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# Zubov's method for interconnected systems

Fabian Wirth

Institute of Mathematics  
University of Würzburg

SADCO - Kick off meeting  
Paris, March, 1–2 2011.

joint work with:  
**Fabio Camilli (L'Aquila) and Lars Grüne (Bayreuth)**

## Zubov's Method

The domain of attraction

Zubov's equation

## Robust domains of attraction

Problem statement

A robust version of Zubov's theorem

Examples

## Interconnected Systems

ISS and Lyapunov functions

## Zubov's Method and Interconnected Systems

# The domain of attraction

Consider a nonlinear system

$$\begin{aligned}\dot{x} &= f(x) & (1) \\ x(0) &= x_0 \in \mathbb{R}^n,\end{aligned}$$

$f$  Lipschitz continuous,  $f(0) = 0$ .

Assume  $x^* = 0$  is **asymptotically stable**.



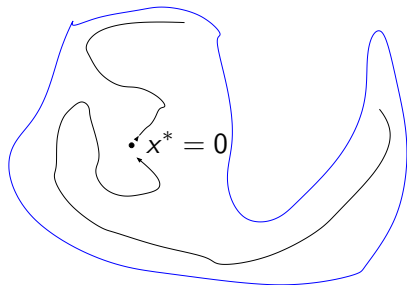
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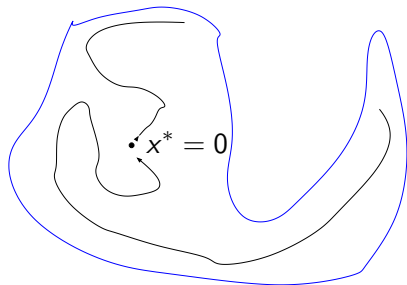
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The **domain of attraction** of 0 is defined by

$$\mathcal{A}(0) := \{x \in \mathbb{R}^n \mid \varphi(t; x) \rightarrow 0, \text{ as } t \rightarrow \infty\}.$$

Here  $\varphi(\cdot; x)$  denotes the solution of (1).

## Zubov's result (1956)

$$\dot{x} = f(x) \quad (1)$$

$$f(0) = 0, \quad x^* = 0 \quad \text{is asymptotically stable.} \quad (2)$$

### Theorem

*A set  $A$  containing  $0$  in its interior is the domain of attraction of (1) if and only if*



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▶  $DV(x) \cdot f(x) = -h(x)(1 - V(x))\sqrt{1 + \|f(x)\|^2}$



## Robust domains of attraction

Consider systems

$$\dot{x}(t) = f(x(t), d(t)), \quad t \in \mathbb{R}$$

with a perturbation term  $d$ . We are interested in robust stability properties. In particular, the **robust domain of attraction**.

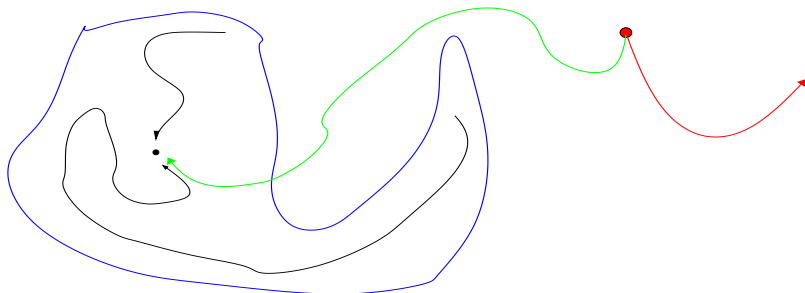


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- $f : \mathbb{R}^n \times D \rightarrow \mathbb{R}^n$  continuous, locally Lipschitz continuous in  $x$ , uniformly in  $d$
- $D \subset \mathbb{R}^m$  compact, convex,  $d(t) \in D$  a.e.



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$$\mathcal{A}_D(0) := \{x \in \mathbb{R}^n \mid \phi(t; x, d) \rightarrow 0 \forall d \in \mathcal{D}\}$$





## A robust version of Zubov's theorem

**Theorem:**[Zubov's theorem for perturbed systems, SICON 2001]  
 Under suitable growth conditions on  $g$  there is a unique viscosity solution of

$$\begin{cases} \inf_{d \in D} \{-Dv(x)f(x, d) - (1 - v(x))g(x, d)\} = 0 \\ v(0) = 0 \end{cases}$$

The robust domain of attraction satisfies

$$\mathcal{A}_D(0) = v^{-1}([0, 1]).$$



# Example

$$\dot{x}_1 = -x_1 + x_2$$

$$\dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$$

Fixed points:

$[0, 0]$ , unstable

$[-2.5505, -2.5505]$ , asymptotically stable

$[-7.4495, -7.4495]$ , asymptotically stable.



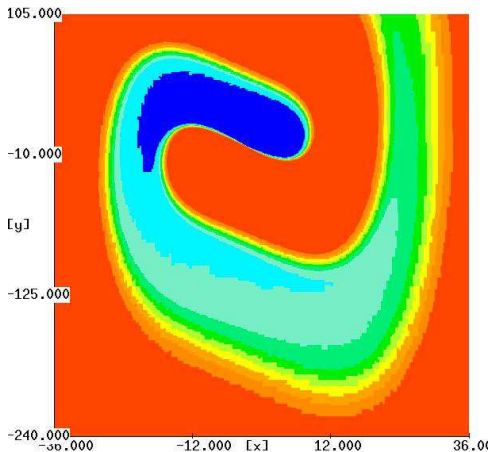
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$$\dot{x}_1 = -x_1 + x_2$$

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$$D = \{0\}$$



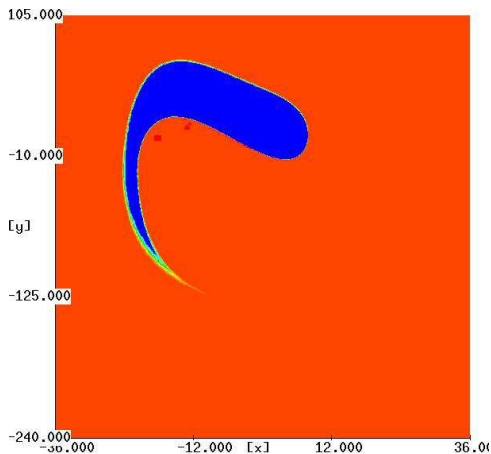
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$$D = [-0.02, 0.02]$$



# Drawbacks

- ▶ Solution of the PDE is obtained by discretisation of the state space.
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## Idea:

View systems in larger dimension as interconnection of low-dimensional systems.

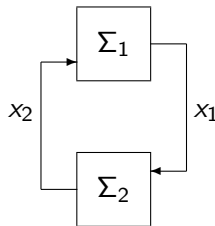
## Interconnection of two systems

Consider

$$\Sigma_1 : \dot{x}_1 = f_1(x_1, x_2)$$

$$\Sigma_2 : \dot{x}_2 = f_2(x_1, x_2)$$

$$f_i : \mathbb{R}^{N_1+N_2+N_u} \rightarrow \mathbb{R}^{N_i}$$



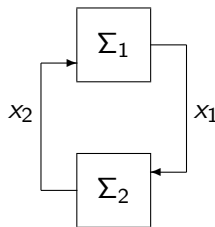
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with two **Lyapunov functions** such that

$$V_1(x_1) > \gamma_{12}(V_2(x_2)) \quad \Rightarrow \quad \dot{V}_1 < -\alpha_1(\|x_1\|)$$

$$V_2(x_2) > \gamma_{21}(V_1(x_1)) \quad \Rightarrow \quad \dot{V}_2 < -\alpha_2(\|x_2\|)$$





# Input-to-state stability (ISS) — Lyapunov version



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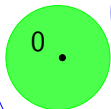
$$V > \gamma(\|u\|) \implies \dot{V} < 0$$



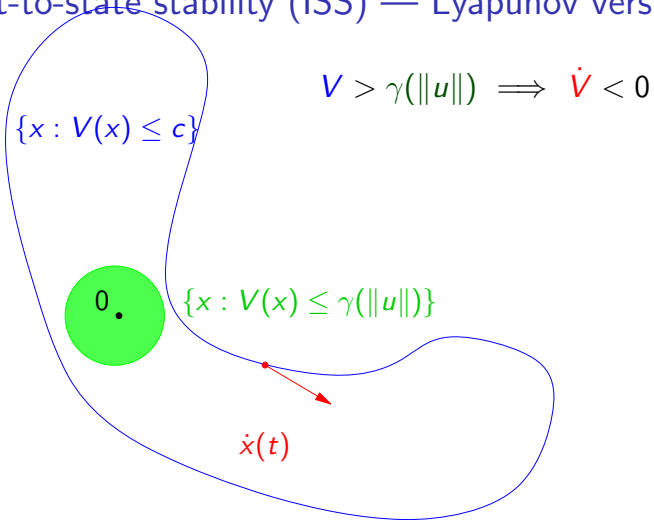
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$\{x : V(x) \leq c\}$

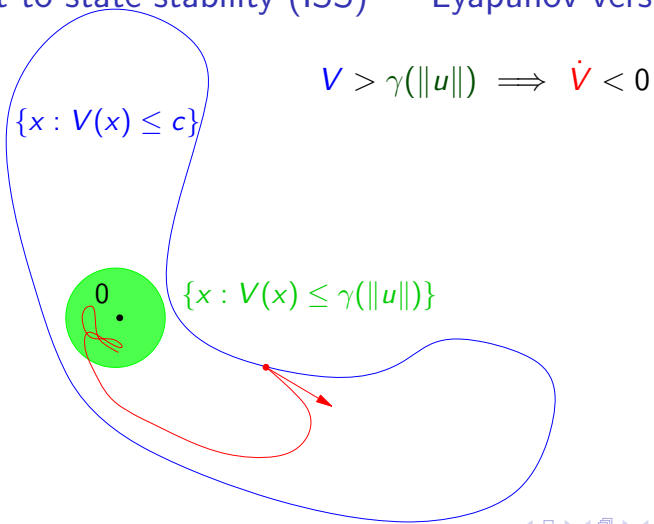
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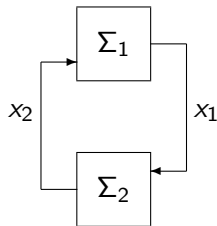
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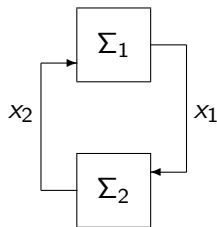
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## The ISS small gain theorem

### Theorem (Jiang, Mareels, Wang 1996)

If there exist  $\mathcal{K}_\infty$ -functions  $\rho_1, \rho_2$  such that

$$(\text{id} + \rho_1) \circ \gamma_{12} \circ (\text{id} + \rho_2) \circ \gamma_{21} < \text{id},$$

then

$$\dot{x} = f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$$

with  $x = (x_1, x_2)^\top$  is asymptotically stable in  $(x_1^*, x_2^*) = (0, 0)$ .

See also [Jiang, Teel, Praly 1994] [Grüne 2002] and [Dashkovskiy, Rüffer, W. 2007, 2009].





# Goal

Can we compute ISS Lyapunov functions using a Zubov approach ?

## Zubov and ISS Lyapunov functions

Choose a gain  $\gamma \in \mathcal{K}_\infty$  and define

$$\begin{aligned}\tilde{f}_\gamma &: \mathbb{R}^n \times B(0, 1) \rightarrow \mathbb{R}^n \\ (x, d) &\mapsto f(x, \gamma(\|x\|)d)\end{aligned}$$

$$\dot{x} = f(x, \gamma(\|x\|)d) := \tilde{f}_\gamma(x, d), \quad (3)$$

### Proposition

Let  $\gamma \in \mathcal{K}_\infty$  be locally Lipschitz on  $(0, \infty)$ .

If  $V$  is a robust Lyapunov function for (3) it is an ISS Lyapunov function for

$$\dot{x} = f(x, d)$$

with Lyapunov gain  $\gamma^{-1}$ .



## Outline of procedure

- (i) For each of the subsystems  $i = 1, 2$  choose  $\gamma_i \in \mathcal{K}_\infty$  and compute the robust Lyapunov function  $v_i$ .



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- (vi) Use this function to estimate the domain of attraction.





## Conclusions

- ▶ An approach for the computation of lower estimates for the domain of attraction of interconnected systems has been presented.
- ▶ The idea still needs a lot of tuning:
  - ▶ which  $\gamma$  do you choose
  - ▶ is  $\gamma \in \mathcal{K}_\infty$  really necessary ? (Angeli and Astolfi, 2007) says clearly that this is not the case
  - ▶ Do state transformations help ?

