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PDE methods for multiscale control and differential games

Martino Bardi

Department of Pure and Applied Mathematics
University of Padua, Italy

Kickoff Meeting I.T.N. SADCO
ENSTA, Paris, March 3rd, 2011

Plan

- Two-scale systems: examples and motivations
- The Hamilton-Jacobi approach to Singular Perturbations
- Representation of the limit control problem:
 - Uncontrolled fast variables
 - Homogenization in low dimensions

Two-scale systems

Dynamical systems with two groups of variables evolving on different time-scales (x_s, y_τ) , $\tau = s/\varepsilon$, $0 < \varepsilon \ll 1$, governed by ODEs

$$\begin{aligned}\dot{x}_s &= f(x_s, y_s) & x_s &\in \mathbf{R}^n, \\ \dot{y}_s &= \frac{1}{\varepsilon} g(x_s, y_s) & y_s &\in \mathbf{R}^m,\end{aligned}$$

or by Stochastic DEs

$$\begin{aligned}dx_s &= f(x_s, y_s) ds + \sigma(x_s, y_s) dW_s \\ dy_s &= \frac{1}{\varepsilon} g(x_s, y_s) ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s) dW_s\end{aligned}$$

Hope to simplify the model in the limit $\varepsilon \rightarrow 0$:

a **Singular Perturbation** problem.

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Hope to simplify the model in the limit $\varepsilon \rightarrow 0$:

a **Singular Perturbation** problem.

The theory is classical for Ordinary Differential Equations, see Levinson and Tychonov 1952, O'Malley's book 1974, and has a large literature also for systems **with controls**,

$$\begin{aligned}\dot{x}_s &= f(x_s, y_s, \alpha_s) \\ \dot{y}_s &= \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s),\end{aligned}$$

α_s a measurable control function taking values in a given set A , see Kokotović - Khalil - O'Reilly book 1986 (deterministic case) Bensoussan 1988, Kushner 1990 also stochastic case:

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There are many different models in Physics, Engineering, Finance,....

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Example 1: Mechanical system with Large Damping

The **large time behavior** of

$$\ddot{X} = F(X, t) - \frac{\dot{X}}{\varepsilon}$$

is described by $x(s) := X(s/\varepsilon)$ that solves

$$\varepsilon^2 \ddot{x} = F(x, s/\varepsilon) - \dot{x}.$$

In the autonomous case ($F = F(x)$) this writes

$$\begin{aligned}\dot{x}_s &= y_s \\ \dot{y}_s &= \frac{F(x_s) - y_s}{\varepsilon^2},\end{aligned}$$

The limit is the **Quasi-Static approximation**

$$\dot{x} = F(x).$$

F. Hoppensteadt, "Quasi-Static state analysis..." Courant L. N. 2010

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Example 2: Control systems with stable fast variables

In Example 1 the formal limit

$$\dot{x}_s = f(x_s, y_s), \quad 0 = g(x_s, y_s)$$

is correct: it fits in the **Reduced Order Method**. For control system the ROM gives in the limit the DIFFERENTIAL-ALGEBRAIC system

$$\dot{x}_s = f(x_s, y_s, \alpha_s), \quad 0 = g(x_s, y_s, \alpha_s),$$

provided that (roughly speaking) the "fast subsystem" (with frozen x)

$$\dot{y}_\tau = g(x, y_\tau, \alpha_\tau)$$

has an **equilibrium** regime with an **asymptotically stabilizing feedback**, see Kokotović - Khalil - O'Reilly book 1986.

Examples in Engineering: high gain feedback, cheap control,....

BUT, many other models do NOT have this stability property!

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Example 3: Control systems in oscillating media

The **homogenization** problem

$$\dot{x}_s = f\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s\right), \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right)$$

by setting $y_s = x_s/\varepsilon$ can be written as

$$\dot{x}_s = f(x_s, y_s, \alpha_s)$$

$$\dot{y}_s = \frac{1}{\varepsilon} f(x_s, y_s, \alpha_s)$$

$$J = \int_0^t l(x_s, y_s, \alpha_s) ds + h(x_t, y_t)$$

that is a Singular Perturbation problem with $g = f$.

Example 4: Financial models

The evolution of a stock S with **stochastic volatility** σ is

$$d \log S_s = \gamma ds + \sigma(y_s) dW_s$$

$$dy_s = \frac{1}{\varepsilon}(m - y_s) + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

see Fouque - Papanicolaou - Sircar, book 2000, for empirical data and many examples.

Merton portfolio optimization problem with stochastic volatility: invest β_s in the stock S_s , $1 - \beta_s$ in a bond with interest rate r . Then the wealth x_s evolves as

$$dx_s = (r + (\gamma - r)\beta_s)x_s ds + x_s\beta_s\sigma(y_s) dW_s$$

$$dy_s = \frac{1}{\varepsilon}(m - y_s) ds + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s$$

Problem: maximize the expected utility at time t , $E[h(x_t)]$, h increasing concave function.

Example 5: Control systems on thin structures

State constraint on the z variables

$$\begin{aligned}\dot{x}_s &= f(x_s, z_s, \alpha_s) \\ \dot{z}_s &= g(x_s, z_s, \alpha_s) \quad |z_s| \leq \varepsilon,\end{aligned}$$

by setting $y_s = z_s/\varepsilon$ becomes

$$\begin{aligned}\dot{x}_s &= f(x_s, \varepsilon y_s, \alpha_s) \\ \dot{y}_s &= \frac{1}{\varepsilon} g(x_s, \varepsilon y_s, \alpha_s) \quad |y_s| \leq 1.\end{aligned}$$

This can be used to justify models of optimal control on **graphs** or **networks**.

The H-J approach to Singular Perturbations

General control system with TWO controllers

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s) ds + \sigma(x_s, y_s, \alpha_s, \beta_s) dW_s, \quad x_s \in \mathbf{R}^n, \alpha_s \in A, \beta_s \in B,$$

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$$x_0 = x, \quad y_0 = y$$

Cost-payoff functional (α minimizes, β_s maximizes)

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s) ds + h(x_t, y_t)$$

Lower value function, where $\Gamma(t)$ are the nonanticipating strategies, $\mathcal{B}(t)$ the open-loop controls

$$u^\varepsilon(t, x, y) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E [J^\varepsilon(t, x, y, \alpha[\beta], \beta)]$$

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H-J-Bellman-Isaacs equation for the SP problem

Dynamic Programming method:

in the **deterministic** case $\sigma, \nu \equiv 0$, the value function solves

$$(CP_\varepsilon) \quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t} + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 & \text{in } (0, +\infty) \times \mathbf{R}^n \times \mathbf{R}^m, \\ u^\varepsilon(0, x, y) = h(x, y) & \text{in } \mathbf{R}^n \times \mathbf{R}^m, \end{cases}$$

$$H(x, y, p, q) = \min_{\beta \in B} \max_{\alpha \in A} \{-p \cdot f(x, y, \alpha, \beta) - q \cdot g(x, y, \alpha, \beta) - l(x, y, \alpha, \beta)\}$$

in the viscosity sense.

In the **stochastic** case $\sigma \neq 0$ or $\nu \neq 0$ the H-J-B-I equation is

$$\frac{\partial u^\varepsilon}{\partial t} + \min_{\beta \in B} \max_{\alpha \in A} [\mathcal{L}_{\alpha, \beta}^\varepsilon u^\varepsilon - l(x, y, \alpha, \beta)] = 0 \quad \text{in } (0, +\infty) \times \mathbf{R}^n \times \mathbf{R}^m,$$

$\mathcal{L}_{\alpha, \beta}^\varepsilon$ = infinitesimal generator of the process with constant controls α, β

it is of 2nd order involving also $D_{xx}^2, D_{yy}^2/\varepsilon, D_{xy}^2/\sqrt{\varepsilon}$.

This H-J-B-I PDE is (degenerate) parabolic.

Again, u^ε is the **unique viscosity solution** of the Cauchy problem.

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Plan of the method:

- 1 pass to the limit as $\varepsilon \rightarrow 0$ in the PDE
 - 2 associate to the limit PDE a "limit control problem"
- 1 was developed in the papers

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O. A. - M. B. - C. MARCHI : J. D. E. '07, '08 (more than 2 scales)
under boundedness assumptions on the fast state variables,
and for unbounded but uncontrolled fast variables in

M. B. - A. CESARONI - L. MANCA : SIAM J. Financial Math. 2010

M. B. - A. CESARONI : European J. Control 2011

Methods are related to **HOMOGENIZATION** of H- J equations:

P.L.Lions- Papanicolaou- Varadhan 1986, L.C.Evans 1989, ...

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- 1 Search *effective Hamiltonian* \bar{H} and *effective initial data* \bar{h} s. t.

$$u^\varepsilon(t, x, y) \rightarrow u(t, x) \quad \text{as } \varepsilon \rightarrow 0,$$

u solution of

$$(\overline{\text{CP}}) \quad \begin{cases} \frac{\partial u}{\partial t} + \bar{H}(x, D_x u, D_{xx}^2 u) = 0 & \text{in } (0, +\infty) \times \mathbf{R}^n, \\ u(0, x) = \bar{h}(x) & \text{in } \mathbf{R}^n \end{cases}$$

- 2 Interpret the effective Hamiltonian \bar{H} as the Bellman-Isaacs Hamiltonian for a new *effective system*

$$\dot{x}_s = \bar{f}(x_s, \eta_s, \theta_s) \quad x_s \in \mathbf{R}^n, \eta_s \in E(x_s), \theta_s \in \Theta(x_s)$$

and *effective* cost functional

$$\bar{J}(t, x, \eta, \theta) := \int_0^t \bar{l}(x_s, \eta_s, \theta_s) ds + \bar{h}(x_t).$$

This is a *variational limit* of the initial $n + m$ -dimensional problem.
Step 2 is largely OPEN !

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Definition of \overline{H} : deterministic case $\sigma \equiv 0, \nu \equiv 0$

Consider the fast subsystem with frozen x and $\varepsilon = 1$

$$(FS) \quad \dot{y}_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau), \quad y_0 = y$$

and the family of value functions in \mathbf{R}^m with parameters $x, p \in \mathbf{R}^n$

$$w(t, y; x, p) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t L(y_\tau, \alpha[\beta]_\tau, \beta_\tau; x, p) d\tau,$$

$$L(y, \alpha, \beta; x, p) := p \cdot f(x, y, \alpha, \beta) + l(x, y, \alpha, \beta)$$

Say (FS) is **ERGODIC** if, for all x, p ,

$$\lim_{t \rightarrow +\infty} \frac{w(t, y; x, p)}{t} = \text{constant (in } y), \text{ uniformly in } y$$

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- Example: (FS) uncontrolled, i.e.

$$\dot{y}_\tau = g(x, y_\tau),$$

and **ergodic** in the classical sense with a **UNIQUE INVARIANT MEASURE** μ_x . Then

$$-\bar{H}(x, p) = \int_{\mathbf{R}^m} \min_{\alpha \in A} L(y, \alpha; x, p) d\mu_x(y).$$

- NO explicit formula for \bar{H} in general!
- Definition of \bar{H} in general non-deterministic case: fast subsystem

$$dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau) d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau) dW_\tau, \quad y_0 = y,$$

$L = \text{trace} (M\sigma\sigma^T(x, y, \alpha, \beta)) / 2 + \dots$ (M a $n \times n$ symmetric matrix)

$$w(t, y; x, p, M) = \inf_{\alpha} \sup_{\beta} E \left[\int_0^t L d\tau \right], \quad \lim_{t \rightarrow +\infty} w/t =: -\bar{H}(x, p, M)$$

- Example: (FS) uncontrolled, i.e.

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and **ergodic** in the classical sense with a **UNIQUE INVARIANT MEASURE** μ_x . Then

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- NO explicit formula for \bar{H} in general!
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Weak convergence theorem

Meta-Theorem

Fast subsystem (FS) ergodic \implies

$$u^\varepsilon(t, x, y) \rightarrow u(t, x) \quad \text{as } \varepsilon \rightarrow 0,$$

(in the sense of relaxed semilimits, or weak viscosity limits),
and u solves (in viscosity sense)

$$\frac{\partial u}{\partial t} + \bar{H}(x, D_x u, D_{xx}^2 u) = 0$$

This is in fact a Theorem if the fast variables y live on the torus \mathbb{T}^M (i.e., all data are \mathbb{Z}^m -periodic in y), or in all \mathbf{R}^M but the process

$$dy_\tau = g(x, y_\tau) d\tau + \nu(x, y_\tau) dW_\tau$$

is an UN-controlled NON-degenerate diffusion.

The effective initial data \bar{h}

For the Fast Subsystem with frozen x

$$(FS) \quad dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau) d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau) dW_\tau, \quad y_0 = y,$$

consider now the value function

$$v(t, y; x) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in B(t)} E[h(x, y_t)]$$

Definition: (FS) is **STABILIZING** (to a constant) for the cost h if, $\forall x$,

$$\lim_{t \rightarrow +\infty} v(t, y; x) = \text{constant (in } y), \text{ uniformly in } y =: \bar{h}(x)$$

Convergence Theorem at $t = 0$

Fast subsystem (FS) stabilizing for the cost $h \implies$

$$\lim_{t \rightarrow 0} \lim_{\varepsilon \rightarrow 0} u^\varepsilon(t, x, y) = \bar{h}(x)$$

(in the sense of relaxed semilimits, or weak viscosity limits).

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Main convergence theorem

Corollary

Fast subsystem (FS) ERGODIC and STABILIZING \implies

$\exists \bar{H}$ and \bar{h} continuous, \bar{H} degenerate elliptic, such that

$u^\varepsilon(t, x, y) \rightarrow u(t, x)$ as $\varepsilon \rightarrow 0$, u solution of

$$(\overline{\text{CP}}) \quad \begin{cases} \frac{\partial u}{\partial t} + \bar{H}(x, D_x u, D_{xx}^2 u) = 0 & \text{in } (0, +\infty) \times \mathbf{R}^n, \\ u(0, x) = \bar{h}(x) & \text{in } \mathbf{R}^n. \end{cases}$$

If moreover, \bar{H} is regular enough w.r.t. x , then $(\overline{\text{CP}})$ has a **unique** solution and

$$u^\varepsilon \rightarrow u \quad \text{locally uniformly.}$$

Conclusion

The initial $(n + m)$ -dimensional H-J-B-I equation is split into

- two m -dimensional ergodic-type problems (one for \bar{H} and one for \bar{h}),
- a n -dimensional "effective" PDE

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The next steps

Main remaining questions:

- 1 when is (FS) ergodic and stabilizing ?
- 2 can we find effective dynamics $\bar{f}, \bar{\sigma}$, running cost \bar{l} , and control constraints E, Θ associated to \bar{H} ?

Some answers:

- 1 for bounded fast variables (related to homogenization):
 - uniformly nondegenerate fast subsystem (FS),
 - deterministic fast subsystem (FS) controllable by one player from each point to any other point in a uniformly bounded time,
 - under nonresonance conditions on the torus;

for unbounded fast variables: uncontrolled diffusion processes with a unique invariant measure;

- 2 several examples but no general recipe.

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The case of uncontrolled fast variables

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s) ds + \sigma(x_s, y_s) dW_s$$

$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s) ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s) dW_s,$$

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s) ds + h(x_t, y_t)$$

Assume there exists a unique **invariant measure** μ_x of

$$(FS) \quad dy_\tau = g(x, y_\tau) d\tau + \nu(x, y_\tau) dW_\tau,$$

Denote $\langle \phi \rangle(x) := \int \phi(x, y) d\mu_x(y)$. Then effective H and h are

$$\bar{h}(x) = \langle h \rangle(x)$$

$$\bar{H}(x, p, M) = \left\langle \min_{\beta \in B} \max_{\alpha \in A} \left\{ -\text{trace}(M\sigma\sigma^T)/2 - f \cdot p - l \right\} \right\rangle$$

Corollary

For split systems and cost, i.e.,

$$f = f_0(x, y) + f_1(x, \alpha, \beta), \quad l = l_0(x, y) + l_1(x, \alpha, \beta),$$

the linear averaging of the data is the correct limit, i.e.,

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon(t, x, y) = u(t, x) :=$$

$$\inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E \left[\int_0^t \langle l \rangle(x_s, \alpha[\beta]_s, \beta_s) ds + \langle h \rangle(x_t) \right],$$

$$dx_s = \langle f \rangle(x_s, \alpha[\beta]_s, \beta_s) ds + \langle \sigma \sigma^T \rangle^{1/2}(x_s) dW_s$$

Proof: $\bar{H}(x, p, M) = -\text{trace}(M \langle \sigma \sigma^T \rangle) / 2 + \min_B \max_A \{ -\langle f \rangle \cdot p - \langle l \rangle \}$.

A similar result was proved by Kushner (book, 1990) for a single controller by probabilistic methods.

In general, for system or cost **NOT split**,

$$\overline{H}(x, p, M) = \langle \min_B \max_A \{ \dots \} \rangle \neq \min_B \max_A \langle \{ \dots \} \rangle$$

and the limit control problem is not obvious.

For some classical problems we derived the explicit form of the effective control problem:

- Merton portfolio optimization with stochastic volatility,
- Ramsey model of optimal economic growth with (fast) random parameters,
- Vidale - Wolfe advertising model with random parameters,
- advertising game in a duopoly with Lanchester dynamics and random parameters.

Often they involve a **nonlinear average** of some parameter.

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We can give a general **representation formula** for the effective control problem by **enlarging the control set**.

Assume for simplicity $\sigma \equiv 0$ (deterministic slow subsystem) and a single controller (B singleton).

Define $E_x := L^1((\mathbf{R}^m, \mu_x); A) \supset A$ and for $\tilde{\alpha} \in E_x$

$$\bar{f}(x, \tilde{\alpha}) := \int_{\mathbf{R}^m} f(x, y, \tilde{\alpha}(y)) d\mu_x(y), \quad \bar{l}(x, \tilde{\alpha}) := \int_{\mathbf{R}^m} l(x, y, \tilde{\alpha}(y)) d\mu_x(y).$$

Then

$$\bar{H}(x, p) = \left\langle \max_{\alpha \in A} \{-f \cdot p - l\} \right\rangle = \max_{\tilde{\alpha} \in E_x} \left\{ -\bar{f}(x, \tilde{\alpha}) \cdot p - \bar{l}(x, \tilde{\alpha}) \right\}$$

and the effective control problem is

$$\begin{aligned} \dot{x}_s &= \bar{f}(x_s, \tilde{\alpha}_s), \quad x_0 = x, \quad \tilde{\alpha}_s \in E_{x_s} \\ \min \quad \bar{J}(t, x, \tilde{\alpha}.) &= \int_0^t \bar{l}(x_s, \tilde{\alpha}_s) ds + \langle h \rangle(x_t). \end{aligned}$$

A one-dimensional homogenization problem

$$\dot{x}_s = g(x_s) \alpha_s, \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right), \quad -1 \leq \alpha_s \leq 1, \quad g > 0.$$

Here $\dot{y}_s = \frac{1}{\varepsilon} g(x_s) \alpha_s$ depends on the control.

The value function $u^\varepsilon(t, x)$ solves the H-J equation

$$\frac{\partial u^\varepsilon}{\partial t} + g(x) \left| \frac{\partial u^\varepsilon}{\partial x} \right| = l\left(x, \frac{x}{\varepsilon}\right), \quad u^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right).$$

For $l(x, \cdot), h(x, \cdot)$ 1-periodic and $\min l(x, \cdot) = 0$ the limit PDE is

$$\frac{\partial u}{\partial t} + \left(g(x) \left| \frac{\partial u}{\partial x} \right| - \langle l \rangle(x) \right)^+ = 0, \quad u(0, x) = \min_{y \in [0,1]} h(x, y).$$

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Homogenization of a 2-D differential game

$$\dot{x}_s = g_1(x_s, z_s)\alpha_s, \quad -1 \leq \alpha_s \leq 1, \quad g_1 > 0$$

$$\dot{z}_s = g_2(x_s, z_s)\beta_s, \quad -1 \leq \beta_s \leq 1, \quad g_2 > 0$$

$$J^\varepsilon(t, x, z, \alpha, \beta) := \int_0^t [l_1(x_s, z_s, \frac{x_s}{\varepsilon}) + l_2(x_s, z_s, \frac{z_s}{\varepsilon})] ds + h(x_t, z_t, \frac{x_s}{\varepsilon}, \frac{z_s}{\varepsilon})$$

$$\min_{[0,1]} l_1(x, z, \cdot) = 0, \quad \max_{[0,1]} l_2(x, z, \cdot) = 0$$

and assume h has a saddle

$$\min_{\xi \in [0,1]} \max_{\eta \in [0,1]} h(x, z, \xi, \eta) = \max_{\eta \in [0,1]} \min_{\xi \in [0,1]} h(x, z, \xi, \eta) =: h_S(x, z)$$

The limit differential game has the same system and controls, and the effective cost-payoff

$$\bar{J} = \int_0^t [|\alpha_s| \langle l_1 \rangle(x_s, z_s) + |\beta_s| \langle l_2 \rangle(x_s, z_s)] ds + h_S(x_t, z_t)$$

joint work with G.Terrone.

Work in progress on [homogenization](#) of control systems and games:

- the last two examples extend to n -dimensional systems provided the oscillations are at 1-dimensional scale;
- an "abstract" representation of some effective control problems can be obtained by the [limit occupational measures](#) studied by Artstein, Gaitsgory, Borkar, ..., (joint work with Gabriele TERRONE);
- homogenization of [deterministic differential games](#) is wide open: there are easy examples of non-convergence of the value functions and only a few known cases of convergence (see also Cardaliaguet '09)

Thanks for your attention !