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# On the construction of Patchy Feedbacks for stabilization and optimization problems

**Fabio ANCONA**

University of Padova

ITN SADCO - Kick off Meeting

Ensta Paris Tech, Paris

March 3 - 4, 2011

# Outline

- 1 Introduction
  - General Setting
  - Main Difficulties
- 2 Patchy Vector Fields
  - Definition and Basic Properties
  - Robustness of Patchy Vector Fields
- 3 Patchy Feedbacks
  - Definition
  - Stabilizing Patchy Feedback
  - Nearly Optimal Patchy Feedback
  - ISS Patchy Feedback

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Consider a control system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad (1)$$

- $f$  smooth with sublinear growth

$$|f(x, u)| \leq c(1 + |x|) \quad \forall x, u$$

- (I) **Asymptotic Feedback Stabilization:** Seek feedback control  $u = U(x)$  s.t. all trajectories of

$$\dot{x} = g(x) \doteq f(x, U(x)) \quad (2)$$

approach the origin as  $t \rightarrow \infty$  (and the origin is Lyapunov stable).

- (II) **Optimal Feedback Control:** Given compact set  $K \subset \mathbb{R}^n$ , seek feedback control  $u = U(x)$  s.t. all sol'ns of (1) reach given target set  $S$  in minimum time.

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- Smooth feedback controls that solve (I)-(II) in general may not exist

(I) Topological obstructions may prevent the existence of smooth (or even continuous) stabilizing feedback

(SUSSMANN '79; SONTAG & SUSSMANN '80; BROCKETT '83; CORON '90)

Ex: Brockett's nonholonomic integrator

$$\left. \begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned} \right\} \doteq f(x, u) \quad \begin{aligned} &(x_1, x_2, u_1, u_2) \rightarrow f(x, u) \\ &\text{not open at zero :} \\ &(0, 0, x_3) \notin \text{Im}(f) \quad \text{when } x_3 \neq 0 \end{aligned}$$

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- Discontinuous stabilizing (or optimal) feedback in general are not robust w.r.t. (inner and outer) perturbations

$$\dot{x} = f(x, U(x + e_1(t))) + e_2(t) \quad (3)$$

( $e_1$  : measurement error,  $e_2$  : external disturbance)

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## Two possible approaches

in dealing with stabilization and optimization problems for

$$\dot{x} = f(x, U(x)).$$

1. Allow **arbitrary discontinuous feedback** controls  $u = U(x)$  and work with weak or generalized solutions of discontinuous ODEs.
2. Use a particular class of feedback controls  $U(x)$  whose **discontinuities** are **sufficiently tame** so to guarantee the existence of **Carathéodory solutions**

$$x(t) = \bar{x} + \int_0^t f(x(s), U(x(s))) ds.$$

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## How to interpret solutions to discontinuous ODEs ?

- Define weaker concept of solution

For example, one can consider Krasovskii sol'ns, i.e. trajectories of the differential inclusion

$$\dot{x} \in G(x) \doteq \bigcap_{\varepsilon > 0} \overline{\text{co}} \{ g(y); |y - x| < \varepsilon \}, \quad g(y) \doteq f(x, U(x)). \quad (4)$$

**Drawback:** too many sol'ns are obtained. Not all enjoy desired properties.

- Introduce suitable generalized solutions:

"Sample-and-hold-solutions" & "Euler solutions": successfully implemented for problems (I)-(II)

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S-a-h solutions & Euler solutions of  $\dot{x} = f(x, U(x))$ 

**S-a-h solution**  $x(t)$  recursively defined as follows.

1. fix a time mesh  $t_0 = 0 < t_1 < \dots < t_i < \dots$ , with size  $\delta \doteq \sup |t_{i+1} - t_i|$ ;
2. sample  $U(x(t_i)) \doteq u_i$  and compute  $x(t) \doteq x_{i+1}(t)$ ,  $t \in ]t_i, t_{i+1}]$  solving

$$\dot{x}_{i+1} = f(x_{i+1}, u_i), \quad x_{i+1}(t_i) = x_i(t_i) \quad \text{on } ]t_i, t_{i+1}].$$

**Euler solution** is a (uniform) limit of S-a-h solutions as size  $\delta \rightarrow 0$ .

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## An alternative approach

1. Introduce a class of discontinuous, piecewise smooth vector fields “**Patchy Vector Fields**” (PVF) which admit (forward) Carathéodory solutions, and analyze their properties.
2. Construct piecewise smooth stabilizing or optimal feedback  $U = U(x)$  so that corresponding vector field  $g(x) \doteq f(x, U(x))$  belongs to such class of PVF.
3. Establish **robustness properties** of stabilizing or optimal feedback in this class, relative to **external disturbances** and **internal measurement errors**.

(A. & BRESSAN '99-2008; BRESSAN & PRIULI 2008; A. & PRIULI 2011)

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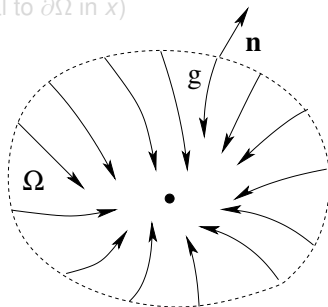
- **Patches**

$(\Omega, g)$  is said to be a **patch** if:

- $\Omega \subset \mathbb{R}^n$  is open domain with (piecewise) smooth boundary  $\partial\Omega$
- $g$  smooth vector field on a neighborhood of  $\partial\Omega$ , that points strictly inward at each point  $x \in \partial\Omega$

$$\langle g(x), \mathbf{n}(x) \rangle < 0 \quad \forall x \in \partial\Omega$$

$\mathbf{n}(x)$  outer normal to  $\partial\Omega$  in  $x$



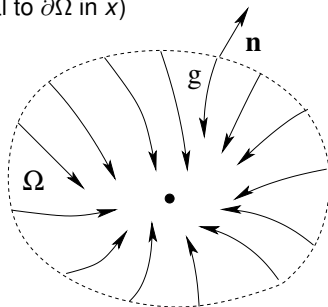
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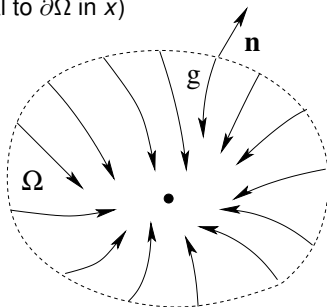
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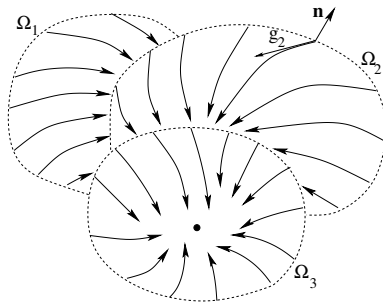
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• Patchy Vector Fields

locally finite covering in terms of patches  $(\Omega_i, g_i)$ ,  $i \in \mathbb{Z}$ .  
 $g : \Omega \rightarrow \mathbb{R}^n$  is said to be a **patchy vector field** if:

$$g(x) = g_i(x) \iff x \in \Omega_i \setminus \bigcup_{j>i} \Omega_j$$

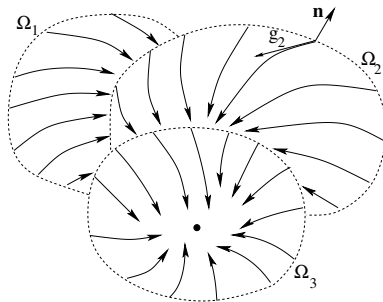


$$\implies g(x) = g_{i^*(x)}(x), \quad i^*(x) \doteq \max\{i \in \mathbb{Z} : x \in \Omega_i\}$$

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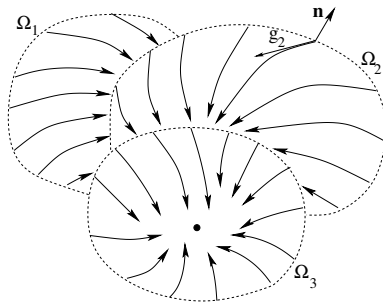


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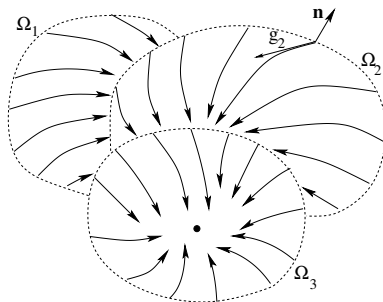


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# Properties of Patchy Vector Fields

$$\dot{x} = g(x) = g_{j^*(x)}(x)$$

**ODEs with patchy right hand side** (A. & BRESSAN '99)

Let  $g$  be a patchy vector field on  $\Omega \subset \mathbb{R}^n$ . Then the following holds.

i) For every  $\bar{x} \in \Omega$

$$\begin{cases} \dot{x} = g(x) \\ x(0) = \bar{x} \end{cases}$$

has

at least 1 (local) forward  
at most 1 backward

Carathéodory soln

- ii) sol'ns  $x(t)$  are piecewise smooth, with derivatives having finite set of jumps on any compact time interval. The map  $t \mapsto i_*(x(t))$  is **piecewise constant, non-decreasing**.
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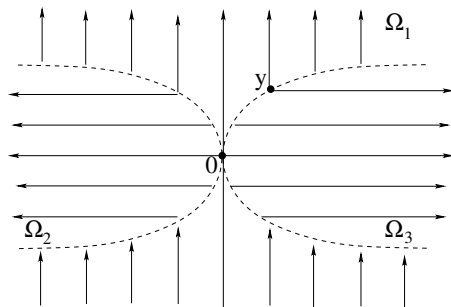
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## Properties of Patchy Vector Fields ... continued

**Note:** in general, the set of Carathéodory solutions to a Cauchy problem is **not connected**, not acyclic.

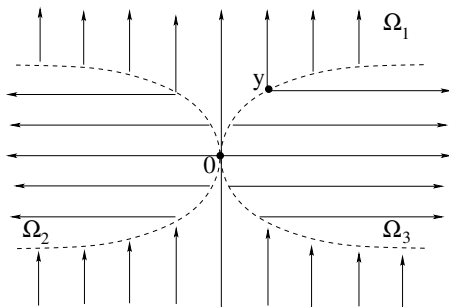
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## Robustness w.r.t. (small) inner and outer perturbations

$$\dot{y} = g(y + e_1(t)) + e_2(t) \quad (5)$$

**Flow stability of a Patchy Vector Field** (A. & BRESSAN 2003, 2004)

Let  $g$  be a patchy vector field on  $\Omega \subset \mathbb{R}^n$ . Given any compact subset  $C_0 \subset \Omega$ , and any  $T, \varepsilon > 0$ , there exists  $\delta > 0$  such that the following holds.

If  $y : [0, T] \mapsto \Omega$  is a solution of the perturbed system (5), with  $y(0) \in C_0$ , and

$$\|e_1\|_{BV} \leq \delta, \quad \|e_2\|_{L^1} \leq \delta,$$

then there exists a solution  $x : [0, T] \mapsto \Omega$  of the unperturbed equation

$$\dot{x} = g(x),$$

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$$|x(t) - y(t)| < \varepsilon \quad \forall t \in [0, T].$$

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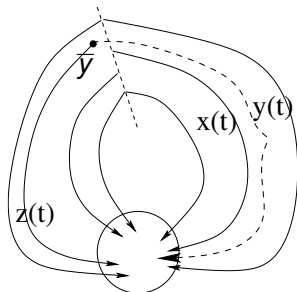
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**Note:** Since Cauchy problem for PVF lacks of forward uniqueness & continuous dependence, one cannot expect to have single sol'n's of PVF be stable w.r.t. perturbations.

Here we have **stability property** involving whole Carathéodory **solution set**: upper semicontinuity property of the solution set.





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$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in K \subset \mathbb{R}^m$$

A piecewise smooth map  $U : \Omega \rightarrow \mathbb{R}^m$  is a **patchy feedback** control on open domain  $\Omega \subset \mathbb{R}^n$  if:

- there exist  $U_i : \Omega_i \rightarrow \mathbb{R}^m$ ,  $i \in \mathbb{Z}$ , smooth maps, with  $\{\Omega_i\}_{i \in \mathbb{Z}}$  locally finite covering of  $\Omega$ ;
- $(\Omega_i, g_i)$ , with

$$g_i(x) \doteq f(x, U_i(x)), \quad x \in \Omega_i,$$

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$$U(x) = U_i(x) \quad \forall x \in \Omega_i \setminus \bigcup_{j>i} \Omega_j.$$

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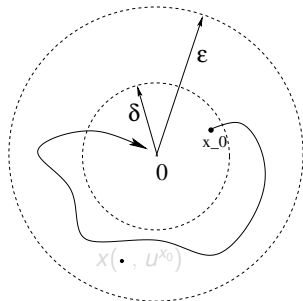
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$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in K \subset \mathbb{R}^m \quad (6)$$

### Stabilization by Patchy Feedback (A. & BRESSAN 1999)

**Assume:** (6) is globally asymptotically controllable to the origin:

1. **Attractiveness:**  $\forall x_0 \in \mathbb{R}^n \exists$  admissible open-loop control  $u^{x_0}(t)$  s.t. the trajectory  $x(\cdot, u^{x_0})$  starting at  $x_0$  is defined  $\forall t \geq 0$  and satisfies  $x(t, u^{x_0}) \rightarrow 0$  as  $t \rightarrow \infty$ .
2. **Lyapunov stability:**  $\forall \varepsilon > 0 \exists \delta > 0$  s.t.  $\forall x_0 \in \mathbb{R}^n, |x_0| \leq \delta$ ,  $\exists$  admissible control  $u^{x_0}(t)$  as in 1., with  $|x(t, u^{x_0})| \leq \varepsilon, \forall t \geq 0$ .

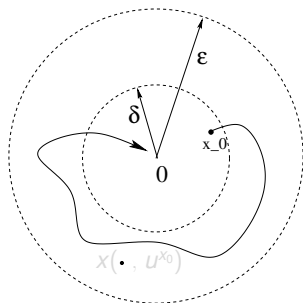


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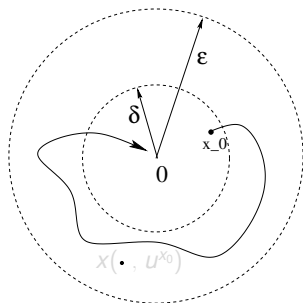


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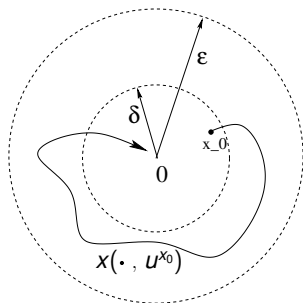


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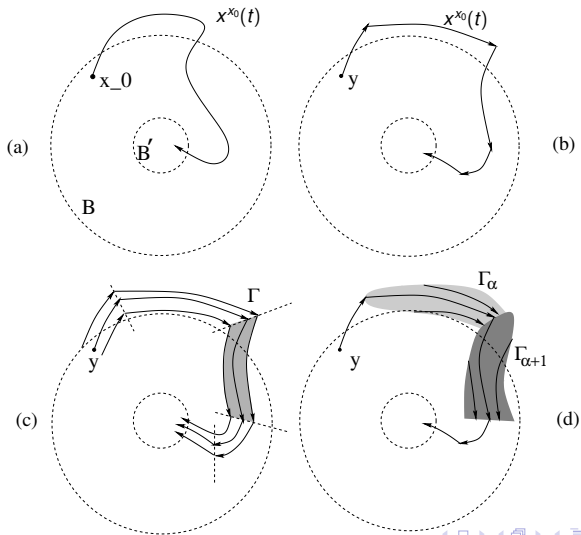
## Stabilization by Patchy Feedback ... continued

**Then:** there exists a **patchy feedback** control  $U(x)$  that makes **asymptotically stable** the closed-loop system

$$\dot{x} = f(x, U(x)) \quad (7)$$

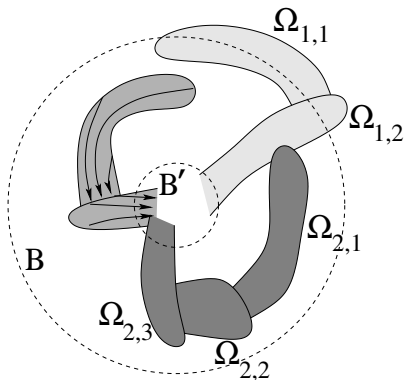
every Carathéodory solution of (7) either tends asymptotically to the origin, or reaches the origin in finite time. System enjoys (7) Lyapunov stability.

## Construction of a stabilizing patchy feedback by patching together open-loop controls



## Semi-global practical stabilization by a patchy feedback

Construct a patchy feedback that steers every point of a ball  $B$  inside a smaller ball  $B' \subsetneq B$ , within finite time.



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- $f$  smooth with sublinear growth
- $T(y) \doteq$  minimum time to steer the system from  $y$  to the origin

### Nearly Time Optimal Patchy Feedback

(A. & BRESSAN 2007; BRESSAN & PRIULI 2008)

Given  $\varepsilon > 0$ ,  $\tau > 0$ , there exists a sub-optimal patchy feedback  $u = U(x)$  defined on sub-level set

$$R_\tau = \{y : T(y) \leq \tau\},$$

such that  $\forall y \in R_\tau$ ,  $|y| > \varepsilon$ , any (Carathéodory) solution of

$$\dot{x} = f(x, U(x)) \quad x(0) = y,$$

reaches the  $\varepsilon$ -ball  $B_\varepsilon = \{|x| \leq \varepsilon\}$  within time  $T(y) + \varepsilon$ .

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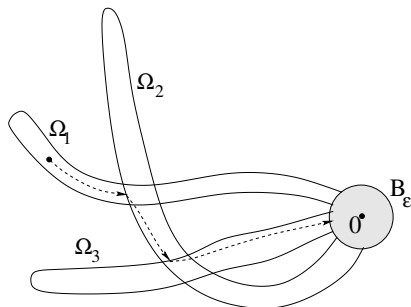
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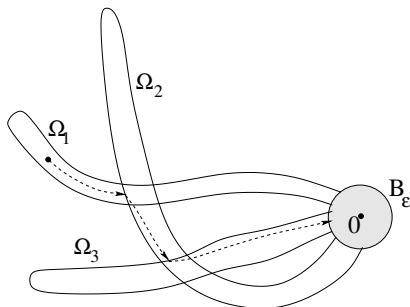
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**Alternative construction:** work with the value function

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# Construction of Nearly Time Optimal Patchy Feedback

- Approximate the minimum time function  $T(\cdot)$  with a semi-concave, piecewise quadratic function:

$$\tilde{T}(x) = \min_{j=1, \dots, N} \left\{ a_j + \mathbf{b}_j \cdot x \right\} + M \cdot |x|^2.$$

- Set of points where  $\tilde{T}$  not differentiable is contained in finite union of hyperplanes

$$\Gamma_{i,j} \doteq \left\{ a_i + \mathbf{b}_i \cdot x = a_j + \mathbf{b}_j \cdot x \right\}.$$

- Level sets  $\tilde{T}(x) = \text{const.}$  contained in union of spherical surfaces.

# Construction of Nearly Time Optimal Patchy Feedback

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$$\tilde{T}(x) = \min_{j=1,\dots,N} \left\{ a_j + \mathbf{b}_j \cdot x \right\} + M \cdot |x|^2.$$

- Set of points where  $\tilde{T}$  not differentiable is contained in finite union of hyperplanes

$$\Gamma_{i,j} \doteq \left\{ a_i + \mathbf{b}_i \cdot x = a_j + \mathbf{b}_j \cdot x \right\}.$$

- Level sets  $\tilde{T}(x) = \text{const.}$  contained in union of spherical surfaces.

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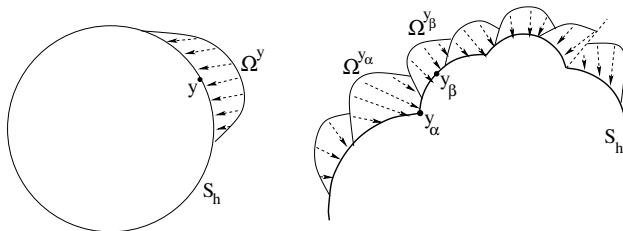
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## Construction of Nearly Time Optimal Patchy Feedback ... continued

- Construct the patchy feedback by induction, progressively working on larger sub-level sets of  $\tilde{T}$



# Outline

- 1 Introduction
  - General Setting
  - Main Difficulties
- 2 Patchy Vector Fields
  - Definition and Basic Properties
  - Robustness of Patchy Vector Fields
- 3 Patchy Feedbacks
  - Definition
  - Stabilizing Patchy Feedback
  - Nearly Optimal Patchy Feedback
  - **ISS Patchy Feedback**



**ISS Stability:** robustness of stabilization in presence of **large external perturbations** (SONTAG 1989)

- In presence of large external disturbances, solutions remain in a ball around the origin of the same size of the  $L^\infty$ -strength of the perturbation
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Consider affine system

$$f_0(x) + \sum_{i=1}^m u_i f_i(x) \quad x \in \mathbb{R}^n$$

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**ISS Stabilizing Patchy Feedback for Carathéodory solns**  
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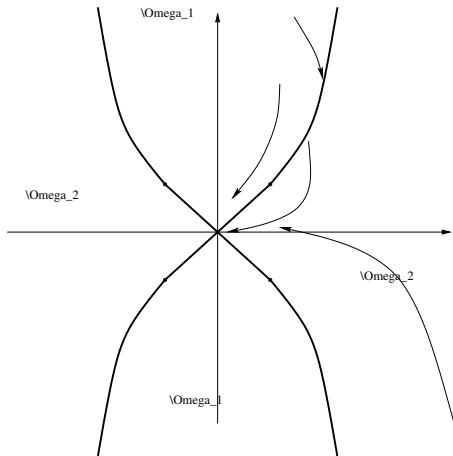
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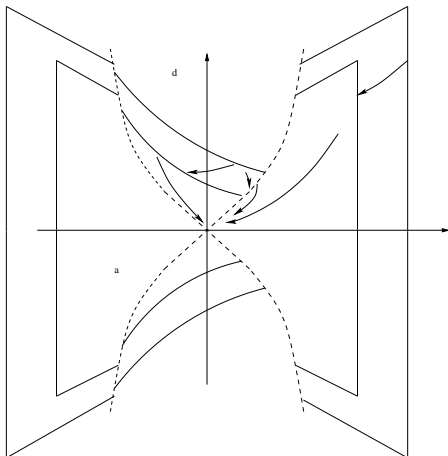
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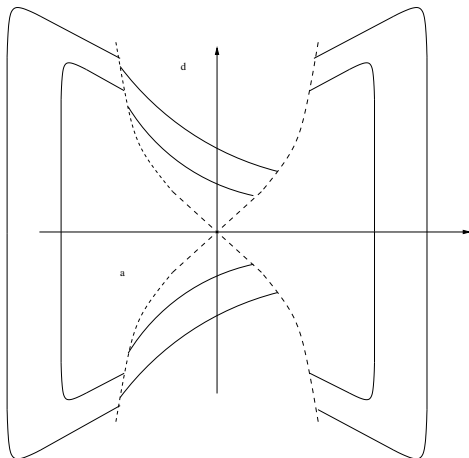
## Stabilizing Patchy Feedback for Nonholonomic Integrator



## ISS Stabilizing Patchy Feedback for Nonholonomic Integrator



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## Future directions

- Construction of stabilizing or sub-optimal patchy feedback for control **problems with constraints**.
- Construction of stabilizing or sub-optimal **hybrid** patchy feedback.  
(GOEBEL, PRIEUR, TEEL)
- Introduction of **numeric schemes** to solve HJ or HJB equations, based on patchy decomposition technique.  
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et finalement....

**Merci de votre attention!**