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Zubov's method for interconnected systems

Fabian Wirth

Institute of Mathematics
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SADCO - Kick off meeting
Paris, March, 1–2 2011.

joint work with:
Fabio Camilli (L'Aquila) and Lars Grüne (Bayreuth)

Zubov's Method

The domain of attraction

Zubov's equation

Robust domains of attraction

Problem statement

A robust version of Zubov's theorem

Examples

Interconnected Systems

ISS and Lyapunov functions

Zubov's Method and Interconnected Systems

The domain of attraction

Consider a nonlinear system

$$\begin{aligned}\dot{x} &= f(x) & (1) \\ x(0) &= x_0 \in \mathbb{R}^n,\end{aligned}$$

f Lipschitz continuous, $f(0) = 0$.

Assume $x^* = 0$ is **asymptotically stable**.



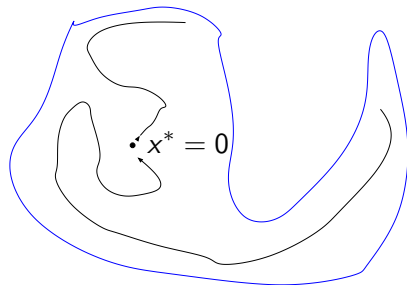
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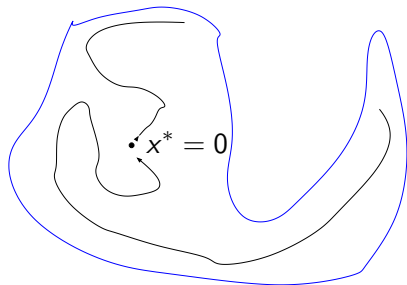
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The **domain of attraction** of 0 is defined by

$$\mathcal{A}(0) := \{x \in \mathbb{R}^n \mid \varphi(t; x) \rightarrow 0, \text{ as } t \rightarrow \infty\}.$$

Here $\varphi(\cdot; x)$ denotes the solution of (1).

Zubov's result (1956)

$$\dot{x} = f(x) \quad (1)$$

$$f(0) = 0, \quad x^* = 0 \quad \text{is asymptotically stable.} \quad (2)$$

Theorem

A set A containing 0 in its interior is the domain of attraction of (1) if and only if



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▶ $DV(x) \cdot f(x) = -h(x)(1 - V(x))\sqrt{1 + \|f(x)\|^2}$



Robust domains of attraction

Consider systems

$$\dot{x}(t) = f(x(t), d(t)), \quad t \in \mathbb{R}$$

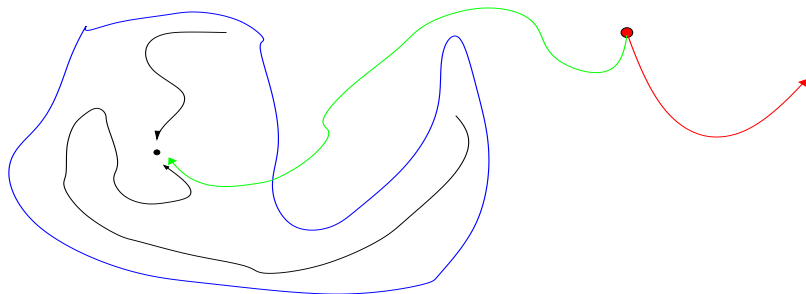
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$$\mathcal{A}_D(0) := \{x \in \mathbb{R}^n \mid \phi(t; x, d) \rightarrow 0 \forall d \in \mathcal{D}\}$$



A robust version of Zubov's theorem

Theorem:[Zubov's theorem for perturbed systems, SICON 2001]
 Under suitable growth conditions on g there is a unique viscosity solution of

$$\begin{cases} \inf_{d \in D} \{-Dv(x)f(x, d) - (1 - v(x))g(x, d)\} = 0 \\ v(0) = 0 \end{cases}$$

The robust domain of attraction satisfies

$$\mathcal{A}_D(0) = v^{-1}([0, 1]).$$

Example

$$\dot{x}_1 = -x_1 + x_2$$

$$\dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$$

Fixed points:

$[0, 0]$, unstable

$[-2.5505, -2.5505]$, asymptotically stable

$[-7.4495, -7.4495]$, asymptotically stable.



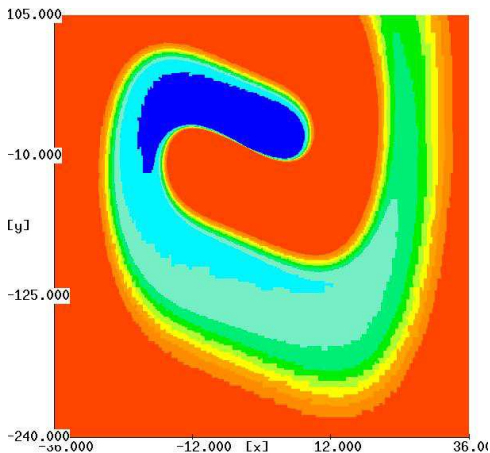
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$$D = \{0\}$$

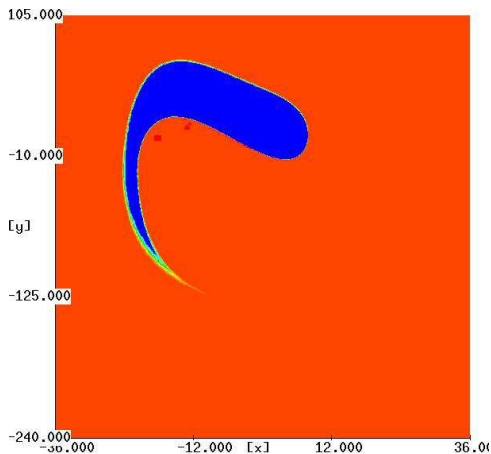


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$$D = [-0.02, 0.02]$$



Drawbacks

- ▶ Solution of the PDE is obtained by discretisation of the state space.
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Idea:

View systems in larger dimension as interconnection of low-dimensional systems.

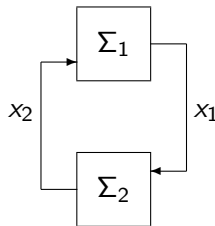
Interconnection of two systems

Consider

$$\Sigma_1 : \dot{x}_1 = f_1(x_1, x_2)$$

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$$f_i : \mathbb{R}^{N_1+N_2+N_u} \rightarrow \mathbb{R}^{N_i}$$



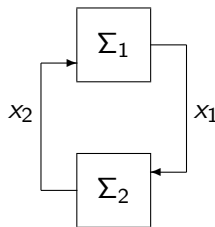
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with two **Lyapunov functions** such that

$$V_1(x_1) > \gamma_{12}(V_2(x_2)) \Rightarrow \dot{V}_1 < -\alpha_1(\|x_1\|)$$

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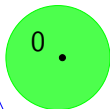
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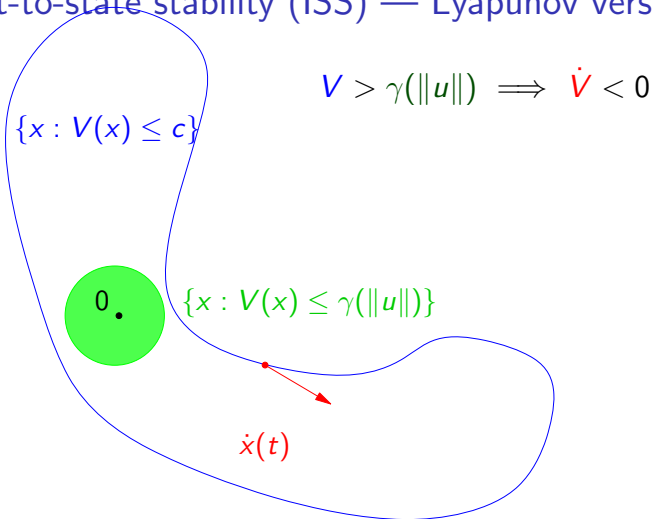
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$\{x : V(x) \leq c\}$

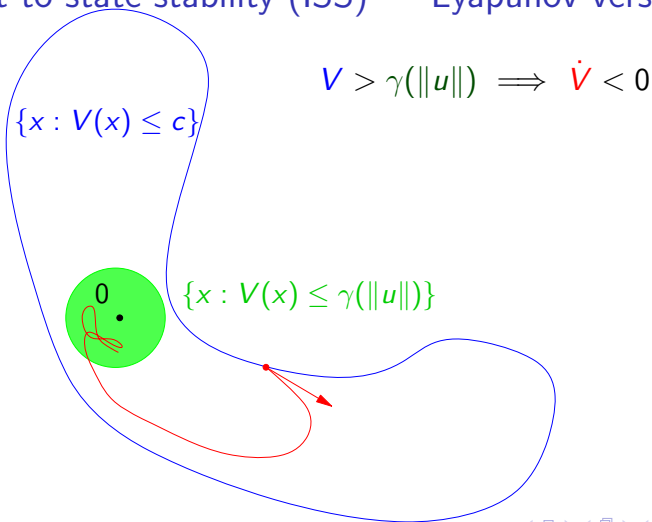
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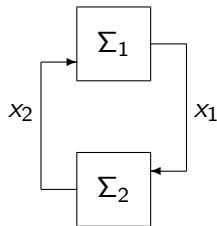
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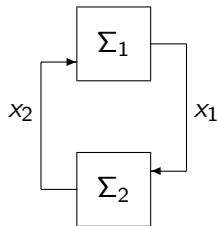
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The ISS small gain theorem

Theorem (Jiang, Mareels, Wang 1996)

If there exist \mathcal{K}_∞ -functions ρ_1, ρ_2 such that

$$(\text{id} + \rho_1) \circ \gamma_{12} \circ (\text{id} + \rho_2) \circ \gamma_{21} < \text{id},$$

then

$$\dot{x} = f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

with $x = (x_1, x_2)^\top$ is asymptotically stable in $(x_1^*, x_2^*) = (0, 0)$.

See also [Jiang, Teel, Praly 1994] [Grüne 2002] and [Dashkovskiy, Rüffer, W. 2007, 2009].



Goal

Can we compute ISS Lyapunov functions using a Zubov approach ?

Zubov and ISS Lyapunov functions

Choose a gain $\gamma \in \mathcal{K}_\infty$ and define

$$\begin{aligned} \tilde{f}_\gamma &: \mathbb{R}^n \times B(0, 1) \rightarrow \mathbb{R}^n \\ (x, d) &\mapsto f(x, \gamma(\|x\|)d) \end{aligned}$$

$$\dot{x} = f(x, \gamma(\|x\|)d) := \tilde{f}_\gamma(x, d), \quad (3)$$

Proposition

Let $\gamma \in \mathcal{K}_\infty$ be locally Lipschitz on $(0, \infty)$.

If V is a robust Lyapunov function for (3) it is an ISS Lyapunov function for

$$\dot{x} = f(x, d)$$

with Lyapunov gain γ^{-1} .



Outline of procedure

- (i) For each of the subsystems $i = 1, 2$ choose $\gamma_i \in \mathcal{K}_\infty$ and compute the robust Lyapunov function v_i .



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- (vi) Use this function to estimate the domain of attraction.



Conclusions

- ▶ An approach for the computation of lower estimates for the domain of attraction of interconnected systems has been presented.
- ▶ The idea still needs a lot of tuning:
 - ▶ which γ do you choose
 - ▶ is $\gamma \in \mathcal{K}_\infty$ really necessary ? (Angeli and Astolfi, 2007) says clearly that this is not the case
 - ▶ Do state transformations help ?

