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Marc Quincampoix

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# Optimal control problems with horizon tending to infinity and lacking controllability assumption

Marc Quincampoix  
Université de Brest

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M. Q., & J. Renault, On Existence of a limit value in some non expansive optimal control problems, (*submitted*) (2009)

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## An Optimal Control Problem

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$$V_t(y_0) := \inf_{u \in \mathcal{U}} \frac{1}{t} \int_{s=0}^t h(y(s, u, y_0), u(s)) ds,$$

where  $s \mapsto y(s, u, y_0)$  denotes the solution to

$$y'(s) = g(y(s), u(s)), \quad y(0) = y_0.$$

$g : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$  Lipschitz,  $U$  compact,  $g$   $h$  bounded.

**PROBLEM** : Existence of a limit of  $V_t(y_0)$  as  $t \rightarrow +\infty$ .

No ergodicity condition here (**Lions-Papanicolaou-Varadhan, Arisawa-Lions, Bettiol, Alvarez-Bardi Capuzzo-Dolcetta, Artstein-Gaitsgory, Fathi...**) The limit may depend on the initial condition

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2. *A controllability approach*
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## Introduction

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**Definition 1** *The problem  $\Gamma(y_0) := (\Gamma_t(y_0))_{t>0}$  has a limit value if*

$$V(y_0) := \lim_{t \rightarrow \infty} V_t(y_0) = \lim_{t \rightarrow \infty} \inf_{u \in \mathcal{U}} \frac{1}{t} \int_{s=0}^t h(y(s, u, y_0), u(s)) ds.$$

**Definition 2** *The problem  $\Gamma(y_0)$  has a uniform value if it has a limit value  $V(y_0)$  and if:*

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t} \int_{s=0}^t h(y(s, u, y_0), u(s)) ds \leq V(y_0) + \varepsilon.$$

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## Examples

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- **Example 1:** here  $y \in \mathbb{R}^2$  (seen as the complex plane  $i^2 = -1$ ), there is no control

$$y'(t) = i y(t),$$

$$V_t(y_0) \xrightarrow{t \rightarrow \infty} \frac{1}{2\pi|y_0|} \int_{|z|=|y_0|} h(z) dz,$$

and since there is no control, the value is uniform.

- **Example 2:** in the complex plane again, but now  $g(y, u) = i y u$ , where  $u \in U$  a given bounded subset of  $\mathbb{R}$ , and  $h$  is continuous in  $y$ .

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## Assumptions and Notations

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**The function  $h : \mathbb{R}^d \times U \rightarrow \mathbb{R}$  is measurable and bounded**

$$\left\{ \begin{array}{l} \exists L \geq 0, \forall (y, y') \in \mathbb{R}^{2d}, \forall u \in U, \|g(y, u) - g(y', u)\| \leq L\|y - y'\| \\ \exists a > 0, \forall (y, u) \in \mathbb{R}^d \times U, \|g(y, u)\| \leq a(1 + \|y\|) \end{array} \right.$$

**(HK)  $\exists$  a compact invariant set  $K$  for the control system**

**Average cost induced by  $u$  between 0 and  $t$  by:**

$$\gamma_t(y_0, u) := \frac{1}{t} \int_0^t h(y(s, u, y_0), u(s)) ds, \quad V_t(y_0) = \inf_{u \in \mathcal{U}} \gamma_t(y_0, u).$$

$$\text{for } m \geq 0, \quad \gamma_{m,t}(y_0, u) := \frac{1}{t} \int_m^{m+t} h(y(s, u, y_0), u(s)) ds,$$

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## A Classical Controlability Approach

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Suppose that  $\exists T > 0, \forall (y_1, y_2) \in K, \exists t \leq T, \forall u \in \mathcal{U}, \exists v \in \mathcal{U}, \|y(t, u, y_1) - y(t, v, y_2)\| = 0$ .

Then for any  $t \geq T$  and any  $\Psi \in C(K)$  the maps

$$y_0 \mapsto V_t^\Psi(y_0) := \inf_{u \in \mathcal{U}} \int_{s=0}^t h(y(s, u, y_0)) ds + \Psi(y(t, u, y_0)),$$

are equicontinuous with a modulus of continuity which does not depend on  $t$  and  $\Psi$  (but only on the Lipschitz constants of  $h$  and  $f$ ).



Thus  $V_t^\Psi$  is more regular than  $\Psi$ . This also could be obtained and generalized using HJB results with coercive concave hamiltonians.

• **Example 3:**  $g(y, u) = -y + u$ , where  $u \in U$  a given bounded subset of  $\mathbb{R}^d$ , and  $h$  is continuous in  $y$ .

• **Example 4:** in  $\mathbb{R}^2$ . The initial state is  $y_0 = (0, 0)$  and  $U = [0, 1]$ , and the cost is  $h(y) = 1 - y_1(1 - y_2)$ .

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1 - y_1(s)) \\ u^2(s)(1 - y_1(s)) \end{pmatrix}.$$

One can easily observe that the reachable set  $G(y_0) \subset [0, 1]^2$ .

If  $u = \varepsilon > 0$  constant,  $y_1(t) = 1 - \exp(-\varepsilon t)$  and  $y_2(t) = \varepsilon y_1(t)$ . So we have  $V_t(y_0) \xrightarrow[t \rightarrow \infty]{} 0$ . **Existence of a Uniform Value**

**No ergodicity :**

$$\{y \in [0, 1]^2, \lim_{t \rightarrow \infty} V_t(y) = \lim_{t \rightarrow \infty} V_t(y_0)\} = [0, 1] \times \{0\},$$

**and starting from  $y_0$  it is possible to reach no point in  $(0, 1] \times \{0\}$ .**

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## A first result in Nonexpansive case

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Denote by  $G(y_0) := \{y(t, u, y_0), t \geq 0, u \in \mathcal{U}\}$  the reachable set

**Theorem 3**  $h(y, u) = h(y)$  *only depends on the state,*

$G(y_0)$  *is bounded (invariant),*

$\forall (y_1, y_2) \in G(y_0)^2, \quad \sup_{u \in U} \inf_{v \in U} \langle y_1 - y_2, g(y_1, u) - g(y_2, v) \rangle \leq 0.$

**Then**  $\Gamma(y_0)$  *has a limit value*  $V_t(y_0) \xrightarrow[t \rightarrow +\infty]{} V^*(y_0)$ . **The convergence of**  $(V_t)_t$  *to*  $V^*$  **is uniform over**  $G(y_0)$ , **and the value of**  $\Gamma(y_0)$  *is uniform.*

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## A Crucial Technical Lemma

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We define  $V^-(y_0) := \liminf_{t \rightarrow +\infty} V_t(y_0)$ ,

$V^+(y_0) := \limsup_{t \rightarrow +\infty} V_t(y_0)$ .

**Lemma 4** *For every  $m_0$  in  $\mathbb{R}_+$ , we have:*

$$\sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \geq V^+(y_0) \geq V^-(y_0) \geq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0).$$

**Definition 5**

$$V^*(y_0) = \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0).$$

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## Sketch of the proof of the first result

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**Lemma 6**  $\forall T > 0, \forall \varepsilon > 0, \forall (y_1, y_2) \in G(y_0)^2, \forall u \in \mathcal{U}, \exists v \in \mathcal{U},$

$$\forall t \in [0, T], \|y(t, u, y_1) - y(t, v, y_2)\| \leq \|y_1 - y_2\| + \varepsilon.$$

**Proposition 7**  $\forall \varepsilon > 0, \exists m_0, \sup_{t>0} \inf_{m \leq m_0} V_{m,t}(y_0) \leq \sup_{t>0} \inf_{m \geq 0} V_{m,t}(y_0) + 2\varepsilon$

- $(V_T(y_0))_{T>0}$  is equicontinuous (Lemma 6 + continuity of  $h$ )
- Define  $G^m(y_0) := \{y(t, u, y_0), t \leq m, u \in \mathcal{U}\}$  the reachable set in time  $m$ .

$\forall \varepsilon, \exists m_0, \forall z \in G(y_0), \exists z' \in G^{m_0}(y_0)$  such that  $\|z - z'\| \leq \varepsilon$ .

- We have  $\inf_{m \geq 0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G(y_0)\}$ , and  $\inf_{m \leq m_0} V_{m,t}(y_0) = \inf\{V_t(z), z \in G^{m_0}(y_0)\}$ . By steps 1 and 2  $\inf\{V_t(z), z \in G^{m_0}(y_0)\} \leq \inf\{V_t(z), z \in G(y_0)\} + 2\varepsilon$ .

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## Generalizations

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**Theorem 8**  $\exists C^1 \Delta : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$ , *vanishing on the diagonal* ( $\Delta(y, y) = 0$ ) *and symmetric* ( $\Delta(y_1, y_2) = \Delta(y_2, y_1)$ )  
 $h(y, u) = h(y)$  *only depends on the state,*  
 $G(y_0)$  *is bounded (invariant),*  
 $\forall (y_1, y_2) \in G(y_0)^2, \forall u \in U, \exists v \in U.$

$$\langle g(y_1, u), \frac{\partial}{\partial y_1} \Delta(y_1, y_2) \rangle + \langle g(y_2, v), \frac{\partial}{\partial y_2} \Delta(y_1, y_2) \rangle \leq 0$$

**Then**  $\Gamma(y_0)$  *has a limit value*  $V_t(y_0) \xrightarrow[t \rightarrow +\infty]{} V^*(y_0)$ . **The convergence of**  $(V_t)_t$  *to*  $V^*$  *is uniform over*  $G(y_0)$ , *and the value of*  $\Gamma(y_0)$  *is uniform.*

• **This result can be applied to example 4, with  $\Delta(y_1, y_2) = \|y_1 - y_2\|_1$  ( $L^1$ -norm). In this example, we have for each  $y_1, y_2$  and  $u$ :  $\Delta(y_1 + tg(y_1, u), y_2 + tg(y_2, u)) \leq \Delta(y_1, y_2)$  as soon as  $t \geq 0$  is small enough.**

**Example 4: in  $\mathbb{R}^2$ . The initial state is  $y_0 = (0, 0)$  and  $U = [0, 1]$ , and the cost is  $h(y) = 1 - y_1(1 - y_2)$ .**

$$y'(s) = g(y(s), u(s)) = \begin{pmatrix} u(s)(1 - y_1(s)) \\ u^2(s)(1 - y_1(s)) \end{pmatrix}.$$



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## Further Generalizations

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**Theorem 9 (H1)**  *$h$  is uniformly continuous in  $y$  on  $\bar{Z}$  uniformly in  $u$ . And for each  $y$  in  $\bar{Z}$ , either  $h$  does not depend on  $u$  or the set  $\{(g(y, u), h(y, u)) \in \mathbb{R}^d \times [0, 1], u \in U\}$  is closed.*

**(H2):**  $\exists \Delta : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$ , *vanishing on the diagonal ( $\Delta(y, y) = 0$ ) and symmetric ( $\Delta(y_1, y_2) = \Delta(y_2, y_1)$ ), and a uniformly continuous function  $\hat{\alpha} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$  s.t.  $\hat{\alpha}(t) \xrightarrow{t \rightarrow 0} 0$  satisfying:*

a)  $\forall$  **sequence**  $(z_n)_n \subset Z$ ,  $\forall \varepsilon > 0$ ,  $\exists n$ ,  $\liminf_p \Delta(z_n, z_p) \leq \varepsilon$ .

b)  $\forall (y_1, y_2) \in \bar{Z}^2$ ,  $\forall u \in U$ ,  $\exists v \in U$  **such that**

$D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0$ ,  $h(y_2, v) - h(y_1, u) \leq \hat{\alpha}(\Delta(y_1, y_2))$ .

**Then  $\Gamma(y_0)$  has a uniform value  $\lim_{t \rightarrow \infty} V_t = V^*$ .**

## Remarks

- Although  $\Delta$  may not satisfy the triangular inequality nor the separation property, it may be seen as a “distance” adapted to the problem  $\Gamma(y_0)$ .
- $D \uparrow$  is the contingent epi-derivative (which reduces to the upper Dini derivative if  $\Delta$  is Lipschitz)  $D\uparrow\Delta(z)(\alpha) = \liminf_{t \rightarrow 0^+, \alpha' \rightarrow \alpha} \frac{1}{t}(\Delta(z + t\alpha') - \Delta(z))$ . If  $\Delta$  is differentiable, the condition  $D \uparrow \Delta(y_1, y_2)(g(y_1, u), g(y_2, v)) \leq 0$  just reads:  
 $\langle g(y_1, u), \frac{\partial}{\partial y_1} \Delta(y_1, y_2) \rangle + \langle g(y_2, v), \frac{\partial}{\partial y_2} \Delta(y_1, y_2) \rangle \leq 0$ .

• The assumption: “ $\{(g(y, u), h(y, u)) \in \mathbb{R}^d \times [0, 1], u \in U\}$  closed” could be checked for instance if  $U$  is compact and if  $h$  and  $g$  are continuous with respect to  $(y, u)$ .

•  $H2a)$  is a precompactness condition. It is satisfied as soon as  $G(y_0)$  is bounded. **cf Renault 2008**

• Notice that  $H2$  is satisfied with  $\Delta = 0$  if we are in the trivial case where  $\inf_u h(y, u)$  is constant.

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## On Uniform Value

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**Definition 10**  $\Gamma(y_0)$  *has a uniform value if*  $\exists V(y_0)$  *and if:*

$$\forall \varepsilon > 0, \exists u \in \mathcal{U}, \exists t_0, \forall t \geq t_0, \frac{1}{t} \int_{s=0}^t h(y(s, u, y_0), u(s)) ds \leq V(y_0) + \varepsilon.$$

• **Example 5:** in  $\mathbb{R}^2$ ,  $y_0 = (0, 0)$ , control set  $U = [0, 1]$ ,  $y'(t) = (y_2(t), u(t))$ , and  $h(y_1, y_2) = 0$  if  $y_1 \in [1, 2]$ , = 1 otherwise.

We have  $u(s) = y_2'(s) = y_1''(s)$ ,

**Interpretation:**  $u$  "acceleration",  $y_2$  "speed",  $y_1$  the "position".

If  $u = \varepsilon$  constant, then  $y_2(t) = \sqrt{2\varepsilon y_1(t)} \quad \forall t \geq 0$ .

**Limit Value:**  $V_T(y_0) \xrightarrow{T \rightarrow \infty} 1/2$

**No Uniform Value.**

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## Optimal control with discounted factor $\lambda \rightarrow 0^+$

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We define  $\Theta_\lambda(y_0) := \inf_{u \in \mathcal{U}} \int_{s=0}^{+\infty} \lambda e^{-\lambda s} h(y(s, u, y_0), u(s)) ds,$

**Theorem 11 (Oliu-Barton Vigerol 2010)** *the following uniform limit in  $K$  exists*  $\lim_{\lambda \rightarrow 0^+} \Theta_\lambda(y_0)$

*iff*

*the following uniform limit in  $K$  exists*  $\lim_{t \rightarrow \infty} V_t(y_0)$

Question Application to different concepts of means

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## Open Problems

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Differential Game at horizon  $t$ :

$$V_t(y_0) := \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} \frac{1}{t} \int_{s=0}^t h(y(s, u, v, y_0), u(s), v(s)) ds,$$

where  $s \mapsto y(s, u, y_0)$  denotes the solution to

$$y'(s) = g(y(s), u(s), v(s)), \quad y(0) = y_0.$$

**OPEN PROBLEM** : Existence of a limit of  $V_t(y_0)$  as  $t \rightarrow \infty$ .

Only Partial results:

- When the Hamiltonian is coercive (hence ergodicity and the limit is  $y$  independent) **Alvarez-Bardi ...**
- For nonconvex and non coercive Hamiltonian in  $\mathbb{R}^2$  **Cardaliagu**

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## Averaging Problem for singularly perturbed system

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$$\begin{cases} i) & x'(s) = f(x(s), y(s), u(s)), \quad x(0) = x, \quad s \in [0, T] \\ ii) & \varepsilon y'(s) = g(x(s), y(s), u(s)) \quad y(0) = y, \end{cases} \quad (1)$$

**Change of variable**  $\tau = \frac{t}{\varepsilon}$ ,  $(X(\tau), Y(\tau), U(\tau)) = (x(\varepsilon\tau), y(\varepsilon\tau), u(\varepsilon\tau))$

$$\begin{cases} X'(\tau) = \varepsilon f(X(\tau), Y(\tau), U(\tau)), \quad X(0) = x, \quad \tau \in [0, \frac{T}{\varepsilon}] \\ Y'(\tau) = g(X(\tau), Y(\tau), U(\tau)), \quad Y(0) = y. \end{cases} \quad (2)$$

**Take  $\varepsilon = 0$  in (2). We have the following associated system:**

$$y'(\tau) = g(x, y(\tau), u(\tau)), \quad y(0) = y, \quad (3)$$

$y_x(\cdot, u, y)$  denotes the unique solution of (3).

## Averaging method

We suppose that  $f$  and  $g$  are Lipschitz and there is a compact set  $M \times N$  which is invariant by (1) for all  $\varepsilon$ .

$$A(x, y, S, u) = \frac{1}{S} \int_0^S f(x, y_x(\tau, u, y), u(\tau)) d\tau,$$

$$F(x, y, S) = \{A(x, y, S, u); u \in \mathcal{U}\}$$

**Theorem 12 *Gaitsgory, Grammel*** *If  $\exists \gamma : \mathbb{R} \rightarrow \mathbb{R}_+$  with  $\lim_{S \rightarrow +\infty} \gamma(S) = 0$  and a Lipschitz set-valued map  $\bar{F} : M \rightarrow \mathbb{R}^b$  with compact convex nonempty values such that*

$$d(\text{co cl} F(x, y, S), \bar{F}(x)) \leq \gamma(S), \quad \forall (x, y) \in M \times N, \quad \forall S > 0,$$

*then  $\forall x, y$  the solutions of the differential inclusion*

$$x'(s) \in \bar{F}(x(s)), \quad x(0) = x. \quad (4)$$

*approximate the solutions of the singularly perturbed system (1) in the following sense:*



For any  $\varepsilon > 0$ , and any  $T > 0$  there exists  $M(T, \varepsilon) > 0$  with  $\lim_{\varepsilon \rightarrow 0} M(T, \varepsilon) = 0$  such that

a) For any family of solutions  $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$  to (1) there exists a solution  $x(\cdot)$  to (4) such that

$$\sup_{t \in [0, T]} \|x_\varepsilon(t) - x(t)\| \leq M(T, \varepsilon).$$

b) Conversely fix  $x(\cdot)$  a solution to (4) then for any  $\varepsilon$  small enough there exists a solution  $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$  to (1) such that

$$\sup_{t \in [0, T]} \|x_\varepsilon(t) - x(t)\| \leq M(T, \varepsilon).$$

cf also **Wattbled, M.Q Wattbled ...**

QUESTION ases and conditions where  $\overline{F}$  may depend on  $y$ .

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Thank You for your Attention

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