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# PDE methods for multiscale control and differential games

Martino Bardi

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Kickoff Meeting I.T.N. SADCO  
ENSTA, Paris, March 3rd, 2011

# Plan

- Two-scale systems: examples and motivations
- The Hamilton-Jacobi approach to Singular Perturbations
- Representation of the limit control problem:
  - Uncontrolled fast variables
  - Homogenization in low dimensions

# Two-scale systems

Dynamical systems with two groups of variables evolving on different time-scales  $(x_s, y_\tau)$ ,  $\tau = s/\varepsilon$ ,  $0 < \varepsilon \ll 1$ , governed by ODEs

$$\begin{aligned}\dot{x}_s &= f(x_s, y_s) & x_s &\in \mathbf{R}^n, \\ \dot{y}_s &= \frac{1}{\varepsilon} g(x_s, y_s) & y_s &\in \mathbf{R}^m,\end{aligned}$$

or by Stochastic DEs

$$\begin{aligned}dx_s &= f(x_s, y_s) ds + \sigma(x_s, y_s) dW_s \\ dy_s &= \frac{1}{\varepsilon} g(x_s, y_s) ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s) dW_s\end{aligned}$$

Hope to simplify the model in the limit  $\varepsilon \rightarrow 0$ :

a **Singular Perturbation** problem.

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Hope to simplify the model in the limit  $\varepsilon \rightarrow 0$ :

a **Singular Perturbation** problem.

The theory is classical for Ordinary Differential Equations, see Levinson and Tychonov 1952, O'Malley's book 1974, and has a large literature also for systems **with controls**,

$$\begin{aligned}\dot{x}_s &= f(x_s, y_s, \alpha_s) \\ \dot{y}_s &= \frac{1}{\varepsilon} g(x_s, y_s, \alpha_s),\end{aligned}$$

$\alpha_s$  a measurable control function taking values in a given set  $A$ , see Kokotović - Khalil - O'Reilly book 1986 (deterministic case) Bensoussan 1988, Kushner 1990 also stochastic case:

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Main motivation: **reducing the dimension** of the state space.

There are many different models in Physics, Engineering, Finance,....

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# Example 1: Mechanical system with Large Damping

The **large time behavior** of

$$\ddot{X} = F(X, t) - \frac{\dot{X}}{\varepsilon}$$

is described by  $x(s) := X(s/\varepsilon)$  that solves

$$\varepsilon^2 \ddot{x} = F(x, s/\varepsilon) - \dot{x}.$$

In the autonomous case (  $F = F(x)$  ) this writes

$$\dot{x}_s = y_s$$

$$\dot{y}_s = \frac{F(x_s) - y_s}{\varepsilon^2},$$

The limit is the **Quasi-Static approximation**

$$\dot{x} = F(x).$$

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## Example 2: Control systems with stable fast variables

In Example 1 the formal limit

$$\dot{x}_s = f(x_s, y_s), \quad 0 = g(x_s, y_s)$$

is correct: it fits in the **Reduced Order Method**. For control system the ROM gives in the limit the DIFFERENTIAL-ALGEBRAIC system

$$\dot{x}_s = f(x_s, y_s, \alpha_s), \quad 0 = g(x_s, y_s, \alpha_s),$$

provided that (roughly speaking) the "fast subsystem" (with frozen  $x$ )

$$\dot{y}_\tau = g(x, y_\tau, \alpha_\tau)$$

has an **equilibrium** regime with an **asymptotically stabilizing feedback**, see Kokotović - Khalil - O'Reilly book 1986.

Examples in Engineering: high gain feedback, cheap control,....

BUT, many other models do NOT have this stability property!

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**BUT**, many other models do **NOT have this stability** property!

## Example 3: Control systems in oscillating media

The **homogenization** problem

$$\dot{x}_s = f\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s\right), \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}, \alpha_s\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right)$$

by setting  $y_s = x_s/\varepsilon$  can be written as

$$\dot{x}_s = f(x_s, y_s, \alpha_s)$$

$$\dot{y}_s = \frac{1}{\varepsilon} f(x_s, y_s, \alpha_s)$$

$$J = \int_0^t l(x_s, y_s, \alpha_s) ds + h(x_t, y_t)$$

that is a Singular Perturbation problem with  $g = f$ .

## Example 4: Financial models

The evolution of a stock  $S$  with **stochastic volatility**  $\sigma$  is

$$\begin{aligned}d \log S_s &= \gamma ds + \sigma(y_s) dW_s \\ dy_s &= \frac{1}{\varepsilon}(m - y_s) + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s\end{aligned}$$

see Fouque - Papanicolaou - Sircar, book 2000, for empirical data and many examples.

**Merton portfolio optimization** problem with stochastic volatility: invest  $\beta_s$  in the stock  $S_s$ ,  $1 - \beta_s$  in a bond with interest rate  $r$ . Then the wealth  $x_s$  evolves as

$$\begin{aligned}dx_s &= (r + (\gamma - r)\beta_s)x_s ds + x_s\beta_s\sigma(y_s) dW_s \\ dy_s &= \frac{1}{\varepsilon}(m - y_s) ds + \frac{\nu}{\sqrt{\varepsilon}} d\tilde{W}_s\end{aligned}$$

Problem: maximize the expected utility at time  $t$ ,  $E[h(x_t)]$ ,  $h$  increasing concave function.

## Example 5: Control systems on thin structures

State constraint on the  $z$  variables

$$\begin{aligned}\dot{x}_s &= f(x_s, z_s, \alpha_s) \\ \dot{z}_s &= g(x_s, z_s, \alpha_s) \quad |z_s| \leq \varepsilon,\end{aligned}$$

by setting  $y_s = z_s/\varepsilon$  becomes

$$\begin{aligned}\dot{x}_s &= f(x_s, \varepsilon y_s, \alpha_s) \\ \dot{y}_s &= \frac{1}{\varepsilon} g(x_s, \varepsilon y_s, \alpha_s) \quad |y_s| \leq 1.\end{aligned}$$

This can be used to justify models of optimal control on **graphs** or **networks**.

# The H-J approach to Singular Perturbations

General control system with TWO controllers

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s) ds + \sigma(x_s, y_s, \alpha_s, \beta_s) dW_s, \quad x_s \in \mathbf{R}^n, \alpha_s \in A, \beta_s \in B,$$

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$$x_0 = x, \quad y_0 = y$$

Cost-payoff functional ( $\alpha$  minimizes,  $\beta_s$  maximizes)

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s) ds + h(x_t, y_t)$$

Lower value function, where  $\Gamma(t)$  are the nonanticipating strategies,  $\mathcal{B}(t)$  the open-loop controls

$$u^\varepsilon(t, x, y) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E [J^\varepsilon(t, x, y, \alpha[\beta], \beta)]$$

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# H-J-Bellman-Isaacs equation for the SP problem

Dynamic Programming method:

in the **deterministic** case  $\sigma, \nu \equiv 0$ , the value function solves

$$(CP_\varepsilon) \quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t} + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 & \text{in } (0, +\infty) \times \mathbf{R}^n \times \mathbf{R}^m, \\ u^\varepsilon(0, x, y) = h(x, y) & \text{in } \mathbf{R}^n \times \mathbf{R}^m, \end{cases}$$

$$H(x, y, p, q) = \min_{\beta \in B} \max_{\alpha \in A} \{-p \cdot f(x, y, \alpha, \beta) - q \cdot g(x, y, \alpha, \beta) - l(x, y, \alpha, \beta)\}$$

in the viscosity sense.

In the **stochastic** case  $\sigma \neq 0$  or  $\nu \neq 0$  the H-J-B-I equation is

$$\frac{\partial u^\varepsilon}{\partial t} + \min_{\beta \in B} \max_{\alpha \in A} [\mathcal{L}_{\alpha, \beta}^\varepsilon u^\varepsilon - l(x, y, \alpha, \beta)] = 0 \quad \text{in } (0, +\infty) \times \mathbf{R}^n \times \mathbf{R}^m,$$

$\mathcal{L}_{\alpha, \beta}^\varepsilon$  = infinitesimal generator of the process with constant controls  $\alpha, \beta$

it is of 2nd order involving also  $D_{xx}^2, D_{yy}^2/\varepsilon, D_{xy}^2/\sqrt{\varepsilon}$ .

This H-J-B-I PDE is (degenerate) parabolic.

Again,  $u^\varepsilon$  is the **unique viscosity solution** of the Cauchy problem.

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# Plan of the method:

- 1 pass to the limit as  $\varepsilon \rightarrow 0$  in the PDE
  - 2 associate to the limit PDE a "limit control problem"
- 1 was developed in the papers

O. ALVAREZ - M. B. : SIAM J. Cont. '01, Arch. Rat. Mech. Anal. '03, Memoir A.M.S. '10;

O. A. - M. B. - C. MARCHI : J. D. E. '07, '08 (more than 2 scales)  
under boundedness assumptions on the fast state variables,  
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M. B. - A. CESARONI : European J. Control 2011

Methods are related to **HOMOGENIZATION** of H- J equations:

P.L.Lions- Papanicolaou- Varadhan 1986, L.C.Evans 1989, ...

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Methods are related to **HOMOGENIZATION** of H- J equations:

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- 1 Search *effective Hamiltonian*  $\bar{H}$  and *effective initial data*  $\bar{h}$  s. t.

$$u^\varepsilon(t, x, y) \rightarrow u(t, x) \quad \text{as } \varepsilon \rightarrow 0,$$

$u$  solution of

$$(\overline{\text{CP}}) \quad \begin{cases} \frac{\partial u}{\partial t} + \bar{H}(x, D_x u, D_{xx}^2 u) = 0 & \text{in } (0, +\infty) \times \mathbf{R}^n, \\ u(0, x) = \bar{h}(x) & \text{in } \mathbf{R}^n \end{cases}$$

- 2 Interpret the effective Hamiltonian  $\bar{H}$  as the Bellman-Isaacs Hamiltonian for a new *effective system*

$$\dot{x}_s = \bar{f}(x_s, \eta_s, \theta_s) \quad x_s \in \mathbf{R}^n, \eta_s \in E(x_s), \theta_s \in \Theta(x_s)$$

and *effective* cost functional

$$\bar{J}(t, x, \eta, \theta) := \int_0^t \bar{l}(x_s, \eta_s, \theta_s) ds + \bar{h}(x_t).$$

This is a *variational limit* of the initial  $n + m$ -dimensional problem.  
Step 2 is largely OPEN !



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# Definition of $\overline{H}$ : deterministic case $\sigma \equiv 0, \nu \equiv 0$

Consider the fast subsystem with frozen  $x$  and  $\varepsilon = 1$

$$(FS) \quad \dot{y}_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau), \quad y_0 = y$$

and the family of value functions in  $\mathbf{R}^m$  with parameters  $x, p \in \mathbf{R}^n$

$$w(t, y; x, p) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t L(y_\tau, \alpha[\beta]_\tau, \beta_\tau; x, p) d\tau,$$

$$L(y, \alpha, \beta; x, p) := p \cdot f(x, y, \alpha, \beta) + l(x, y, \alpha, \beta)$$

Say (FS) is **ERGODIC** if, for all  $x, p$ ,

$$\lim_{t \rightarrow +\infty} \frac{w(t, y; x, p)}{t} = \text{constant (in } y), \text{ uniformly in } y$$

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$$w(t, y; \mathbf{x}, \mathbf{p}) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} \int_0^t L(y_\tau, \alpha[\beta]_\tau, \beta_\tau; \mathbf{x}, \mathbf{p}) d\tau,$$

$$L(y, \alpha, \beta; \mathbf{x}, \mathbf{p}) := \mathbf{p} \cdot f(\mathbf{x}, y, \alpha, \beta) + l(\mathbf{x}, y, \alpha, \beta)$$

Say (FS) is **ERGODIC** if, for all  $\mathbf{x}, \mathbf{p}$ ,

$$\lim_{t \rightarrow +\infty} \frac{w(t, y; \mathbf{x}, \mathbf{p})}{t} = \text{constant (in } y), \text{ uniformly in } y$$

$$=: -\bar{H}(\mathbf{x}, \mathbf{p})$$

- Example: (FS) uncontrolled, i.e.

$$\dot{y}_\tau = g(x, y_\tau),$$

and **ergodic** in the classical sense with a **UNIQUE INVARIANT MEASURE**  $\mu_x$ . Then

$$-\bar{H}(x, p) = \int_{\mathbf{R}^m} \min_{\alpha \in A} L(y, \alpha; x, p) d\mu_x(y).$$

- NO explicit formula for  $\bar{H}$  in general!
- Definition of  $\bar{H}$  in general non-deterministic case: fast subsystem

$$dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau) d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau) dW_\tau, \quad y_0 = y,$$

$L = \text{trace} (M\sigma\sigma^T(x, y, \alpha, \beta)) / 2 + \dots$  ( $M$  a  $n \times n$  symmetric matrix)

$$w(t, y; x, p, M) = \inf_{\alpha} \sup_{\beta} E \left[ \int_0^t L d\tau \right], \quad \lim_{t \rightarrow +\infty} w/t =: -\bar{H}(x, p, M)$$

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# Weak convergence theorem

## Meta-Theorem

Fast subsystem (FS) ergodic  $\implies$

$$u^\varepsilon(t, x, y) \rightarrow u(t, x) \quad \text{as } \varepsilon \rightarrow 0,$$

(in the sense of relaxed semilimits, or weak viscosity limits),  
and  $u$  solves (in viscosity sense)

$$\frac{\partial u}{\partial t} + \bar{H}(x, D_x u, D_{xx}^2 u) = 0$$

This is in fact a Theorem if the fast variables  $y$  live on the torus  $\mathbb{T}^M$  (i.e., all data are  $\mathbb{Z}^m$ -periodic in  $y$ ), or in all  $\mathbf{R}^M$  but the process

$$dy_\tau = g(x, y_\tau) d\tau + \nu(x, y_\tau) dW_\tau$$

is an UN-controlled NON-degenerate diffusion.

# The effective initial data $\bar{h}$

For the Fast Subsystem with frozen  $x$

$$(FS) \quad dy_\tau = g(x, y_\tau, \alpha_\tau, \beta_\tau) d\tau + \nu(x, y_\tau, \alpha_\tau, \beta_\tau) dW_\tau, \quad y_0 = y,$$

consider now the value function

$$v(t, y; x) := \inf_{\alpha \in \Gamma(t)} \sup_{\beta \in B(t)} E[h(x, y_t)]$$

**Definition:** (FS) is **STABILIZING** (to a constant) for the cost  $h$  if,  $\forall x$ ,

$$\lim_{t \rightarrow +\infty} v(t, y; x) = \text{constant (in } y), \text{ uniformly in } y =: \bar{h}(x)$$

**Convergence Theorem at  $t = 0$**

Fast subsystem (FS) stabilizing for the cost  $h \implies$

$$\lim_{t \rightarrow 0} \lim_{\varepsilon \rightarrow 0} u^\varepsilon(t, x, y) = \bar{h}(x)$$

(in the sense of relaxed semilimits, or weak viscosity limits).

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# Main convergence theorem

## Corollary

Fast subsystem (FS) ERGODIC and STABILIZING  $\implies$

$\exists \bar{H}$  and  $\bar{h}$  continuous,  $\bar{H}$  degenerate elliptic, such that

$u^\varepsilon(t, x, y) \rightarrow u(t, x)$  as  $\varepsilon \rightarrow 0$ ,  $u$  solution of

$$(\overline{\text{CP}}) \quad \begin{cases} \frac{\partial u}{\partial t} + \bar{H}(x, D_x u, D_{xx}^2 u) = 0 & \text{in } (0, +\infty) \times \mathbf{R}^n, \\ u(0, x) = \bar{h}(x) & \text{in } \mathbf{R}^n. \end{cases}$$

If moreover,  $\bar{H}$  is regular enough w.r.t.  $x$ , then  $(\overline{\text{CP}})$  has a **unique** solution and

$$u^\varepsilon \rightarrow u \quad \text{locally uniformly.}$$

# Conclusion

The initial  $(n + m)$ -dimensional H-J-B-I equation is split into

- two  $m$ -dimensional ergodic-type problems (one for  $\bar{H}$  and one for  $\bar{h}$ ),
- a  $n$ -dimensional "effective" PDE

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# The next steps

Main remaining questions:

- 1 when is (FS) ergodic and stabilizing ?
- 2 can we find effective dynamics  $\bar{f}, \bar{\sigma}$ , running cost  $\bar{l}$ , and control constraints  $E, \Theta$  associated to  $\bar{H}$  ?

Some answers:

- 1 for bounded fast variables (related to homogenization):
  - uniformly nondegenerate fast subsystem (FS),
  - deterministic fast subsystem (FS) controllable by one player from each point to any other point in a uniformly bounded time,
  - under nonresonance conditions on the torus;

for unbounded fast variables: uncontrolled diffusion processes with a unique invariant measure;

- 2 several examples but no general recipe.

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# The case of uncontrolled fast variables

$$dx_s = f(x_s, y_s, \alpha_s, \beta_s) ds + \sigma(x_s, y_s) dW_s$$

$$dy_s = \frac{1}{\varepsilon} g(x_s, y_s) ds + \frac{1}{\sqrt{\varepsilon}} \nu(x_s, y_s) dW_s,$$

$$J^\varepsilon(t, x, y, \alpha, \beta) := \int_0^t l(x_s, y_s, \alpha_s, \beta_s) ds + h(x_t, y_t)$$

Assume there exists a unique **invariant measure**  $\mu_x$  of

$$(FS) \quad dy_\tau = g(x, y_\tau) d\tau + \nu(x, y_\tau) dW_\tau,$$

Denote  $\langle \phi \rangle(x) := \int \phi(x, y) d\mu_x(y)$ . Then effective  $H$  and  $h$  are

$$\bar{h}(x) = \langle h \rangle(x)$$

$$\bar{H}(x, p, M) = \left\langle \min_{\beta \in B} \max_{\alpha \in A} \left\{ -\text{trace}(M\sigma\sigma^T)/2 - f \cdot p - l \right\} \right\rangle$$

## Corollary

For split systems and cost, i.e.,

$$f = f_0(x, y) + f_1(x, \alpha, \beta), \quad I = I_0(x, y) + I_1(x, \alpha, \beta),$$

the linear averaging of the data is the correct limit, i.e.,

$$\lim_{\varepsilon \rightarrow 0} u^\varepsilon(t, x, y) = u(t, x) :=$$

$$\inf_{\alpha \in \Gamma(t)} \sup_{\beta \in \mathcal{B}(t)} E \left[ \int_0^t \langle I \rangle(x_s, \alpha[\beta]_s, \beta_s) ds + \langle h \rangle(x_t) \right],$$

$$dx_s = \langle f \rangle(x_s, \alpha[\beta]_s, \beta_s) ds + \langle \sigma \sigma^T \rangle^{1/2}(x_s) dW_s$$

Proof:  $\bar{H}(x, p, M) = -\text{trace}(M \langle \sigma \sigma^T \rangle) / 2 + \min_B \max_A \{ -\langle f \rangle \cdot p - \langle I \rangle \}$ .

A similar result was proved by Kushner (book, 1990) for a single controller by probabilistic methods.

In general, for system or cost **NOT split**,

$$\overline{H}(x, p, M) = \langle \min_B \max_A \{ \dots \} \rangle \neq \min_B \max_A \langle \{ \dots \} \rangle$$

and the limit control problem is not obvious.

For some classical problems we derived the explicit form of the effective control problem:

- Merton portfolio optimization with stochastic volatility,
- Ramsey model of optimal economic growth with (fast) random parameters,
- Vidale - Wolfe advertising model with random parameters,
- advertising game in a duopoly with Lanchester dynamics and random parameters.

Often they involve a **nonlinear average** of some parameter.

E.g., the limit of Merton problem is still a Merton problem with constant volatility the harmonic average  $\overline{\sigma} := \langle \sigma^{-2} \rangle^{-1/2}$ .

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We can give a general **representation formula** for the effective control problem by **enlarging the control set**.

Assume for simplicity  $\sigma \equiv 0$  (deterministic slow subsystem) and a single controller ( $B$  singleton).

Define  $E_x := L^1((\mathbf{R}^m, \mu_x); A) \supset A$  and for  $\tilde{\alpha} \in E_x$

$$\bar{f}(x, \tilde{\alpha}) := \int_{\mathbf{R}^m} f(x, y, \tilde{\alpha}(y)) d\mu_x(y), \quad \bar{l}(x, \tilde{\alpha}) := \int_{\mathbf{R}^m} l(x, y, \tilde{\alpha}(y)) d\mu_x(y).$$

Then

$$\bar{H}(x, p) = \langle \max_{\alpha \in A} \{-f \cdot p - l\} \rangle = \max_{\tilde{\alpha} \in E_x} \left\{ -\bar{f}(x, \tilde{\alpha}) \cdot p - \bar{l}(x, \tilde{\alpha}) \right\}$$

and the effective control problem is

$$\begin{aligned} \dot{x}_s &= \bar{f}(x_s, \tilde{\alpha}_s), \quad x_0 = x, \quad \tilde{\alpha}_s \in E_{x_s} \\ \min \quad \bar{J}(t, x, \tilde{\alpha}.) &= \int_0^t \bar{l}(x_s, \tilde{\alpha}_s) ds + \langle h \rangle(x_t). \end{aligned}$$

# A one-dimensional homogenization problem

$$\dot{x}_s = g(x_s) \alpha_s, \quad J = \int_0^t l\left(x_s, \frac{x_s}{\varepsilon}\right) ds + h\left(x_t, \frac{x_t}{\varepsilon}\right), \quad -1 \leq \alpha_s \leq 1, \quad g > 0.$$

Here  $\dot{y}_s = \frac{1}{\varepsilon} g(x_s) \alpha_s$  depends on the control.

The value function  $u^\varepsilon(t, x)$  solves the H-J equation

$$\frac{\partial u^\varepsilon}{\partial t} + g(x) \left| \frac{\partial u^\varepsilon}{\partial x} \right| = l\left(x, \frac{x}{\varepsilon}\right), \quad u^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right).$$

For  $l(x, \cdot), h(x, \cdot)$  1-periodic and  $\min l(x, \cdot) = 0$  the limit PDE is

$$\frac{\partial u}{\partial t} + \left( g(x) \left| \frac{\partial u}{\partial x} \right| - \langle l \rangle(x) \right)^+ = 0, \quad u(0, x) = \min_{y \in [0,1]} h(x, y).$$

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# Homogenization of a 2-D differential game

$$\dot{x}_s = g_1(x_s, z_s)\alpha_s, \quad -1 \leq \alpha_s \leq 1, \quad g_1 > 0$$

$$\dot{z}_s = g_2(x_s, z_s)\beta_s, \quad -1 \leq \beta_s \leq 1, \quad g_2 > 0$$

$$J^\varepsilon(t, x, z, \alpha, \beta) := \int_0^t [l_1(x_s, z_s, \frac{x_s}{\varepsilon}) + l_2(x_s, z_s, \frac{z_s}{\varepsilon})] ds + h(x_t, z_t, \frac{x_s}{\varepsilon}, \frac{z_s}{\varepsilon})$$

$$\min_{[0,1]} l_1(x, z, \cdot) = 0, \quad \max_{[0,1]} l_2(x, z, \cdot) = 0$$

and assume  $h$  has a saddle

$$\min_{\xi \in [0,1]} \max_{\eta \in [0,1]} h(x, z, \xi, \eta) = \max_{\eta \in [0,1]} \min_{\xi \in [0,1]} h(x, z, \xi, \eta) =: h_S(x, z)$$

The limit differential game has the same system and controls, and the effective cost-payoff

$$\bar{J} = \int_0^t [|\alpha_s| \langle l_1 \rangle(x_s, z_s) + |\beta_s| \langle l_2 \rangle(x_s, z_s)] ds + h_S(x_t, z_t)$$

joint work with G.Terrone.

Work in progress on **homogenization** of control systems and games:

- the last two examples extend to  $n$ -dimensional systems provided the oscillations are at 1-dimensional scale;
- an "abstract" representation of some effective control problems can be obtained by the **limit occupational measures** studied by Artstein, Gaitsgory, Borkar,..., (joint work with Gabriele TERRONE);
- homogenization of **deterministic differential games** is wide open: there are easy examples of non-convergence of the value functions and only a few known cases of convergence (see also Cardaliaguet '09)

Thanks for your attention !