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► **To cite this version:**

Olivier Ley, Fabio Camilli, Paola Loreti, Vinh Duc Nguyen. Large time behavior of systems of first-order Hamilton-Jacobi equations. SADCO Kick off, Mar 2011, Paris, France. inria-00585692

HAL Id: inria-00585692

<https://inria.hal.science/inria-00585692>

Submitted on 14 Apr 2011

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1. Scalar case
2. Control
3. Result
4. Proof

Large time behavior of systems of first-order Hamilton-Jacobi equations

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Joint work with F. Camilli (Roma), P. Loreti (Roma) and V. Nguyen (Rennes)

SADCO Meeting, Paris, March 3-4 2011

$$\begin{cases} \frac{\partial u_i}{\partial t} + F_i(x, Du_i) + \sum_{j=1}^m d_{ij}(x)u_j = f_i(x) & \mathbb{T}^N \times (0, +\infty) \\ u_i(x, 0) = u_{0,i}(x) & \mathbb{T}^N \end{cases}$$

for $i = 1, \dots, m$.

- ⇒ Periodic setting
- ⇒ Linear coupling
- ⇒ More precise assumptions later

Aim : Study the behavior of $u(x, t) = (u_1(x, t), \dots, u_m(x, t))$
when $t \rightarrow +\infty$

Plan of the talk

Large time
behavior of
systems of
Hamilton-
Jacobi
equations

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- 1 Recall of the scalar case
- 2 Motivations from control
- 3 Assumptions and a result
- 4 Sketch of the proof

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The scalar periodic case

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$$\begin{cases} \frac{\partial u}{\partial t} + F(x, Du) = f(x) & \mathbb{T}^N \times (0, +\infty) \\ u(x, 0) = u_0(x) & \mathbb{T}^N \end{cases}$$

A lot of works : Lions 82, Fathi 98, Namah-Roquejoffre 99,
Barles-Souganidis 00, Davini-Siconolfi 06, Ishii-Mitake 06,07,...

Theorem. [Namah-Roquejoffre 99]

- Periodicity : $F(\cdot, p), f, u_0$ are 1-periodic continuous
- convexity and coercivity of $F(x, \cdot)$
- $F(x, p) \geq F(x, 0) = 0$
- $f(x) \geq 0, \mathcal{A} = \{x \in \mathbb{T}^N : f(x) = 0\} \neq \emptyset$
- regularity : $|F(x, p) - F(y, p)| \leq \omega((1 + |p|)|x - y|)$

Then, for every $u_0 \in \text{Lip}(\mathbb{T}^N)$, there exists $(c, v) \in \mathbb{R} \times \text{Lip}(\mathbb{T}^N)$ such that

- $u(x, t) - ct \rightarrow v(x)$ as $t \rightarrow +\infty$ uniformly in \mathbb{T}^N
- v is solution of $F(x, Dv) = f + c$ in \mathbb{T}^N
- c is the ergodic constant, unique,

$$c = -\min_{\mathbb{T}^N} f = \lim_{t \rightarrow +\infty} -\frac{u(x, t)}{t} = 0$$

Is this theorem true in the case of systems ?

The Aubry set $\mathcal{A} := \{x \in \mathbb{T}^N : f(x) = 0\}$ plays a particular role. What is the equivalent for systems ?

Control interpretation (scalar case)

Dynamics :
$$\begin{cases} \dot{X}(s) = b(X(s), \alpha(s)) & s \geq 0 \\ X(0) = x \end{cases}$$

$\alpha(s)$ control, takes its value in a compact space K .

Value function :

$$V(x, t) = \inf_{\alpha(\cdot)} \left\{ \int_0^t f(X(s)) ds + u_0(X(t)) \right\}$$

Then V is the unique viscosity solution of

$$\begin{cases} \frac{\partial u}{\partial t} + \sup_{\alpha \in K} \{-b(x, \alpha) \cdot Du\} = f(x) \\ u(x, 0) = u_0(x) \end{cases}$$

Optimal trajectories are attracted by $\mathcal{A} = \operatorname{argmin} f$ and

$$\frac{V(x, t)}{t} \underset{t \rightarrow +\infty}{\sim} -c = \min f.$$

Control for systems : piecewise deterministic trajectories with random jumps (1)

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Dynamics :
$$\begin{cases} \dot{X}(s) = b_{\nu(t)}(X(s), \alpha(s)) & s \geq 0 \\ X(0) = x \end{cases}$$

solution : $(X(s), \nu(s))$ with $\nu(s)$ a Markov process with values in $\{1, 2, \dots, m\}$

Transition probabilities :

$$\mathbb{P}(\nu(t+h) = j \mid \nu(t) = i, X(t) = x) = \gamma_{ij}(x)h + o(h)$$

for $j \neq i$.

Value function :

$$V_i(x, t) = \inf_{\alpha(\cdot)} E_{x,i} \left\{ \int_0^t f_{\nu(s)}(X(s)) ds + u_{0,\nu(t)}(X(t)) \right\}$$

Control for systems : piecewise deterministic trajectories with random jumps (2)

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Then $V = (V_1, \dots, V_m)$ is the unique viscosity solution of the system

$$\begin{cases} \frac{\partial u_i}{\partial t} + \sup_{\alpha \in K} \{-b_i(x, \alpha) \cdot Du_i\} + \sum_{j=1}^m \gamma_{ij}(x)(u_i - u_j) = f_i(x) \\ u_i(x, 0) = u_{0,i}(x) \end{cases}$$

for $i = 1, \dots, m$.

For instance **Fleming-Zhang 98**

$$\sum_{j=1}^m \gamma_{ij}(x)(u_i - u_j) = \sum_{j=1}^m d_{ij}(x)u_j$$

with $d_{ii} = \sum_{j \neq i} \gamma_{ij} \geq 0$ and $d_{ij} = -\gamma_{ij} \leq 0$ for $i \neq j$

Assumptions on the Hamiltonian and initial conditions

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The same as in Namah-Roquejoffre Theorem.

For all $i = 1, \dots, m$:

- Periodicity : $F_i(\cdot, p), f_i, u_{0,i}$ are 1-periodic continuous
- convexity and coercivity of $F_i(x, \cdot)$
- $F_i(x, p) \geq F_i(x, 0) = 0$
- $f_i(x) \geq 0$
- regularity : $|F_i(x, p) - F_i(y, p)| \leq \omega((1 + |p|)|x - y|)$

Assumptions on the coupling matrix

$$D(x) = (d_{ij}(x))_{1 \leq i, j \leq m}$$

For all $x \in \mathbb{T}^N$:

- $d_{ii} \geq 0$, $d_{ij} \leq 0$ for $j \neq i$, $\sum_{j=1}^m d_{ij} \geq 0$.

⇒ D is a ***M*-matrix**

Classical assumptions to have a monotone system
 ⇒ maximum principle for the evolution problem

- d_{ij} are periodic in x
- $D(x)$ has non zero coefficients or :
 is an **irreducible** matrix in \mathbb{T}^N

equivalent definitions :

- (i) $\forall \mathcal{I} \subsetneq \{1, \dots, m\}$, $\exists i \in \mathcal{I}$ and $j \notin \mathcal{I}$ s.t. $d_{ij} \neq 0$
- (ii) $\forall i, j \in \{1, \dots, m\}$, there exists
 a "chain" $i = i_0, i_1, \dots, i_n = j$ s.t. $d_{i_{l-1}i_l} \neq 0$.

⇒ means that the coupling is nontrivial

Introduction of a "Aubry set" and assumptions

$$\text{Let } \mathcal{F} = \bigcap_{1 \leq i \leq m} \operatorname{argmin} f_i$$

$$\mathcal{D} = \bigcap_{1 \leq i \leq m} \{x \in \mathbb{T}^N : \sum_{j=1}^m d_{ij}(x) = 0\}$$

We define

$$\mathcal{A} = \mathcal{F} \cap \mathcal{D}.$$

Assume that

- \mathcal{A} is **non empty**
- all the f_i 's have the **same** minimum \bar{f}
(\Rightarrow to simplify we take $\bar{f} = 0$)
- $\mathcal{D} = \mathbb{T}^N$ (\Rightarrow to simplify)

With these assumptions, since $f_i \geq 0$,

$$\mathcal{A} = \mathcal{F} = \{x \in \mathbb{T}^N : \sum_{i=1}^m f_i(x) = 0\}$$

A Lemma on the coupling matrix

Lemma. Suppose that :

- $D(x)$ is an irreducible M -matrix on \mathbb{T}^N
- $\sum_{j=1}^m d_{ij} = 0$ (i.e., $\mathcal{D} = \mathbb{T}^N$)

Then, for all $x \in \mathbb{T}^N$, :

- $D(x)$ is degenerate of rank $m - 1$
- the kernel of $D(x)$ is spanned with $(1, \dots, 1)$
- there exists a **positive** function $\Lambda : \mathbb{T}^N \rightarrow \mathbb{R}^m$ such that

$$D(x)^T \Lambda(x) = 0 \quad (\text{i.e., } \sum_{i=1}^m \Lambda_i(x) d_{ij} = 0)$$

[Proof : Perron-Froebenius+continuous dependence]

For simplicity, we suppose that $\Lambda(x) = (1, \dots, 1)$

Theorem. Under the previous assumptions,
for every $u_0 = (u_{0,1}, \dots, u_{0,m}) \in \text{Lip}(\mathbb{T}^N)$, there exists
 $((c_1, \dots, c_m), (v_1, \dots, v_m)) \in \mathbb{R}^m \times \text{Lip}(\mathbb{T}^N)$ such that

- $u(x, t) - ct \rightarrow v(x)$ as $t \rightarrow +\infty$ uniformly in \mathbb{T}^N
- v is solution of the stationary system

$$F_i(x, Dv_i) + \sum_{j=1}^m d_{ij}(x)v_j = f_i + c_i \quad \text{in } \mathbb{T}^N, i = 1, \dots, m$$

- c is in the kernel of $D(x)$ so $c = (c_1, \dots, c_m)$, with
 $c_1 = -\bar{f} = \lim_{t \rightarrow +\infty} -\frac{u_i(x, t)}{t} = \mathbf{0}$ for all i .
- $v_i = v_j$ on \mathcal{A} .

Comparison result for the stationary system

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$$F_i(x, Dv_i) + \sum_{j=1}^m d_{ij}(x)v_j = f_i \quad \text{in } \mathbb{T}^N, i = 1, \dots, m$$

Theorem. Let u be a bounded subsolution and v a bounded supersolution.

- (Classical case) If, for all i , $\sum_{j=1}^m d_{ij} > 0$ ($\mathcal{D} = \emptyset$)

then $u \leq v$ on \mathbb{T}^N .

- (Degenerate case) If $\sum_{j=1}^m u_j \leq \sum_{j=1}^m v_j$ on \mathcal{A}

then $u \leq v$

⇒ \mathcal{A} may be empty

Theorem.

- (Classical case) there exists a unique viscosity solution to the stationary system.
- (Degenerate case) For all g continuous on \mathcal{A} with compatibility conditions (see [Fathi-Siconolfi 05](#), [Ishii-Mitake 07](#)), there exists a unique viscosity v solution such that $v = g$ on \mathcal{A} .

⇒ \mathcal{A} is a uniqueness set

Idea of the proof of the comparison theorem (1)

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Let $0 < \mu < 1$

$M := \sup_{1 \leq i \leq m} \sup_{\mathbb{T}^N} \{\mu u_i - v_i\}$ is achieved at \bar{x} .

Let $\mathcal{I} = \{i : \max \text{ is achieved for index } i\}$

\Leftrightarrow Case 1 : $\mathcal{I} = \{1, \dots, m\}$ and $\bar{x} \in \mathcal{A}$.

$$\sum_i u_i(\bar{x}) \leq \sum_i v_i(\bar{x}) \quad \text{and} \quad (\mu u_i - v_i)(\bar{x}) = M \text{ for all } i$$

$$\Rightarrow mM = \sum_i (\mu u_i - v_i)(\bar{x}) \leq (1 - \mu)m|u|_\infty$$

$$\Rightarrow M \leq 0 \text{ when } \mu \rightarrow 1$$

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Idea of the proof of the comparison theorem (2)

⇒ Case 2 : $\mathcal{I} = \{1, \dots, m\}$ and $\bar{x} \notin \mathcal{A}$.

⇒ $\exists i$ s.t. $f_i(\bar{x}) > 0$

Subsolution and supersolutions inequalities for equation i :

$$\mu F_i(\bar{x}, \frac{D\mu u_i(\bar{x})}{\mu}) + \sum_j d_{ij} \mu u_j(\bar{x}) \leq \mu f_i(\bar{x})$$

$$F_i(\bar{x}, Dv_i(\bar{x})) + \sum_j d_{ij} v_j(\bar{x}) \geq f_i(\bar{x})$$

Therefore

$$\underbrace{\mu F_i(\bar{x}, \frac{D\mu u_i(\bar{x})}{\mu}) - F_i(\bar{x}, Dv_i(\bar{x}))}_{\geq 0 \text{ (convexity and max.point)}} + \underbrace{\sum_j d_{ij} (\mu u_j - v_j)(\bar{x})}_{=(\sum_j d_{ij})M} \leq \underbrace{(\mu - 1)f_i(\bar{x})}_{< 0}$$

Idea of the proof of the comparison theorem (3)

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⇨ Case 3 : $\mathcal{I} \neq \{1, \dots, m\}$

$D(\bar{x})$ irreducible $\Rightarrow \exists i \in \mathcal{I}, k \notin \mathcal{I}$ s.t. $d_{ik}(\bar{x}) < 0$

Equation for i (as in case 2) leads to

$$\sum_j d_{ij}(\mu u_j - v_j)(\bar{x}) \leq (\mu - 1)f_i(\bar{x})$$

But $k \notin \mathcal{I} \Rightarrow (\mu u_k - v_k)(\bar{x}) \leq M - \delta, \quad \delta > 0$

Therefore $(\sum_j d_{ij})M - \delta d_{ik}(\bar{x}) \leq (\mu - 1)f_i \leq 0$

contradiction

Ergodic problem (1)

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$$\lambda v_i^\lambda + F_i(x, Dv_i^\lambda) + \sum_{j=1}^m d_{ij}(x)v_j^\lambda = f_i \quad \text{in } \mathbb{T}^N, i = 1, \dots, m$$

Theorem. There exists a unique solution which is Lipschitz continuous with constant L independent of λ .

Up to extract, as $\lambda \rightarrow 0$,

$$\begin{aligned} \lambda v^\lambda &\rightarrow -(c_1, \dots, c_m) = -(c_1, \dots, c_1) \in \ker D \\ v^\lambda - v^\lambda(x^*) &\rightarrow v \in \text{Lip}(\mathbb{T}^N) \end{aligned}$$

and (c, v) is solution of

$$F_i(x, Dv_i) + \sum_{j=1}^m d_{ij}(x)v_j = f_i + c_1 \quad \text{in } \mathbb{T}^N, i = 1, \dots, m$$

⇒ In fact $-c_1 = \min f_i = \bar{f} = \mathbf{0}$ here

In a more general context

$$\sum_i \min f_i \leq -mc_1 \leq \min \sum_i f_i$$

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Proof of the convergence theorem

$$u(x, t) - ct \rightarrow v(x) \text{ as } t \rightarrow +\infty \quad (1)$$

Recall that $-c_1 = -\min f_i = 0$

Comparison for the evolution problem : $\exists C \geq |u_0|_\infty$ s.t.
 $v(x) - C \leq u(x, t) \leq v(x) + C$

The function $u^\varepsilon(x, t) = u(x, \frac{t}{\varepsilon})$ is solution to the system

$$\begin{cases} \varepsilon \frac{\partial u_i}{\partial t} + F_i(x, Du_i) + \sum_{j=1}^m d_{ij}(x) u_j = f_i(x) & \mathbb{T}^N \times (0, +\infty) \\ u_i(x, 0) = u_{0,i}(x) & \mathbb{T}^N \end{cases}$$

for $i = 1, \dots, m$.

Then, by stability, the half-relaxed limits

$$\bar{u}(x) = \limsup_{\varepsilon \rightarrow 0}^* u^\varepsilon(x, t) \quad \text{and} \quad \underline{u}(x) = \liminf_{\varepsilon \rightarrow 0}^* u^\varepsilon(x, t)$$

are respectively sub and supersolutions to the **stationary** system.

$$u(x, t) - ct \rightarrow v(x) \text{ as } t \rightarrow +\infty \quad (2)$$

Summing the evolution equations for $1 \leq i \leq m$,

$$\frac{\partial}{\partial t} \left(\sum_i u_i \right) + \underbrace{\sum_i F_i(x, Du_i)}_{\geq 0} + \underbrace{\sum_i \sum_{j=1}^m d_{ij}(x) u_j}_{=\sum_j u_j \sum_{i=1}^m d_{ij}(x)=0} = \sum_i f_i(x)$$

But $\sum_i f_i(x) = 0$ on \mathcal{A}

Therefore

$$\frac{\partial}{\partial t} \left(\sum_i u_i \right) \leq 0 \quad \Rightarrow \quad \sum_i u_i(\cdot, t) \xrightarrow{t \rightarrow +\infty} \phi \text{ uniformly on } \mathcal{A}$$

Proof of the convergence theorem

$$u(x, t) - ct \rightarrow v(x) \text{ as } t \rightarrow +\infty \quad (3)$$

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Lemma. $\bar{u}_i = \underline{u}_i = \bar{u}_j = \underline{u}_j$ on \mathcal{A}

Therefore

$$\sum_i \bar{u}_i = \sum_i \underline{u}_i = \lim_{t \rightarrow +\infty} \sum_i u_i(\cdot, t) = \phi \quad \text{on } \mathcal{A}$$

Comparison theorem for the subsolution \bar{u} and the supersolution \underline{u} implies

$$\bar{u} = \underline{u} \quad \text{on } \mathbb{T}^N.$$