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# Homogenization of geometric Hamilton-jacobi equations and turbulent combustion models

Antonio Siconolfi

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Homogenization of geometric Hamilton-Jacobi  
equations and turbulent combustion models

A. Siconolfi (Rome "La Sapienza")

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Sadco Kick-off Meeting

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Paris, March 3 2011

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Many thanks to Andrea Dall'Aglio for the graphic  
part of the presentation

Methodology

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PDE with viscosity solution methods

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Qualitative analysis of PDE through techniques issued from dynamical systems, multivalued dynamics, control problems

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Metric methods, length functionals

The model

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$V(x)$  is a  $C^1$ -vector field



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$\zeta$ -equation - Turbulent combustion model - Laminar velocity  
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$\zeta$ -equation - Turbulent combustion model - Laminar velocity  
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Geometric equation describing fronts motions

$$u_0 \neq u'_0 \quad \text{but} \quad \{u_0 \leq 0\} = \{u'_0 \leq 0\} =: \Gamma(0)$$

$$\Rightarrow u(x,t) \neq u'(x,t) \quad \text{but} \quad \{u(\cdot,t) \leq 0\} = \{u'(\cdot,t) \leq 0\} =: \Gamma(t)$$

case  $V \equiv 0$

$$H = |P|$$

3

for a given initial datum  $u_0$

$$u(x, t) = \min \{ u_0(y) : |y - x| \leq t \}$$

$$\Gamma(t) = \{ x : d(x, \Gamma(0)) \leq t \}$$

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Case  $|V| < 1$

$$H = |P| + V \cdot P$$



$$Z(x) = \{ H(x, \cdot) \leq 1 \}$$

dual



$$Z^-(x) = V(x) + B$$

$$|v| \geq 1$$

$$|V| \geq 1$$

We lose the coercivity of the Hamiltonian and the sublevel  $Z(x)$  is not any more compact.

$$H(x, -\lambda V(x)) = \lambda |V(x)| - \lambda |V(x)|^2 \quad \forall \lambda > 0$$

$$\Rightarrow -\lambda V(x) \in Z(x) \quad \forall \lambda > 0$$

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Xin - Yu, Cardaliaguet-Nolen-Souganidis 2010

study homogenization problem with  $V$  periodic in  $\mathbb{R}^N \times [0, +\infty[$

$$\begin{cases} u_t + |Du| + V\left(\frac{x}{\varepsilon}, Du\right) = 0 \\ u = u_0 \quad \text{for } t=0 \end{cases}$$

limit for  $\varepsilon \rightarrow 0$ , behavior of the solutions, limit equation.



Interesting point: homogenization depends on conditions  
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(5)

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Interesting point: homogenization depends on conditions about  $\operatorname{div} V$ .

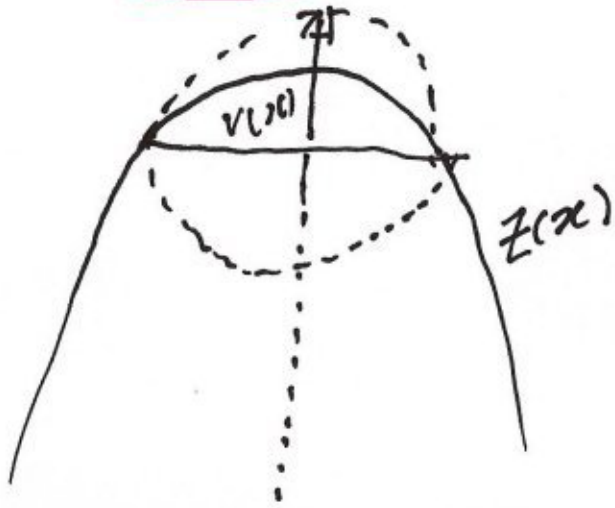
criticism: These investigations use indirect methods not taking into account the geometric features of the problem. The proofs are not self-contained.

I. will just consider the case of autonomous fields, i.e.  $V$  not depending on  $t$ .

# Sublevels in the non convergent case

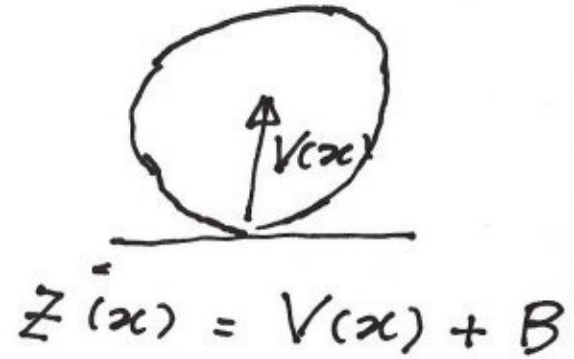
# Sublevels in the non coercive case

$$|V| = 1$$



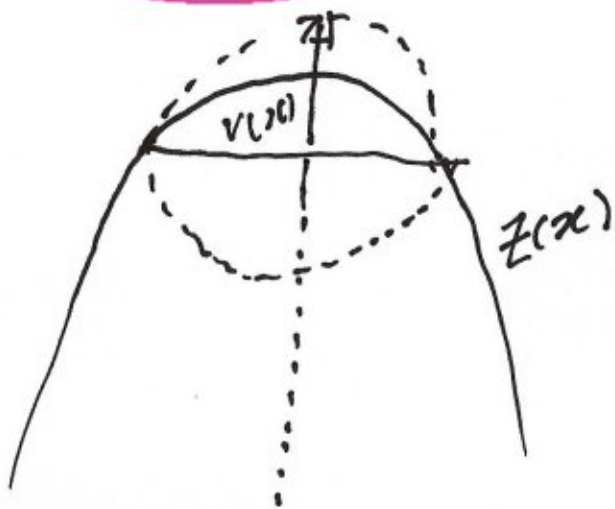
recession cone :  $\{z^\infty(x) = -\lambda V(x) : \lambda \geq 0\}$

dual

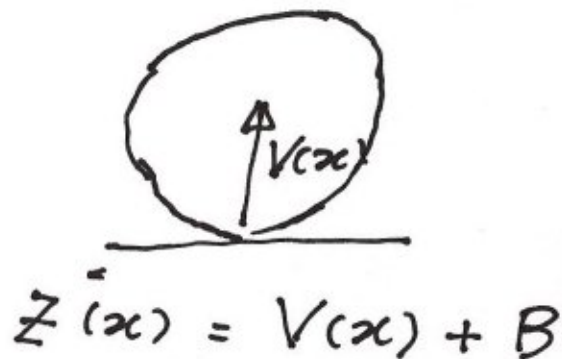


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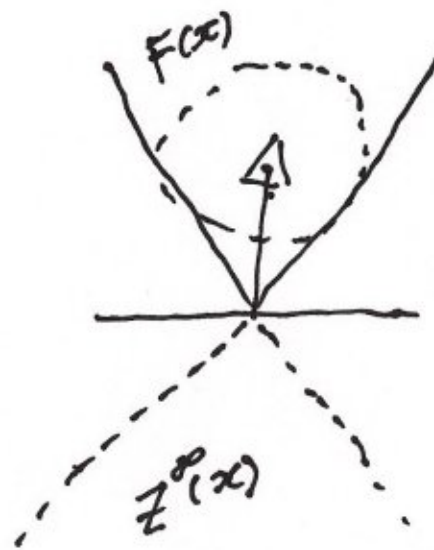


recession cone :  $Z^\infty(x) = \{-\lambda V(x) : \lambda \geq 0\}$

$|V| > 1$



dual



admissible directions = dual of  $Z^\infty(x)$   
 $:= F(x)$

Even in this case one can define an intrinsic distance on  $\mathbb{R}^n$ , possibly infinite and just lower semicontinuous

$S(x, y) =$  minimum time for joining  $x$  to  $y$  through a curve with intrinsic velocity  $\leq 1$

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we can also define a distance on the torus  $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$

$$\tilde{S}([x], [y]) = \inf \{ S(x+z, y) : z \in \mathbb{Z}^N \}$$

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We recover the formulae holding in the eikonal case

$$u(x, t) = \inf \{ u_0(y) : S(y, x) \leq t \}$$

$$\Gamma(t) = \{ y : \inf_{x \in \Gamma(0)} S(x, y) \leq t \}$$

Homogenize the distance  $S$ , if possible

Limit equation of the type  
with  $\bar{H}$  positively homogeneous

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The distance  $S^\varepsilon$  corresponding to  $H(x/\varepsilon, p) = |p| + V(x/\varepsilon) \cdot p$  are

$$S^\varepsilon(x, y) = \varepsilon S\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

study the asymptotic behavior of

$$\lim_n \varepsilon_n S\left(\frac{x}{\varepsilon_n}, \frac{y}{\varepsilon_n}\right) \quad \varepsilon_n \rightarrow 0$$

Among these sequences there are some specials

$v \in \mathbb{Q}^N$ ,  $k \in \mathbb{N}$  st.  $kv \in \mathbb{Z}^N$

$$\lim_n \frac{1}{n} S(0, nk v)$$

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by the subadditive thm

$$\lim_n \frac{1}{n} S(0, nkV) = \inf_n \frac{1}{n} S(0, nkV)$$

at the limit, we get  $f: \mathbb{Q}^N \rightarrow \mathbb{R}^+ \cup \{+\infty\}$

homogeneous, subadditive, lower semicontinuous

Key point: there is homogenization if the asymptotics

$$\lim_{\varepsilon_n} S\left(\frac{x}{\varepsilon_n}, \frac{y}{\varepsilon_n}\right)$$

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This is, in turn, possible if any point of  $\mathbb{R}^N$   
can be projected on the lattice  $\mathbb{Z}^N$  with equibounded  
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(10)

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Notice that  $\mathbb{Z}^N$  corresponds to a point in  $\mathbb{T}^N$

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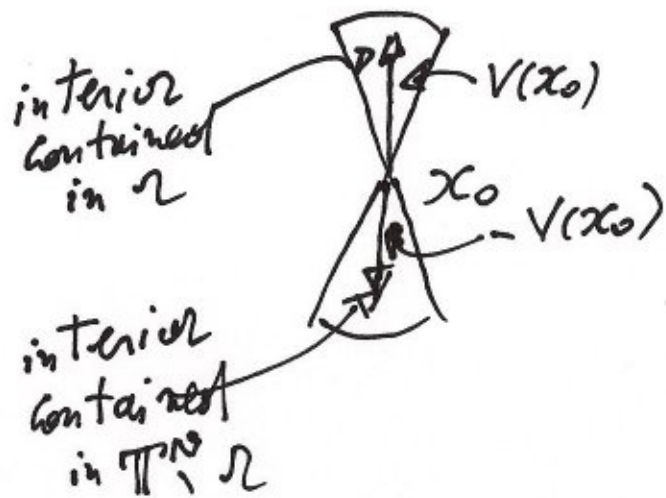
The finiteness property is indeed crucial. This is true if the cone dynamics given by the admissible directions  $F(x)$  does not possess invariant subsets on  $\mathbb{T}^N$ , strictly contained in  $\mathbb{T}^N$ .

In fact, given  $x_0 \in \mathbb{T}^N$ , the set of the points reachable from it through admissible curves, is invariant.

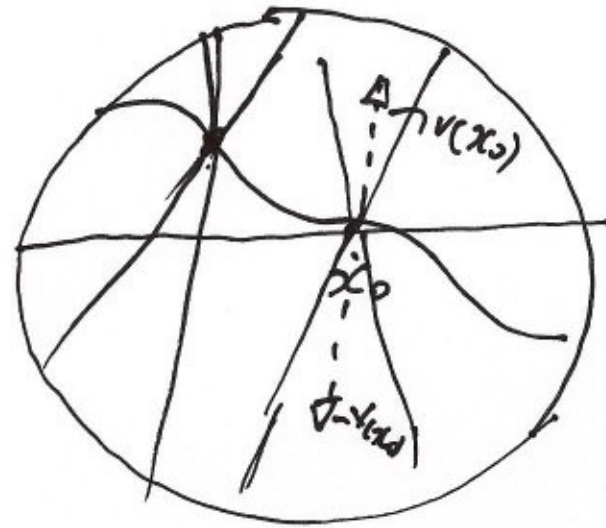
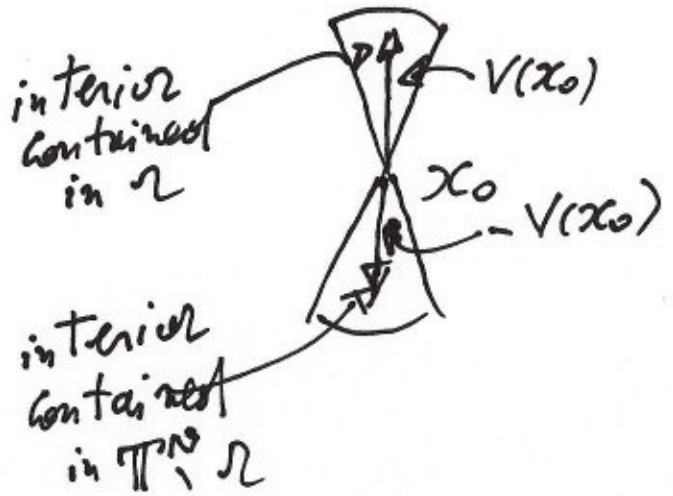
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(12)

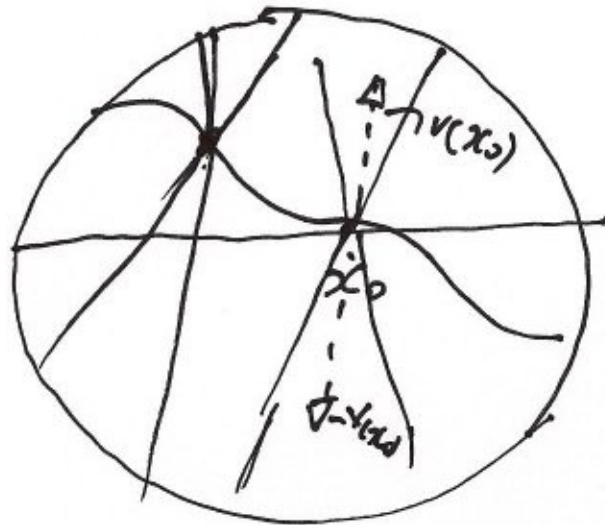
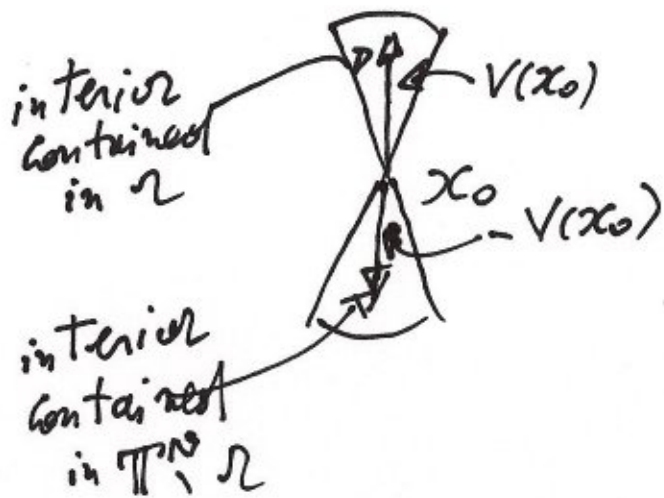


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Fact

$\Omega$  has Lipschitz boundary and

$$\text{Per } \Omega = \left| \int_{\partial \Omega} V \cdot n \, d\mathcal{H}^{N-1} \right|$$

Use Gauss-Green Thm and the isoperimetric inequality to deduce:

if  $\|\operatorname{div} V\|_{L^N(\mathbb{T}^N)} \leq \frac{1}{C}$  then there are no invariant sets  
( $C$  is the isoperimetric constant)

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and so there is homogenization.

Recovering, in the way, the Cardaliaguet-Nolen-Souganidis estimate.