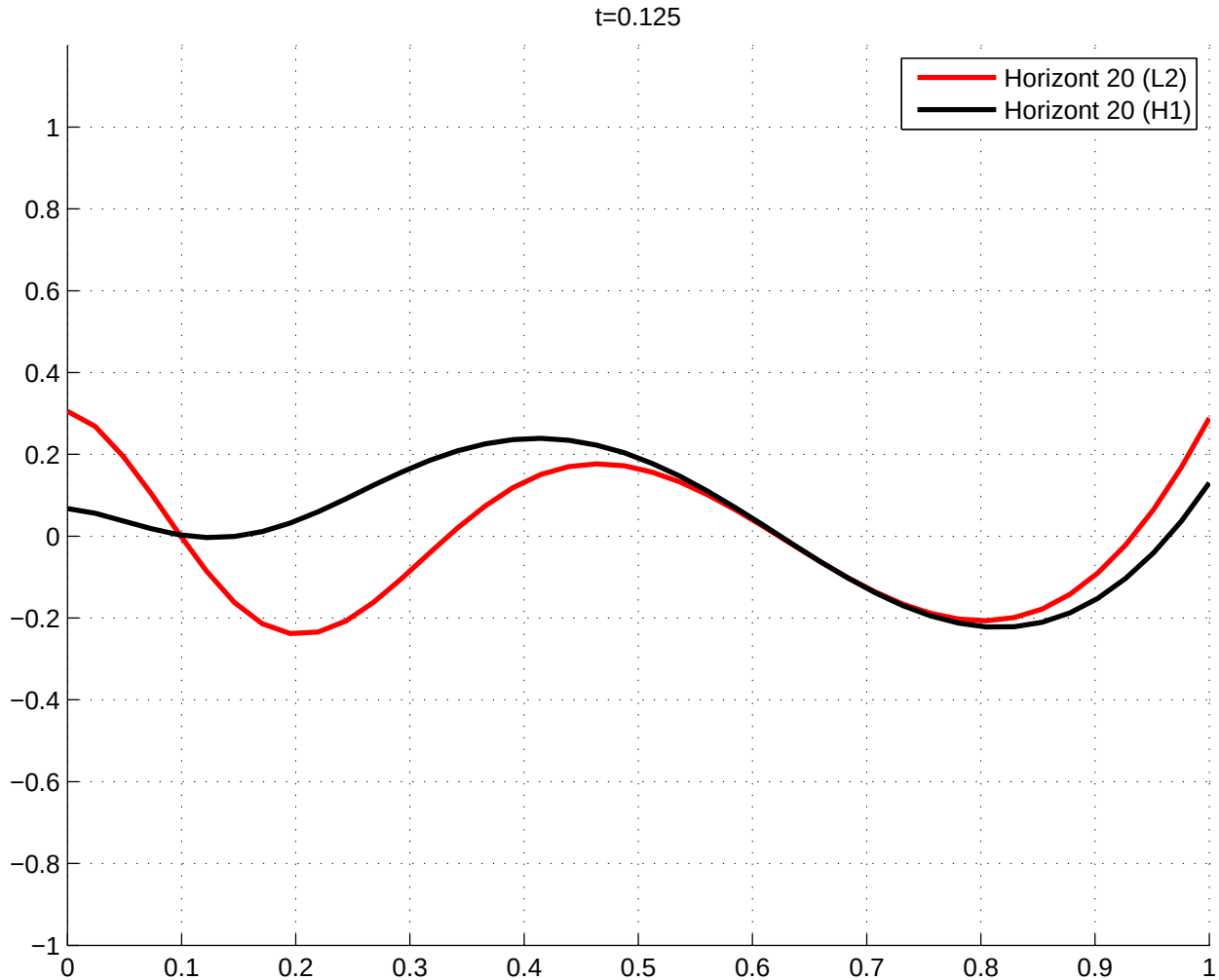

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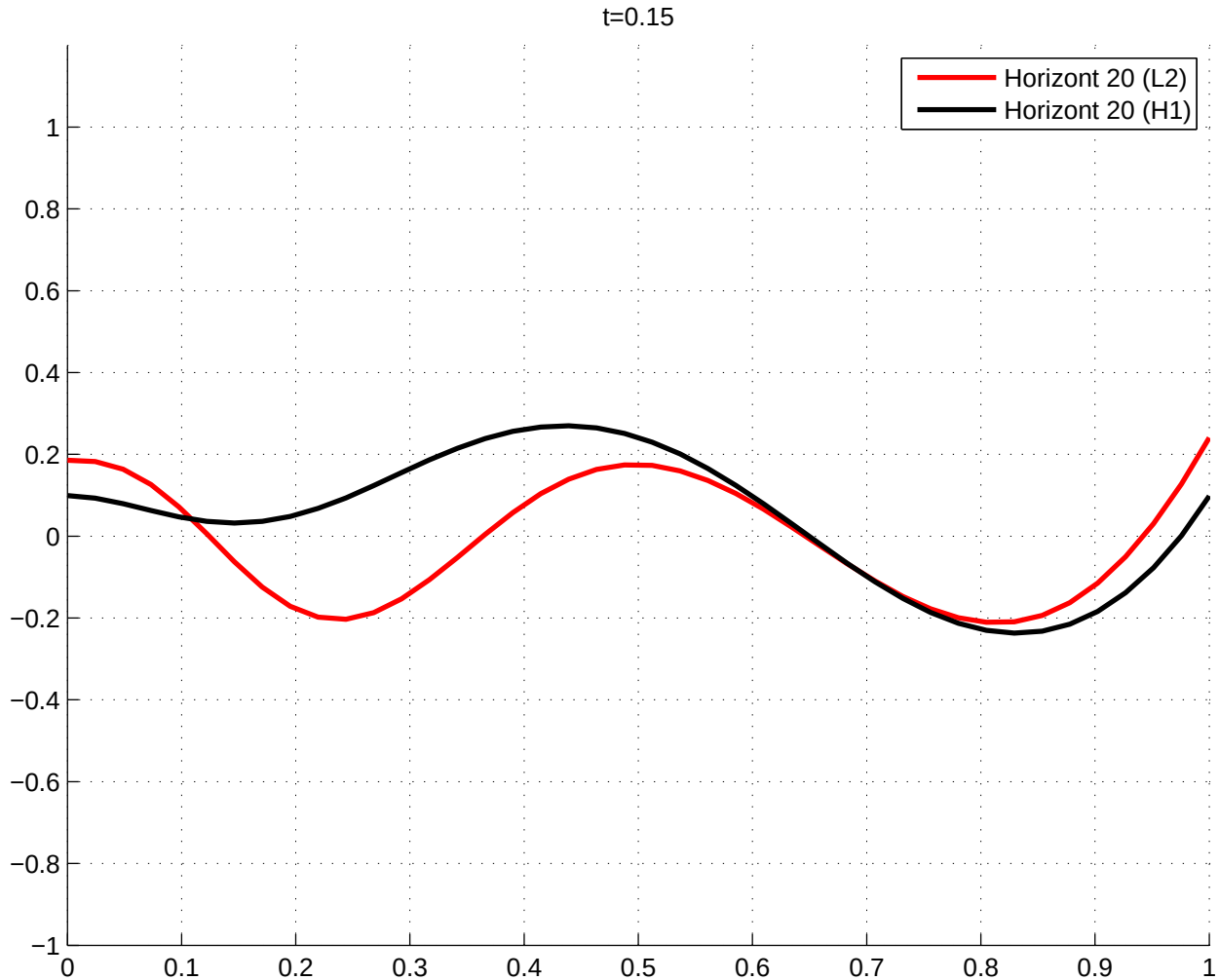

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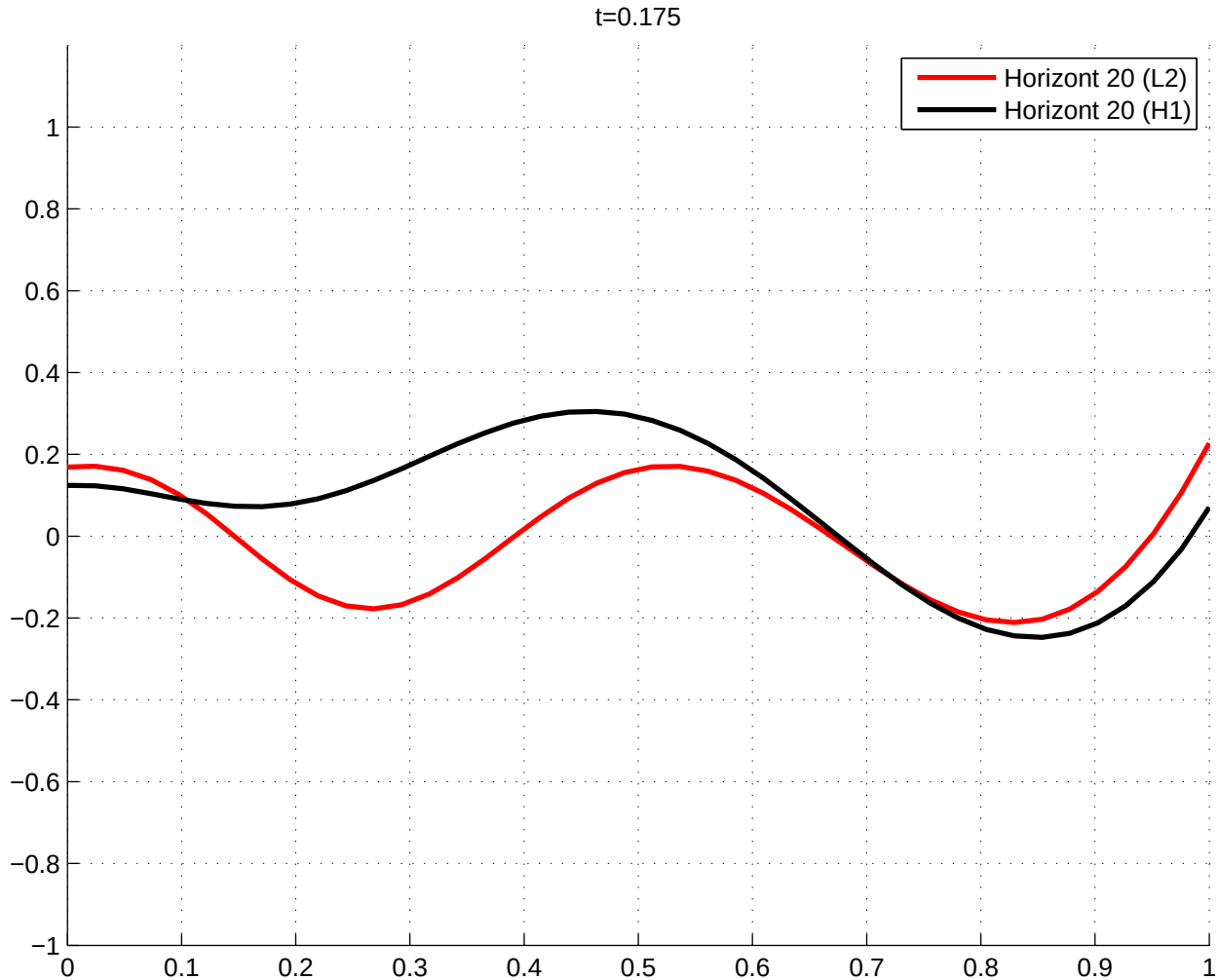

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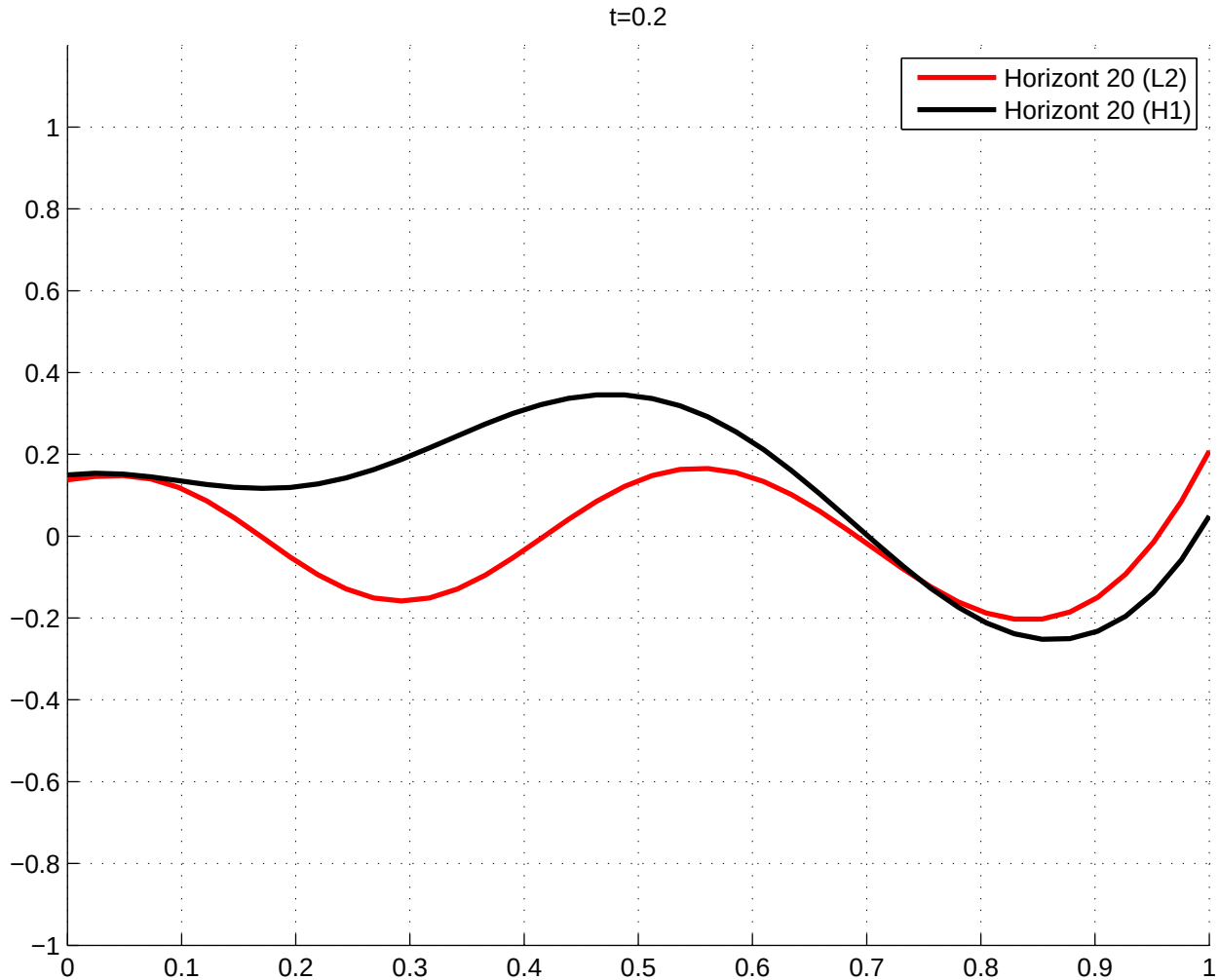

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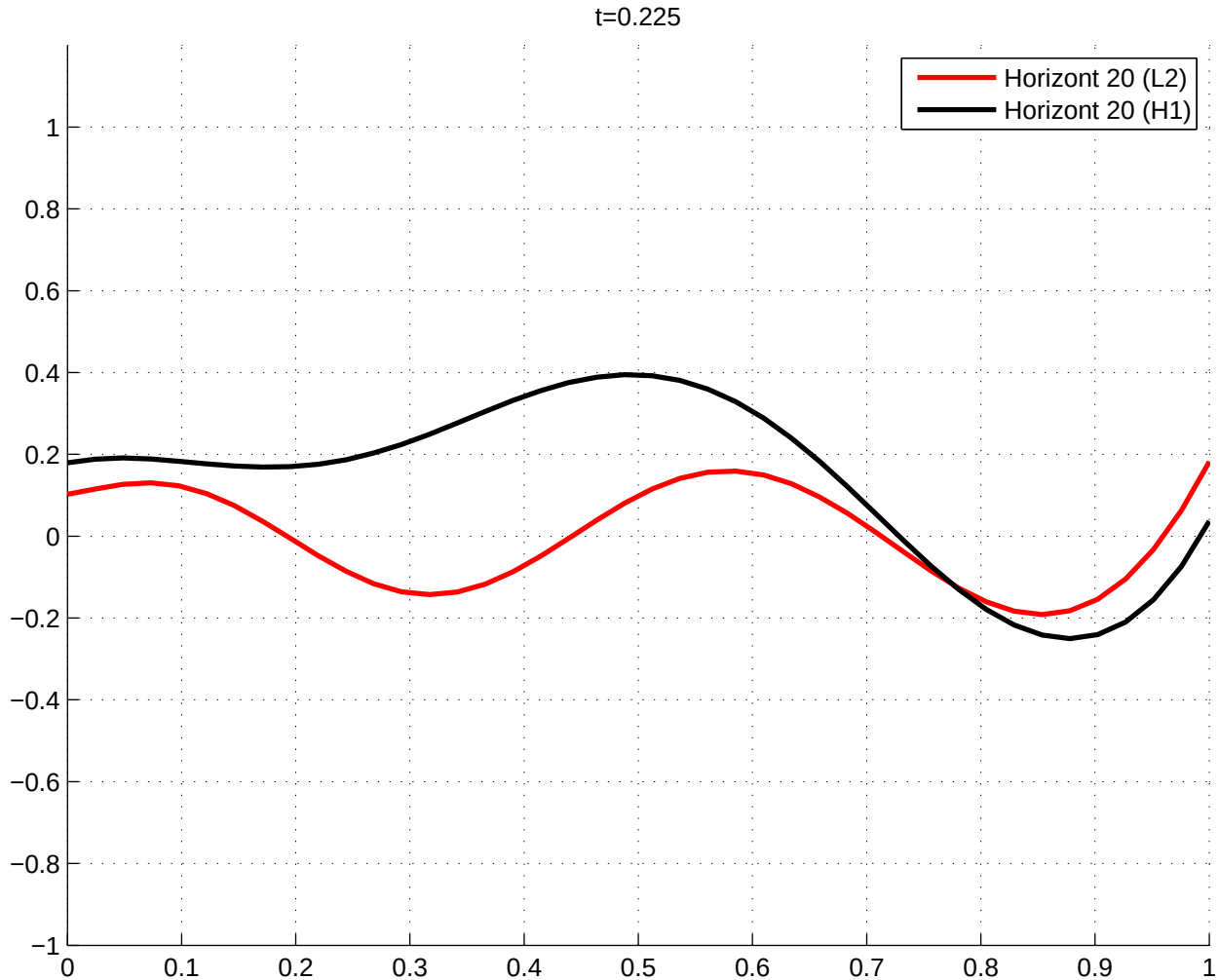

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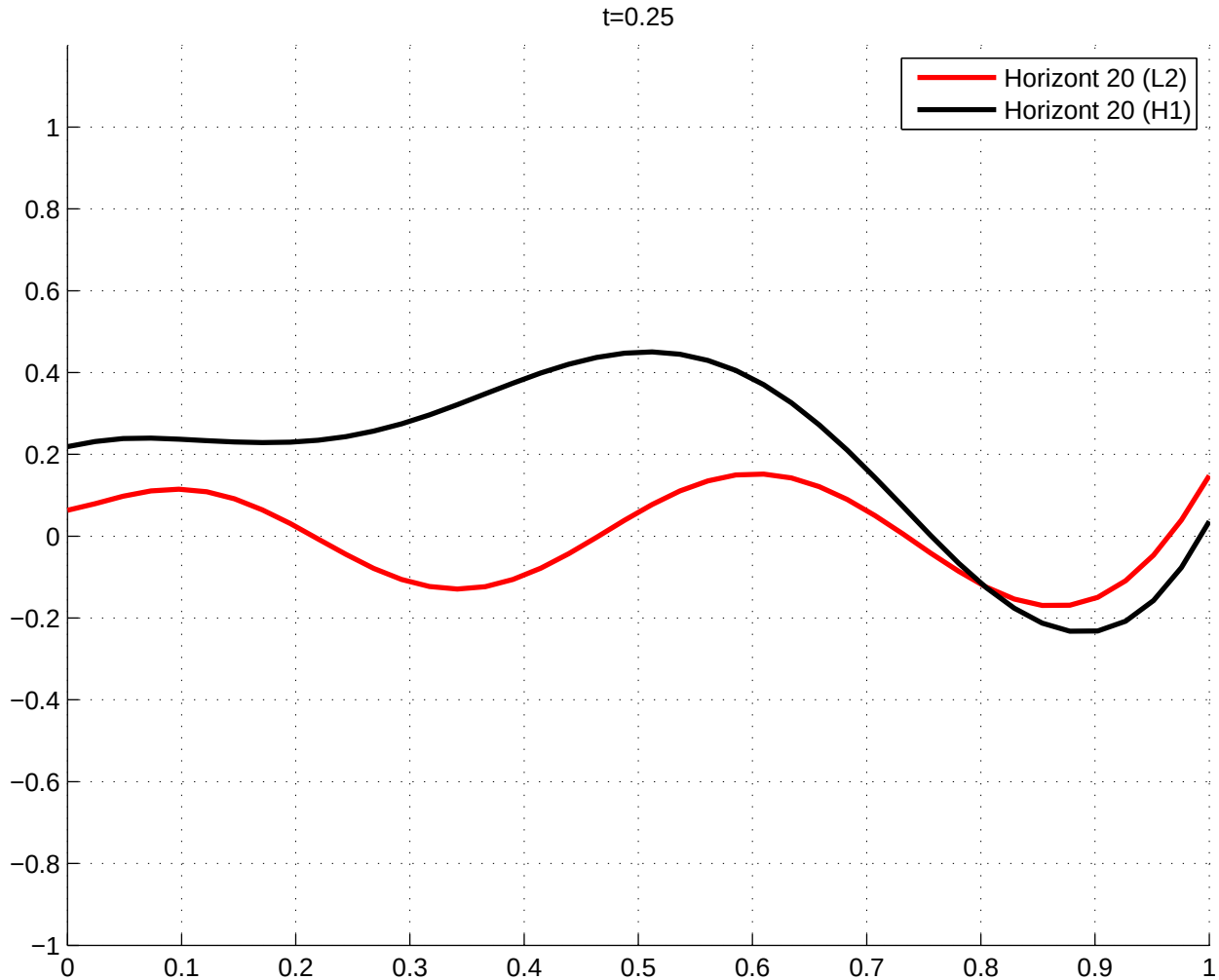

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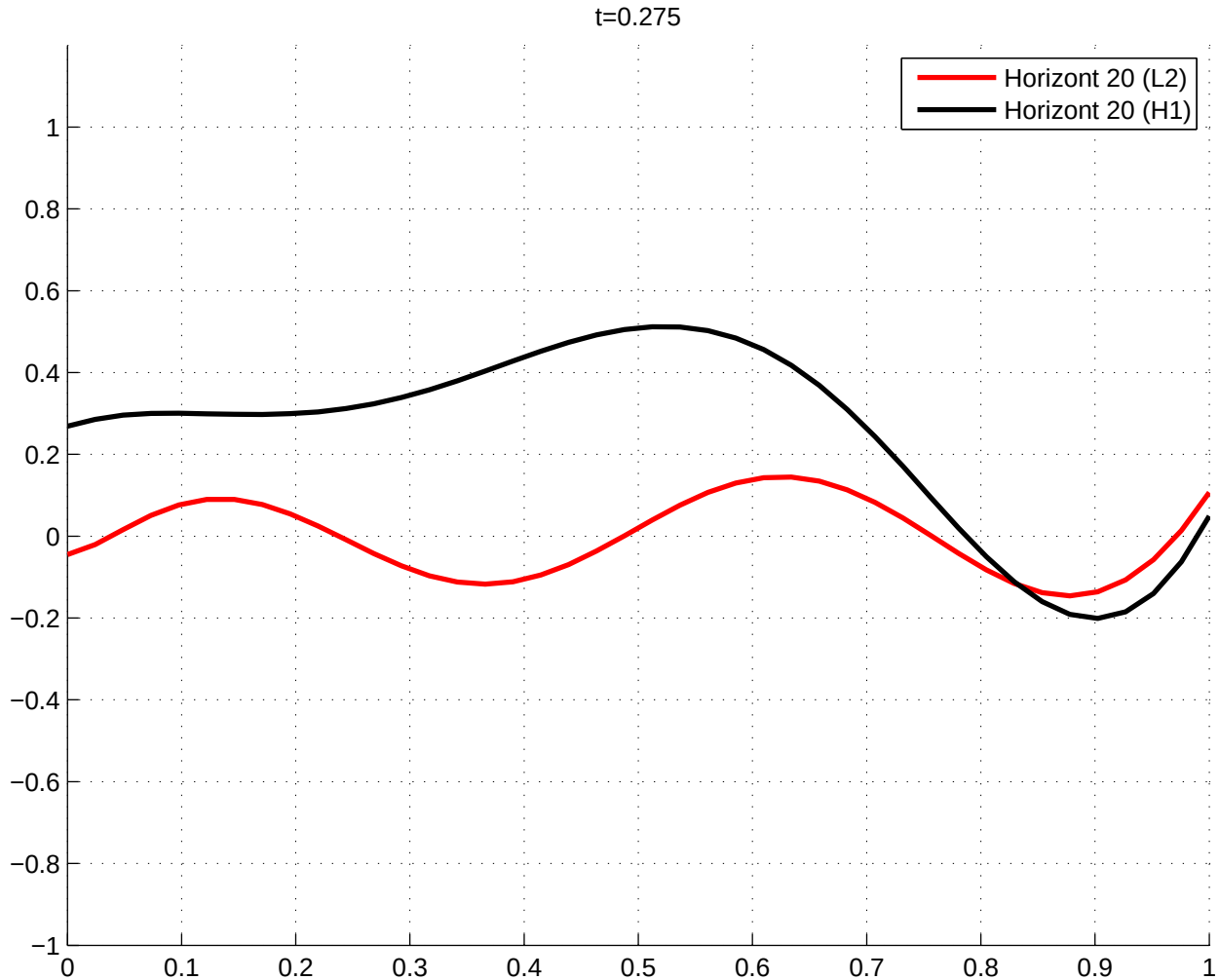

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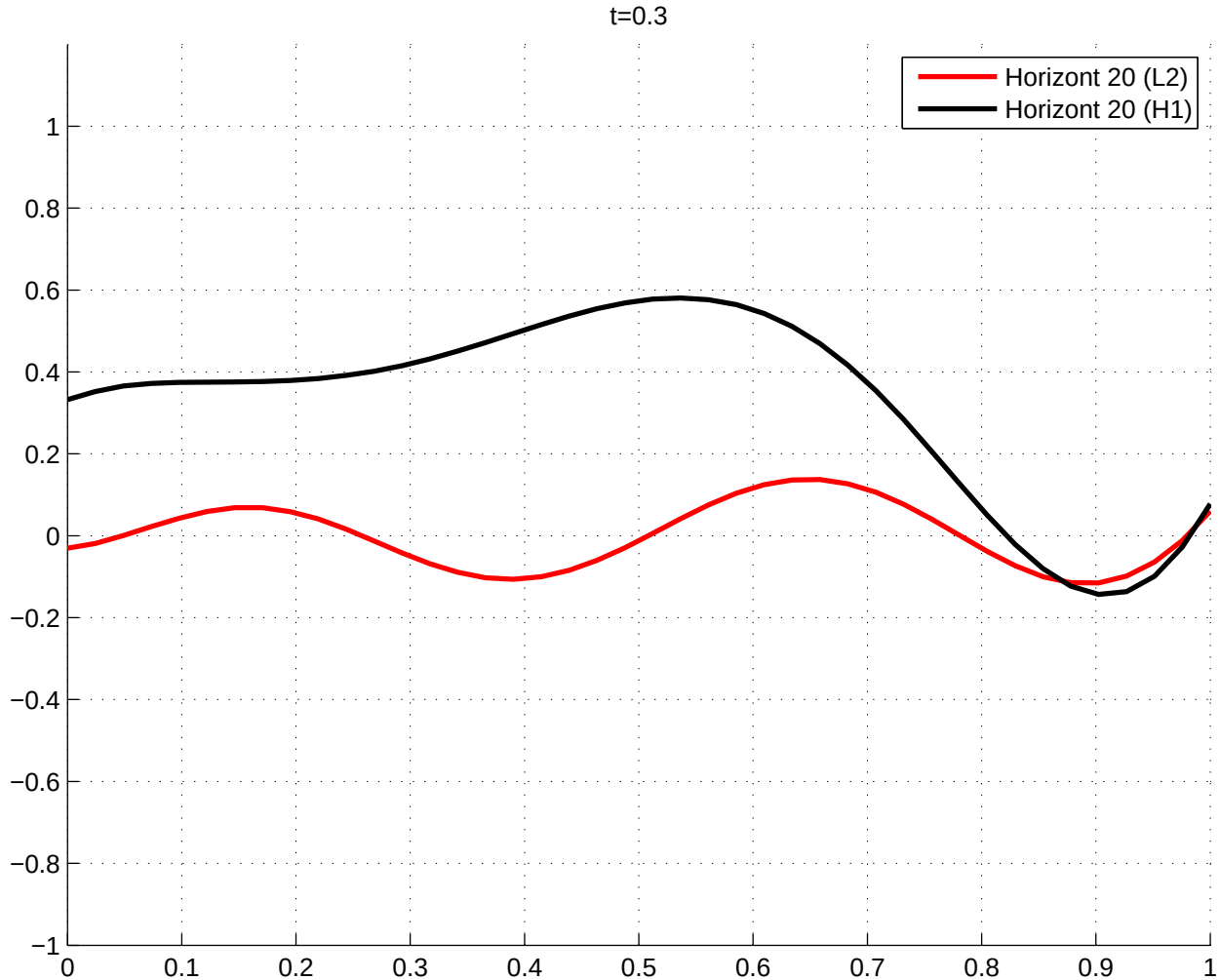

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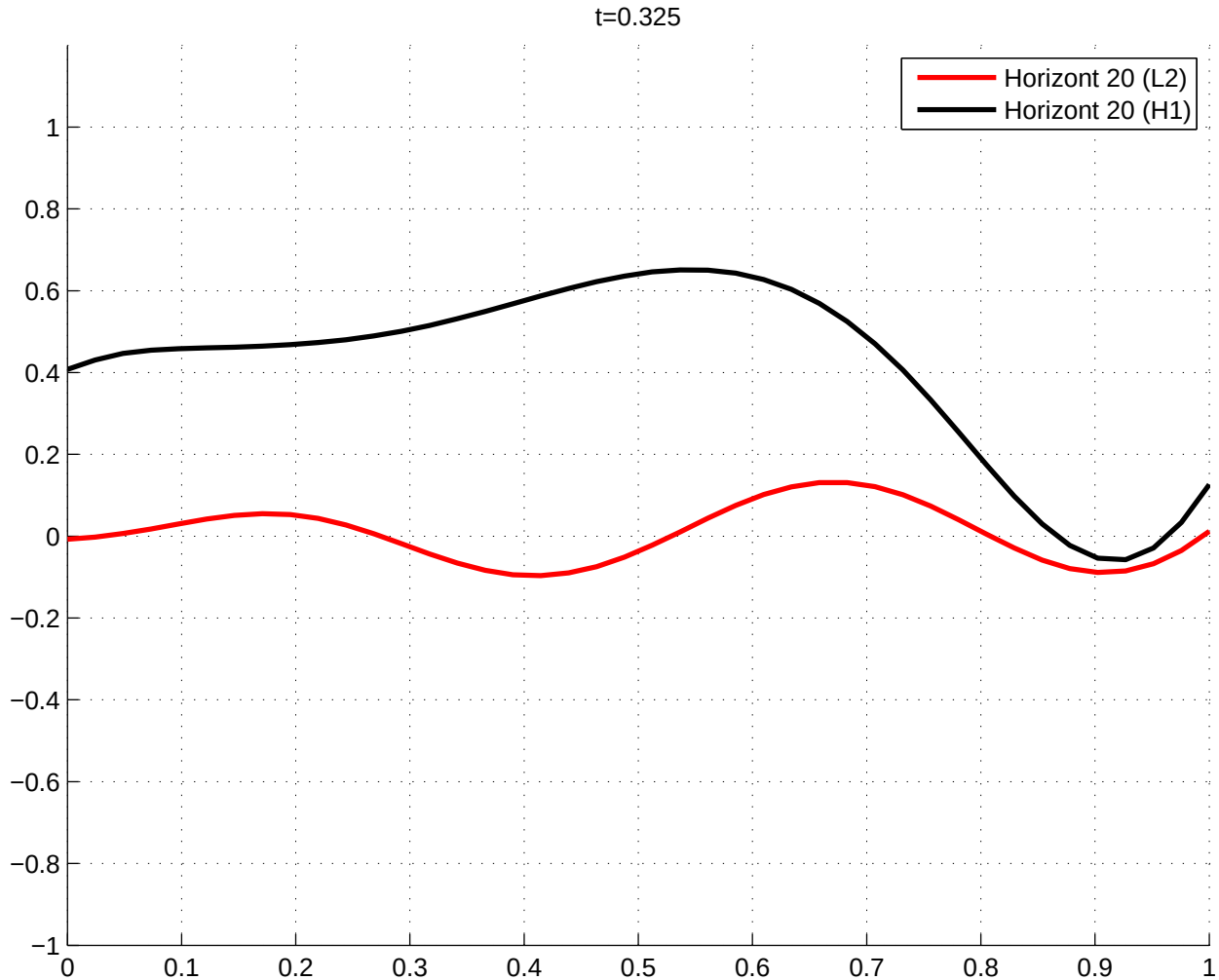

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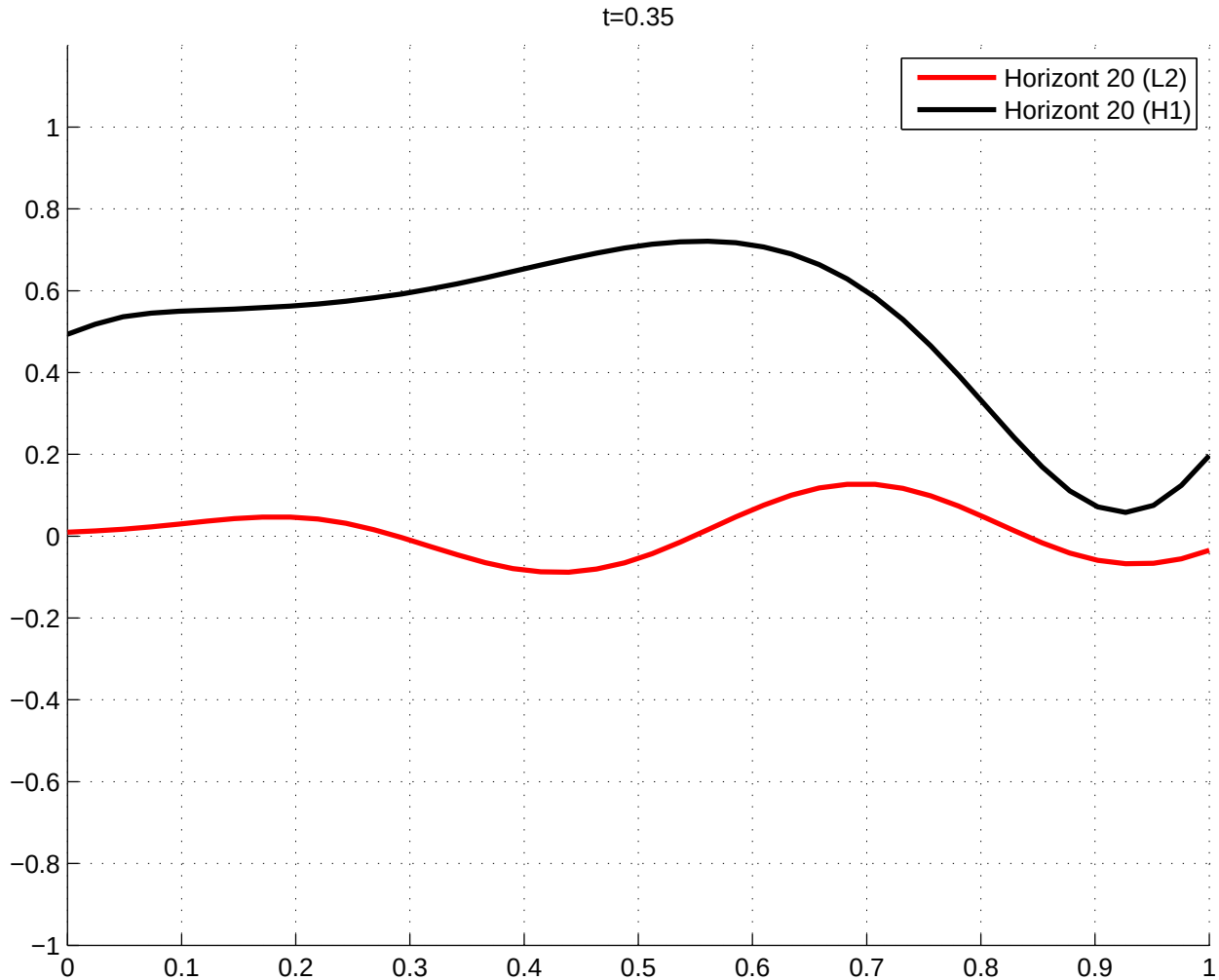

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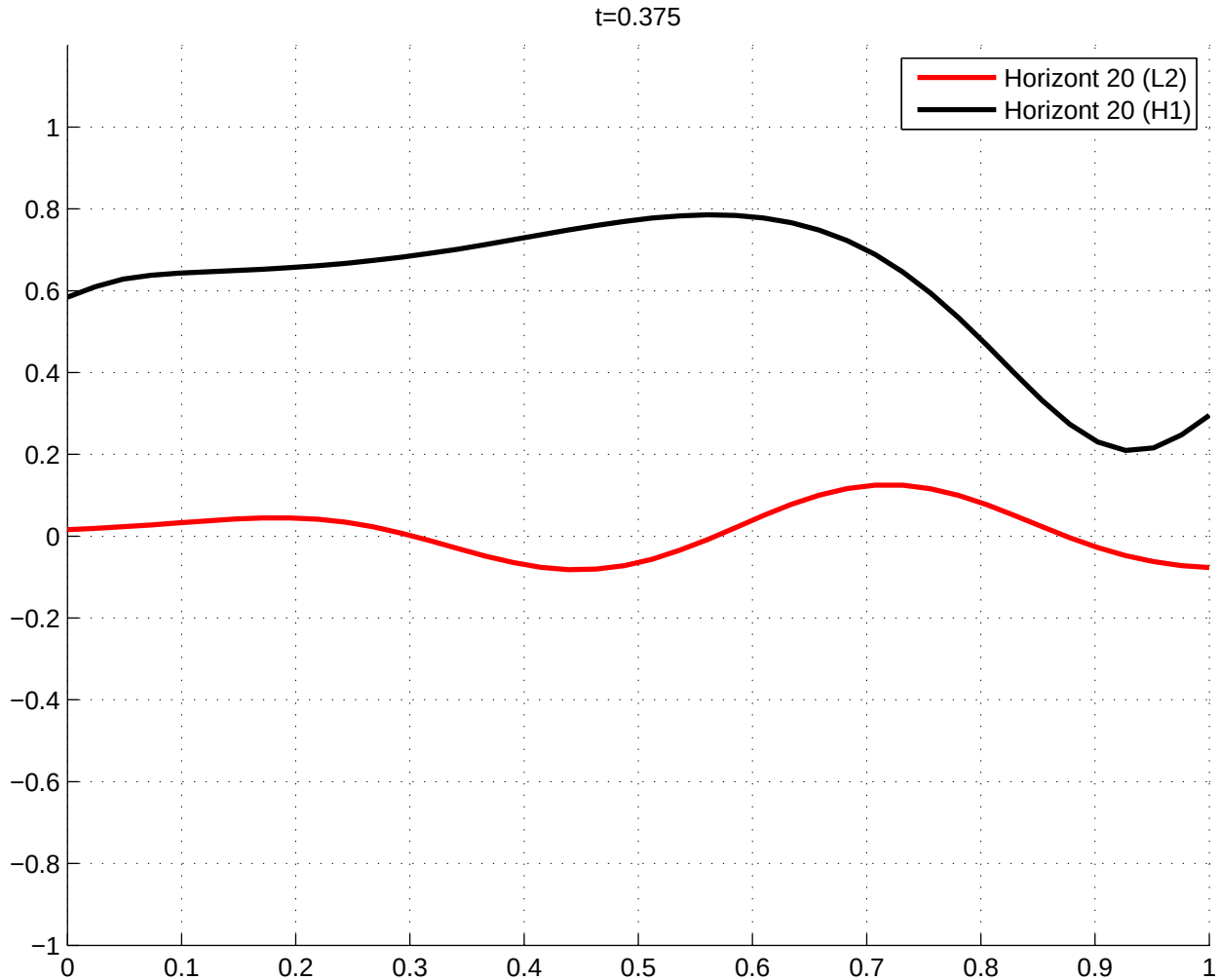

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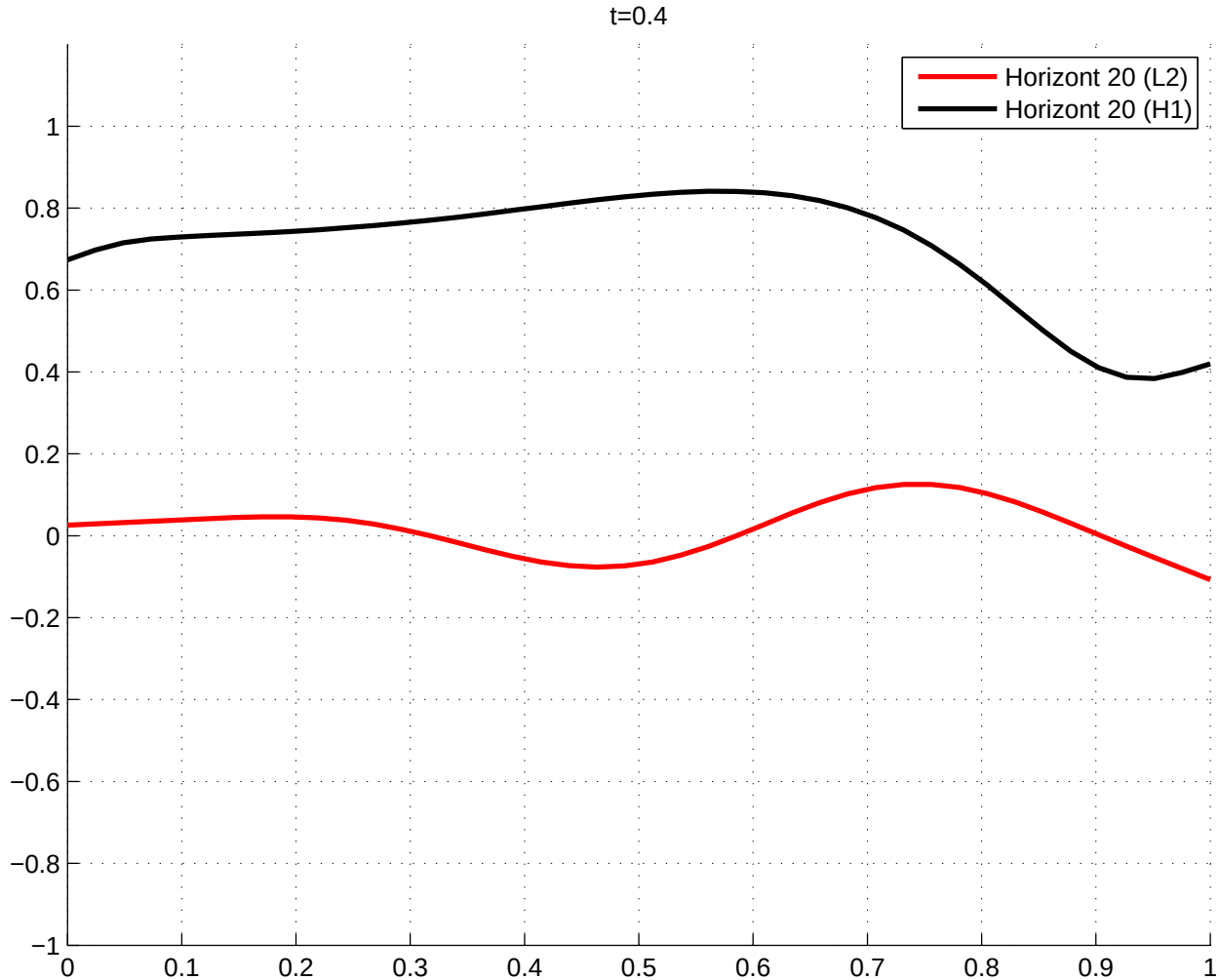

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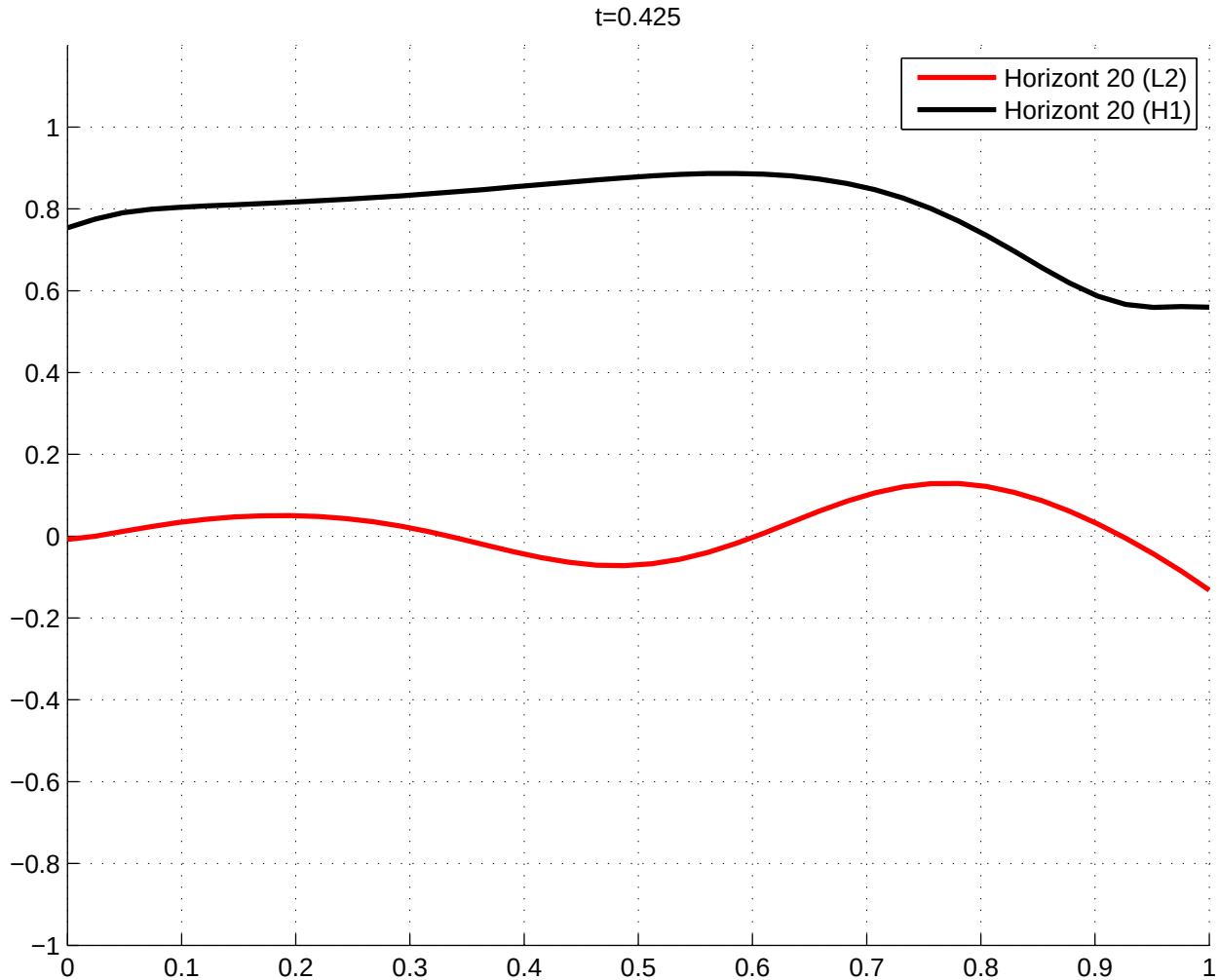

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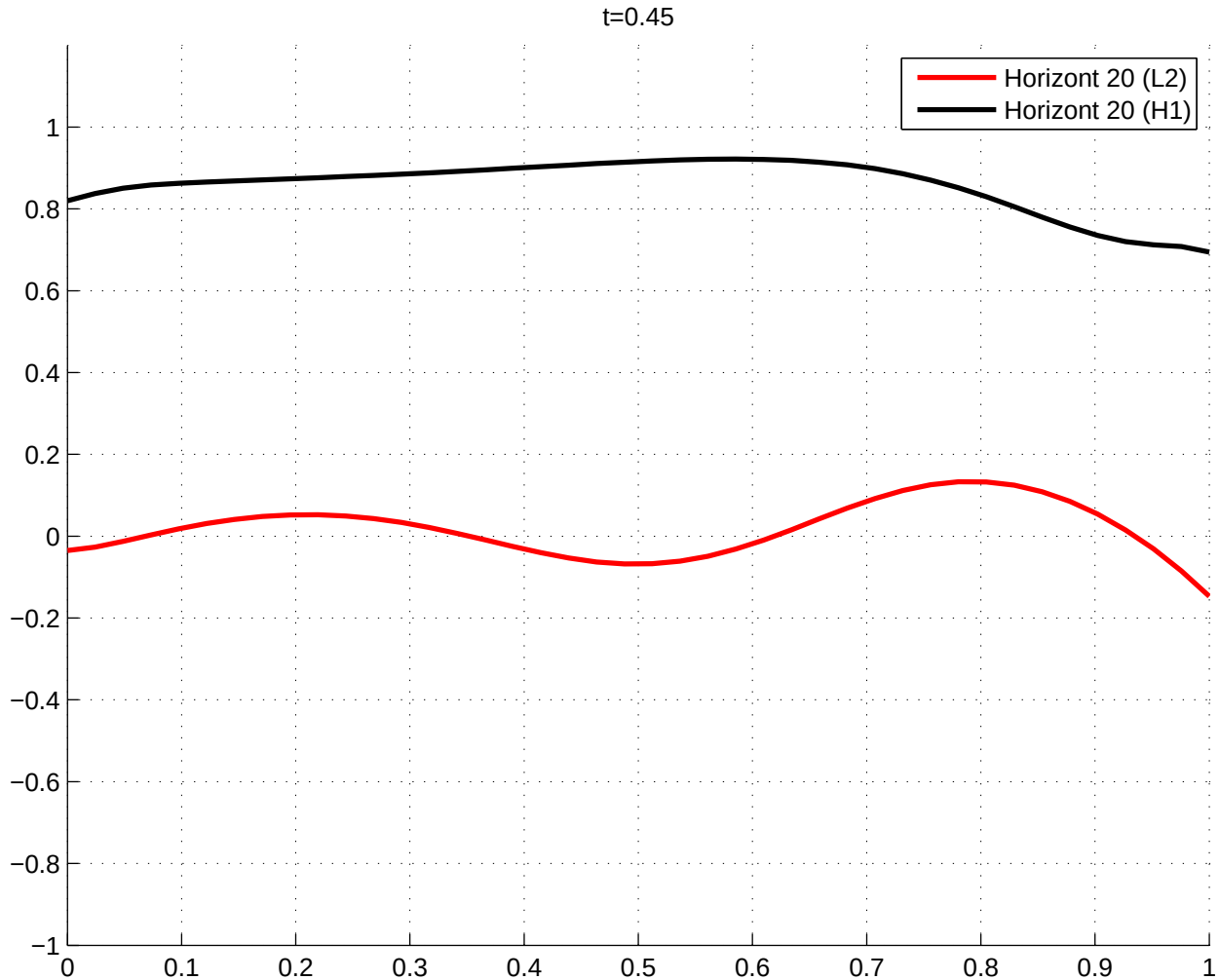

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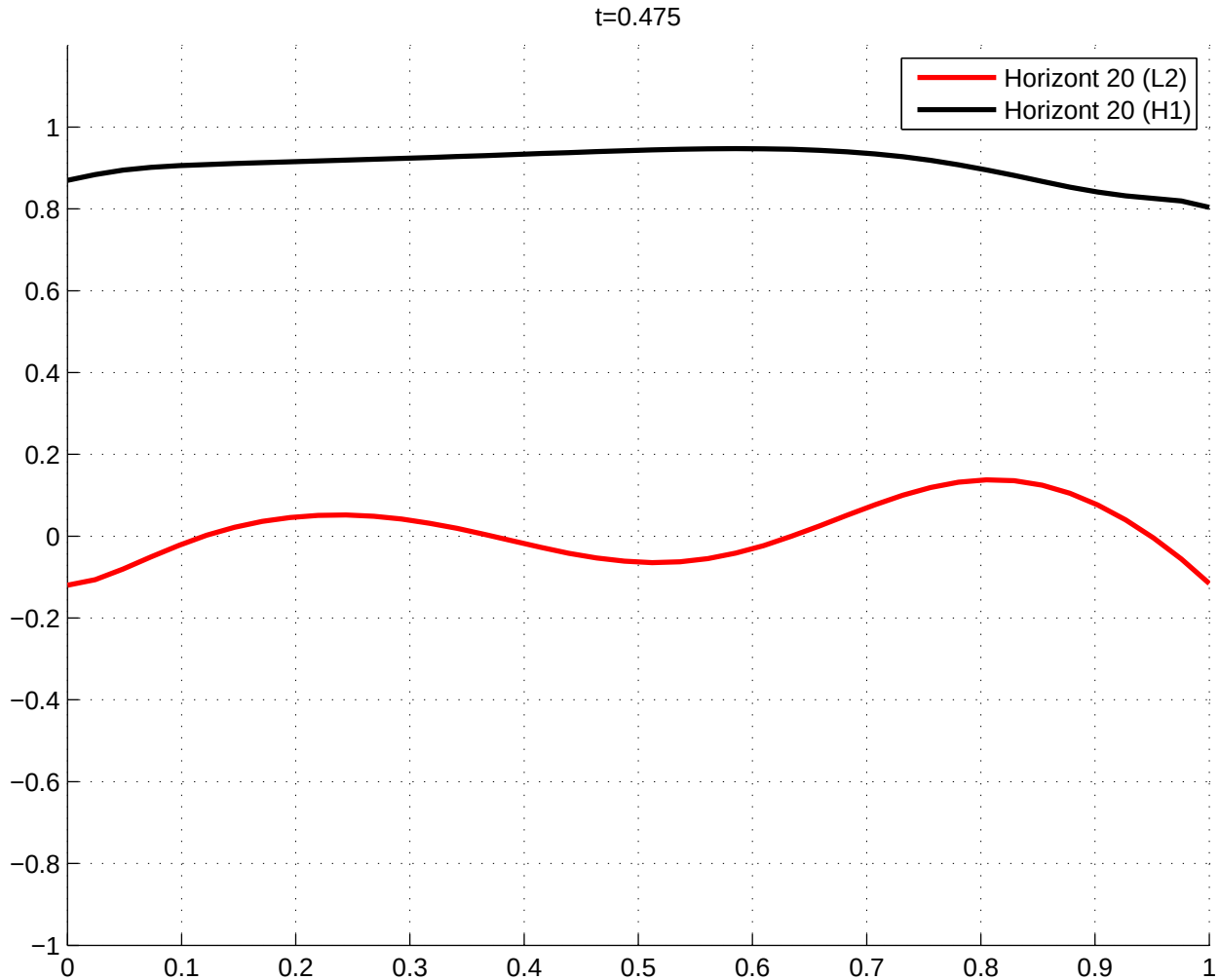

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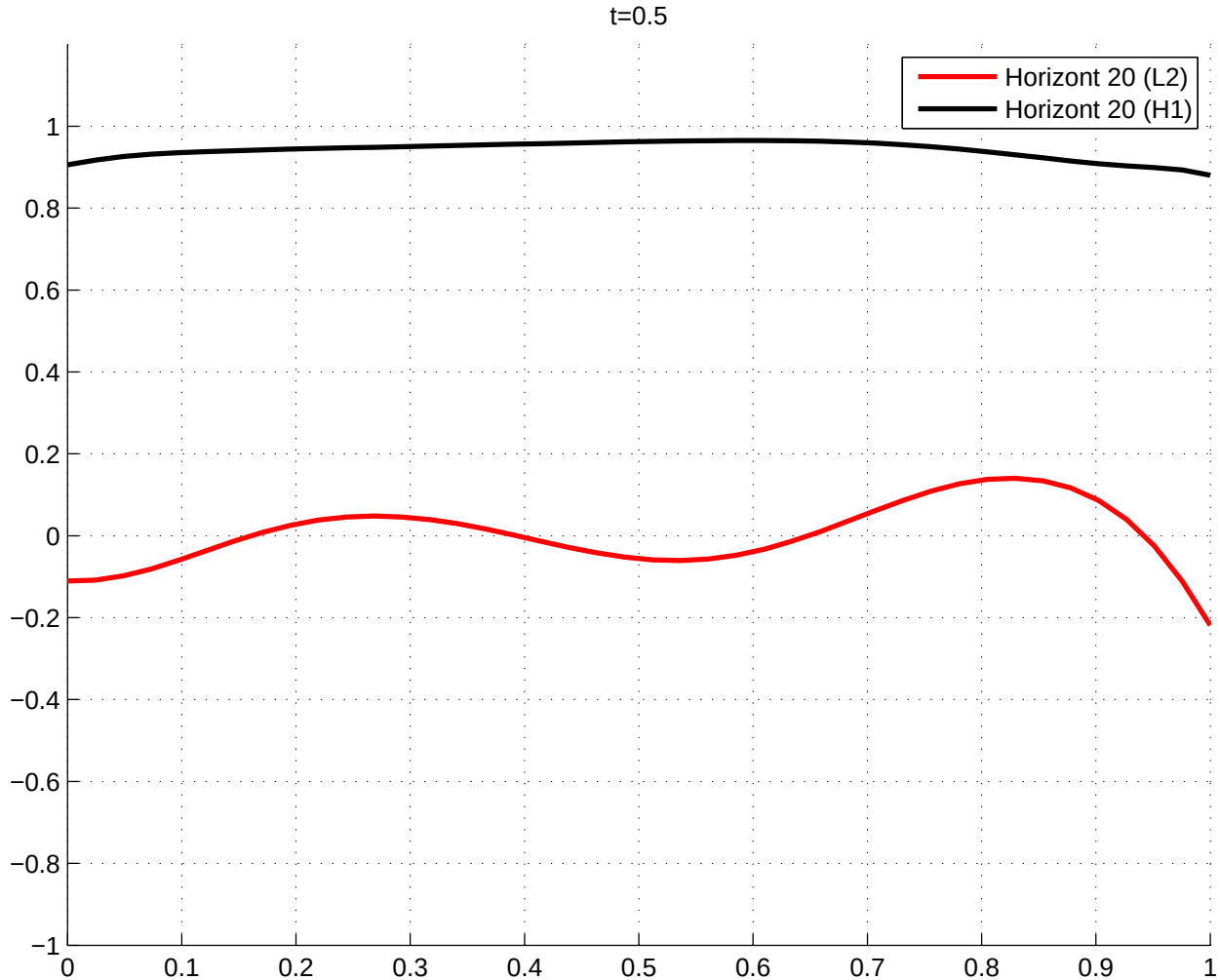

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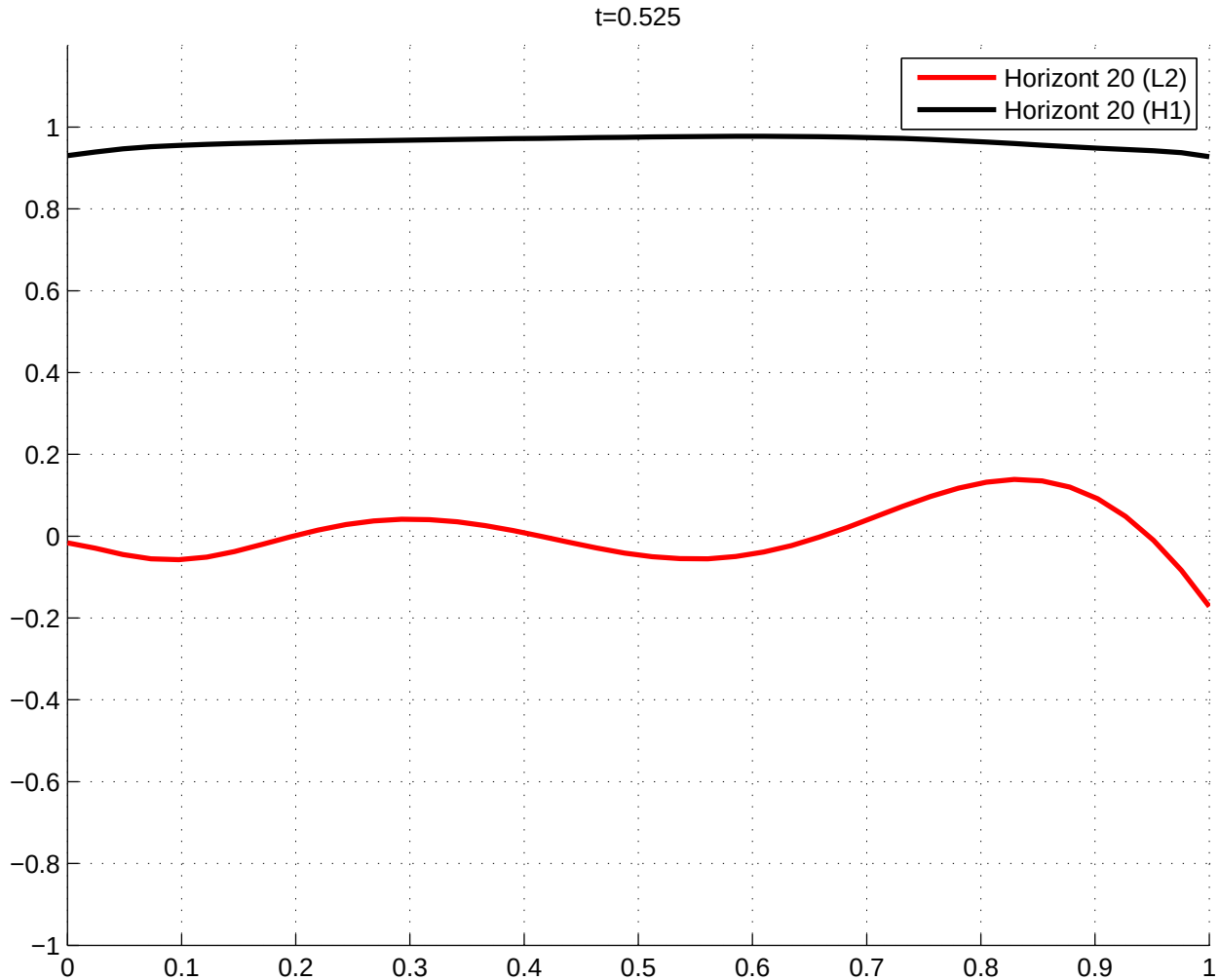

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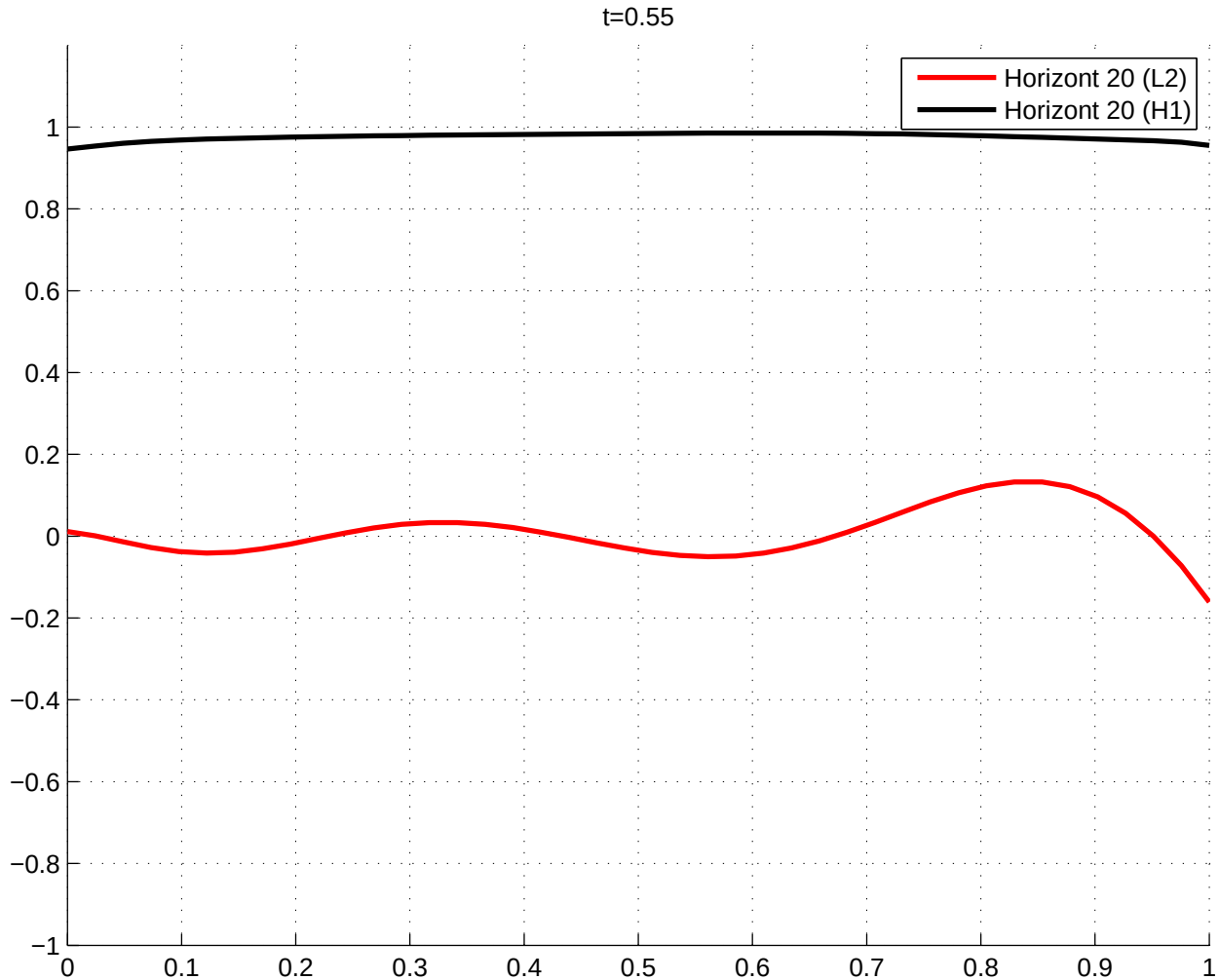

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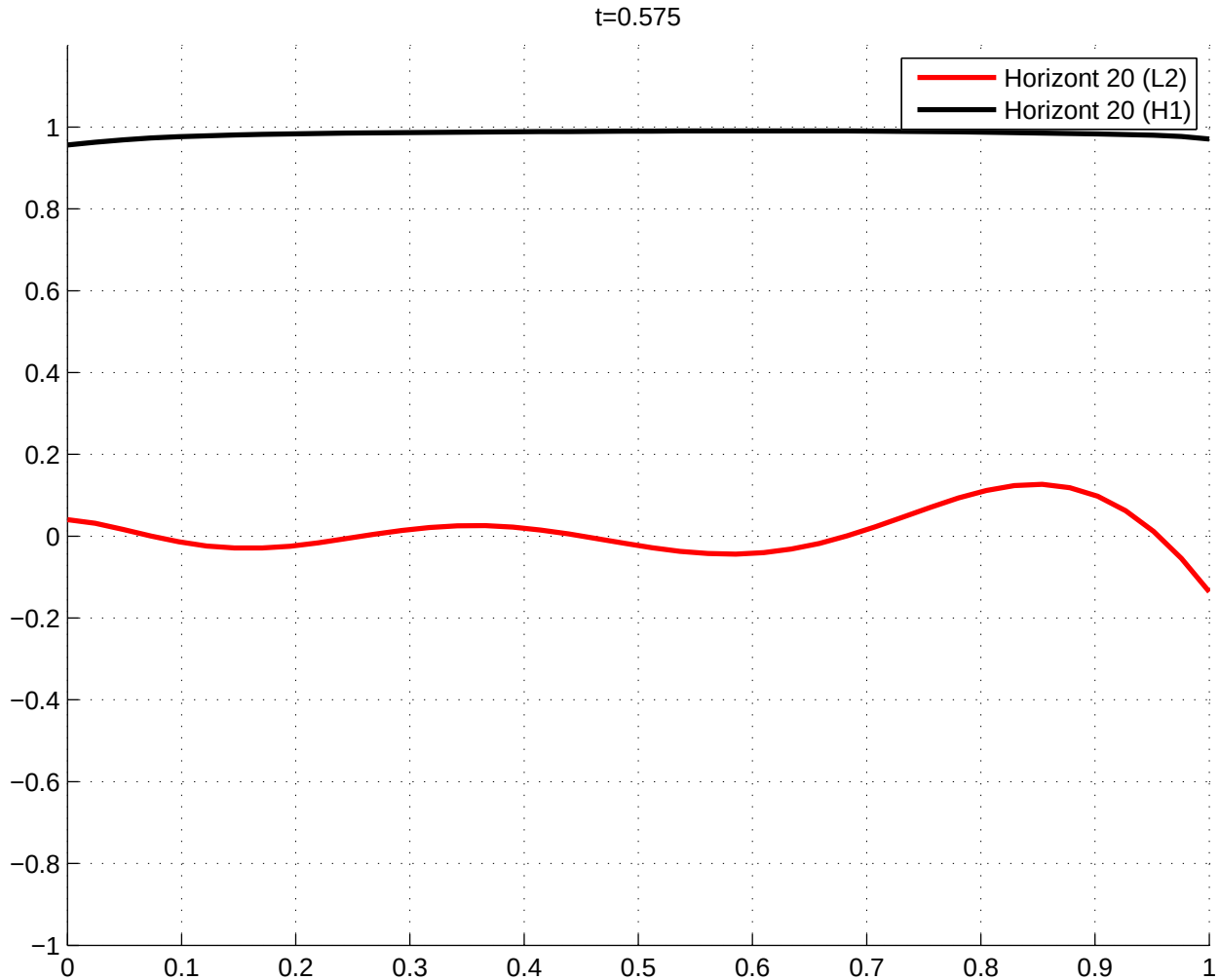

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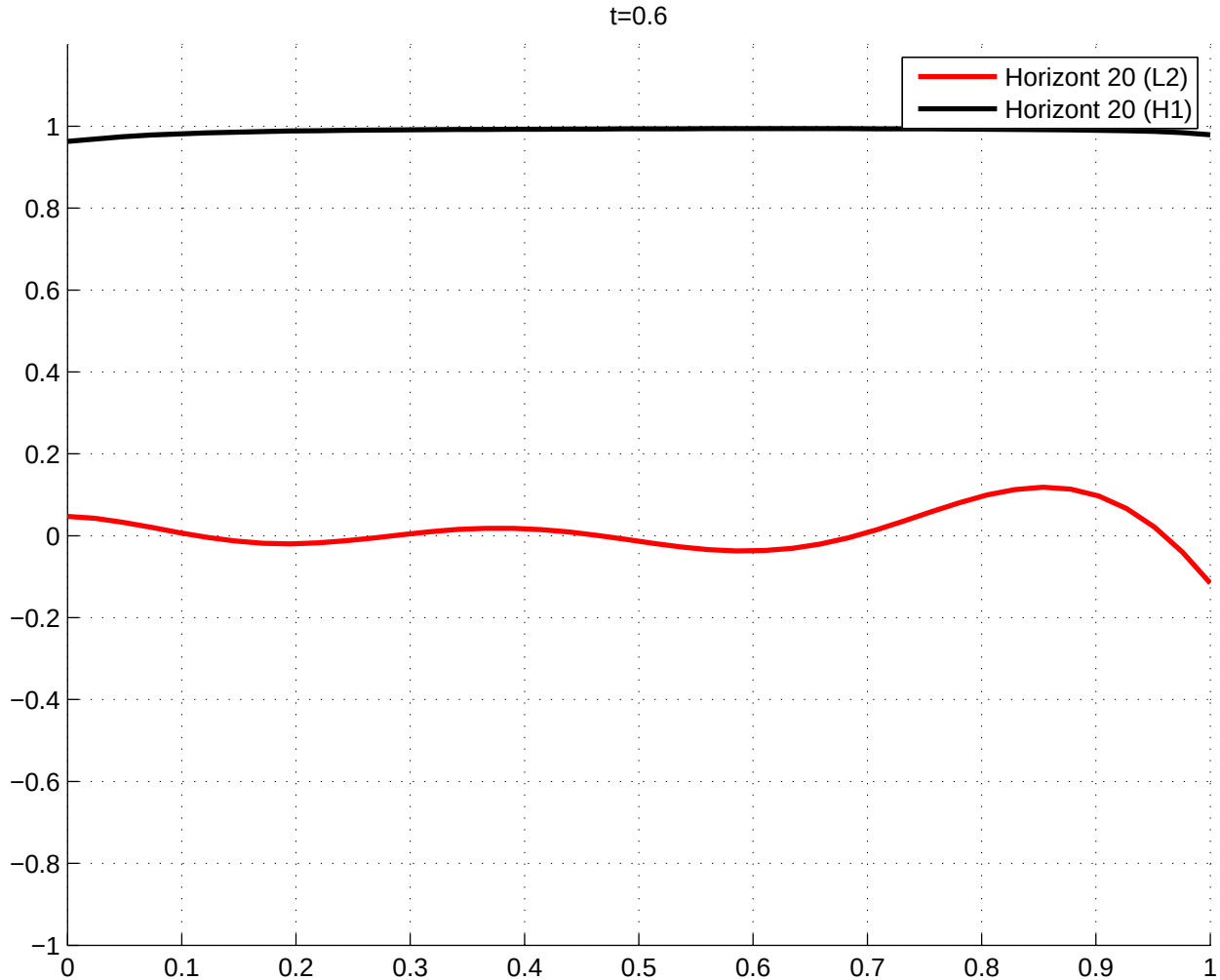

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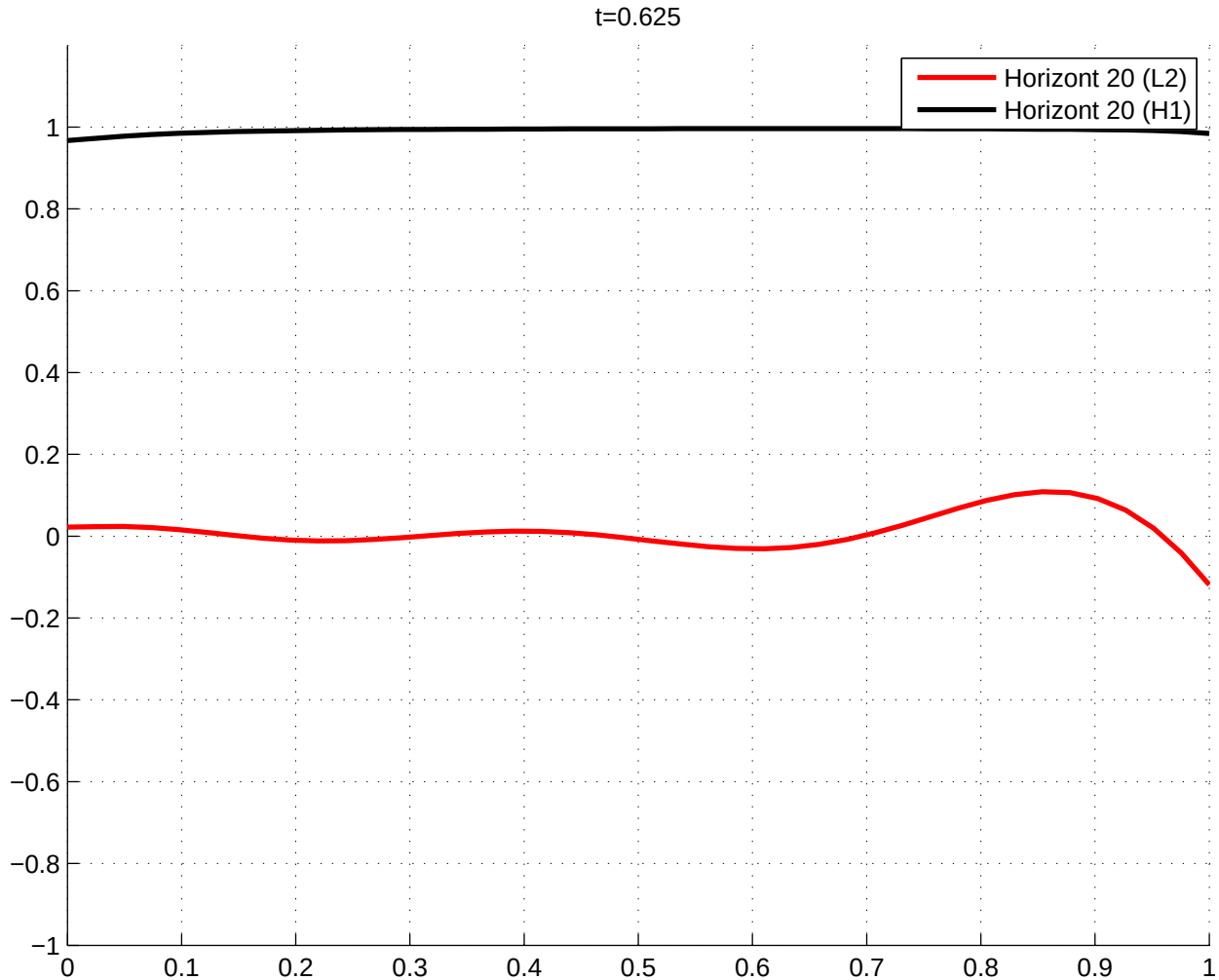

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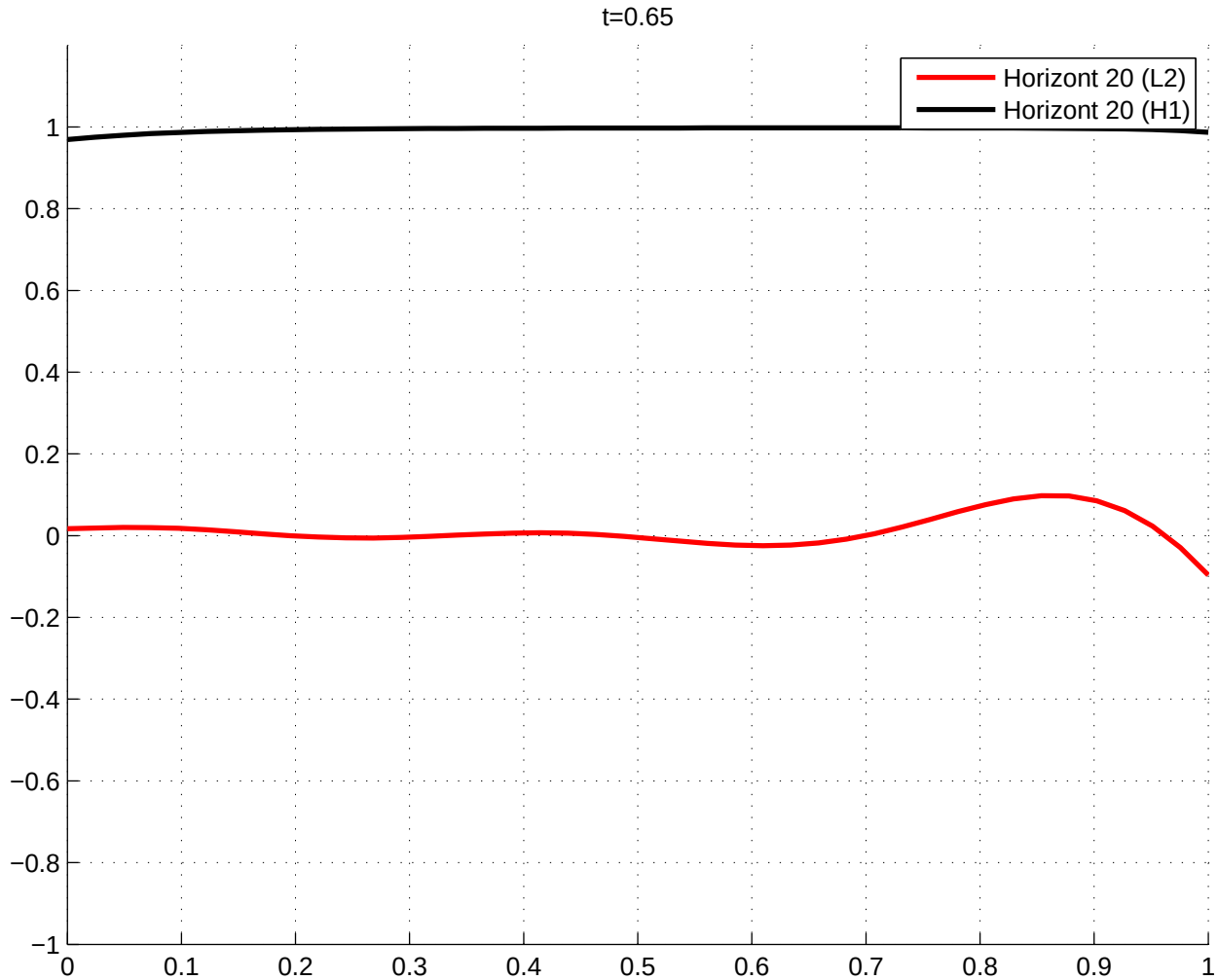

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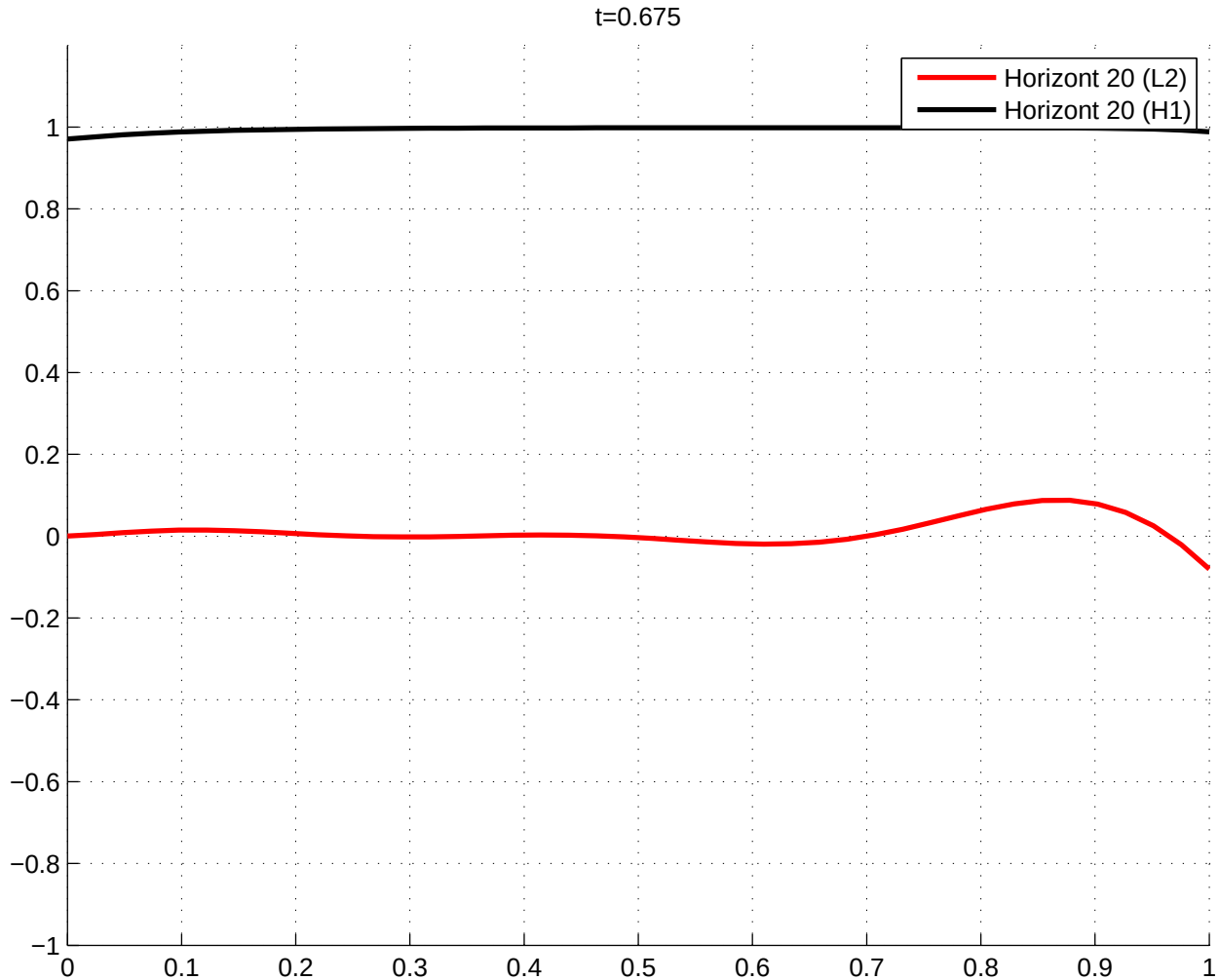

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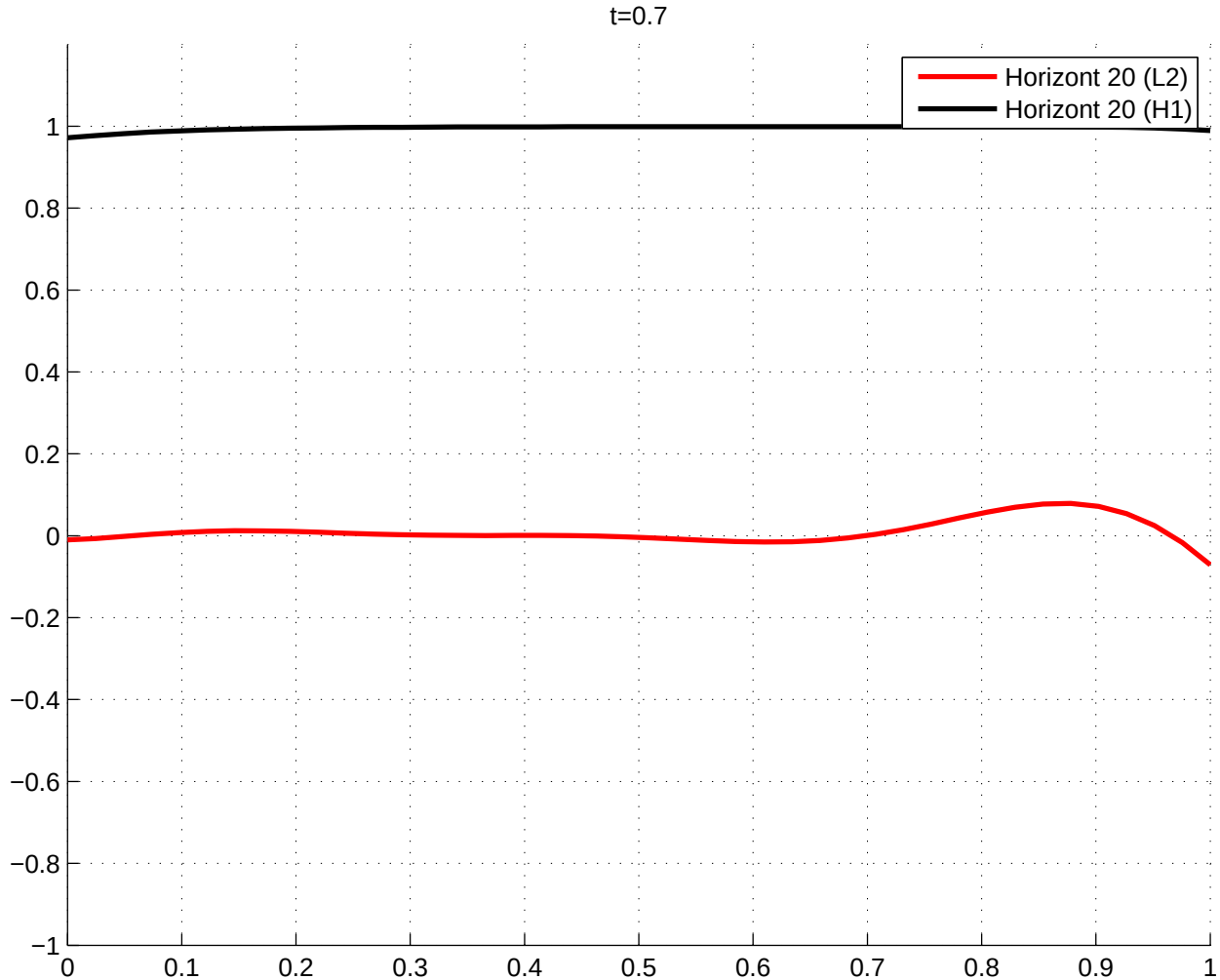

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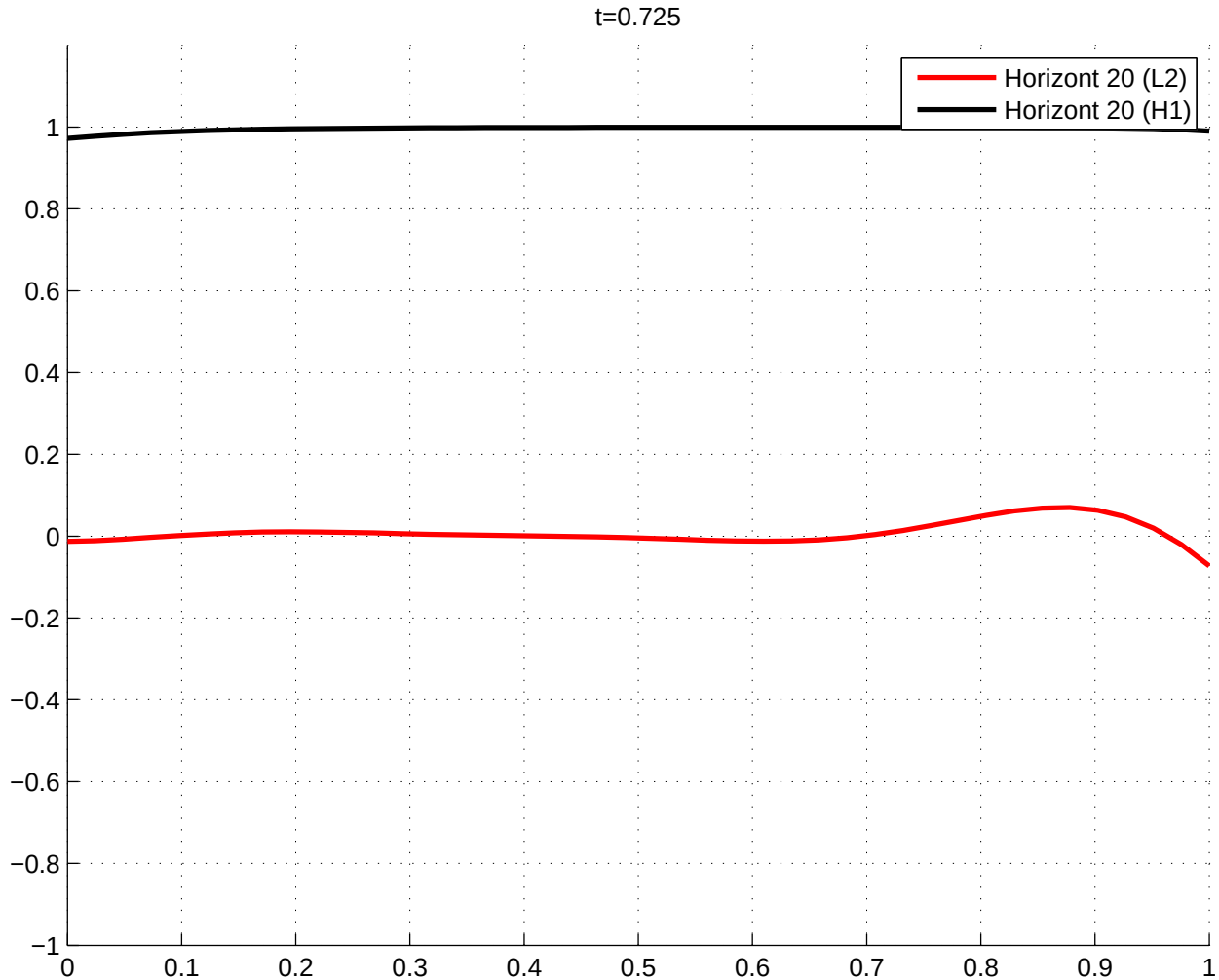

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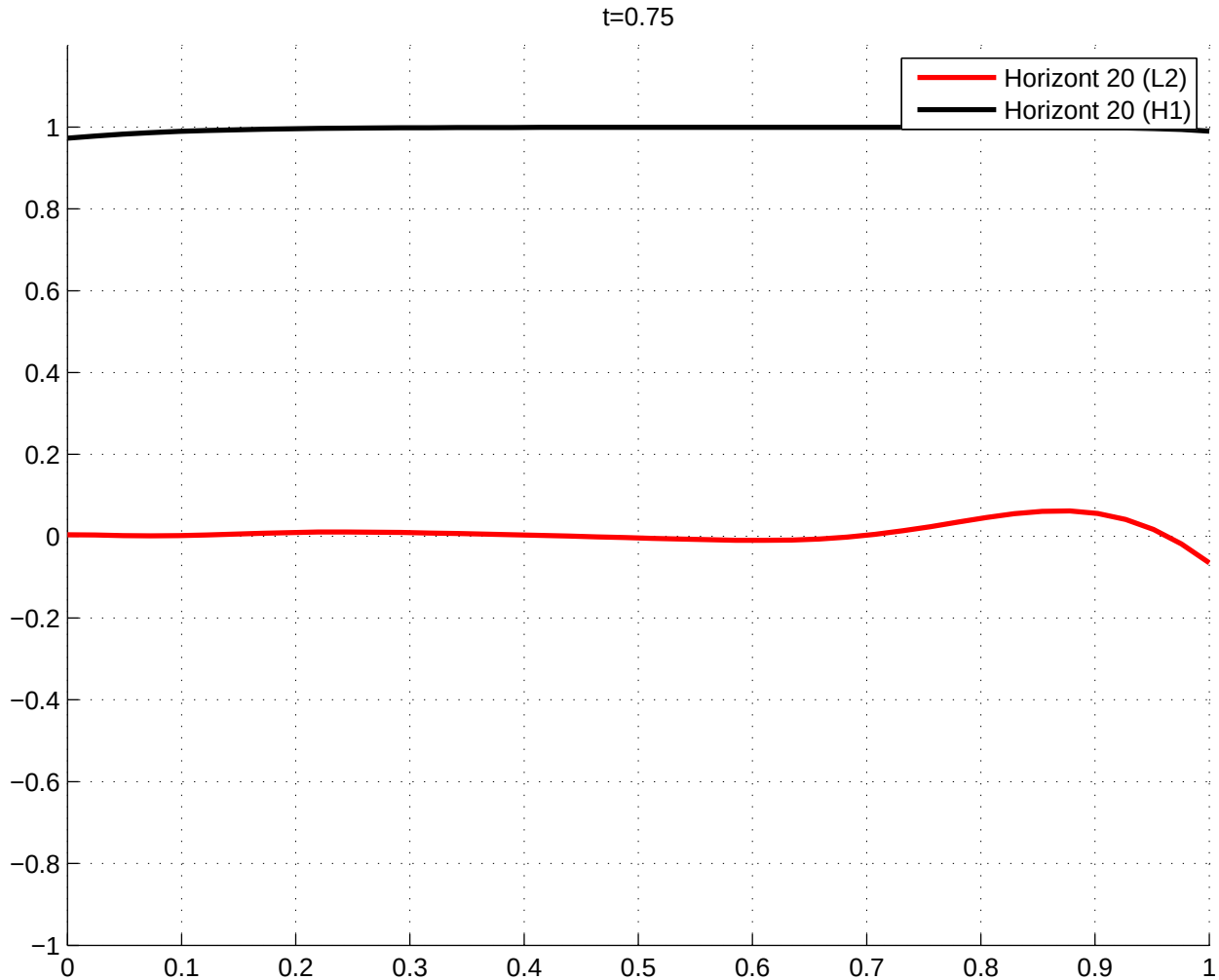

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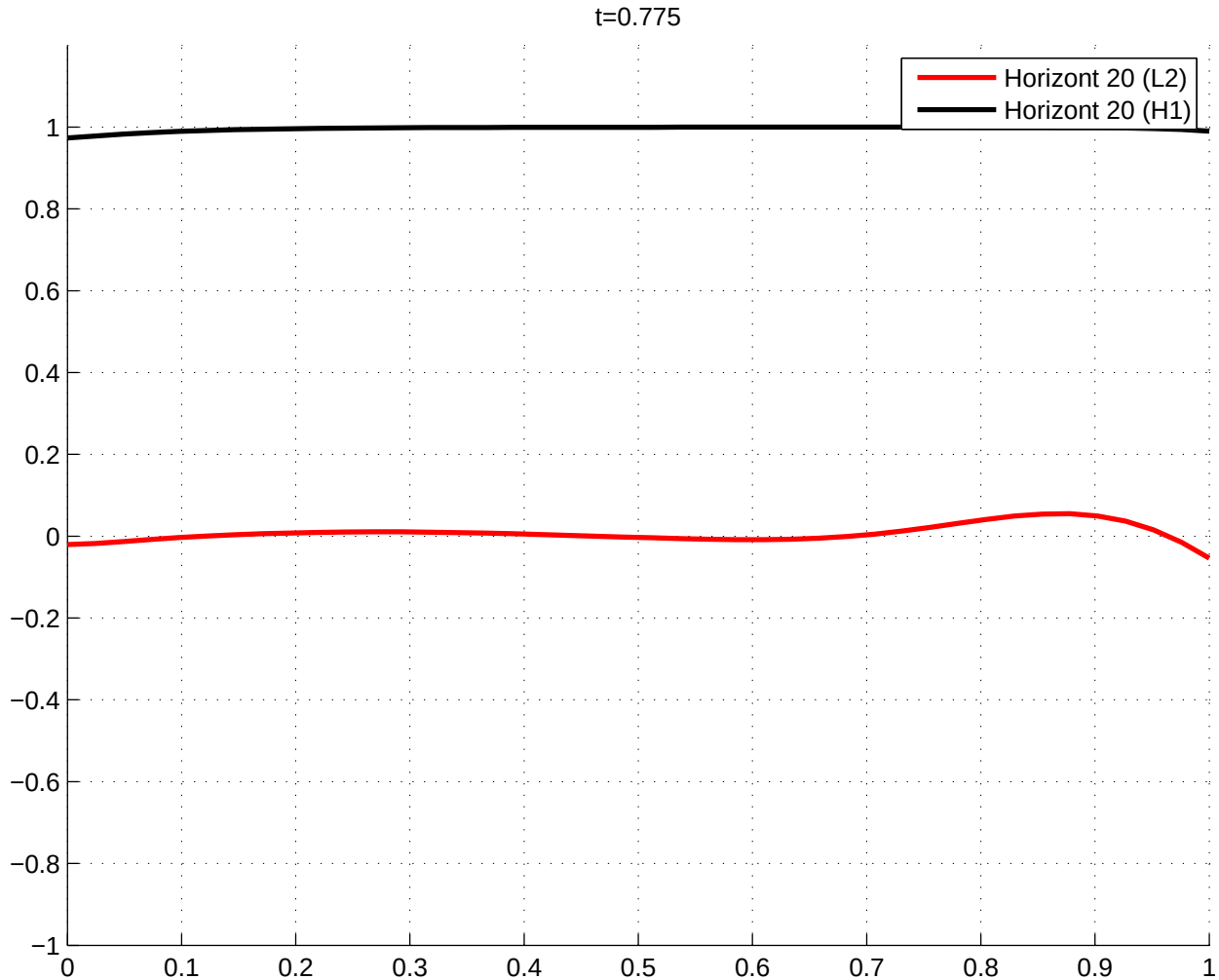

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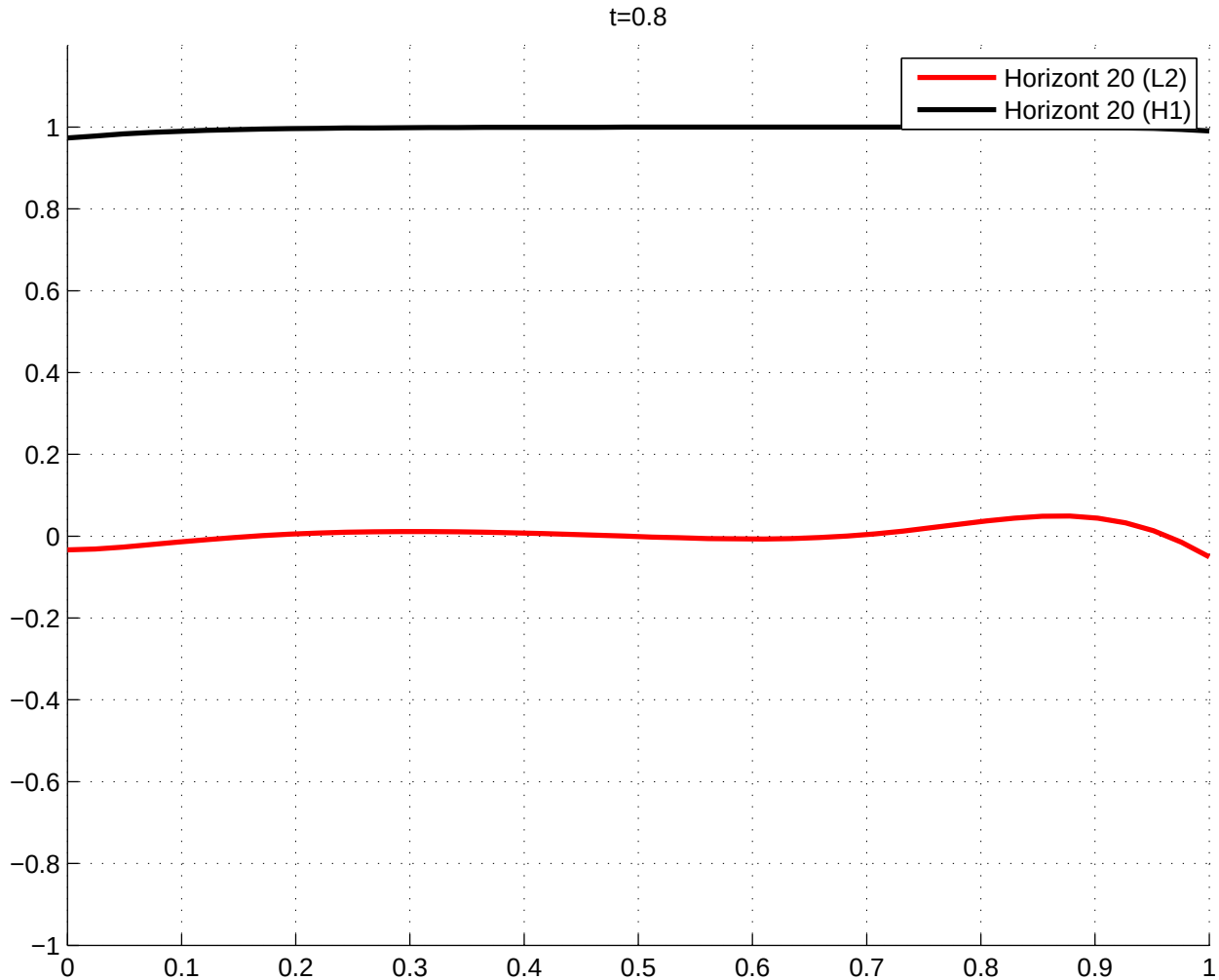

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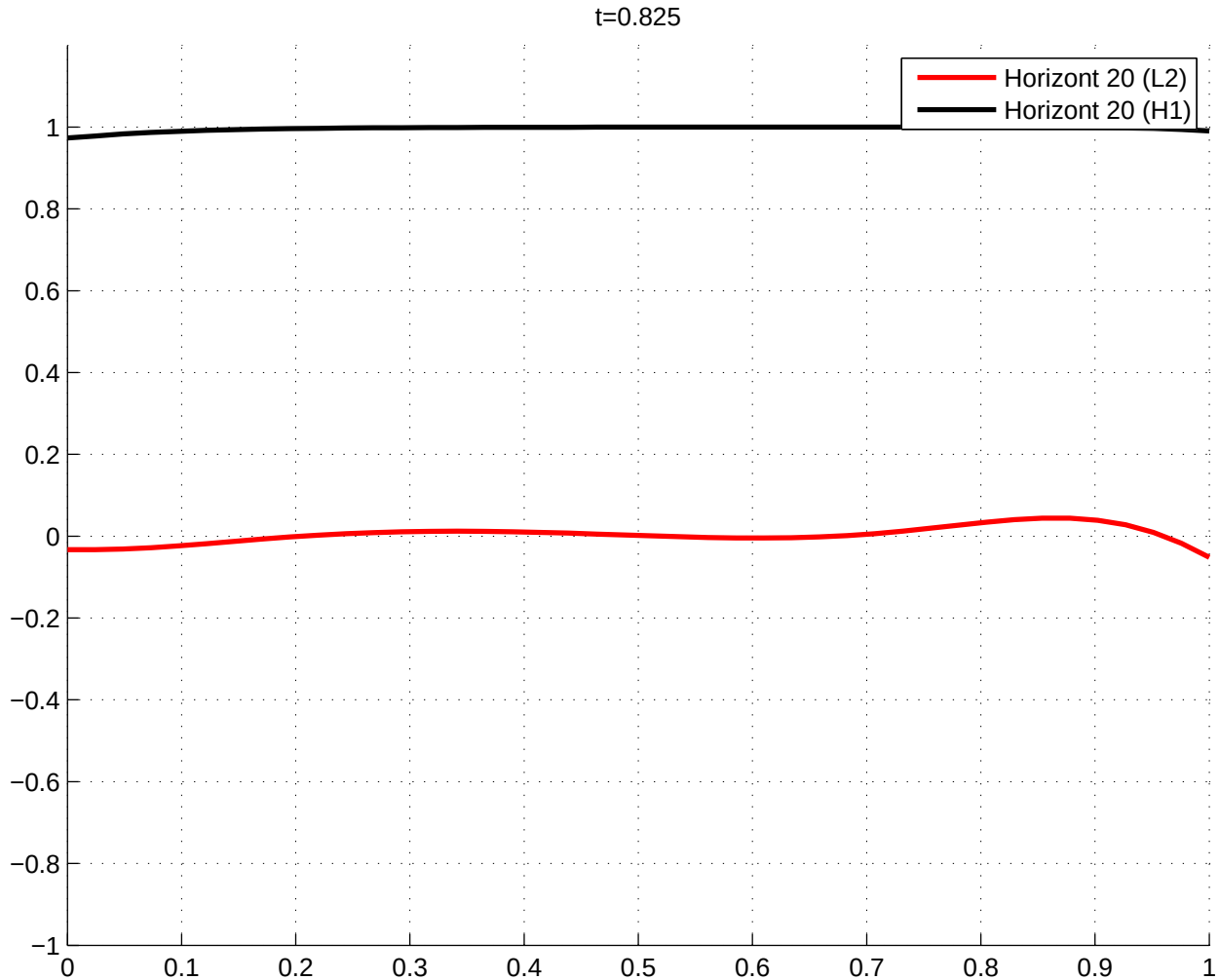

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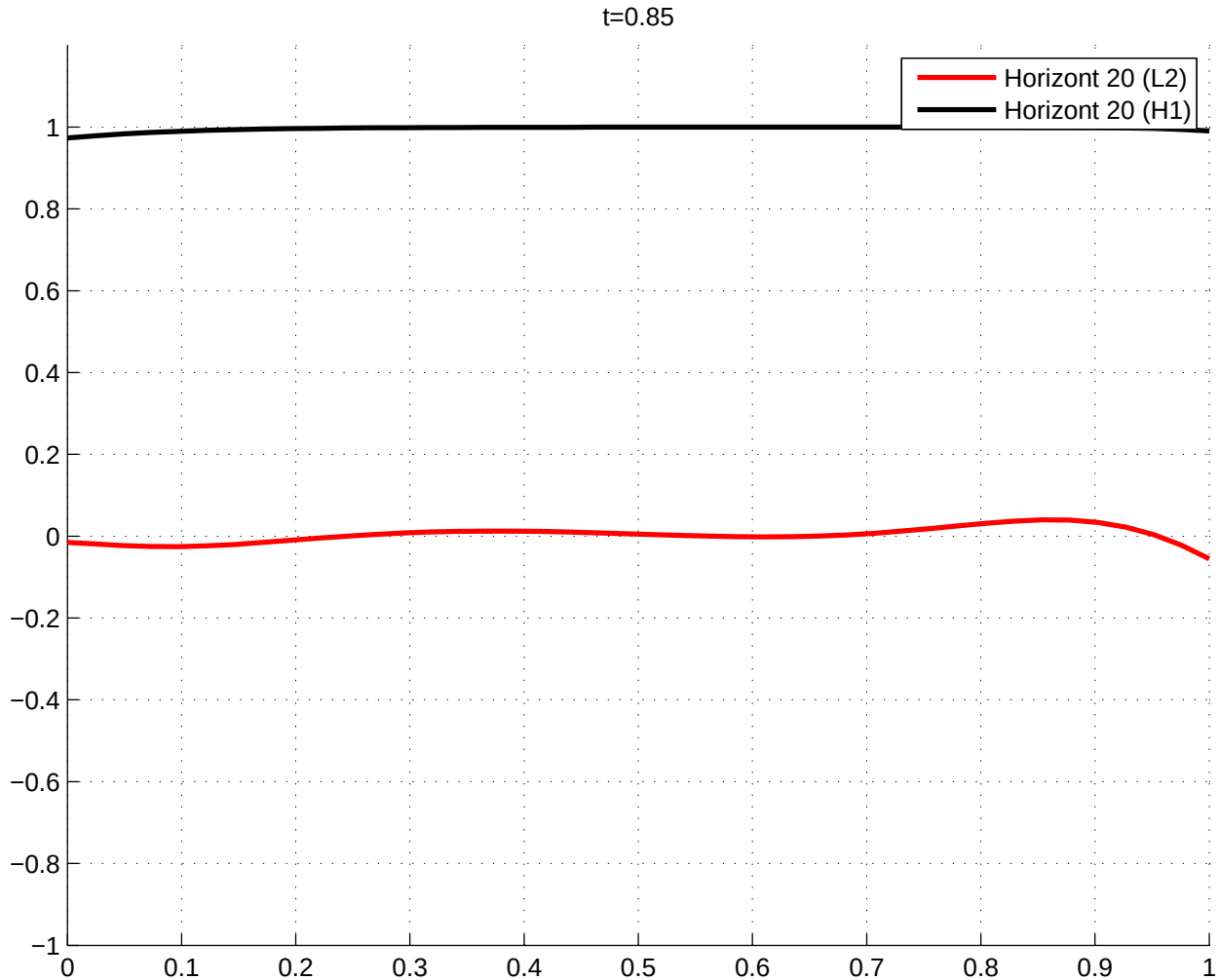

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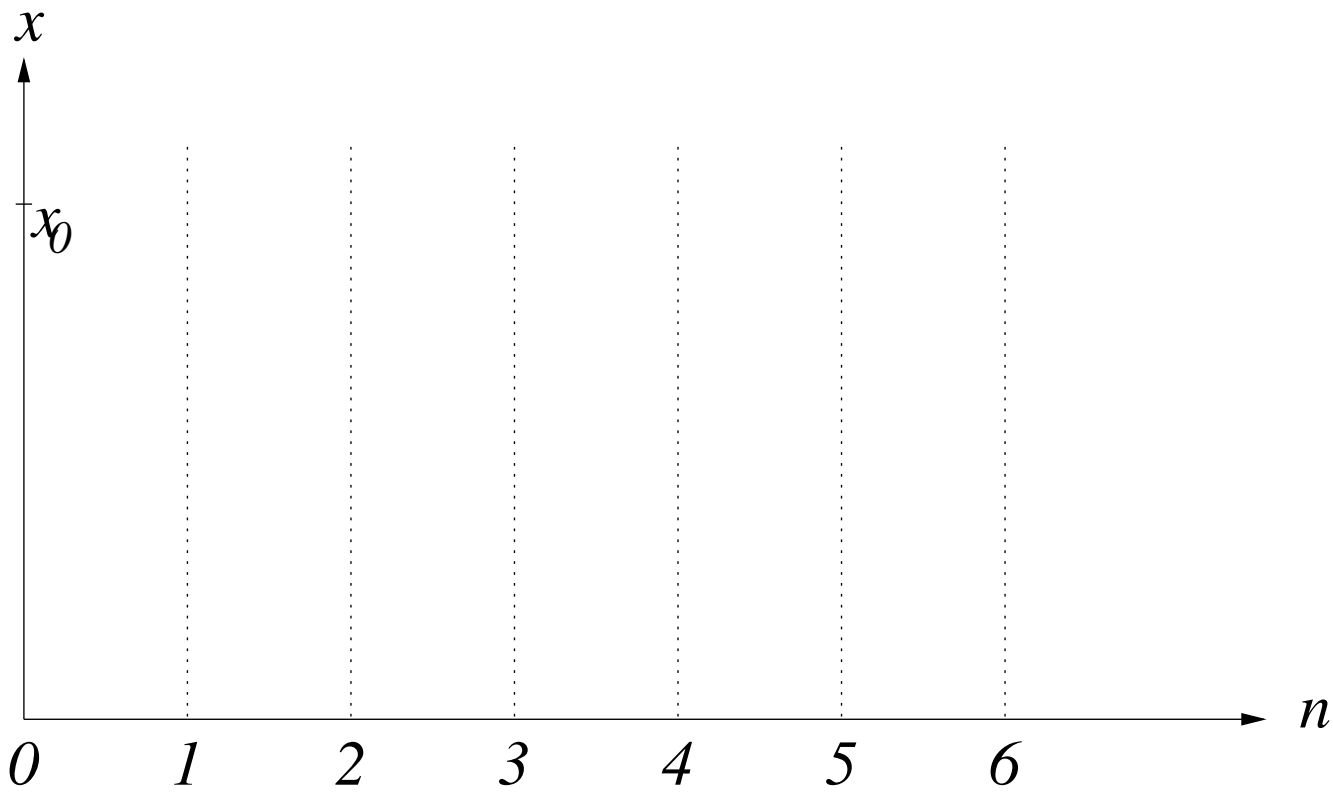
Robustness :$\Leftrightarrow$ the system still approaches/stays within a neighborhood of the stable equilibrium for small $d(n)$

Perturbations in MPC scheme



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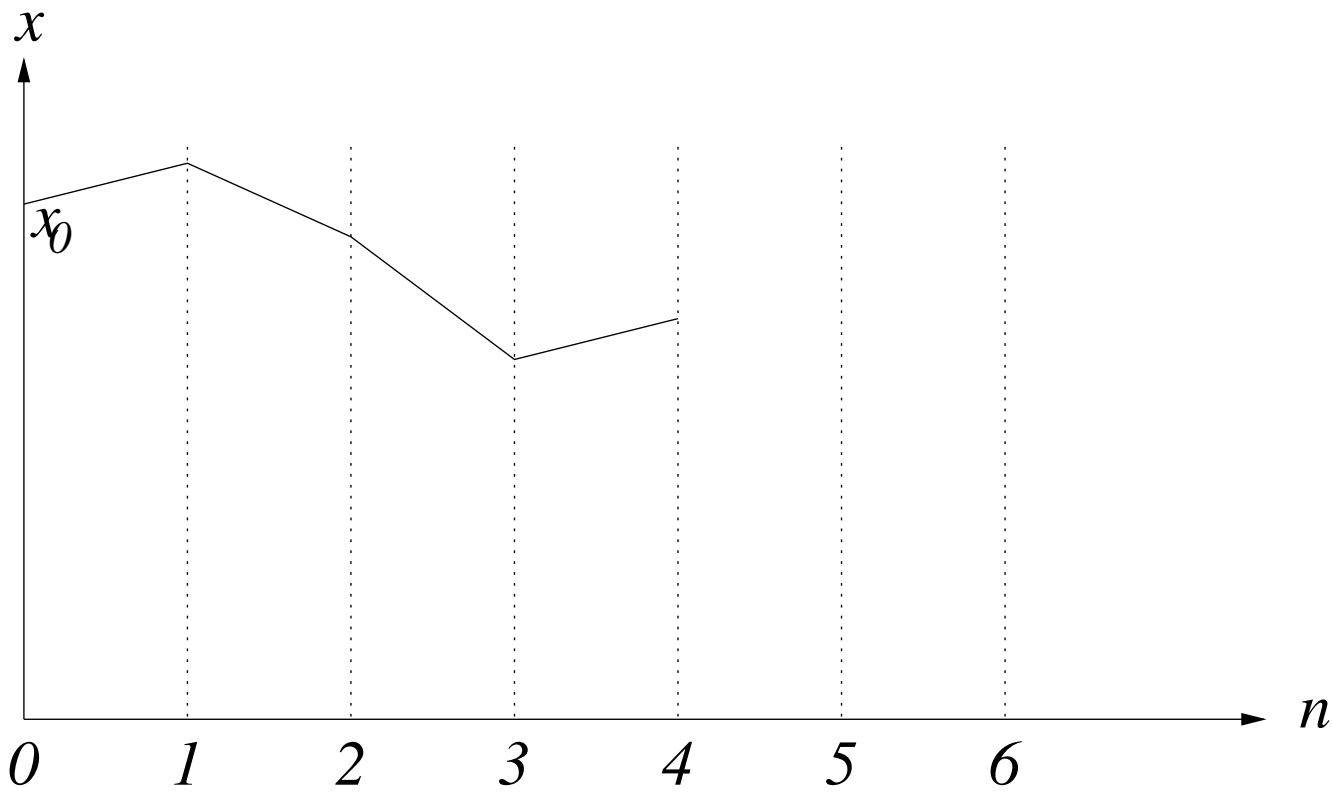

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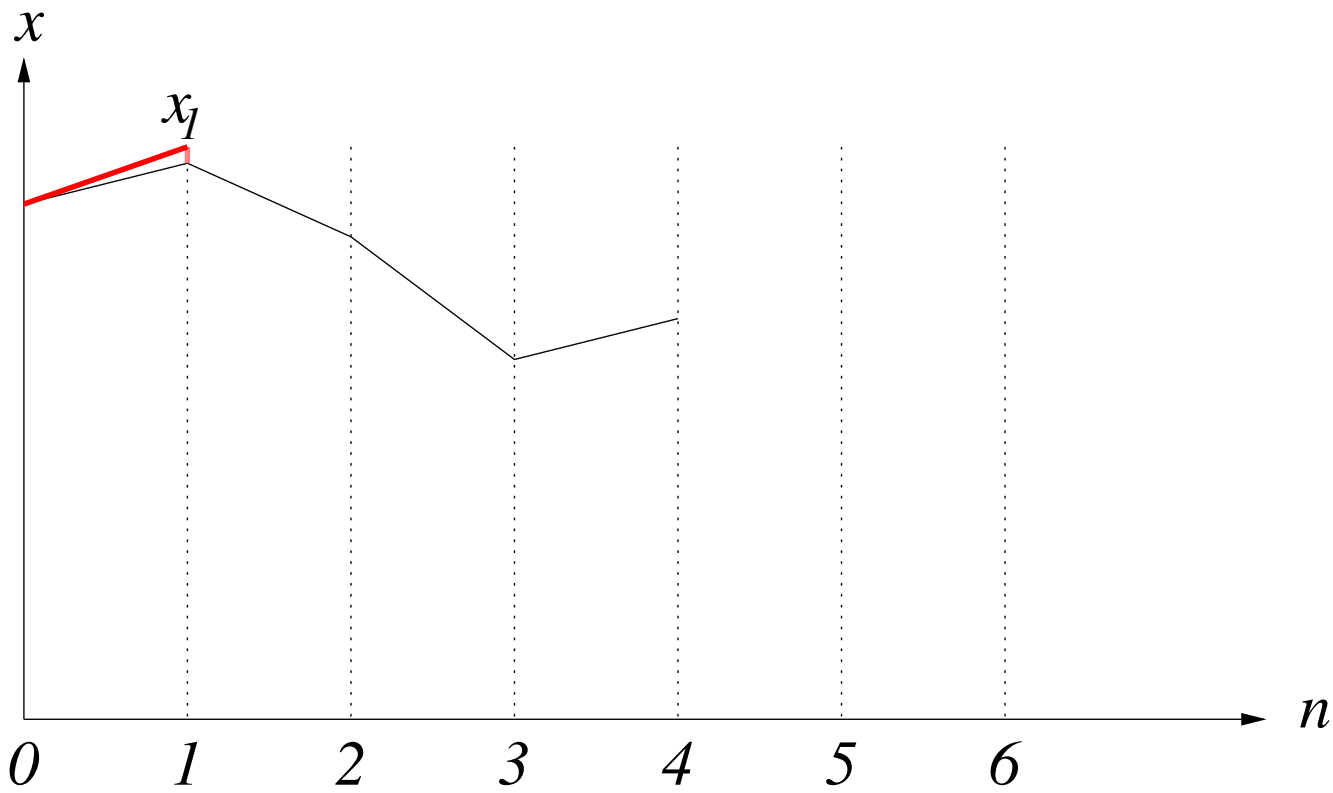

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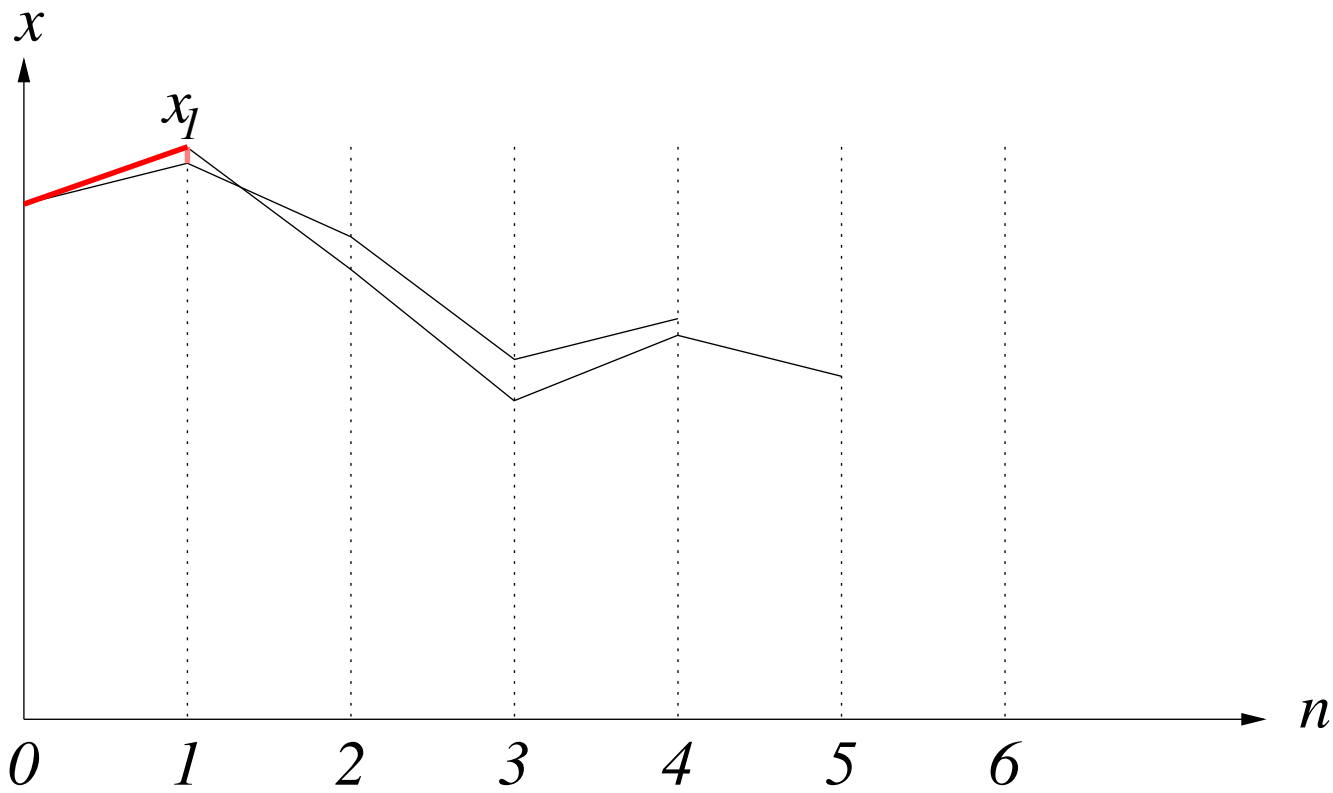

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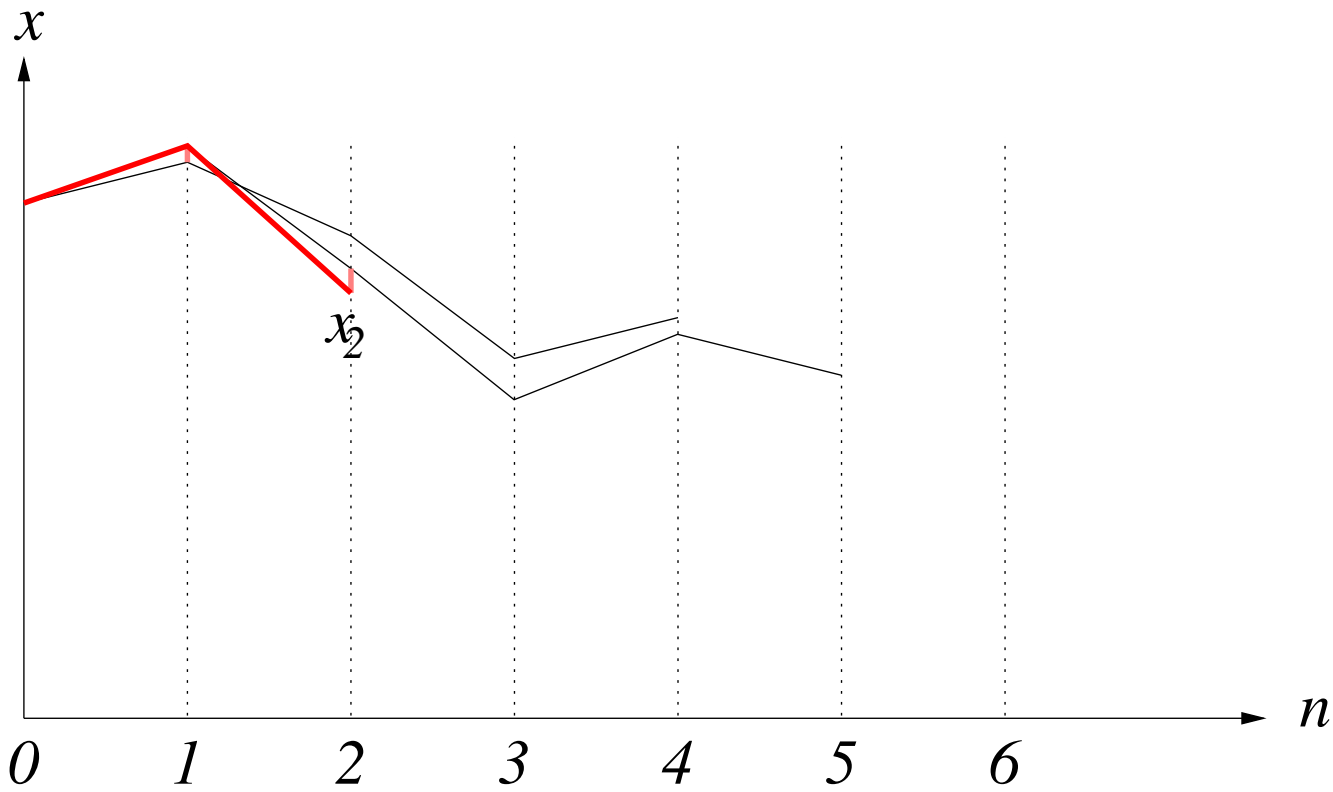

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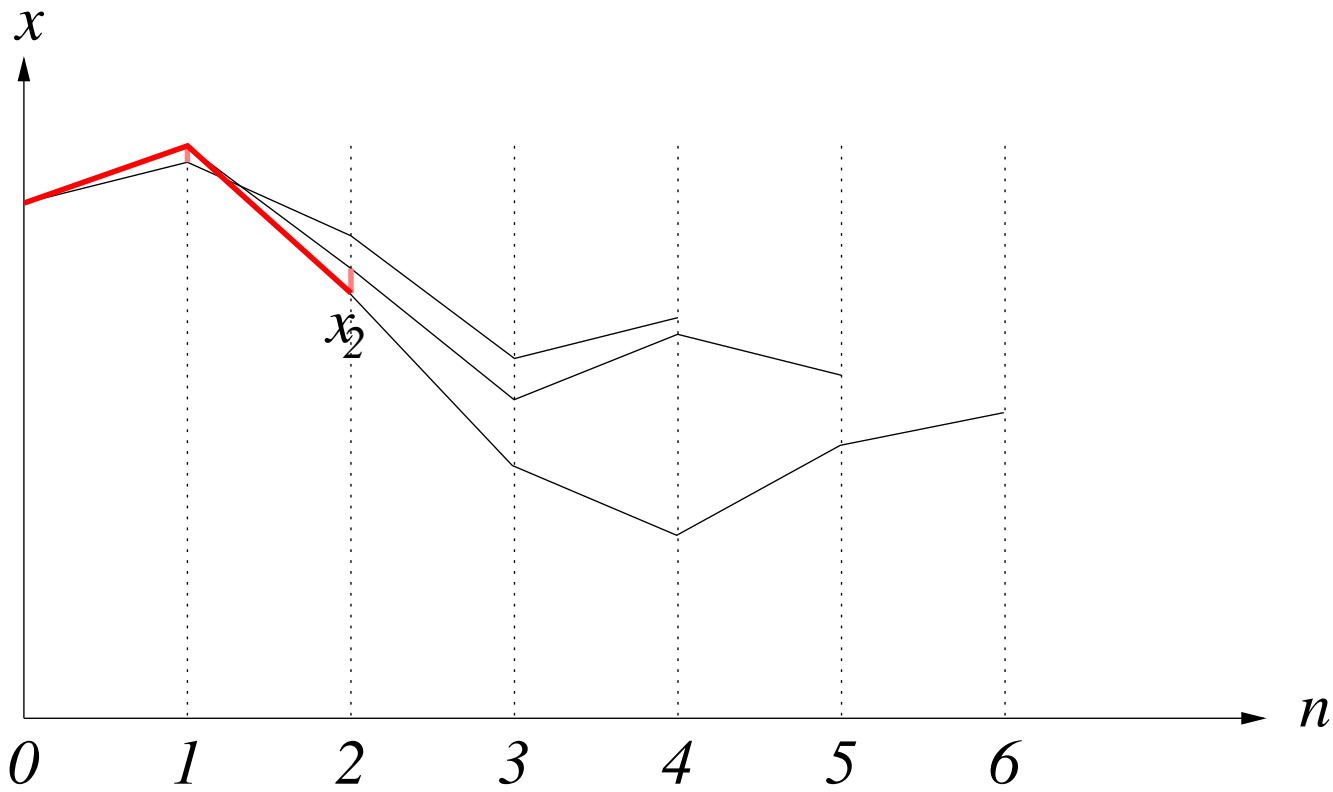

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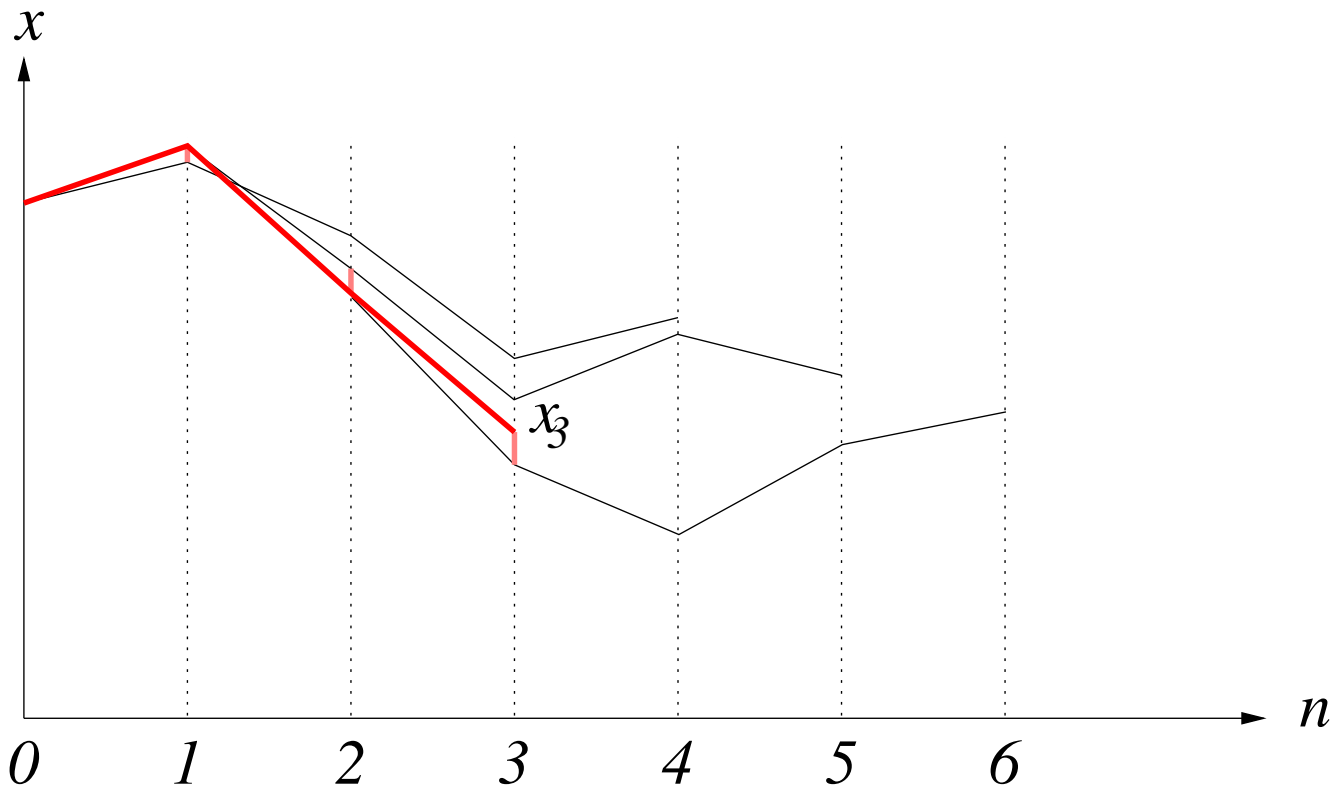

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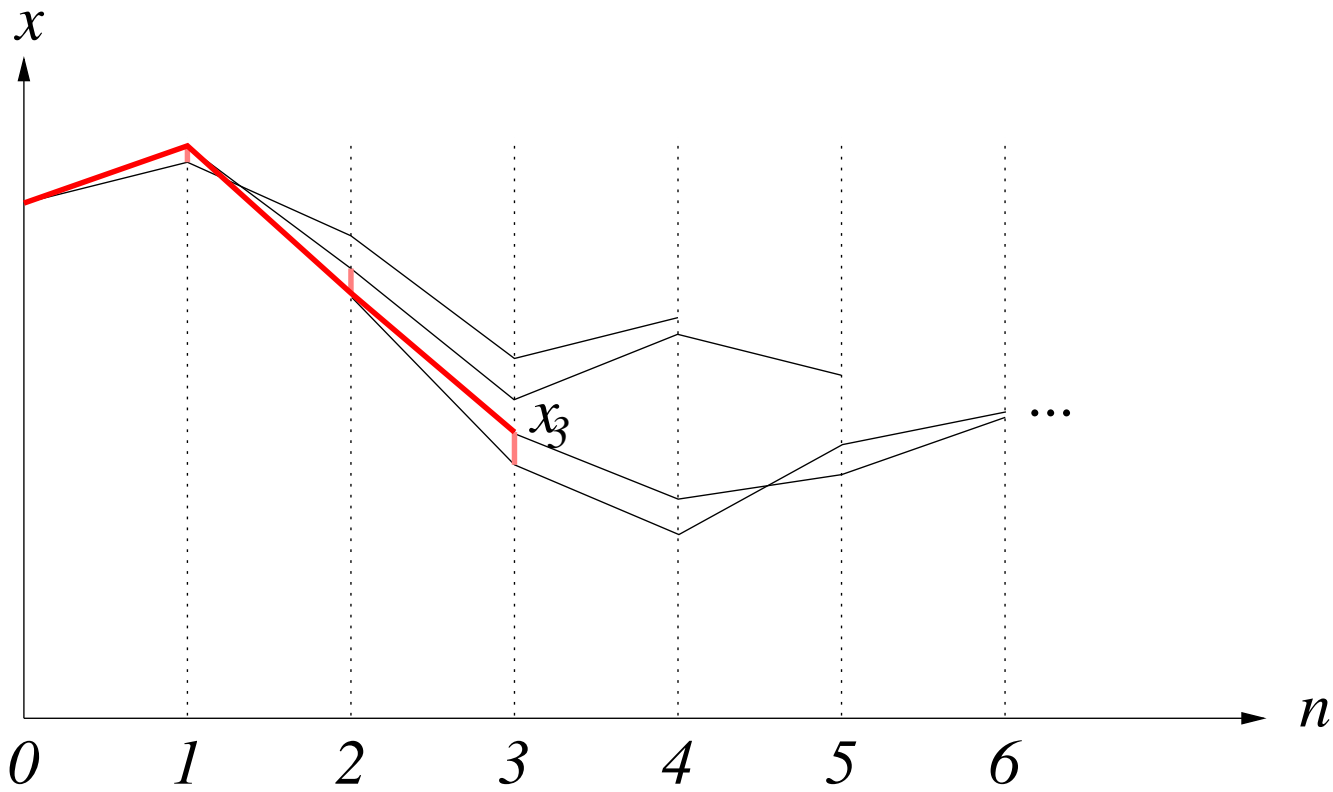

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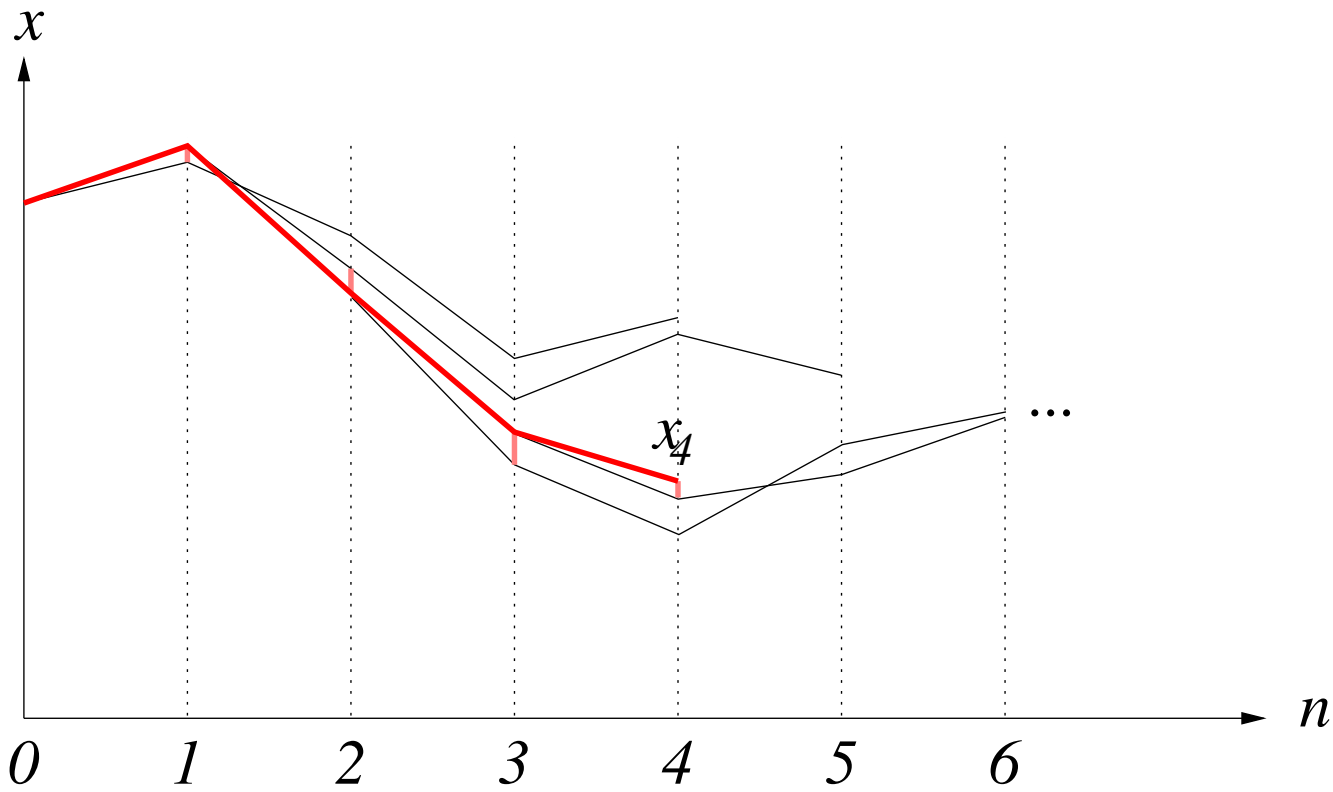

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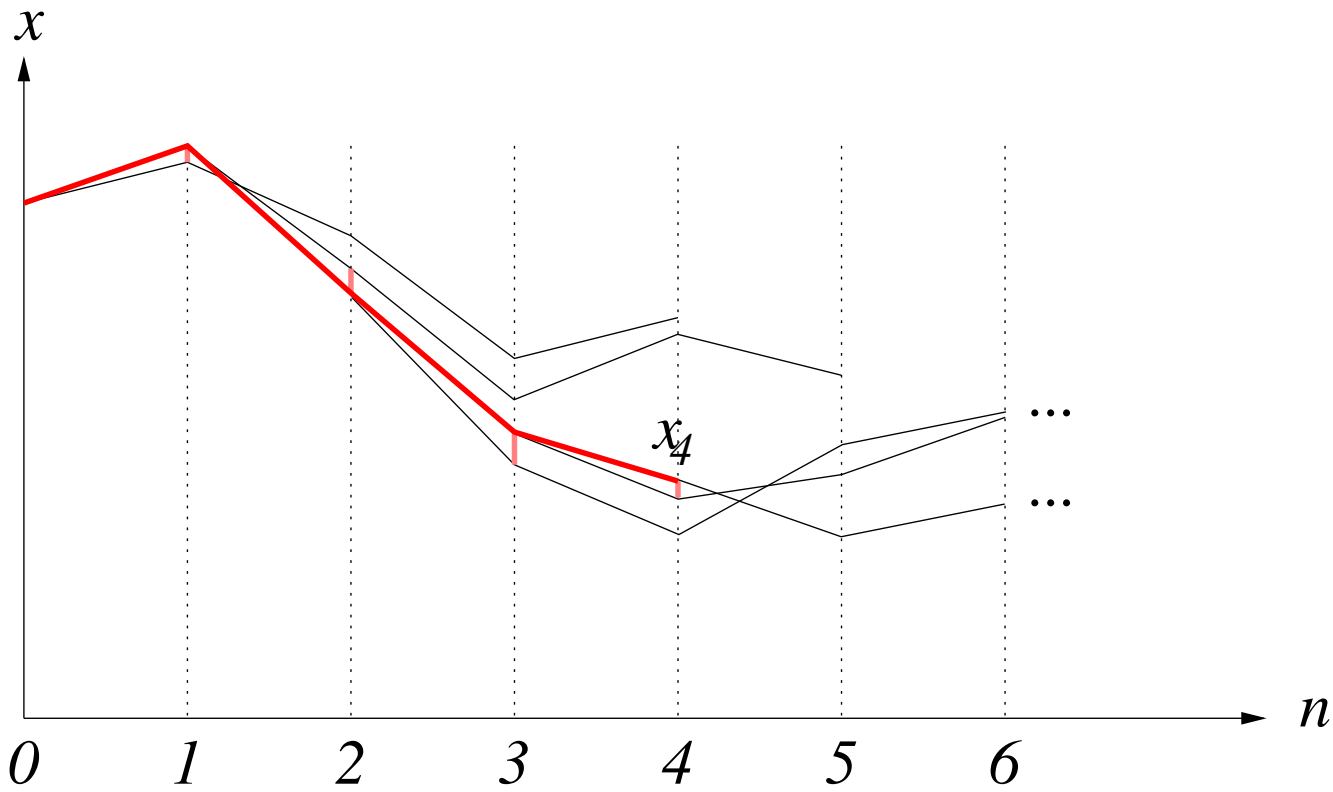

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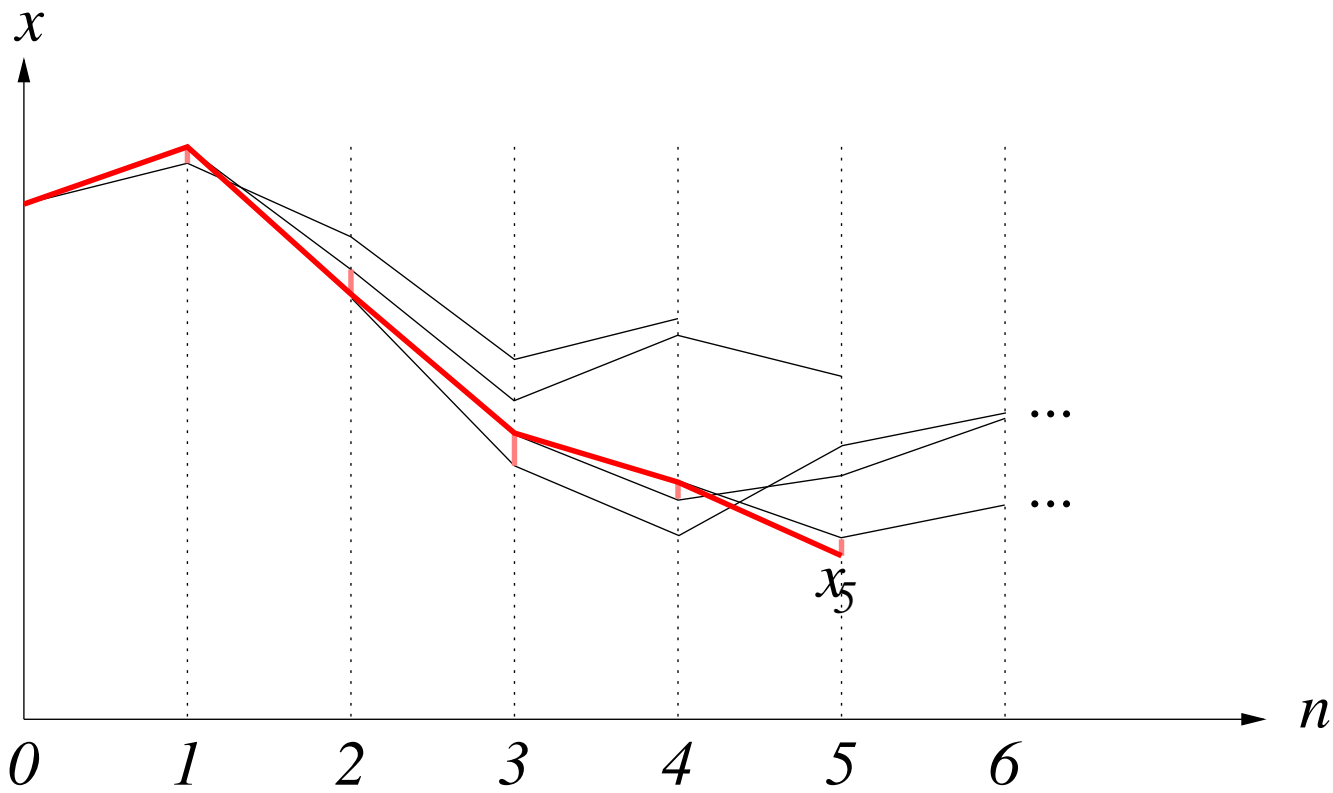

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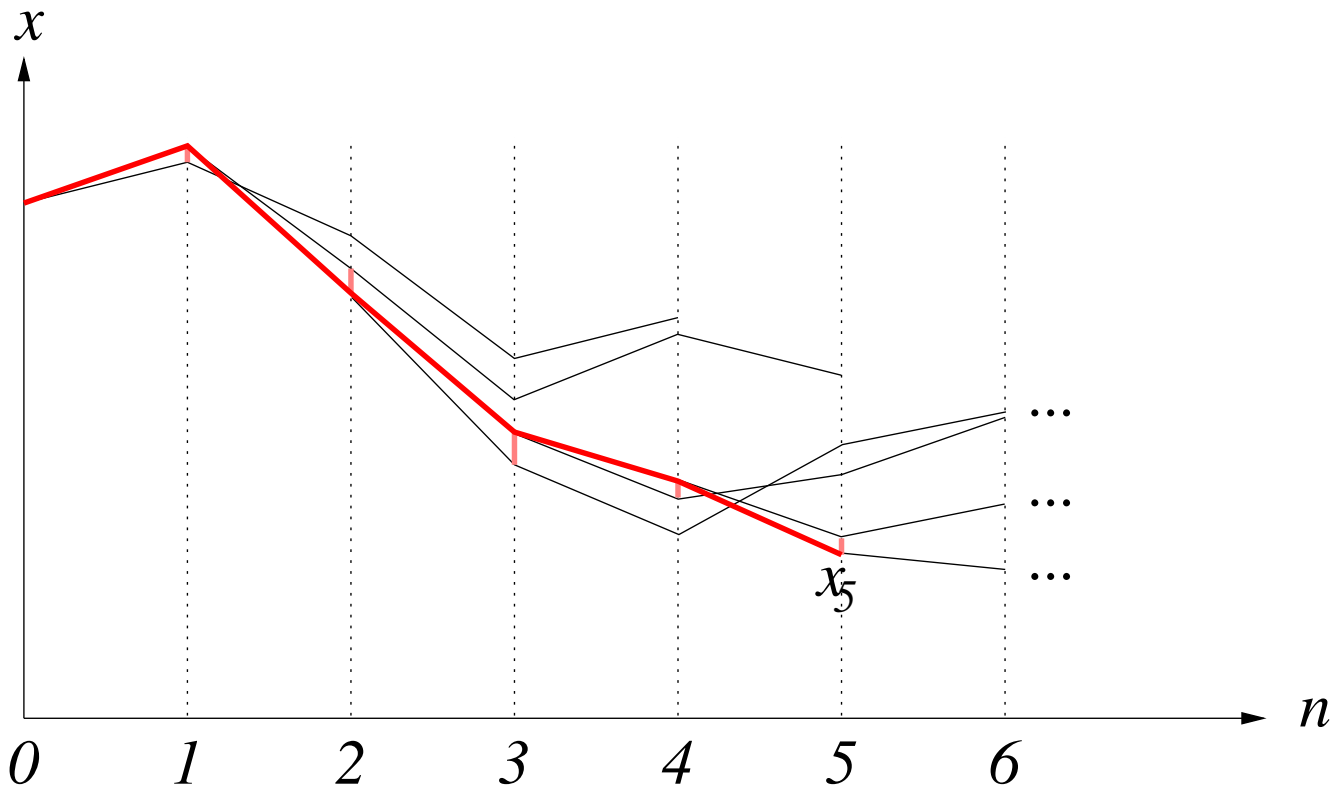

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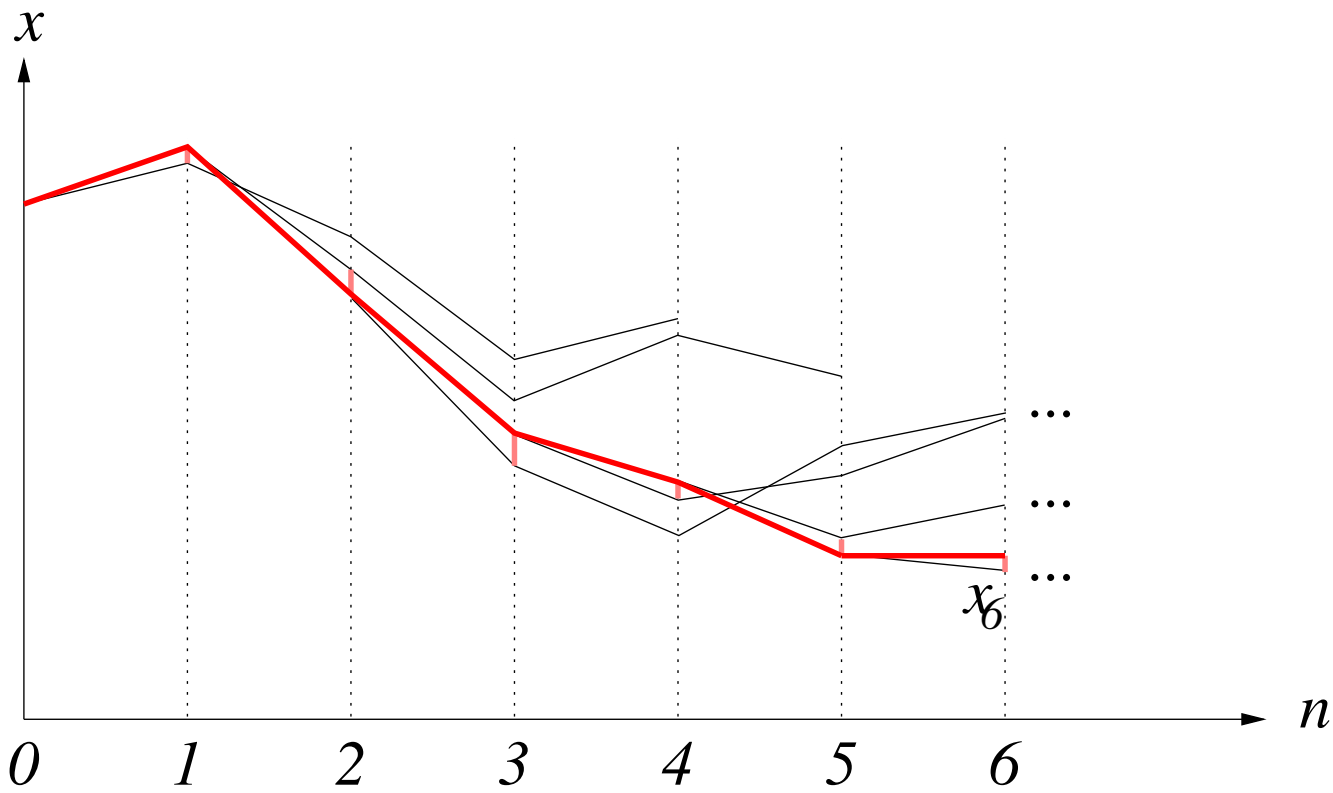

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- a specific construction of tightening state constraints [Michalska/Mayne '93; Limón/Alamo/Camacho '02; Grimm et al. '07; Gr./Pannek '11]

Robustness

Robustness can be ensured, e.g., by

- (uniform) continuity of the optimal value function $V_N(x) = \inf_u J_N(x,u)$, which serves as a Lyapunov function [De Nicolao/Magni/Scattolini '96; Nešić/Teel/Kokotović '99; Gr./Pannek '11]

  (may not hold in presence of state constraints)

- a specific construction of tightening state constraints [Michalska/Mayne '93; Limón/Alamo/Camacho '02; Grimm et al. '07; Gr./Pannek '11]

In the latter case, stability and robustness analysis must be carried out in an integrated way

Reducing the computational load

Back to the unperturbed case:

The computationally most expensive part of an MPC controller is the optimization

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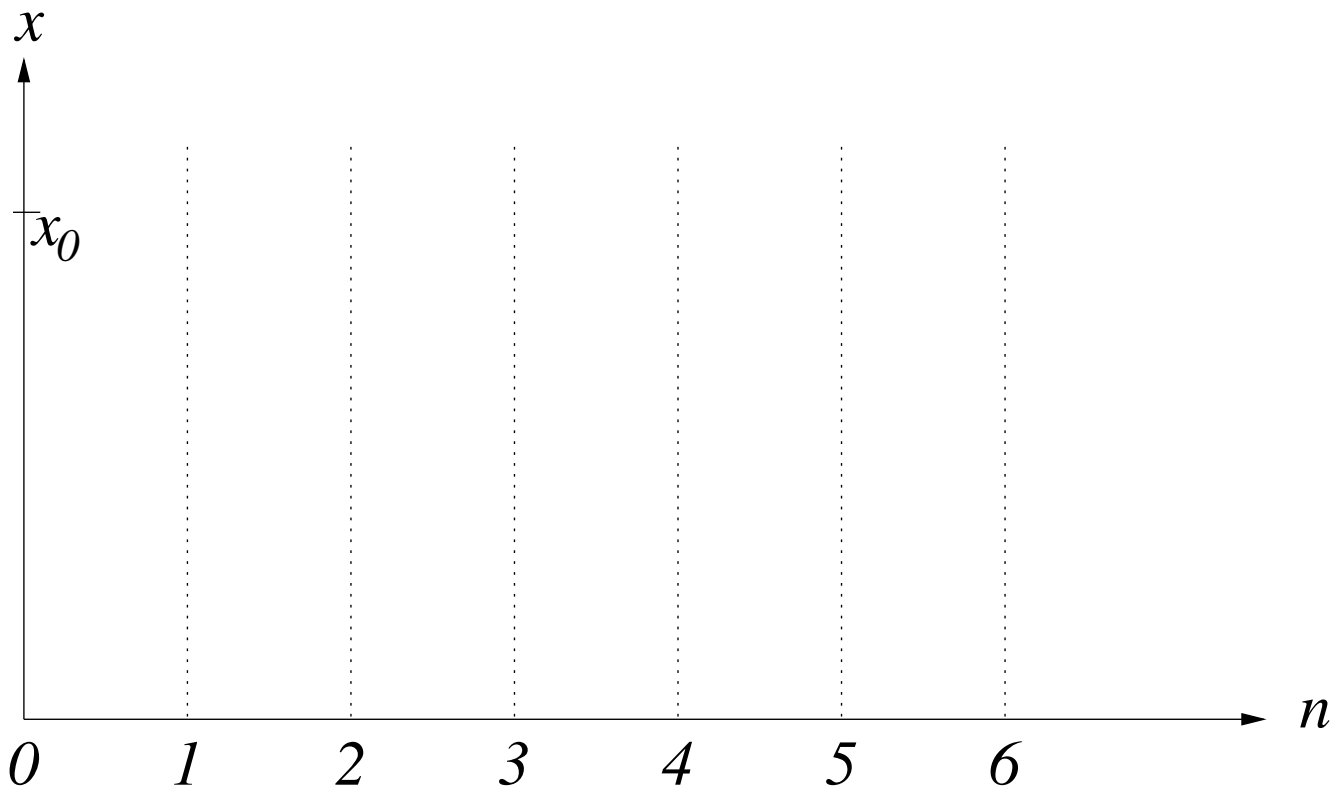
A more systems theoretic approach: perform re-optimization less often

Schematic illustration of the idea



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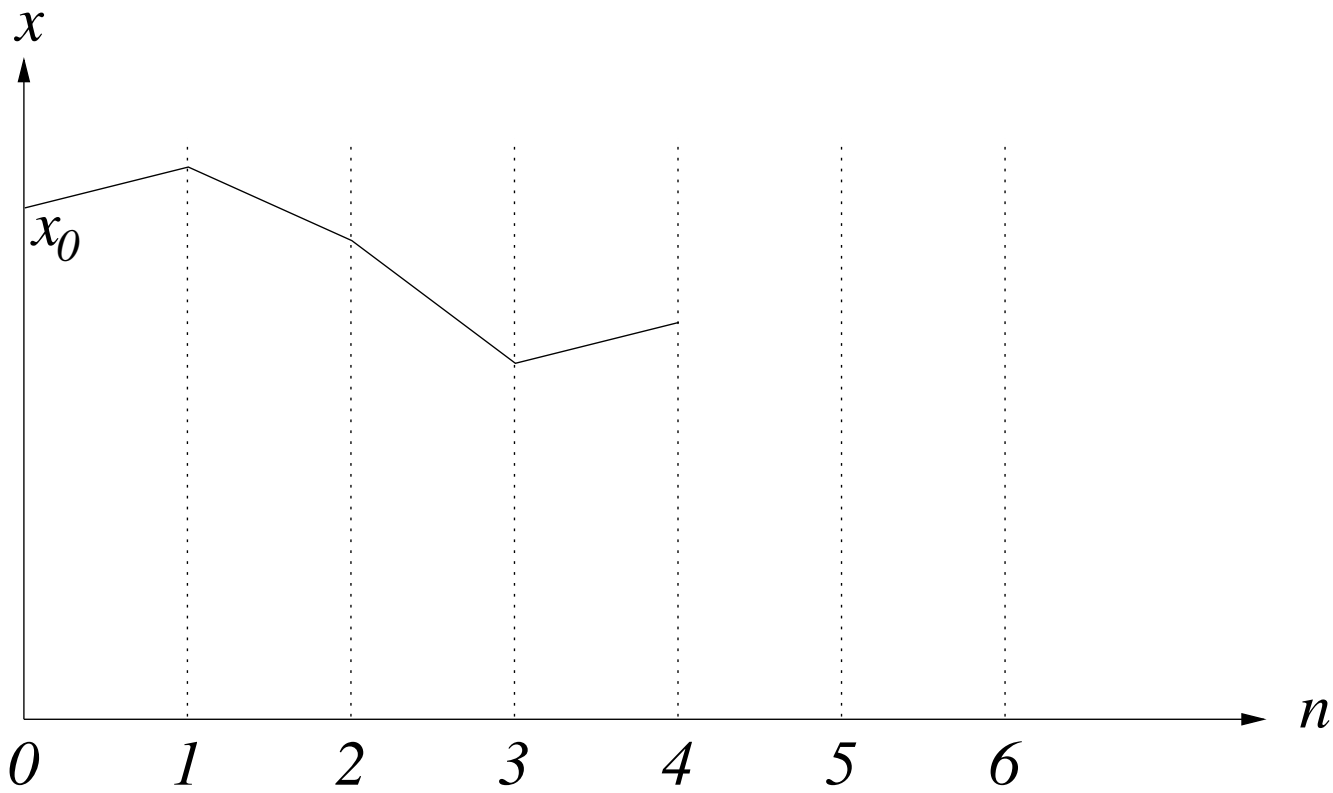

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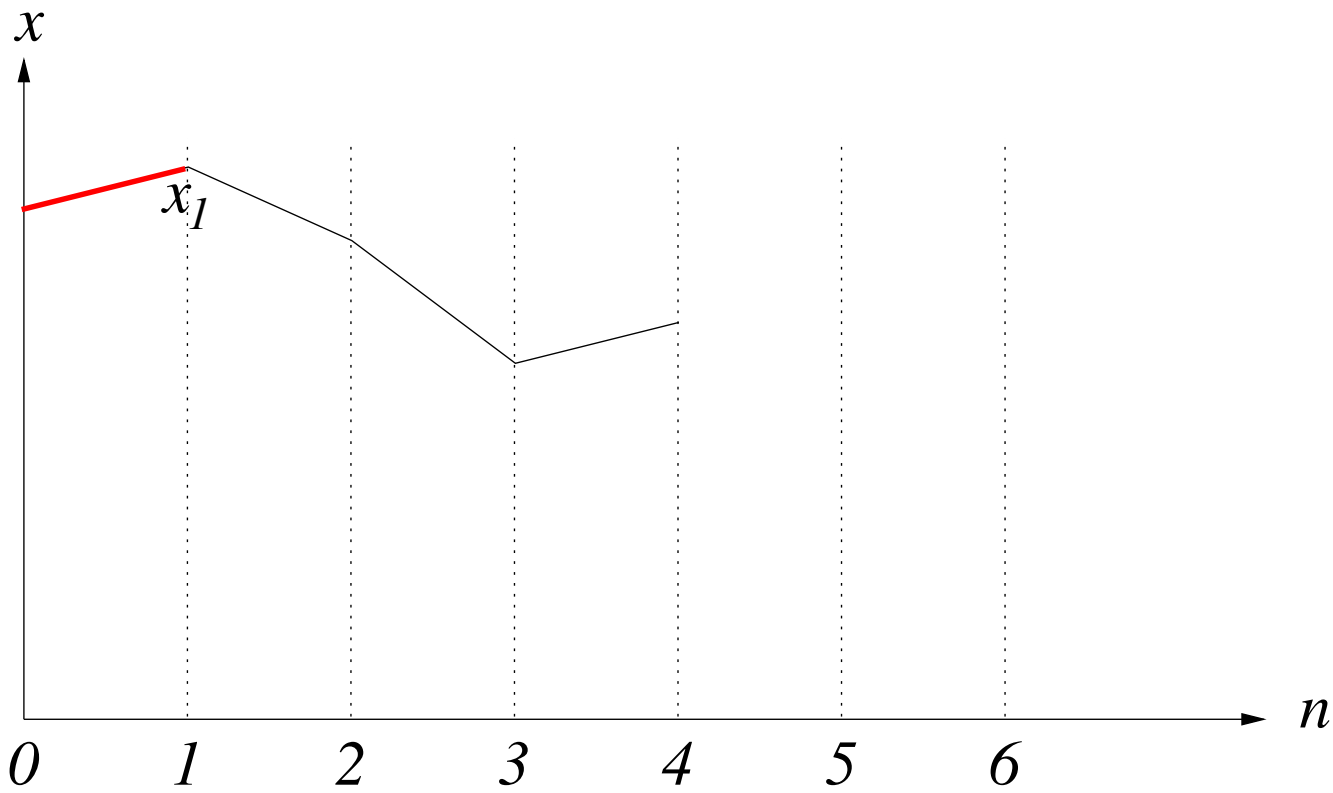

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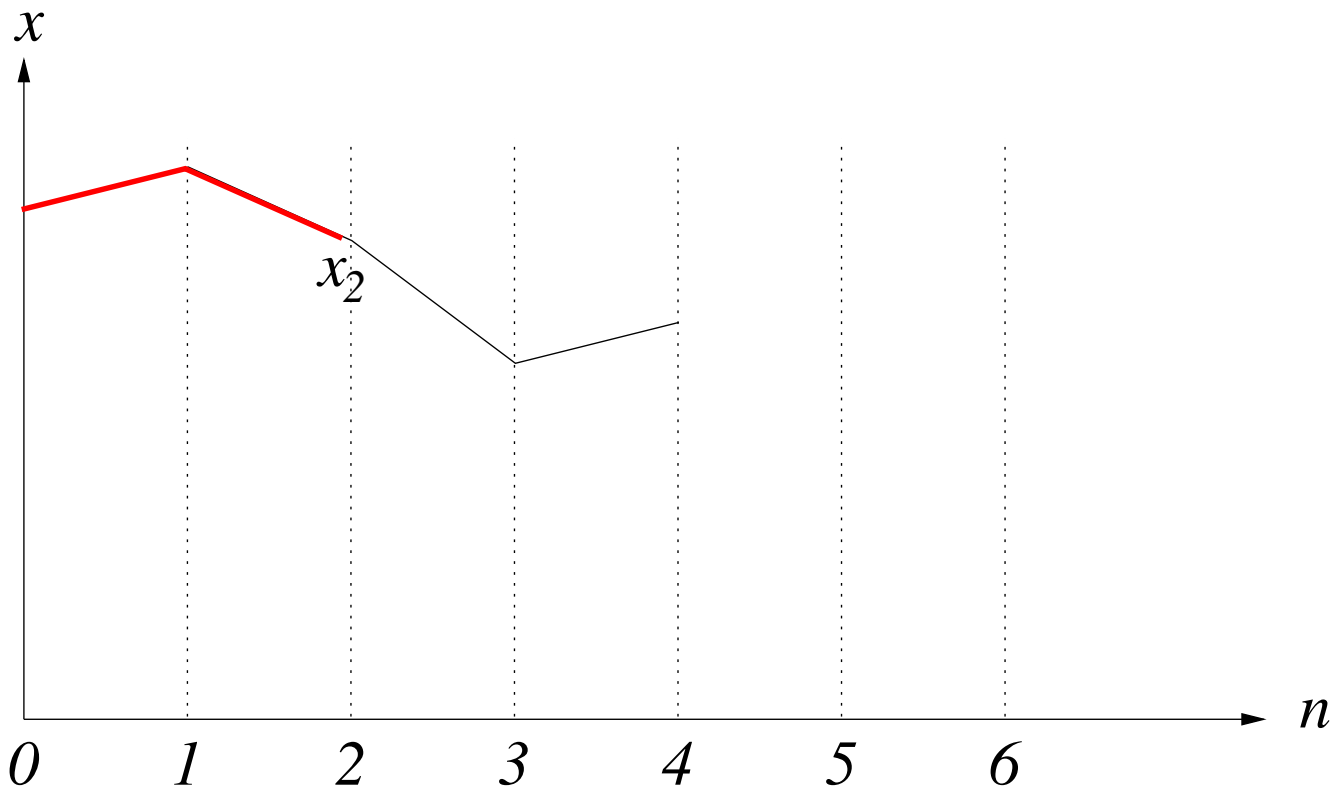

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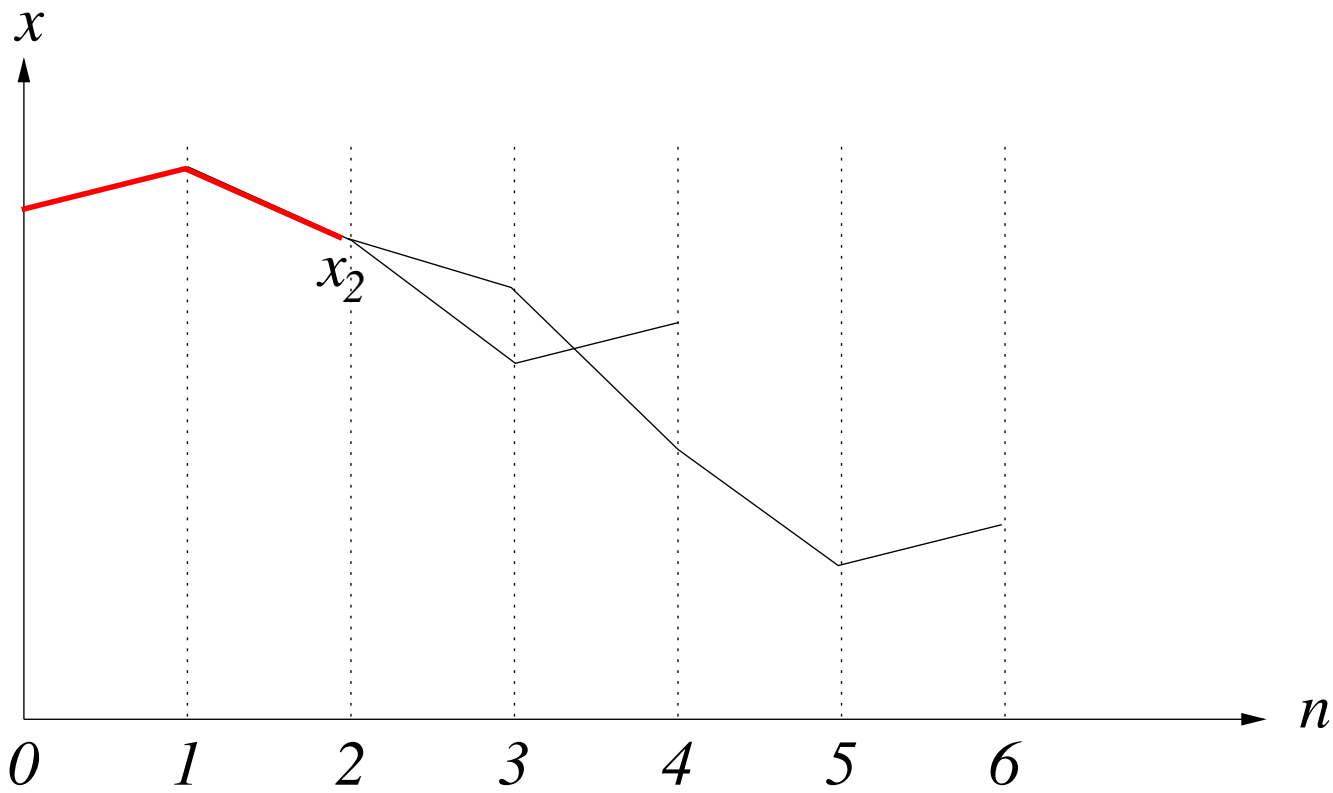

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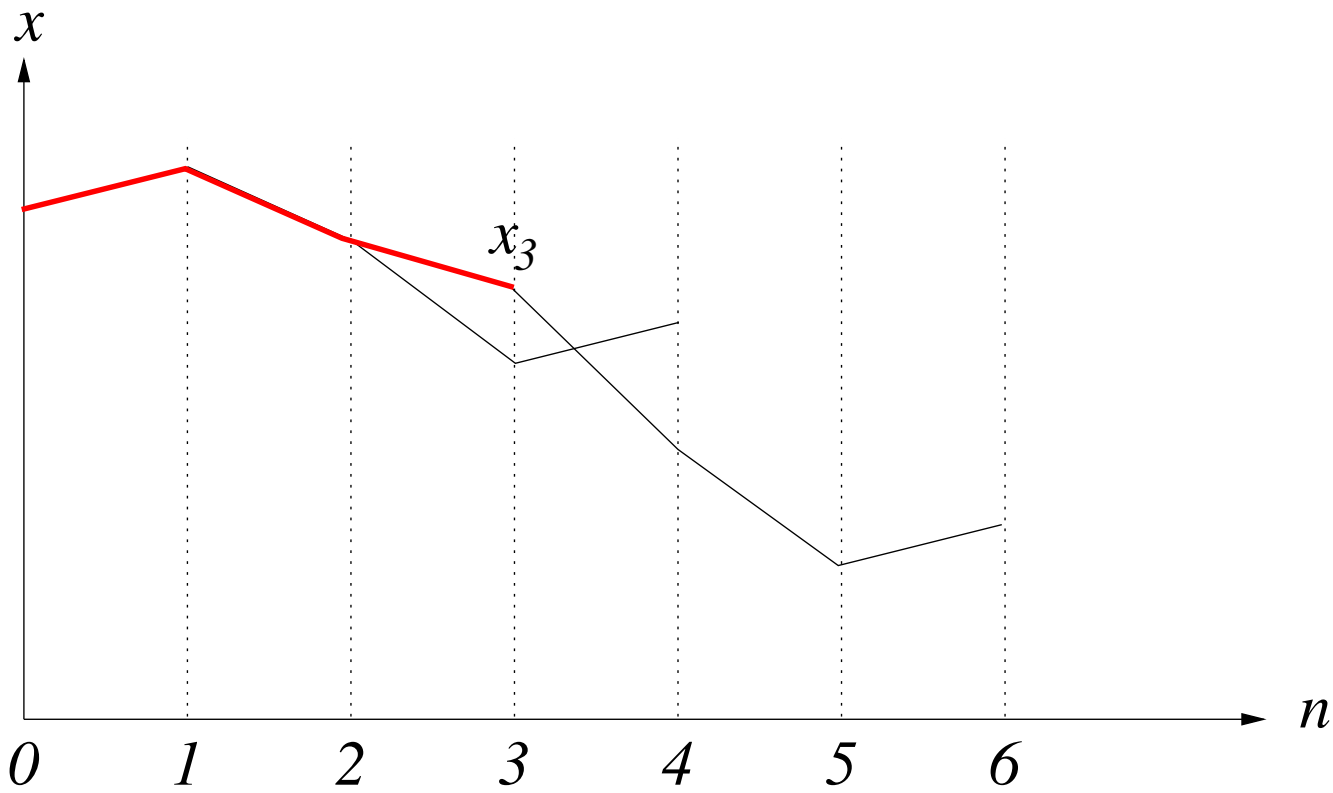

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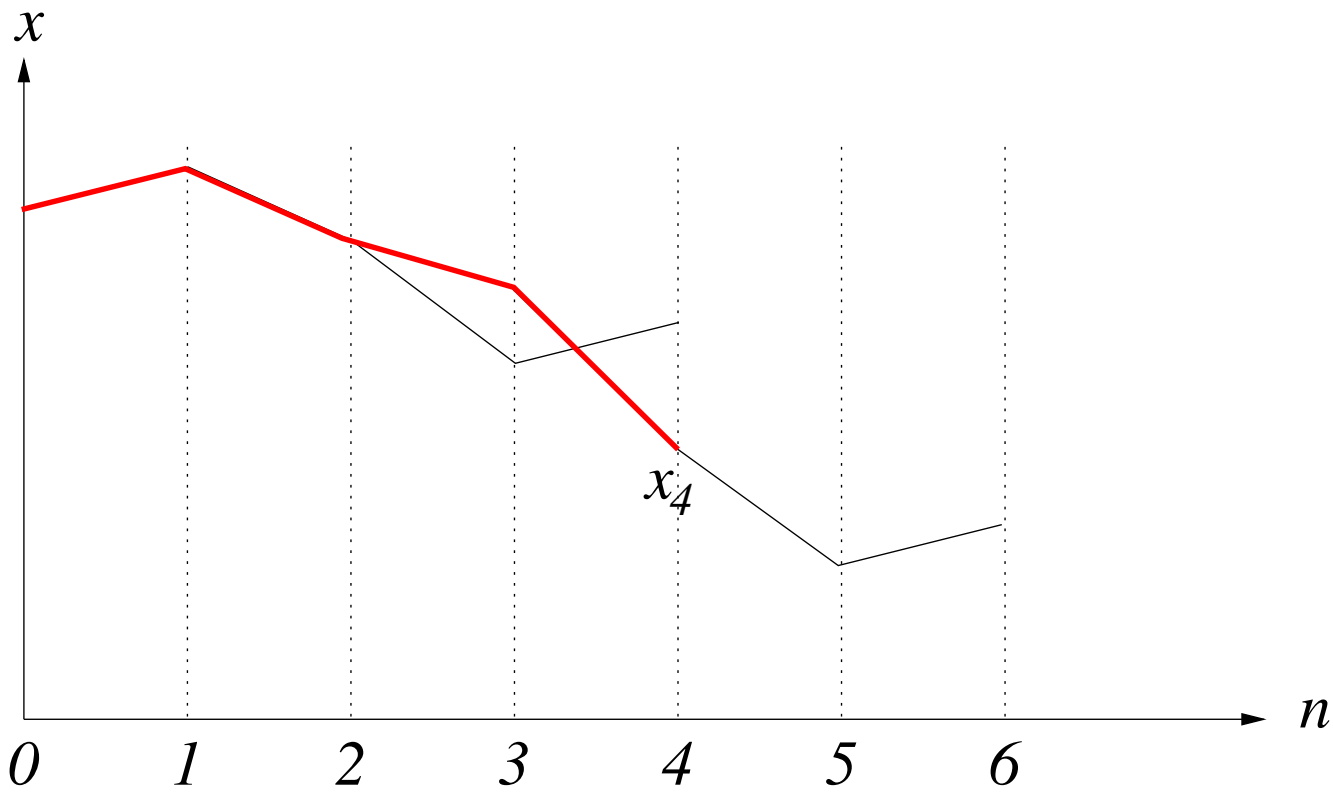

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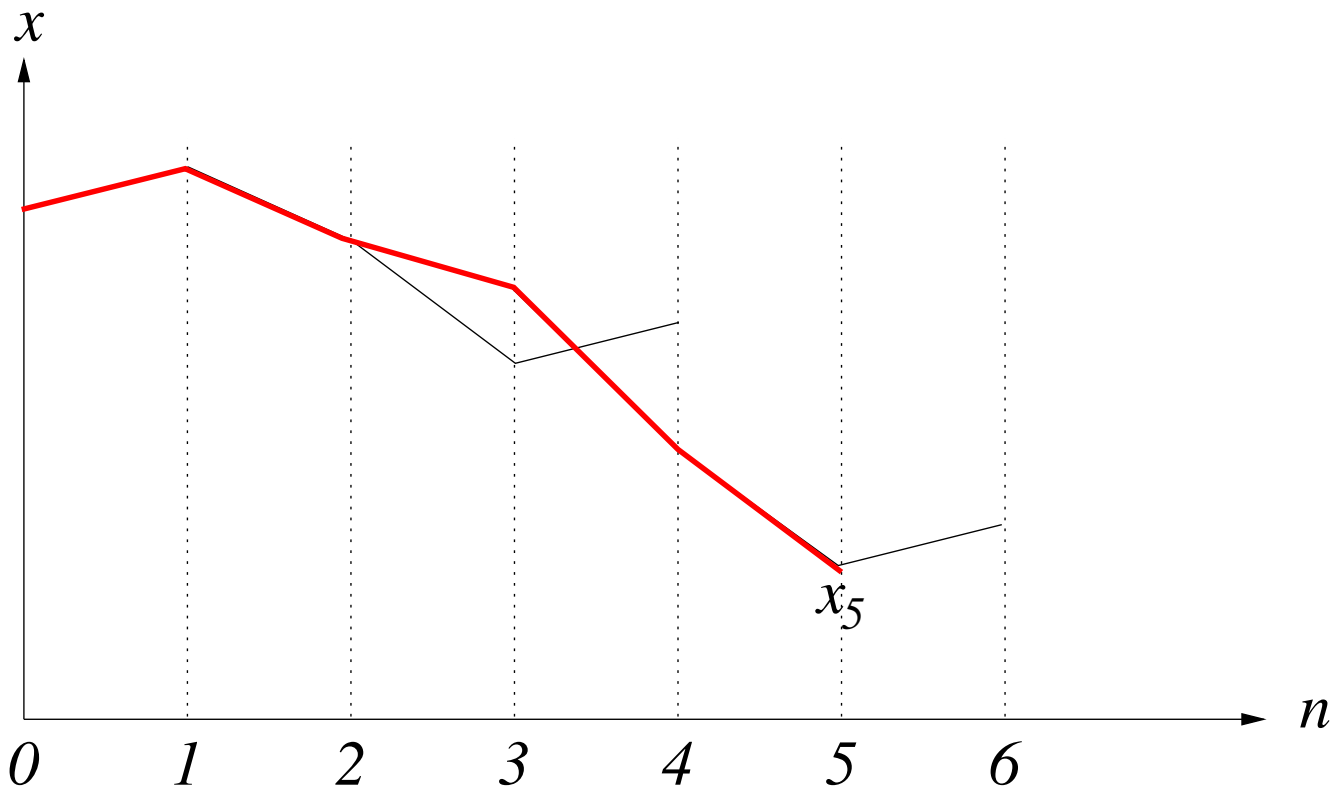

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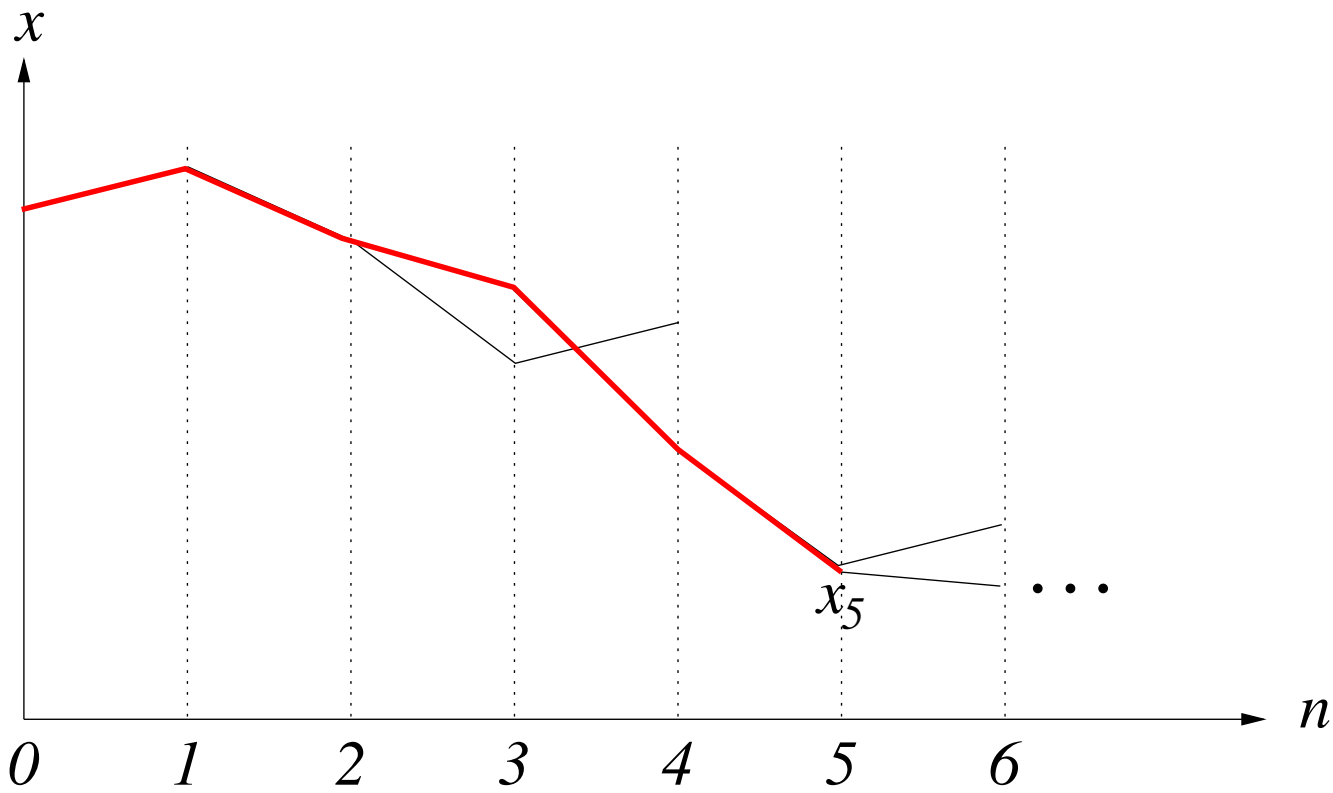

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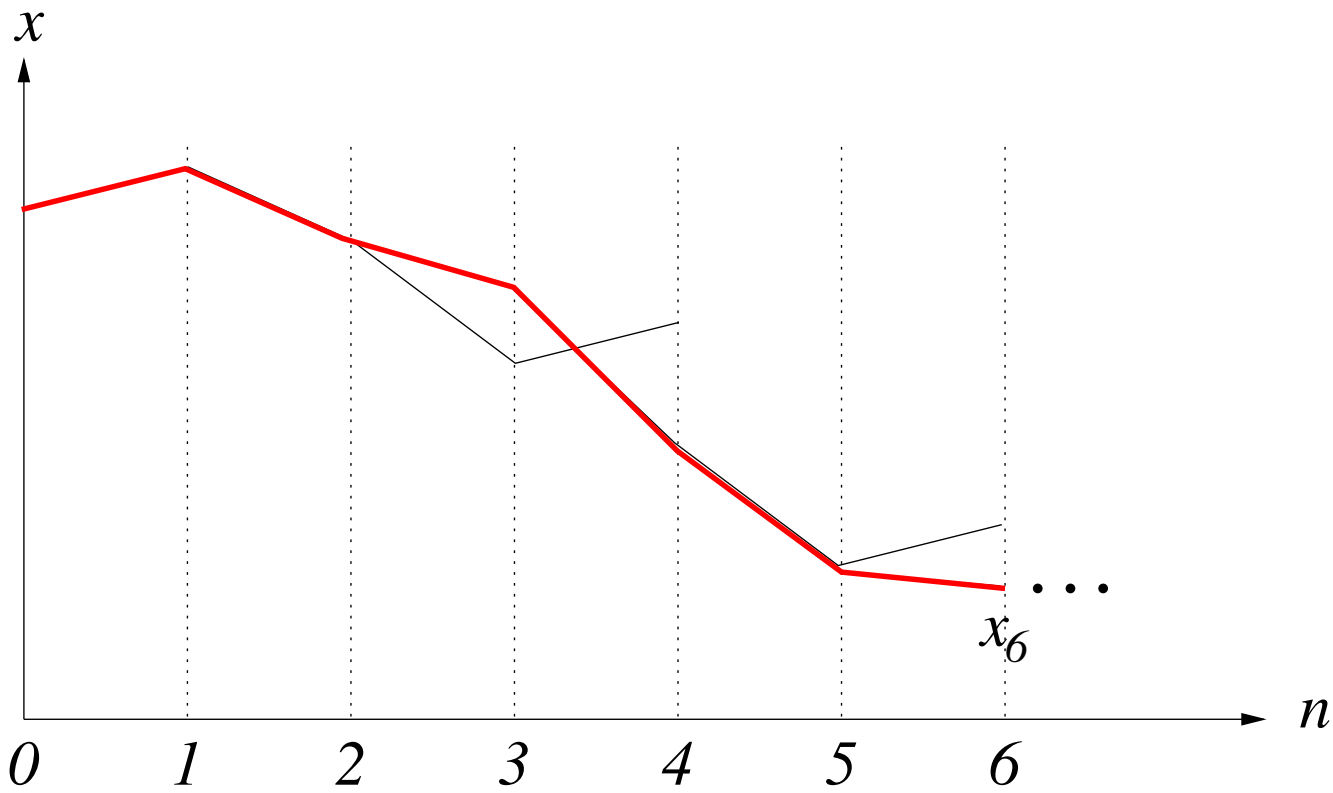

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Stability analysis

Denote the by m_j the number of elements used from the j-th control sequence, called the "control horizon"

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Then the stability and performance analysis extends to time-varying control horizons if we use $\alpha = \min_{m_j} \alpha(m_j)$ where

$$\alpha(m) = 1 - \frac{\prod_{i=m+1}^{N} (\gamma_i - 1) \prod_{i=N-m+1}^{N} (\gamma_i - 1)}{\left(\prod_{i=m+1}^{N} \gamma_i - \prod_{i=m+1}^{N} (\gamma_i - 1) \right) \left(\prod_{i=N-m+1}^{N} \gamma_i - \prod_{i=N-m+1}^{N} (\gamma_i - 1) \right)}$$

with $\gamma_i = \sum_{k=0}^{i-1} C\sigma^k$

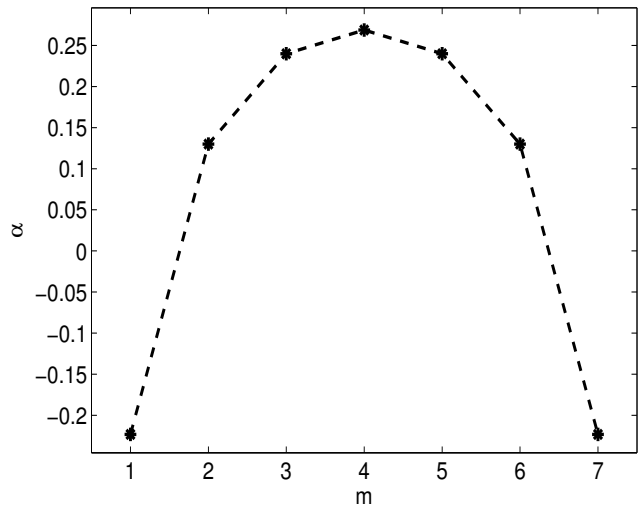
Property of $\alpha(m)$

Theorem: The values $\alpha(m)$ satisfy

$$\alpha(m) = \alpha(N-m), \quad m = 1, \ldots, N-1$$

and

$$\alpha(m) \leq \alpha(m+1), \quad m = 1, \ldots \lceil N/2 \rceil$$



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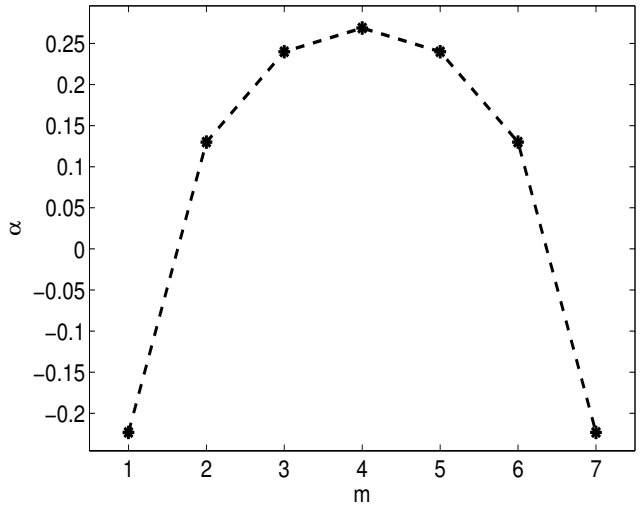

Corollary: If N is such that all C, σ-exponentially controllable systems are stabilized with "classical" MPC ($m = 1$), then they are stabilized for arbitrary varying control horizons $m_i \in \{1, \ldots, N-1\}$

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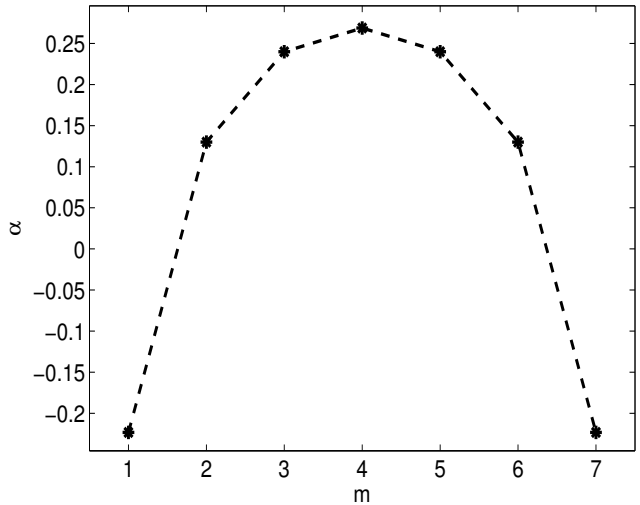

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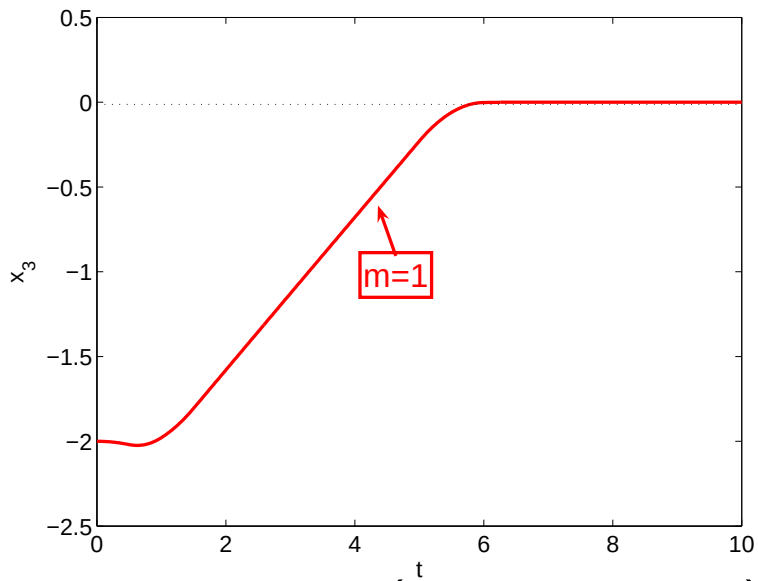
How does $\alpha(m)$ look like for a single system?

Example: linearized inverted pendulum

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ g & -k & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} u, \qquad x_0 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

sampling time $T = 0.5$, $\ell(x,u) = 2\|x\|_1 + 4\|u\|_1$, $N = 11$



x_3 component of trajectory (cart position) for different m

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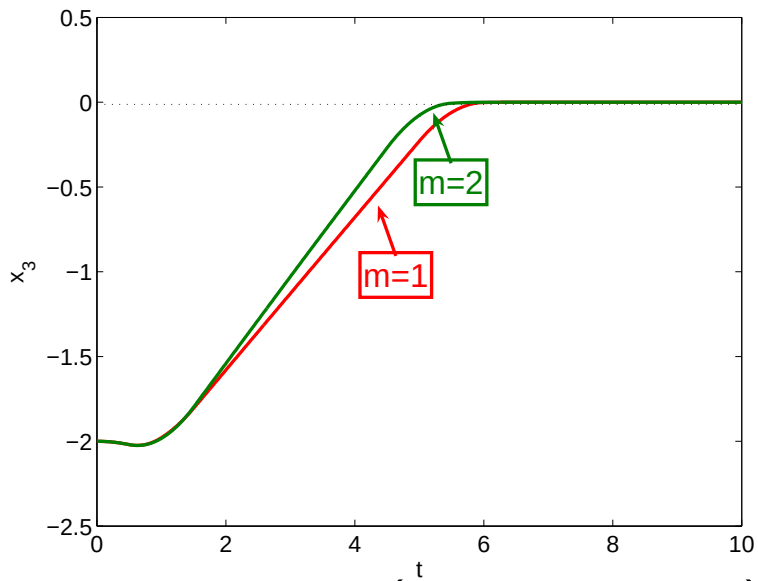

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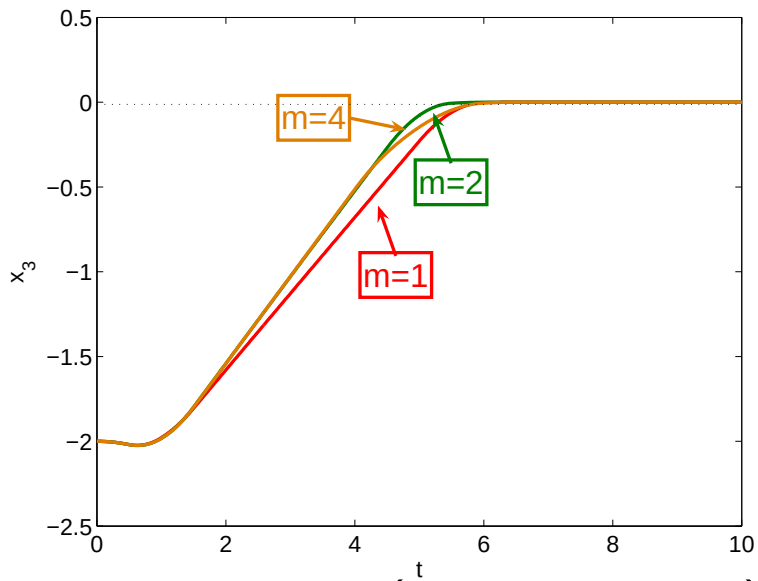

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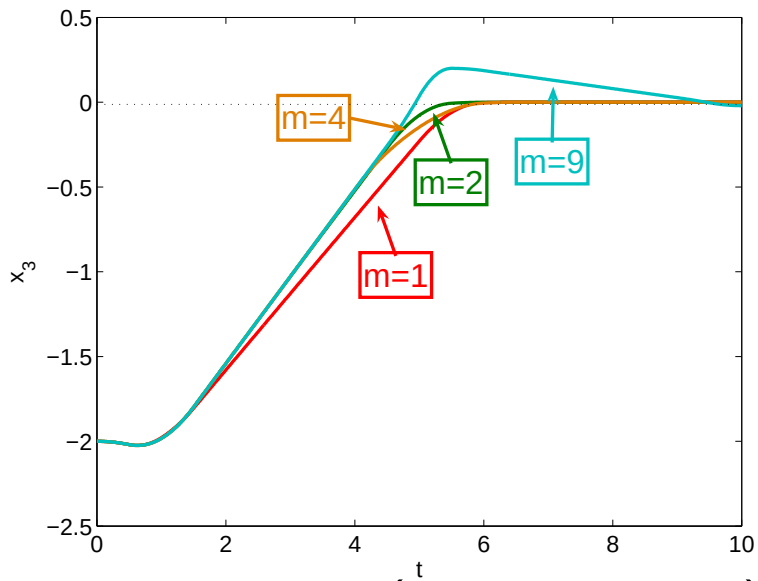

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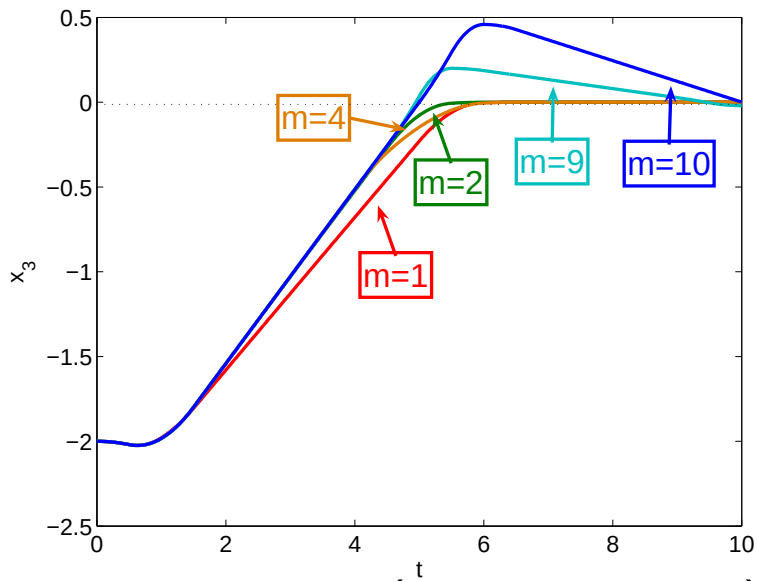

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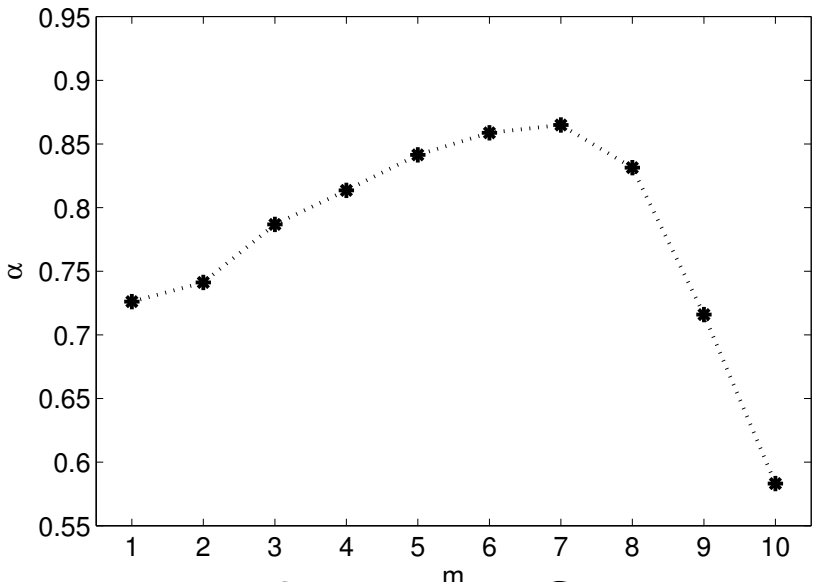
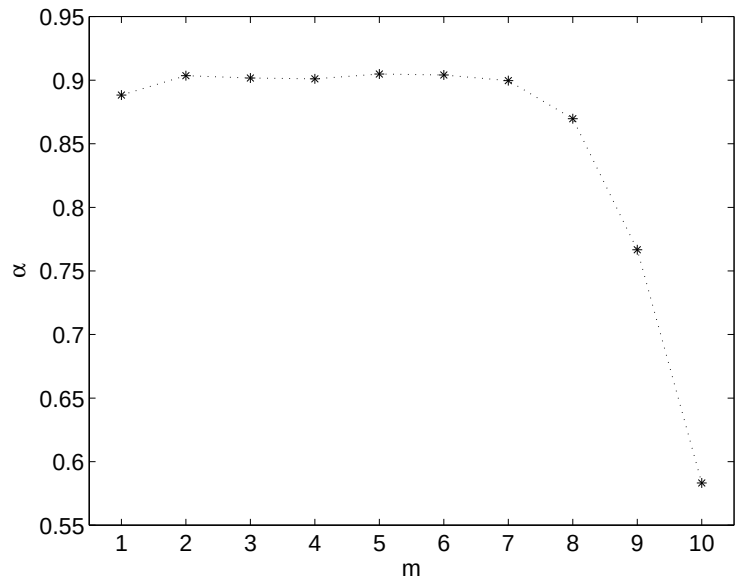


α after 1 MPC step



α at time $n = 20$

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Conclusion:

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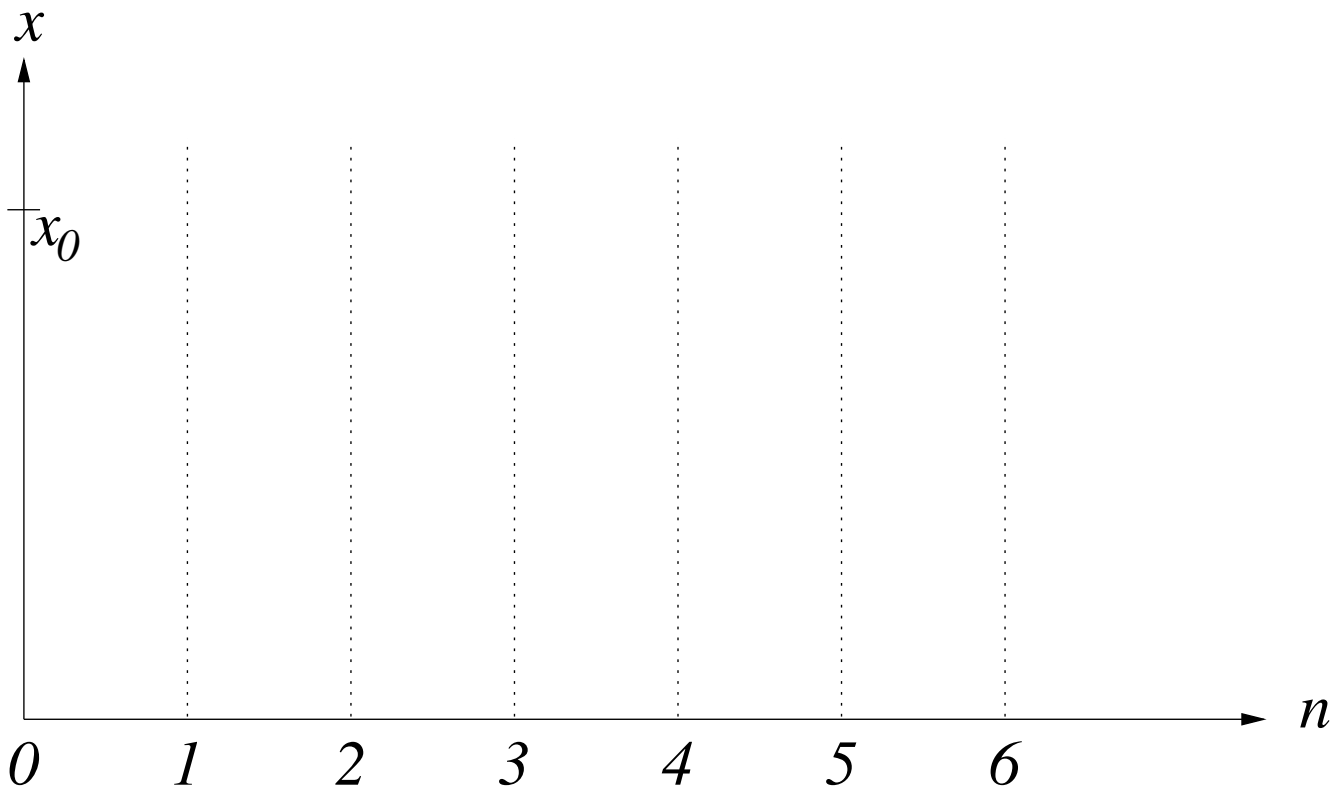

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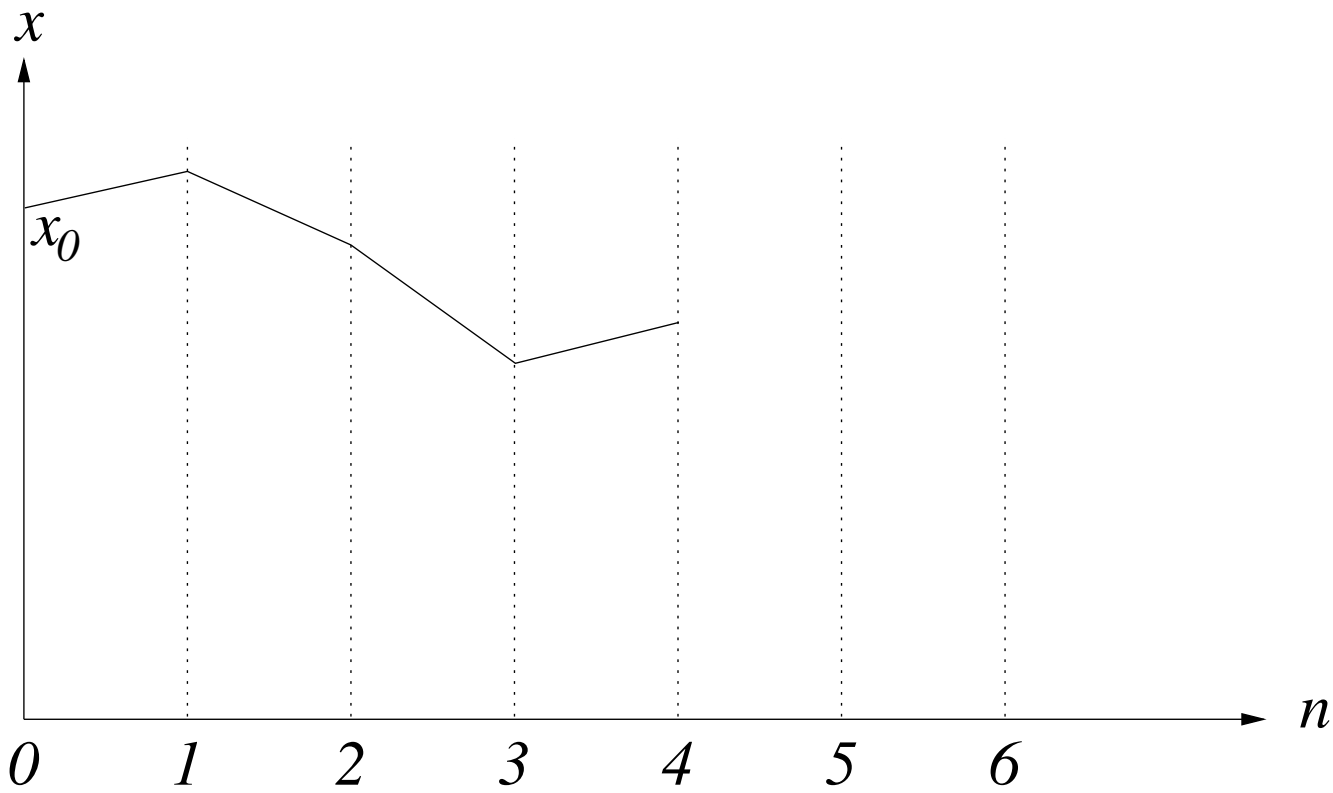

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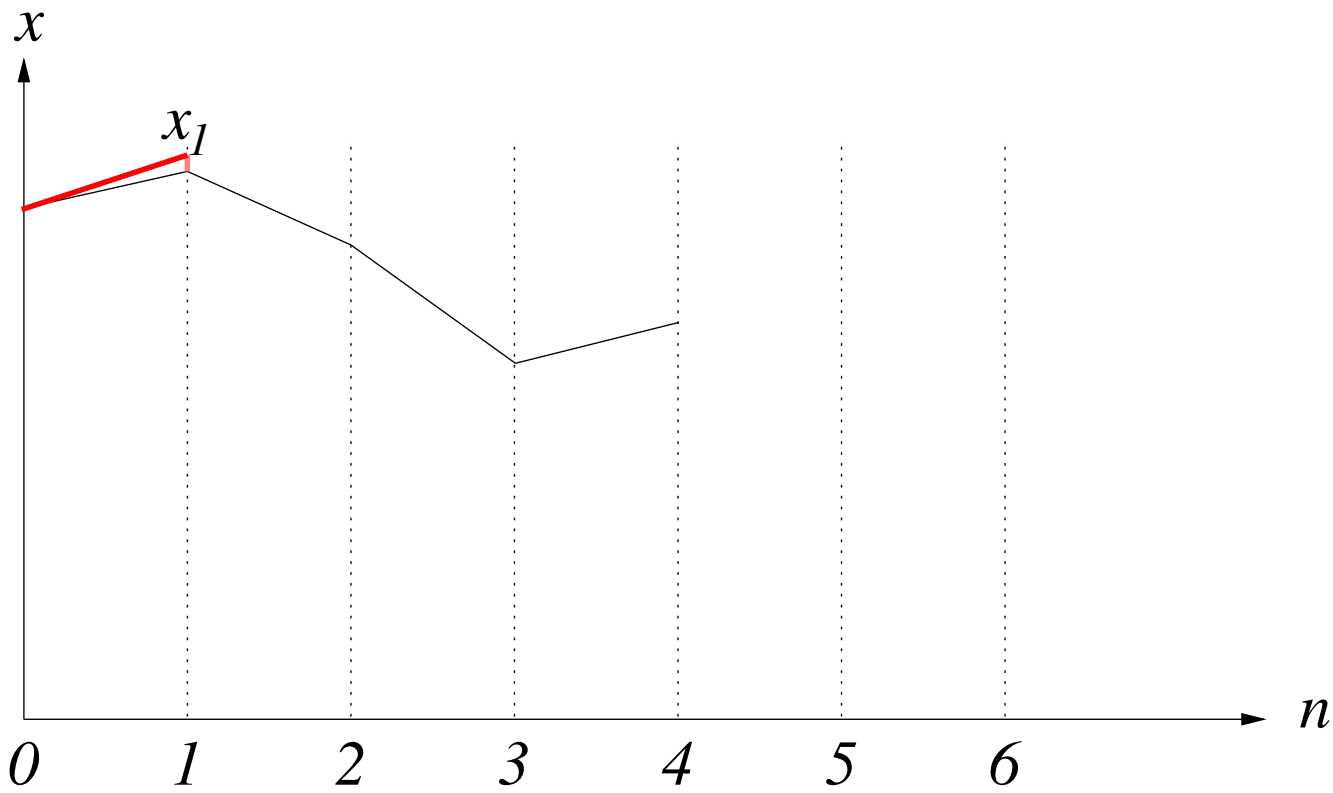

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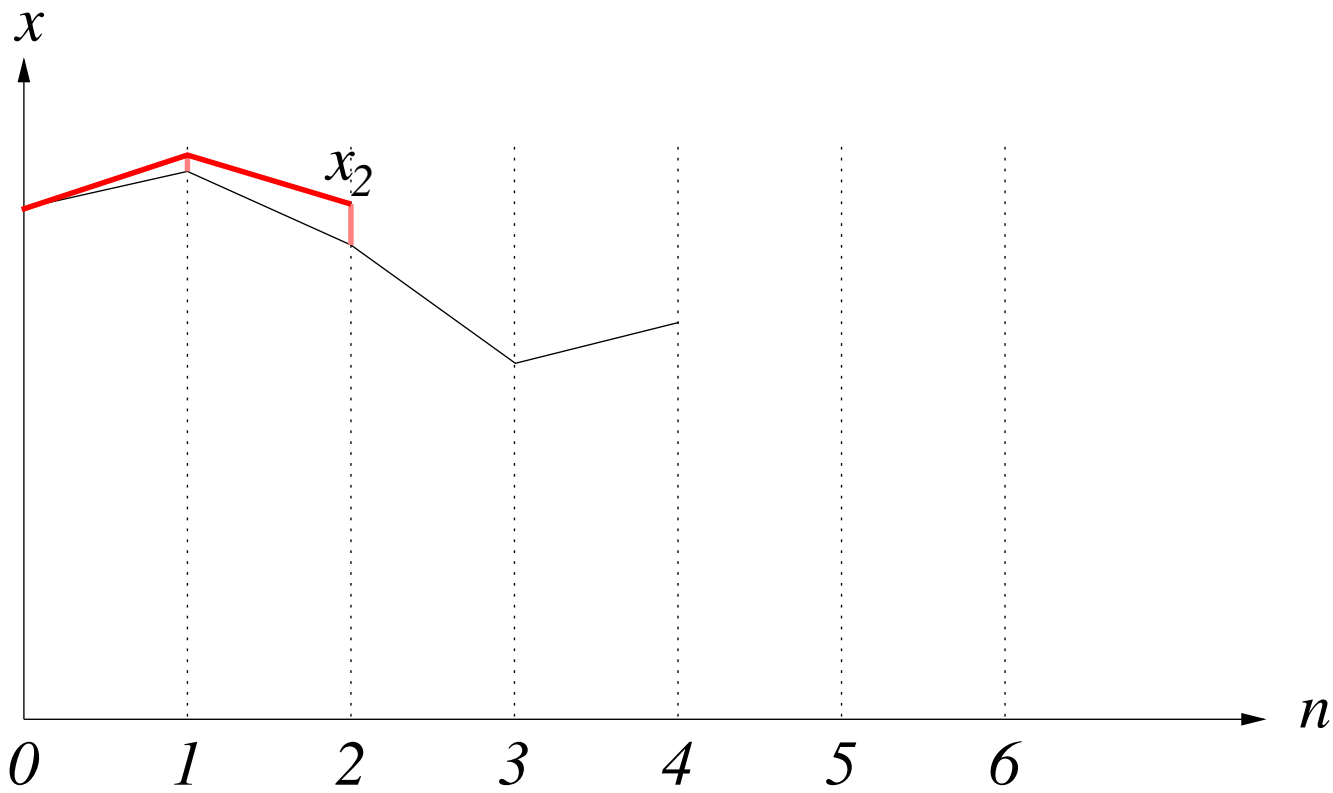

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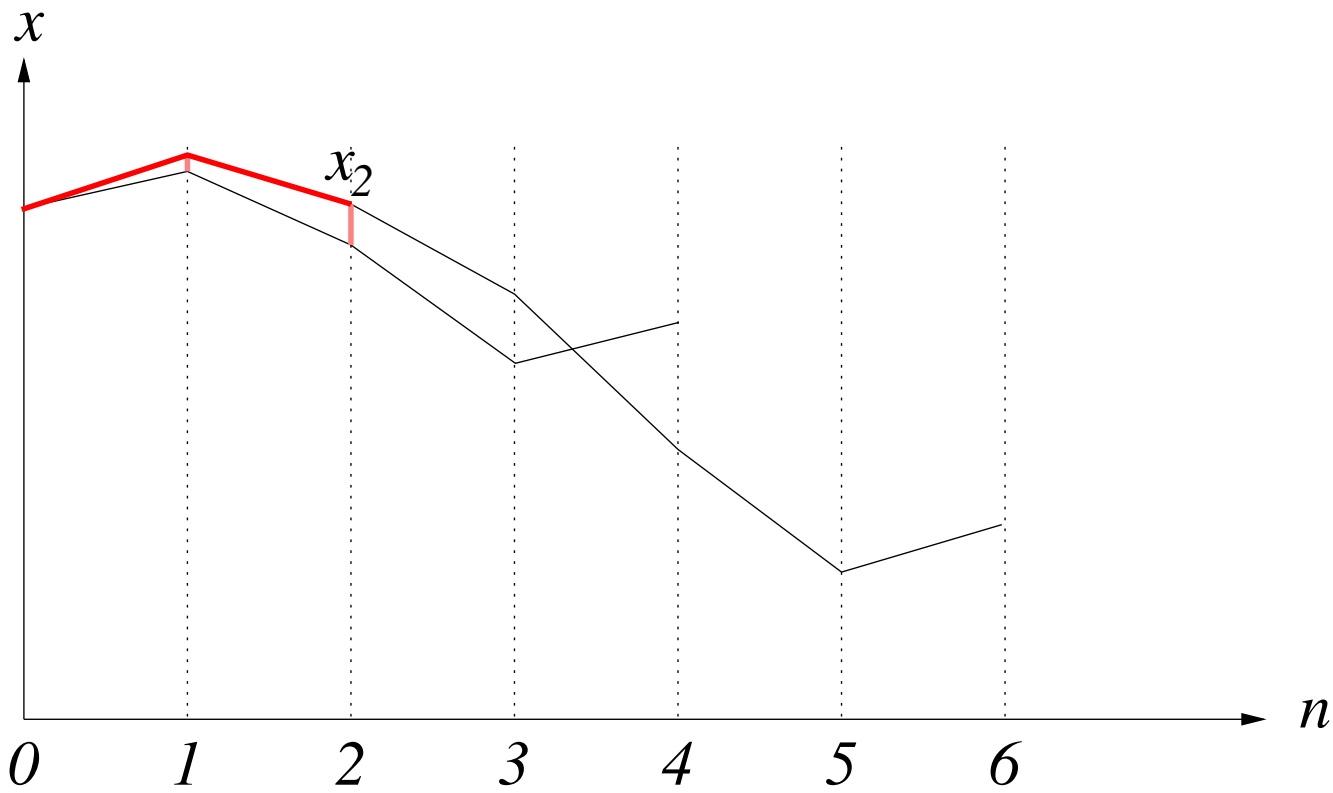

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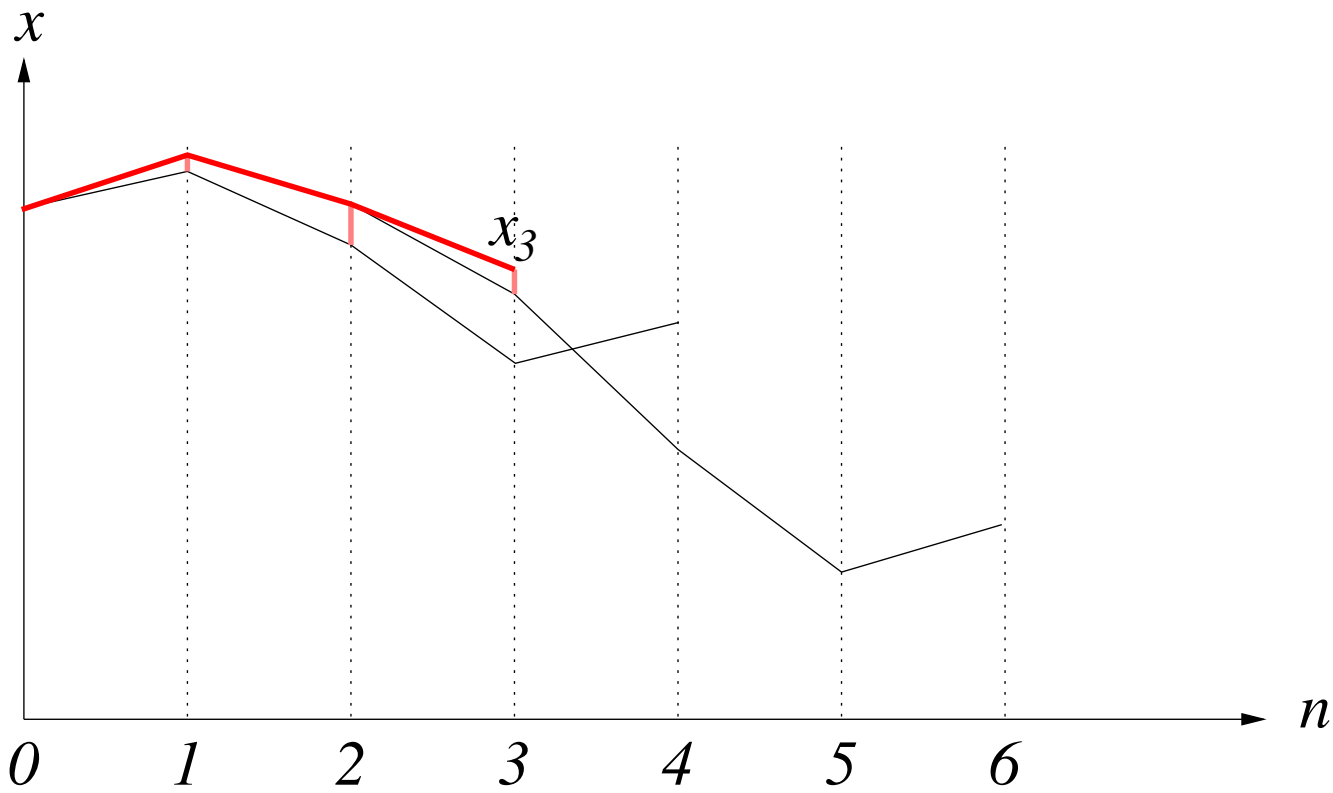

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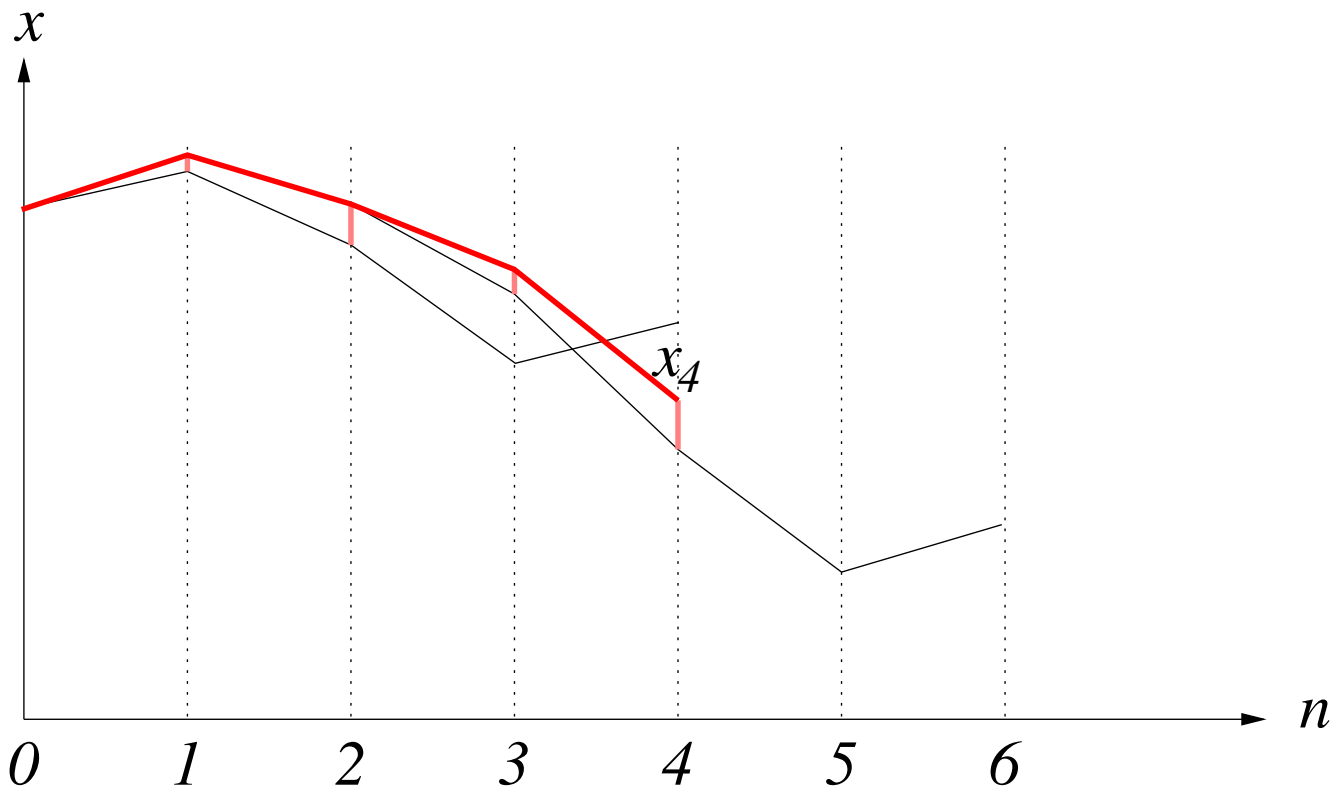

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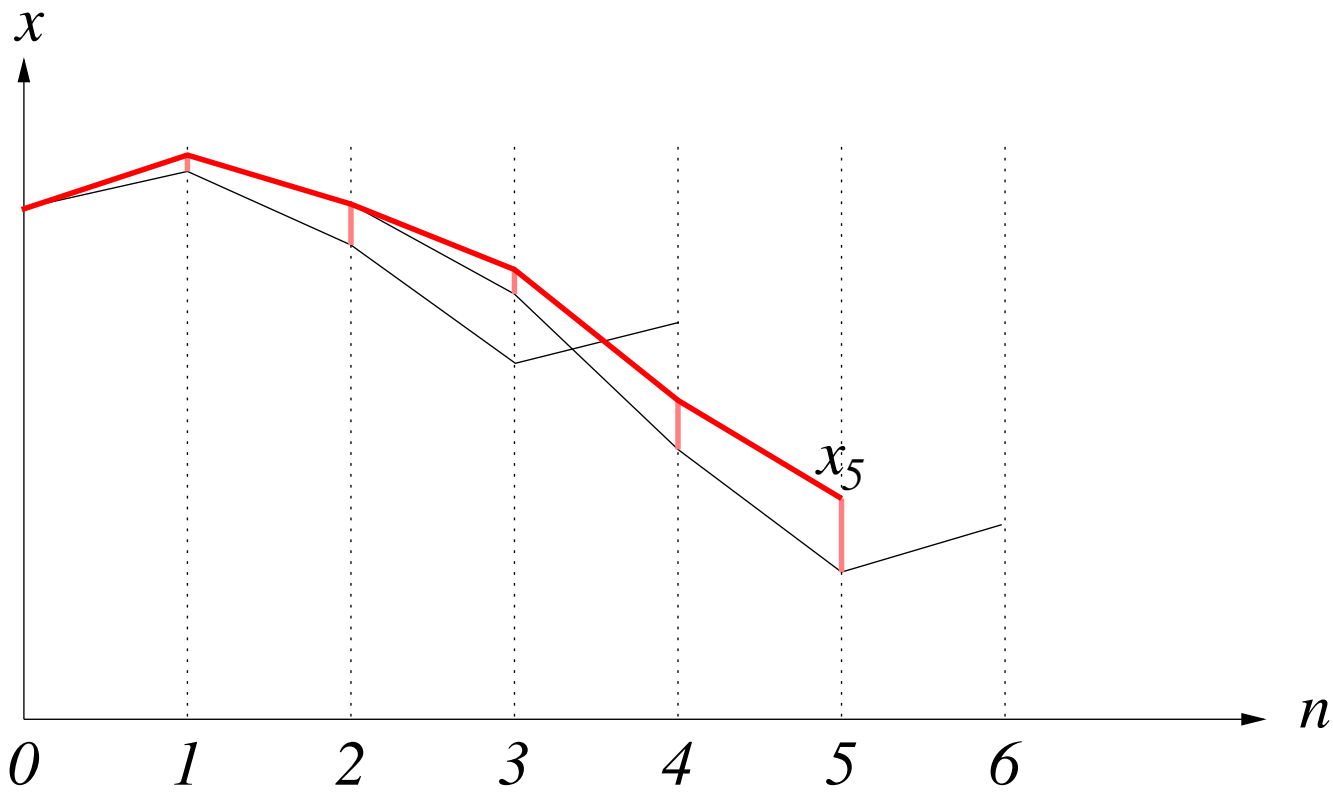

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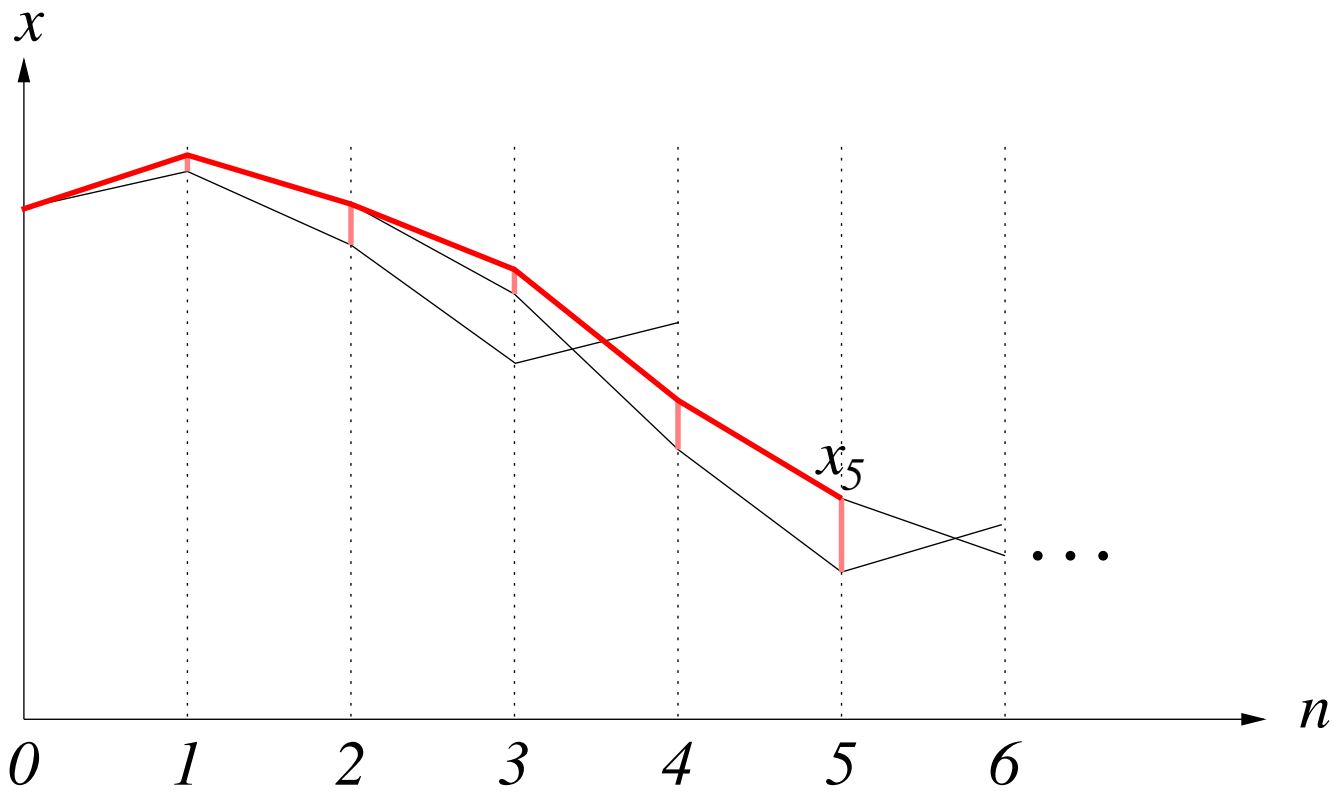

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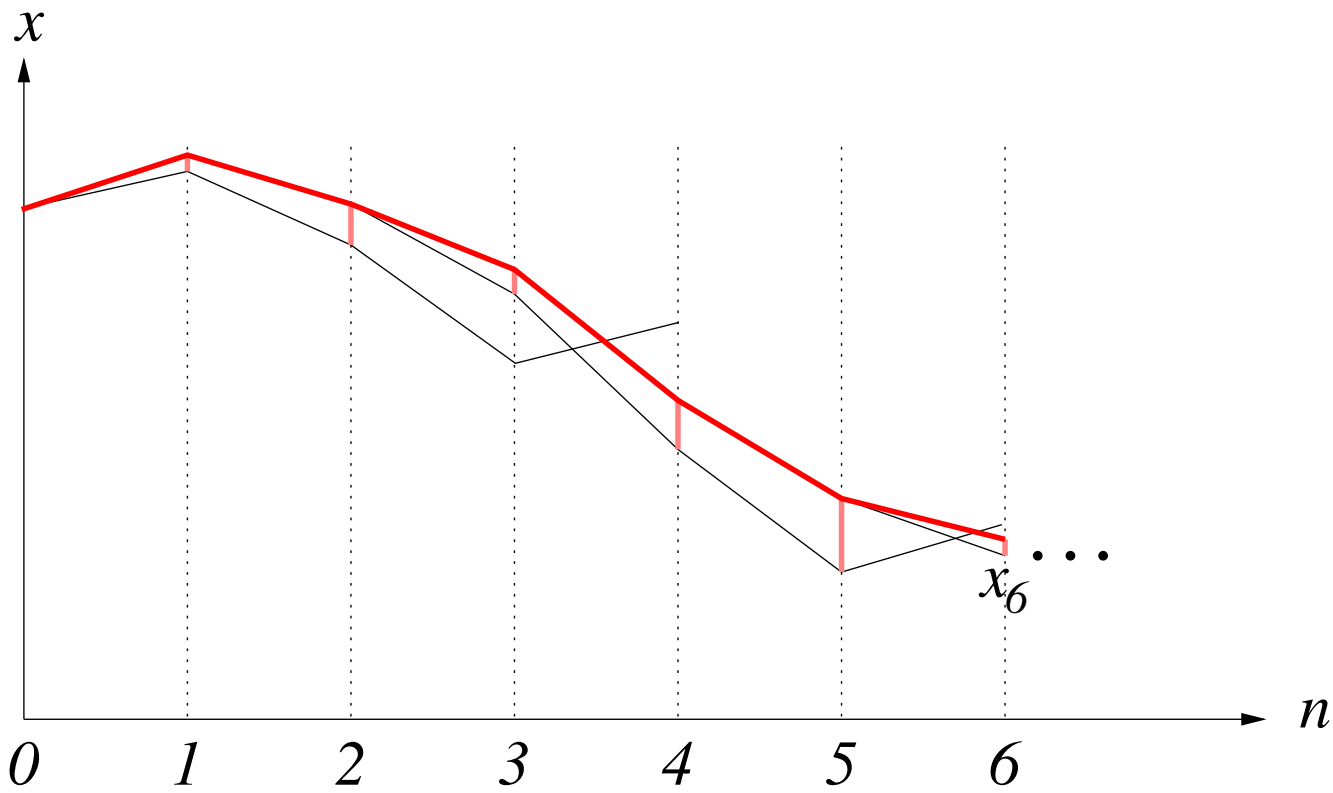

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Remedy:

- use sensitivity based techniques to update the "tails" of the optimal control sequences
- perform an integrated robustness and stability analysis

This will be the starting point for SADCO Task 3.3

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