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Incorporating low-thrust propulsion into the invariant manifolds technique for space trajectories design

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Context

- Space trajectory design using non-Keplerian orbits
 - Transfers to libration point orbits

Aims

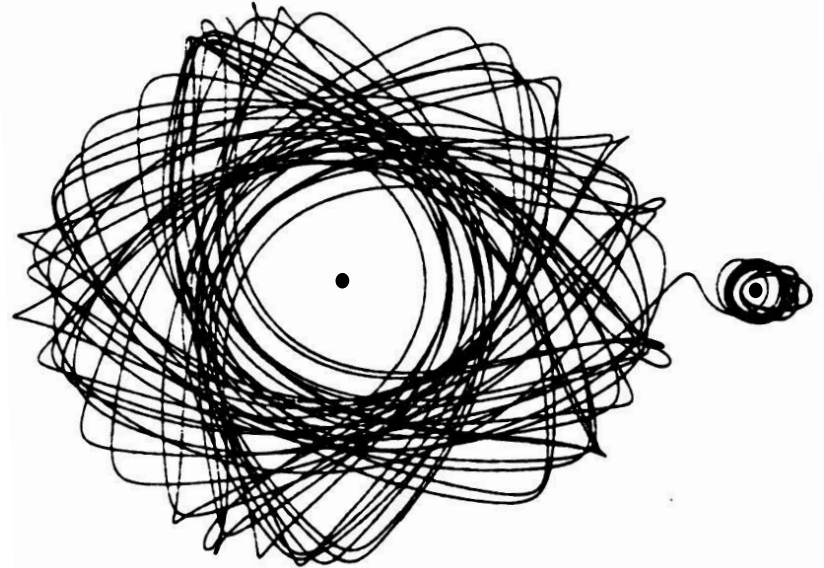
- Improve existing methods
 - Modify “invariant manifolds technique” to include low-thrust propulsion



1. Introduction to **non-Keplerian** orbits
2. Introduction to the **restricted three body problem**
3. Introduction to **equilibrium points, Halo orbits and invariant manifolds**
4. Dynamical system theory in space trajectory design
 - transfers to **Lagrange point orbits**
5. Introduction of **low-thrust** propulsion: controlled RTBP
6. Definition and manipulation of **attainable sets**
7. End-to-end trajectory **optimization**: controlled RFBP
8. Results and comparisons
 - **low-thrust, stable-manifold** transfers to halo orbits
9. Final remarks



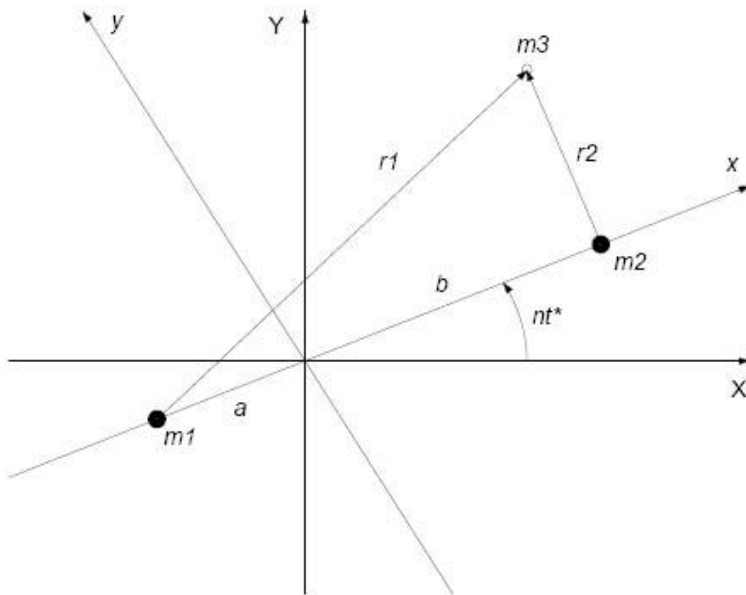
- Solution of n -body problems, $n \geq 3$
 - Unique features
 - Less propellant
 - Accurate solutions
 - Non-integrable systems
 - Chaotic dynamics
 - Long transfer times
 - No systematic approach
-
- How can we design non-Keplerian orbits for space missions?
 - ➔ Exploit knowledge of the **phase space** (e.g., equilibrium points, periodic orbits, invariant manifolds, etc.)





The restricted three-body problem (1)

- Two *primaries* act on a third body (of negligible mass)
- The third body does not affect the motion of the primaries
- The two primaries move in circular orbits around their center of mass (circular restricted three-body problem)



The motion of the primaries is *given*:

$$\begin{aligned} X_1 &= -a \cos nt^*, & X_2 &= b \cos nt^*, \\ Y_1 &= -a \sin nt^*, & Y_2 &= b \sin nt^*. \end{aligned}$$



The restricted three-body problem (2)



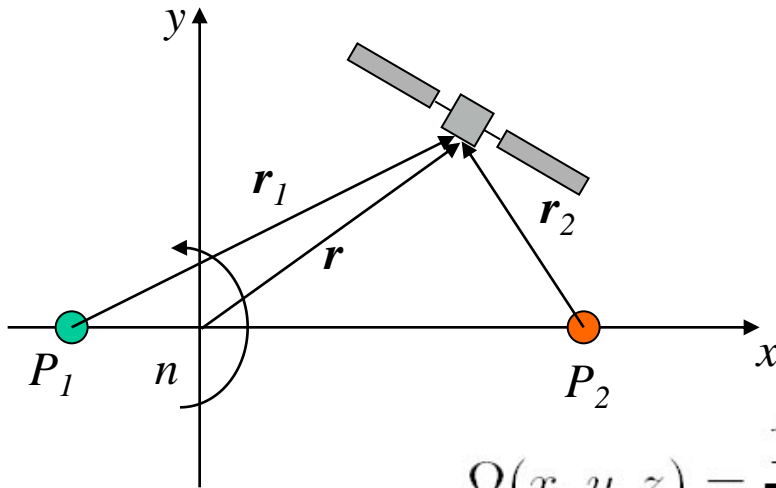
- the problem is *restricted* to the motion of the third body
- Newton equation written in *inertial* frame:

$$\left\{ \begin{array}{l} \frac{d^2 X}{dt^{*2}} \\ \frac{d^2 Y}{dt^{*2}} \\ \frac{d^2 Z}{dt^{*2}} \end{array} \right. = -G \left[\begin{array}{l} \frac{m_1(X + a \cos nt^*)}{R_1^3} + \frac{m_2(X - b \cos nt^*)}{R_2^3} \\ \frac{m_1(Y + a \sin nt^*)}{R_1^3} + \frac{m_2(Y - b \sin nt^*)}{R_2^3} \\ \frac{m_1 Z}{R_1^3} + \frac{m_2 Z}{R_2^3} \end{array} \right],$$



The restricted three-body problem (3)

- Equations written in a *dimensionless* rotating (*synodic*) frame make the dynamical system autonomous



$$\ddot{x} - 2\dot{y} = \Omega_x,$$

$$\ddot{y} + 2\dot{x} = \Omega_y,$$

$$\ddot{z} = \Omega_z,$$

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1 - \mu)$$

$$r_1^2 = (x + \mu)^2 + y^2 + z^2, \quad r_2^2 = (x - 1 + \mu)^2 + y^2 + z^2$$

Conventions:

$$G = 1$$

$$n = 1$$

$$m_1 = 1 - \mu$$

$$x_1 = -\mu$$

$$m_2 = \mu$$

$$x_2 = 1 - \mu$$

$$\mu = m_2 / (m_1 + m_2)$$

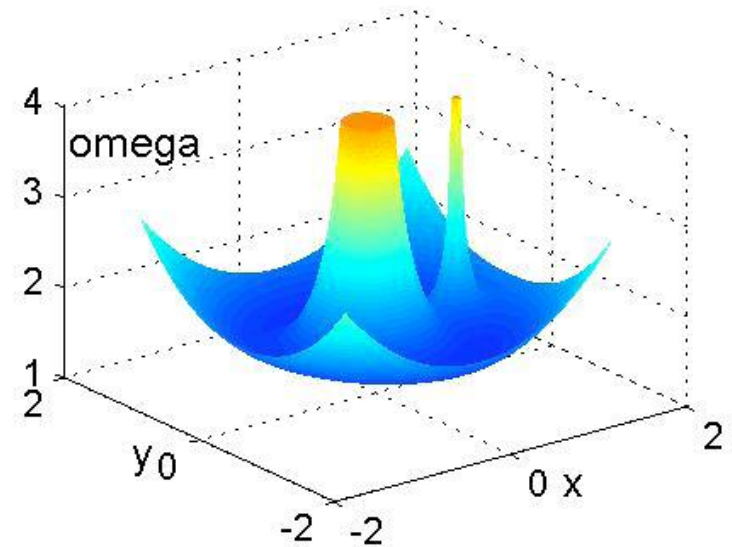
Mass parameter of the R3BP



The restricted three-body problem (4)

- The R3BP has *only one* integral of motion: the Jacobi integral
- The problem is not integrable! Qualitative studies only
- Allowed and forbidden regions are established (Hill's curves)

$$C = 2\Omega(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

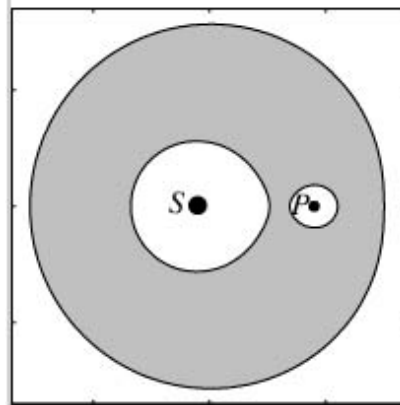


- Given x_0 , then $C_0 = C(x_0)$ and the allowed region will be (x, y, z) such that $\Omega(x, y, z) \geq C_0/2$

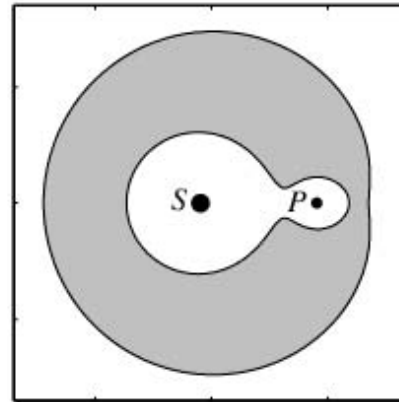


The restricted three-body problem (5)

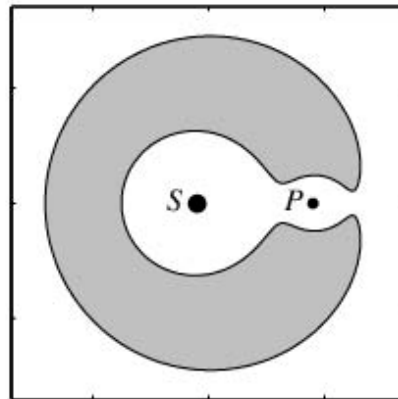
- $C = -2E$
- The higher C , the lower energy!
- For $C < 3$, the S/C is allowed to move in the whole configuration space



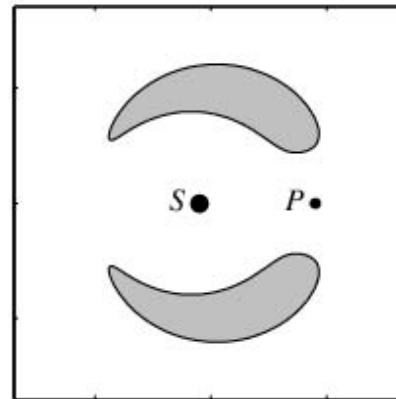
(a) Case 1 : $E < E_1$



(b) Case 2 : $E_1 < E < E_2$



(c) Case 3 : $E_2 < E < E_3$



(d) Case 4 : $E_3 < E < E_4 = E_5$



Equilibrium points (1)

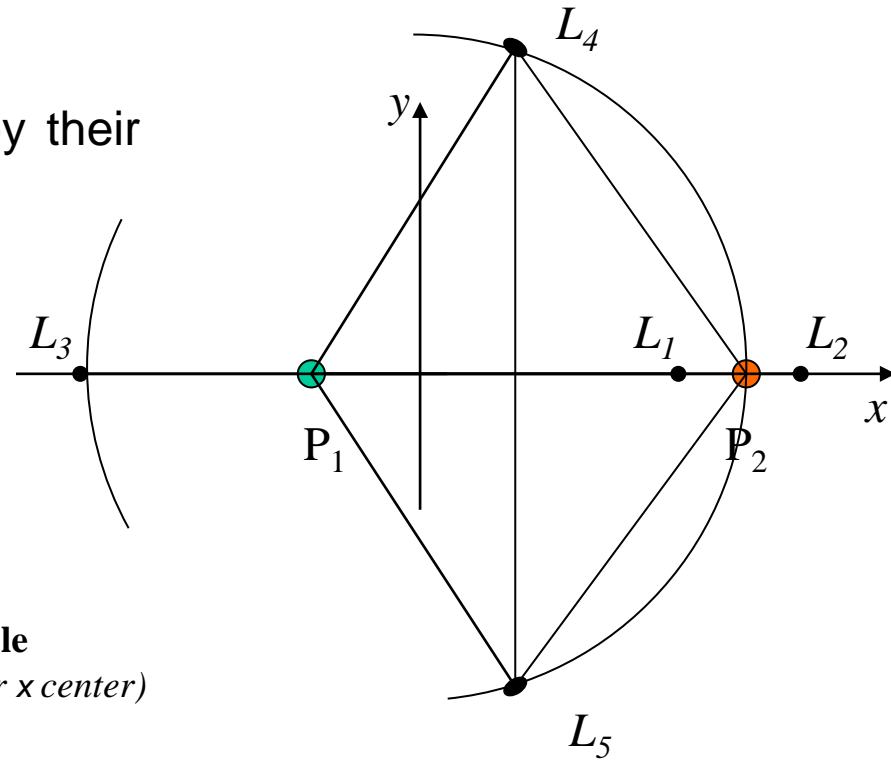
- The differential system has 5 equilibrium points (libration or Lagrangian points).
- The “nature” of the points is given by their linear behaviour.

$$\dot{x} = f(x) \longrightarrow \dot{x} = Ax \longrightarrow \lambda_i$$

$$L1, L2, L3: \begin{cases} \lambda_1 = -\lambda_2 \\ \lambda_{3/4} = \pm i\alpha_1 \longrightarrow \text{Unstable} \\ \lambda_{5/6} = \pm i\alpha_2 \quad (\text{saddle } \times \text{ center } \times \text{ center}) \end{cases}$$

$$L4, L5: \begin{cases} \lambda_{1/2} = \pm i\beta_1 \\ \lambda_{3/4} = \pm i\beta_2 \longrightarrow \text{Stable} \\ \lambda_{5/6} = \pm i\gamma \end{cases} \quad (\text{center } \times \text{ center } \times \text{ center})$$

if $\mu < \mu_0 = 0.0385$



Note: equilibrium is relative to the synodic system!



Equilibrium points (2)

- Collinear points $L1$ and $L2$ are very important for space applications
 - associated to *low energy* values (high values of C)
 - have “fast” stable and unstable dynamics (strong instability)
- *Saddle* part (stable and unstable eigenvalues)
 - Hyperbolic orbit near $L1$ and $L2$
- *Center* part (complex conjugate eigenvalues)
 - Periodic orbits around $L1$ and $L2$
- *Invariant manifolds* of the points
 - $W^s_{L_i}$, stable manifold
 - $W^u_{L_i}$, unstable manifold
- The invariant manifolds of the points have interesting dynamical properties that can be exploited for space applications



Invariant manifolds of equilibrium points (1)

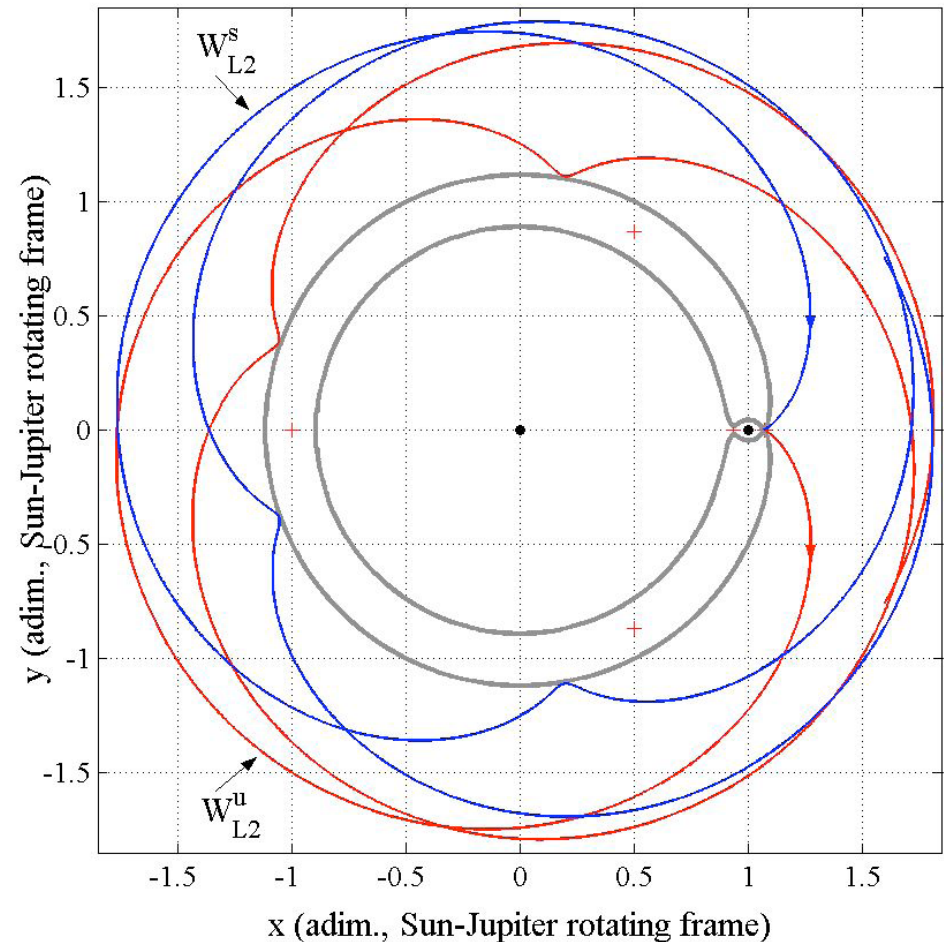
- *Nonlinear* invariant manifolds are defined in the phase space as *1D* invariant subsets such that

$$W_{L_i}^s \equiv \{x \in \mathbb{R}^6 / x(t) \rightarrow L_i, t \rightarrow +\infty\}$$

$$W_{L_i}^u \equiv \{x \in \mathbb{R}^6 / x(t) \rightarrow L_i, t \rightarrow -\infty\}$$

Sun-Jupiter system

- $W_{L_2}^s$ (blue)
- $W_{L_2}^u$ (red)
- The two *1D* manifolds extend for more than *3 AU* far from the equilibrium point.

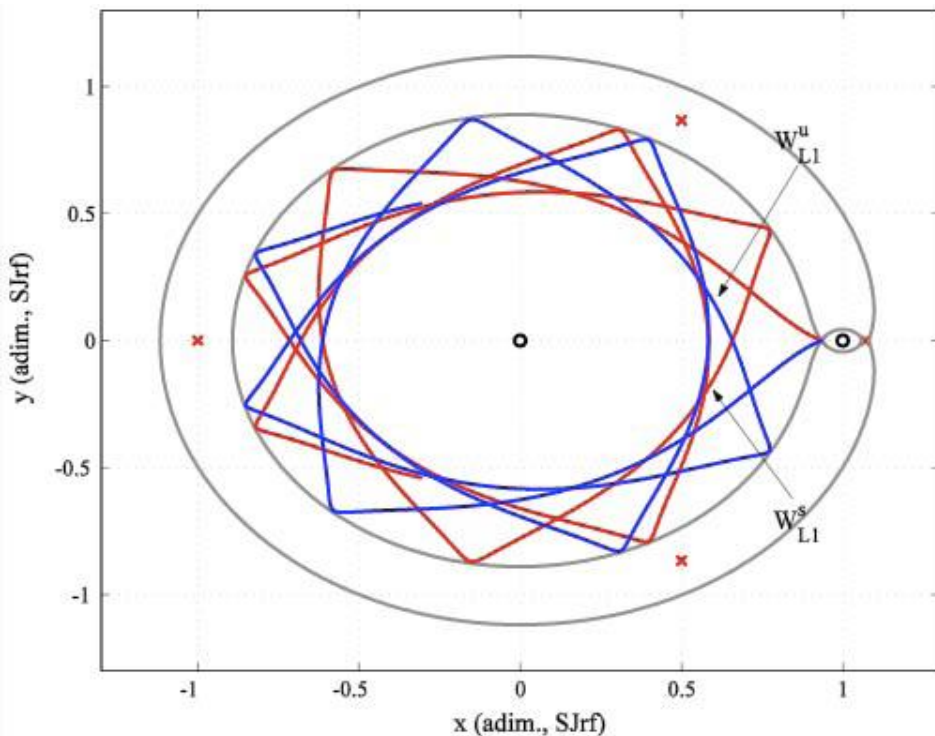




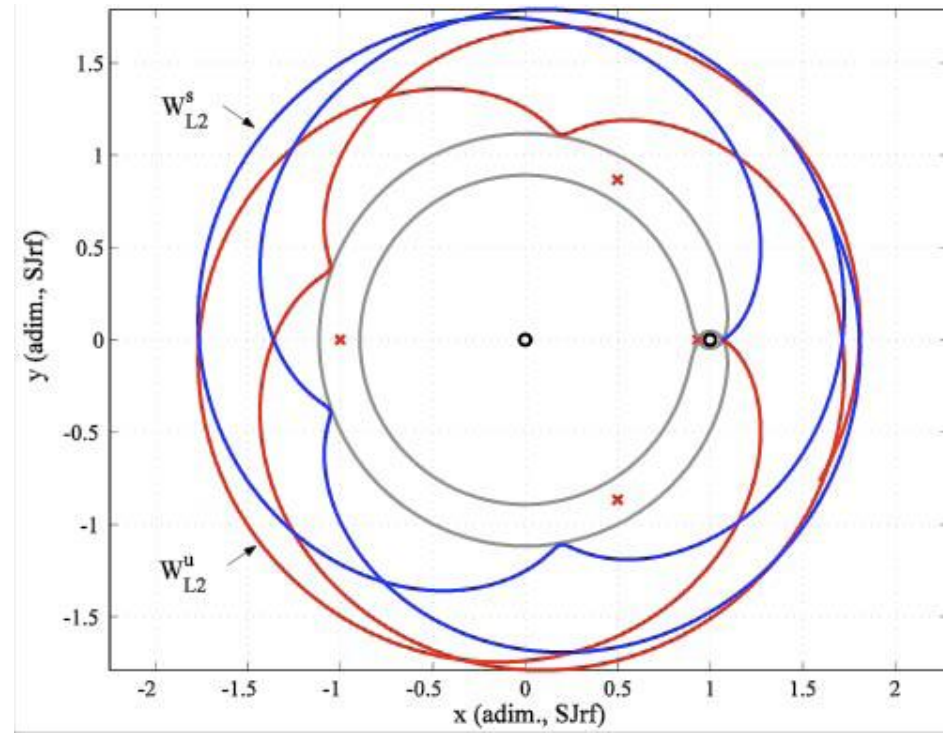
Invariant manifolds of equilibrium points (2)

- If a point lies on the stable (unstable) manifold, in the phase space sense, it will move toward (away) the libration point
- Possibility to have free transfers in the solar system

$W^{s,u}_{L1}$



$W^{s,u}_{L2}$

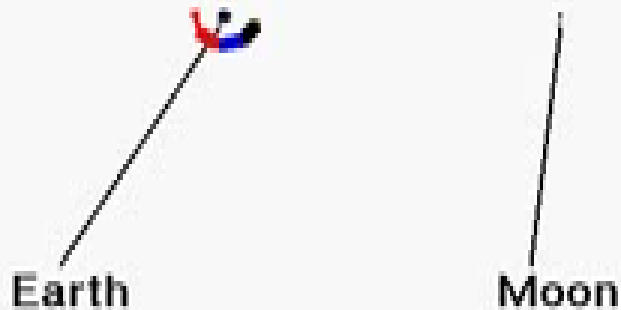




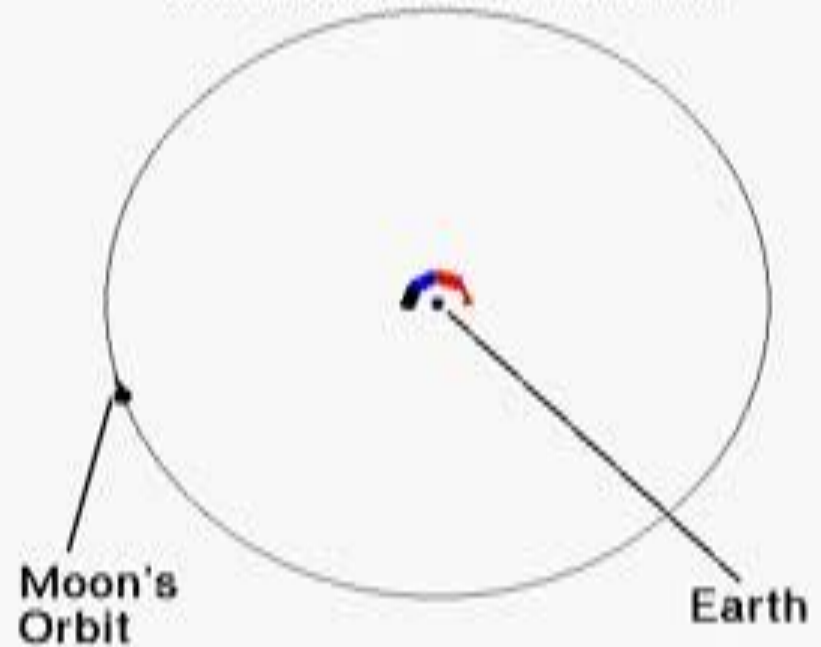
Example – video by Shane Ross, Virginia Tech



**GEO to Moon Orbit Transfer
Seen in Lunar Rotating Frame**



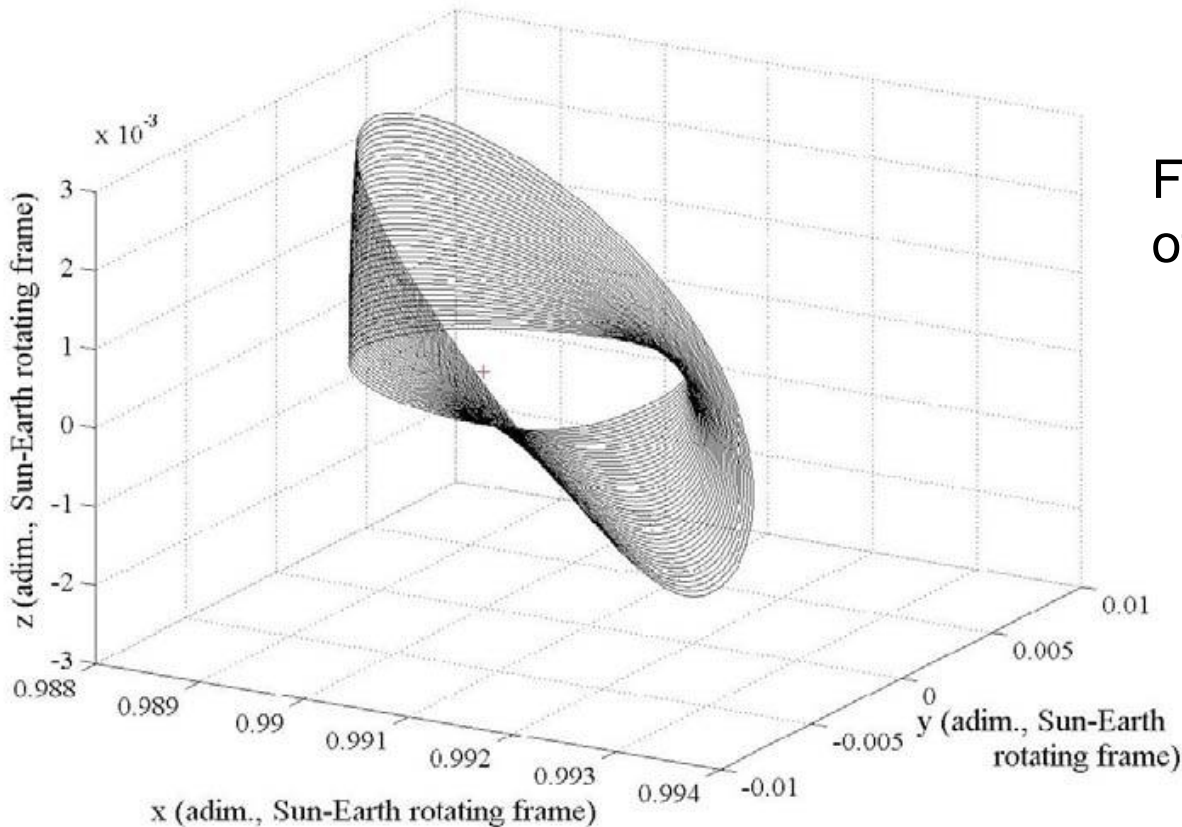
**GEO to Moon Orbit Transfer
Seen in Geocentric Inertial Frame**





Halo orbits (1)

- Large, 3D periodic orbits around collinear Lagrangian points
- $\omega = \nu$, forced by the higher order (nonlinear) terms
- Suitable for many missions (large out-of-plane excursions)
- Obtained through semi-analytical methods



Family of halos around L_1 of the Sun-Earth system



Richardson's method to construct halo orbits:

1. Find a third-order *analytical* first guess solution
2. *Differential correction* under the nonlinear dynamics

The R3BP has the following symmetry

$$S : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \rightarrow (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t)$$

- a trajectory that crosses perpendicularly the $y=0$ plane twice is a periodic orbit

Given the analytic initial guess, it is corrected through

$$\Phi(\mathbf{x}_0 + \Delta\mathbf{x}, T_{1/2} + \Delta t) = \Phi(\mathbf{x}_0, T_{1/2}) + \left[\frac{\partial \Phi(\mathbf{x}_0, T_{1/2})}{\partial \mathbf{x}} \right] \cdot \Delta\mathbf{x} + \frac{\partial \Phi(\mathbf{x}_0, T_{1/2})}{\partial t} \cdot \Delta t$$



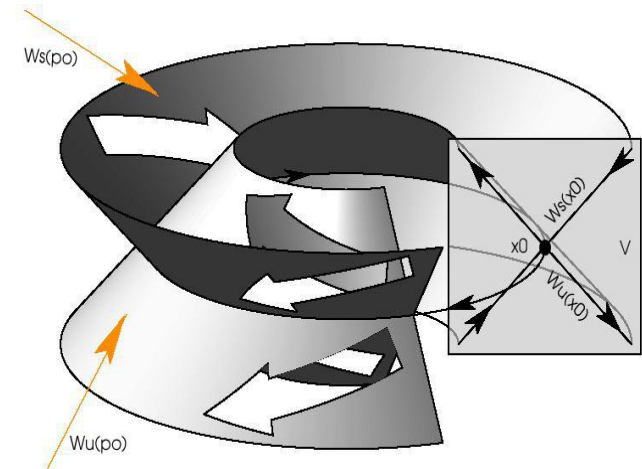
- In a neighborhood of the equilibrium points there are also *invariant manifolds* associated to the periodic orbits!
- These manifolds are more complicated “objects” than the simple manifolds associated to the equilibrium points
- They are due to the *saddle x center x center* part
- They are 2-dimensional subsets!
- Stable manifold (definition)
$$W^s_{Li,p.o.} \equiv \{x \in \mathbb{R}^6 / x(t) \rightarrow p.o. , t \rightarrow +\infty\}$$
- Unstable manifold (definition)
$$W^u_{Li,p.o.} \equiv \{x \in \mathbb{R}^6 / x(t) \rightarrow p.o. , t \rightarrow -\infty\}$$



Invariant manifolds of Halo orbits (2)

- Periodic orbits have stable and unstable manifolds too!
- Viewed as equilibrium points on a Poincaré section

$$\mathbf{x} \mapsto \mathbf{x}_0 + M(\mathbf{x} - \mathbf{x}_0)$$



- They are given by the stable and unstable eigenvalues and eigenvectors of the monodromy matrix M associated to the periodic orbit



$$\mathbf{x}_0$$

- initial condition to generate a periodic orbit (given by the Richardson's method)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad , \quad \mathbf{x}(0) = \mathbf{x}_0$$

- integrate the dynamical system (of order 42) for $0 \leq t \leq T$ to obtain $\mathbf{x} = \mathbf{x}(t)$ and $M = M(t)$

$$\dot{M} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right] M \quad , \quad M(0) = \mathbf{I}$$

$$\mathbf{x}^i = \mathbf{x}(t_i), \quad M^i = M(t_i)$$

- generic point of the orbit

$$\mathbf{v}_s^i$$

- stable eigenvector of M

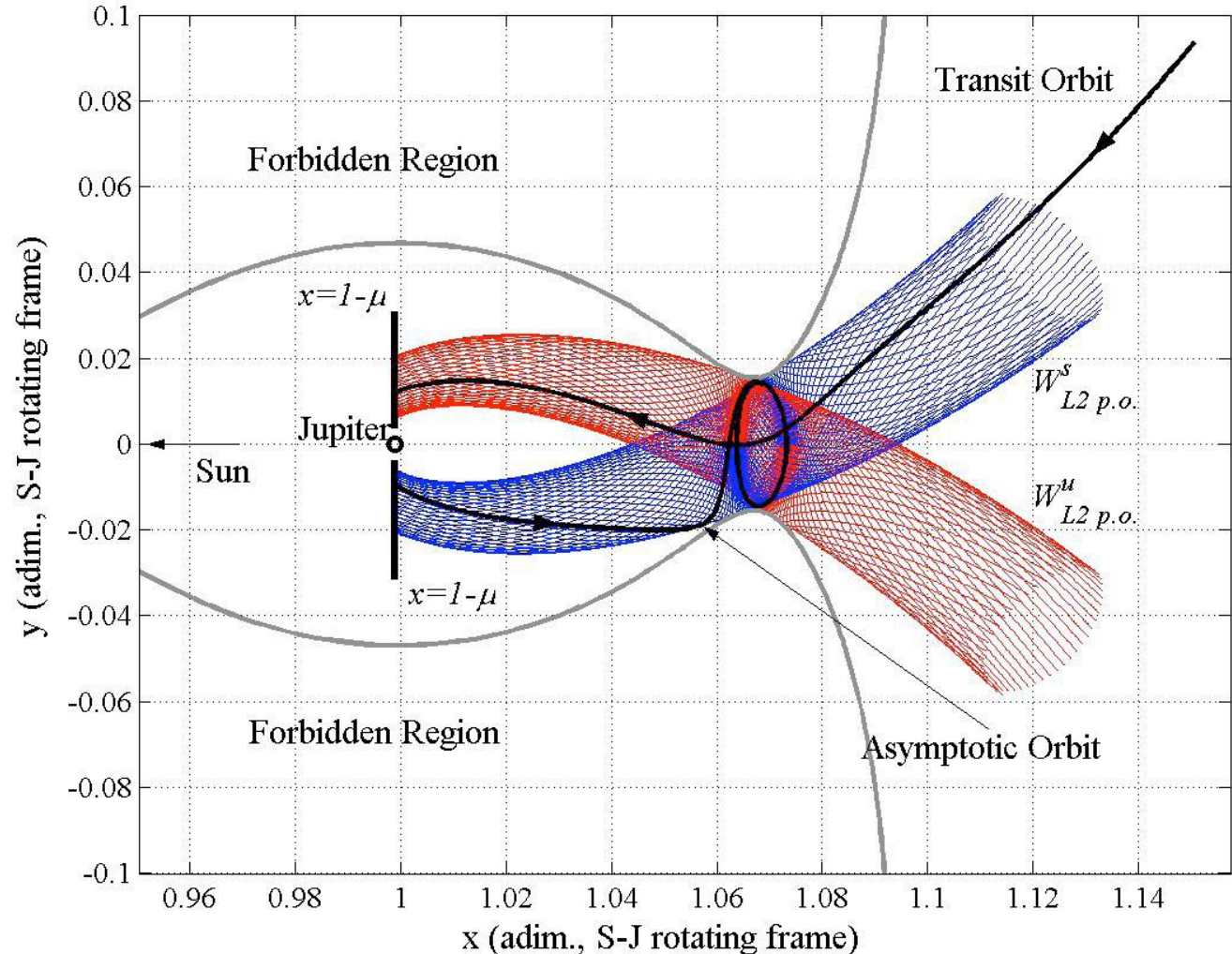
$$\mathbf{x}_{0,s}^i = \mathbf{x}^i \pm \varepsilon \mathbf{v}_s^i$$

- initial condition to integrate backward to get the stable manifold of \mathbf{x}_i



Invariant manifolds of Halo orbits (4)

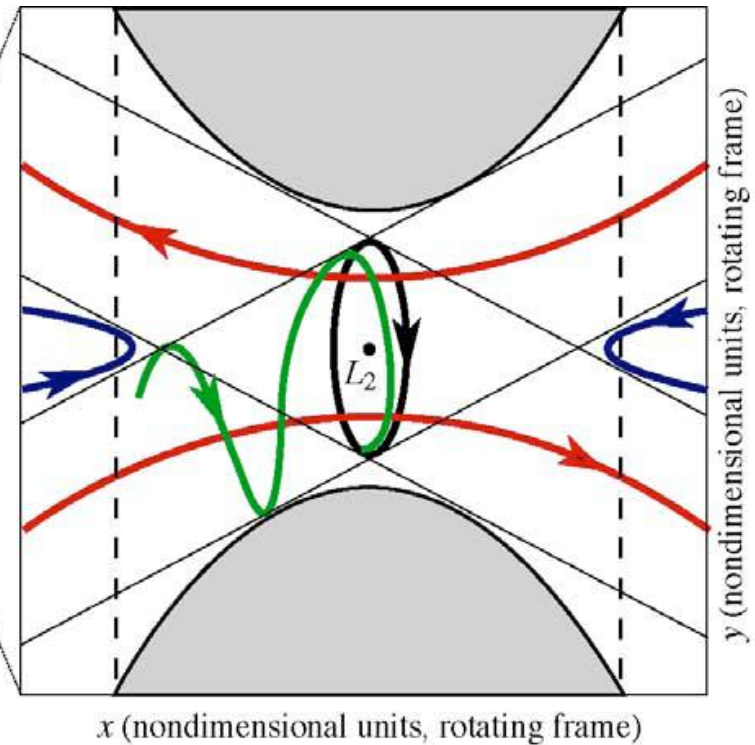
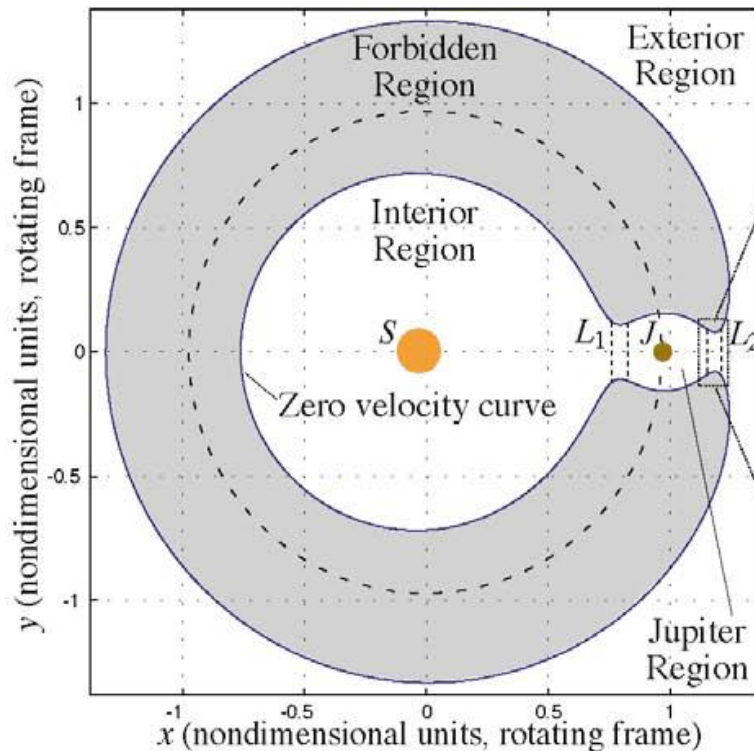
- Example of invariant manifold associated to a periodic orbit around L2 in the Sun-Jupiter system



blue: $W^s_{L2 p.o.}$
red: $W^u_{L2 p.o.}$



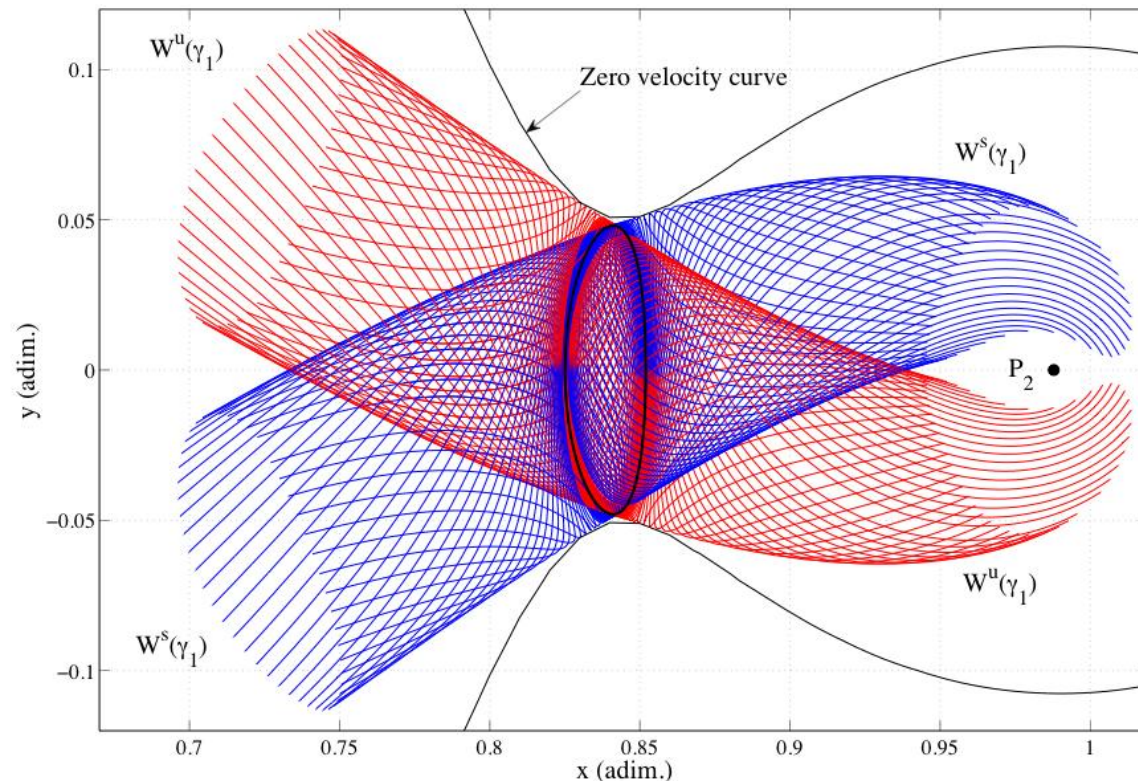
- Complete phase portrait
 - asymptotic orbits
 - periodic orbits
 - *transit orbits* (inside the tubes)
 - non-transit orbits



➤ Asymptotic orbits are useful for the trajectory design of libration point missions



- The phase space structure is exploited in a “natural” way.
- Equilibrium points, periodic orbits, invariant manifolds, etc. are used in place of orbital elements.



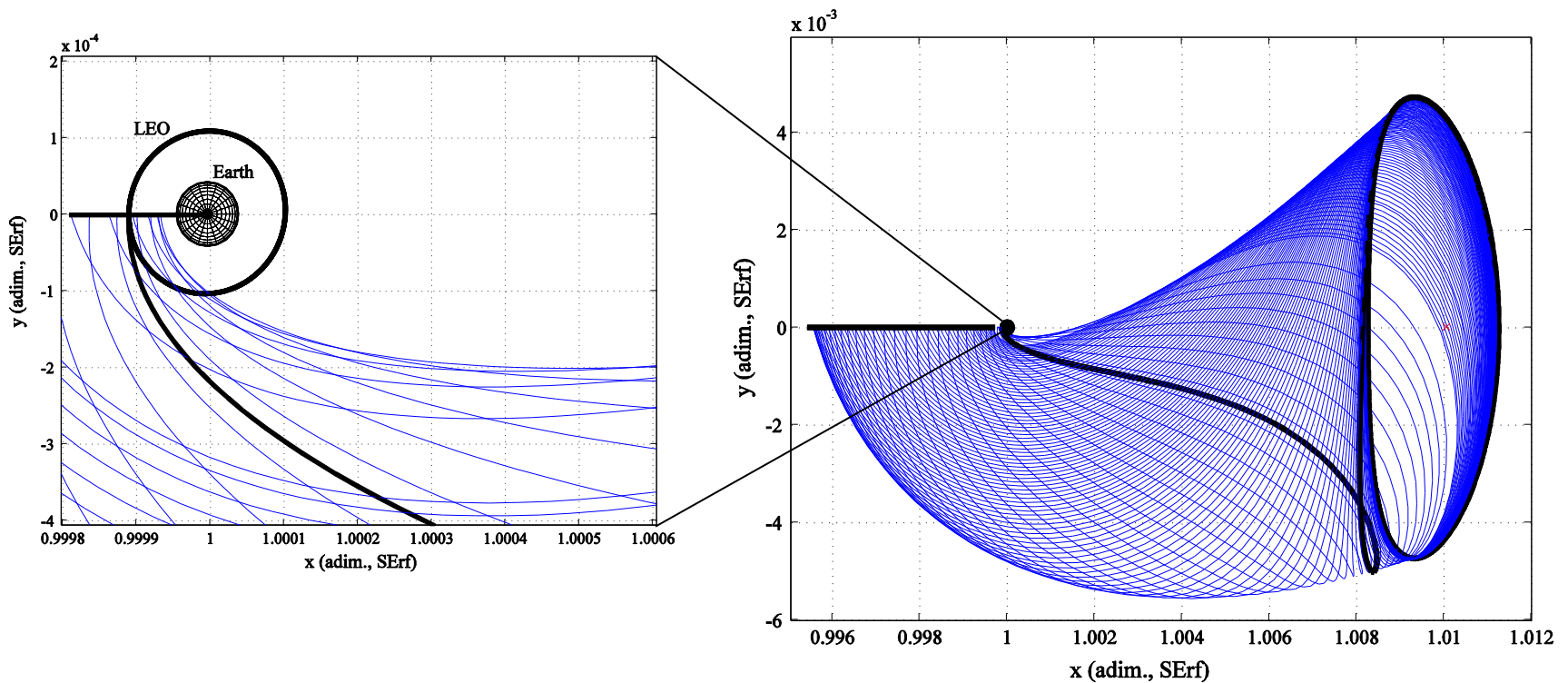
Typical case studies

- Transfers to halo orbits
- Low energy transfers to the Moon



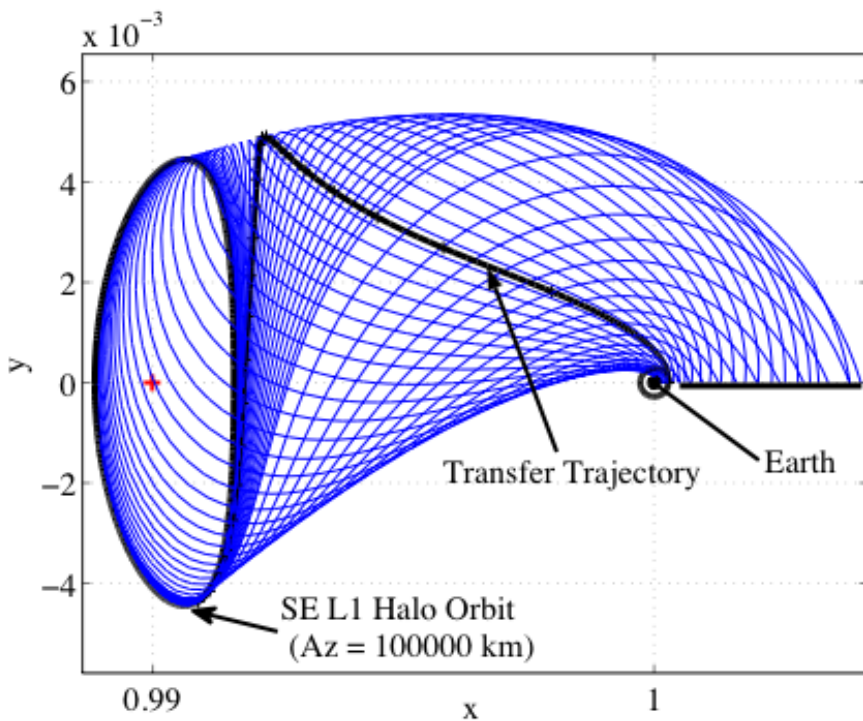
Transfer to Halo orbits (1)

- The concept of **stable manifold** is exploited for a transfer to Halo orbits
 - place the S/C on the stable manifold associated to that orbit
 - *Wait* (the S/C wraps on the final orbit)
 - perform an orbit injection maneuver to stabilize in a halo



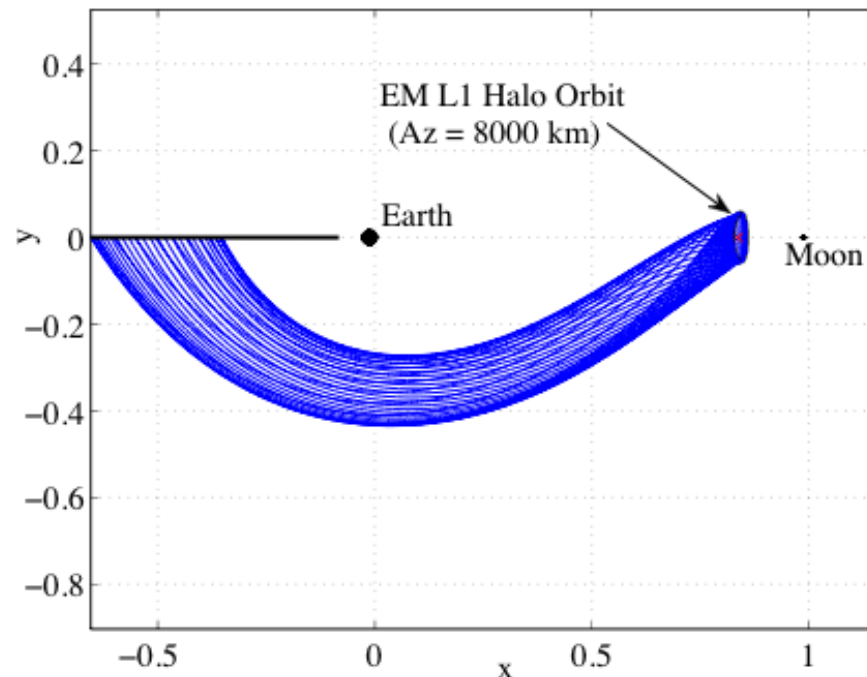


Transfer to Halo orbits (2)



Sun-Earth system

- $\Delta v \simeq 3200$ m/s (from LEO)
- Single-impulse transfer



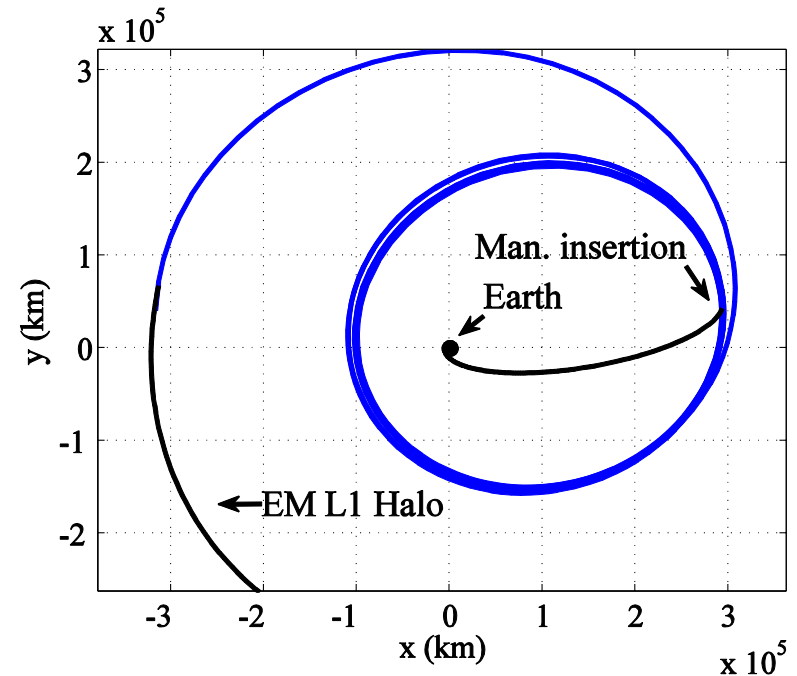
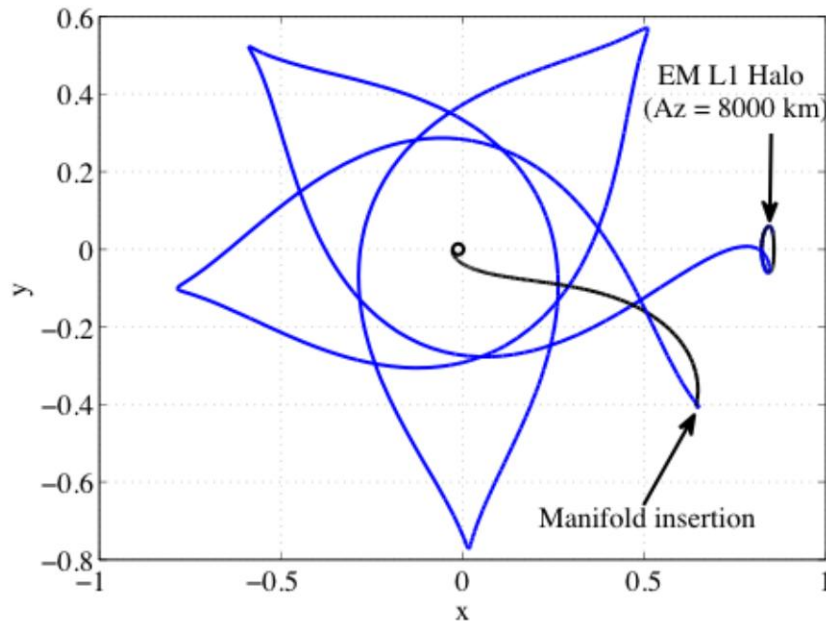
Earth-Moon system

- $\Delta v = ?$
- No single-impulse transfer



Transfer to Halo orbits (3)

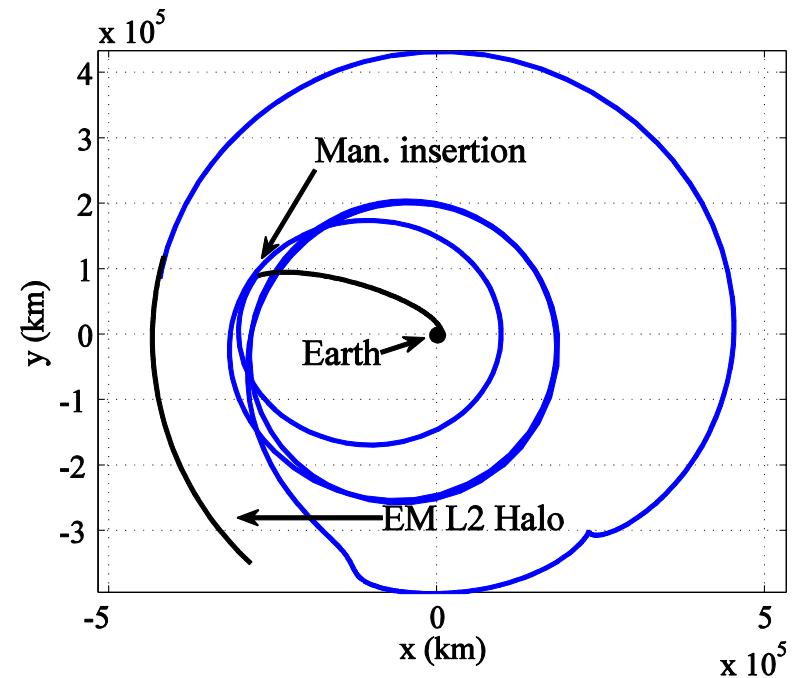
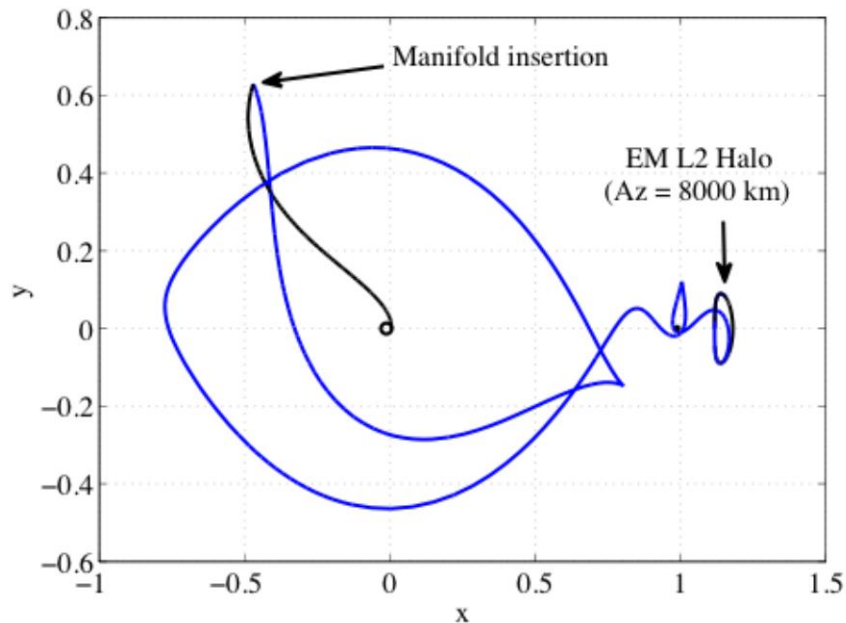
- A point on the stable manifold has to be targeted (two-impulse strategy)
- Transfer to L_1





Transfer to Halo orbits (4)

- A point on the **stable manifold** has to be targeted (two-impulse strategy)
- Transfer to L_2





Transfer to Halo orbits (5)



- In summary

Departing from circular, 200 km LEO

Type	A_z (km)	Δt (days)	Δv (m/s)	I_{sp} (s)	m_p/m_0
L_1	8000	67.3	3659	300	0.711
L_2	8000	65.0	3676	300	0.713



Introduction of Low-Thrust Propulsion (1)

Dynamical system theory allows us to define solutions with

- Unique features (e.g., libration point orbits)
- Less propellant mass (e.g., low energy transfers)

Rocket equation:

$$\frac{m_p}{m_0} = 1 - e^{-\frac{\Delta v}{I_{sp}g_0}}$$

Invariant manifolds technique
is used to reduce Δv

Can we further reduce the propellant mass
using low-thrust, high- I_{sp} systems?

Can we further improve low-energy solutions?



Introduction of Low-Thrust Propulsion (2)



The effect of low-thrust systems is **no longer** modeled as instantaneous velocity changes !



Need to **re-formulate** existing “invariant manifolds technique” to incorporate **low-thrust propulsion** ...



Controlled Restricted Three-Body Problem

Equations of motion including propulsion

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega_3}{\partial x} + \frac{T_x}{m}, \quad \ddot{y} + 2\dot{x} = \frac{\partial \Omega_3}{\partial y} + \frac{T_y}{m}, \quad \ddot{z} = \frac{\partial \Omega_3}{\partial z} + \frac{T_z}{m}, \quad \dot{m} = -\frac{T}{I_{sp} g_0}$$

Present thrust magnitude: $T = \sqrt{T_x^2 + T_y^2 + T_z^2}$

Variations of spacecraft mass, m , are considered through last eq.

The system is singular when $m \rightarrow 0$



$$\mathbf{T}(t) = (T_x(t), T_y(t), T_z(t)), \quad t \in [t_0, t_f] \quad ?$$

The **guidance law** is not given, but rather it is an **unknown** of the optimal control problem ...



$$\mathbf{x}_0 = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, m_0) \quad \text{initial condition}$$

$$\phi_{\mathbf{T}(\tau)}(\mathbf{x}_0, t_0; t) \quad \text{with} \quad \mathbf{T}(\tau), \tau \in [t_0, t] \quad \text{controlled flow}$$

$$\mathbf{x}(t) = \phi_{\bar{\mathbf{T}}}(\mathbf{x}_0, t_0; t) \quad \text{with} \quad \mathbf{T} = \bar{\mathbf{T}} \hat{\mathbf{v}} \quad \text{tangential thrust flow}$$

$$\gamma_{\bar{\mathbf{T}}}(\mathbf{x}_0, t) = \{\phi_{\bar{\mathbf{T}}}(\mathbf{x}_0, t_0; \tau) \mid \tau < t\} \quad \text{tangential thrust orbit}$$

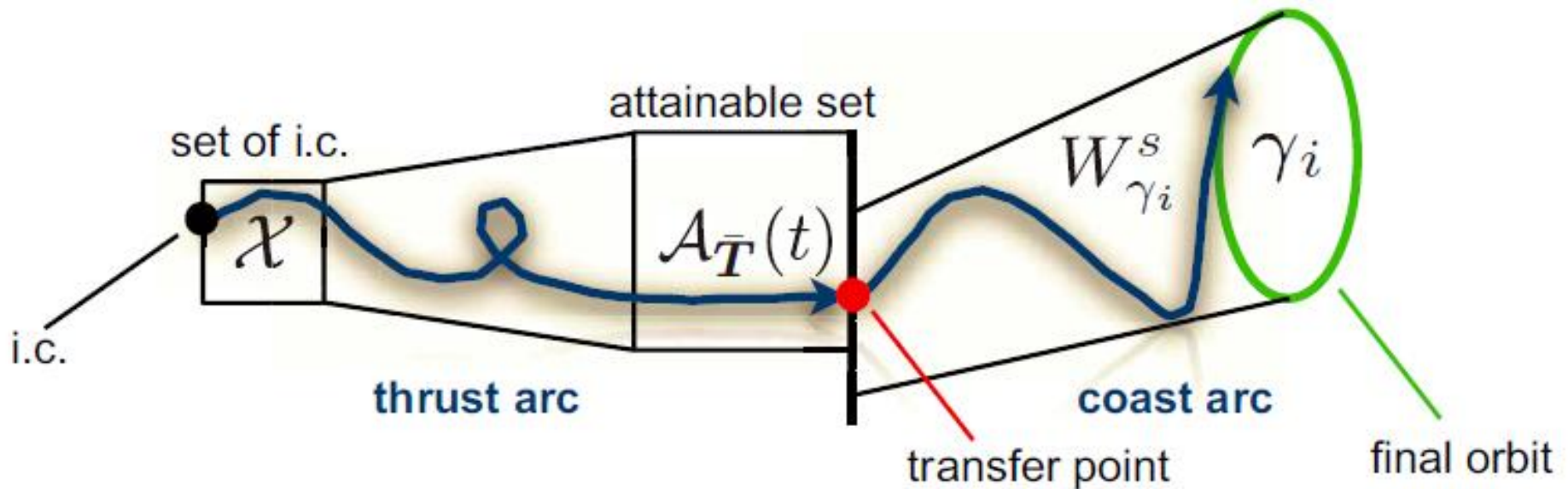
$$\boxed{\mathcal{A}_{\bar{\mathbf{T}}}(t) = \bigcup_{\mathbf{x}_0 \in \mathcal{X}} \gamma_{\bar{\mathbf{T}}}(\mathbf{x}_0, t)} \quad \text{Attainable set}$$



$\mathcal{A}_{\bar{\mathbf{T}}}$ is associated to an admissible set of initial conditions \mathcal{X}

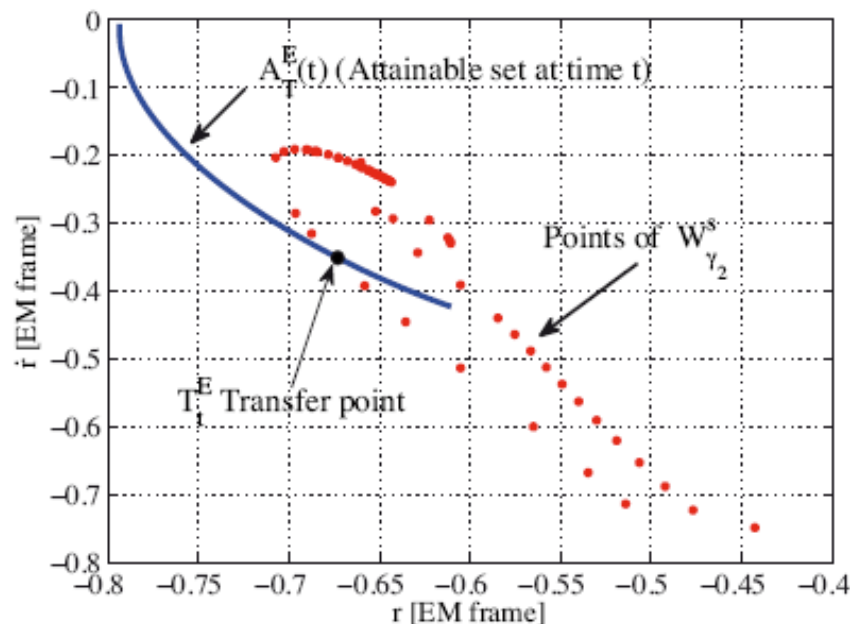
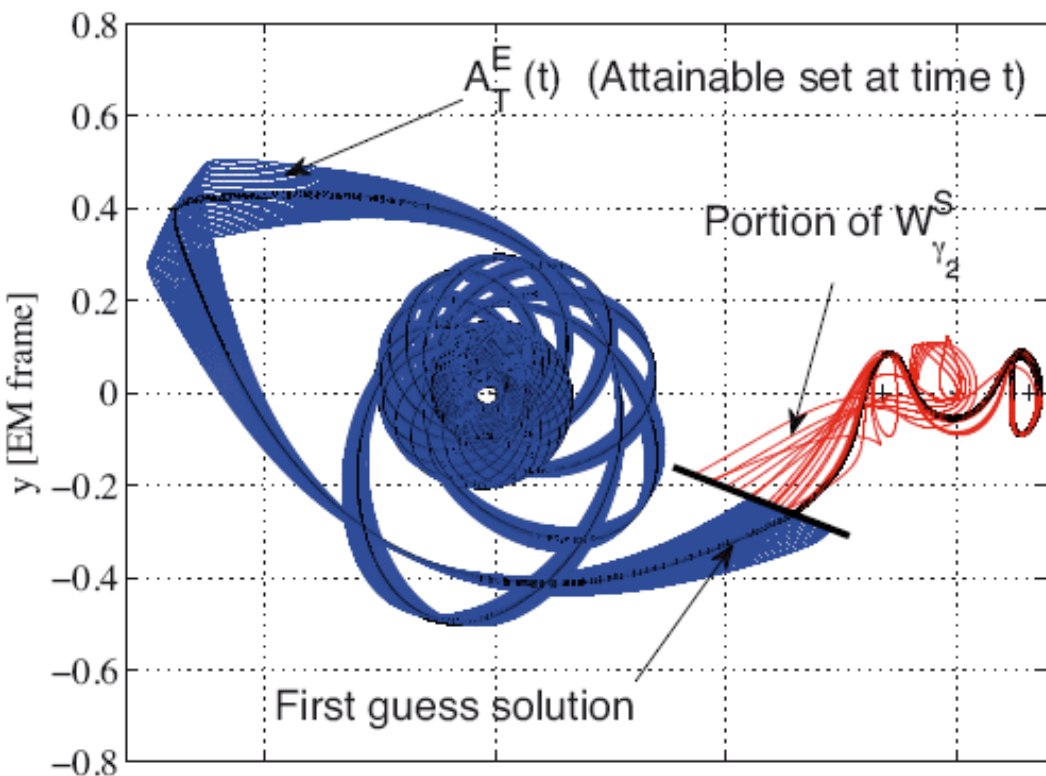


- Contain a set of **tangential low-thrust** orbits associated to admissible initial conditions
- **Tangential thrust** is considered in this stage (preliminary solution later used as first guess of optimal control problem)
- **Attainable sets** are manipulated in the same fashion as **invariant manifolds**





Attainable Sets for Transfers to Halo Orbits



$T = 0.5$ N; $A_z = 8000$ km; $t = 14.8$

- Departure: GTO
- Arrival: L₂ halo

same as before !

$$\mathcal{X}^E = \{x_0(\omega^E) | \omega^E \in [0, 2\pi]\}$$

- Preliminary solution found simply by: $\mathcal{T}_t^E = \mathcal{A}_T^E(t) \cap W_{\gamma_i}^s$
- Small discontinuities are admitted, i.e., $\|x^A - x^W\| \leq \varepsilon$



- Improve preliminary solutions found with attainable sets
- Eliminate possible discontinuities
- More refined dynamical model (controlled, bicircular RFBP):

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega_4}{\partial x} + \frac{T_x}{m}, \quad \ddot{y} + 2\dot{x} = \frac{\partial \Omega_4}{\partial y} + \frac{T_y}{m}, \quad \ddot{z} = \frac{\partial \Omega_4}{\partial z} + \frac{T_z}{m}, \quad \dot{\theta} = \omega_S, \quad \dot{m} = -\frac{T}{I_{sp} g_0}$$

- Includes perturbation of the Sun in the EM model

$$\Omega_4(x, y) = \Omega_3(x, y, \mu_{EM}) + \frac{m_s}{r_s} - \frac{m_s}{\rho^2} (x \cos \theta + y \sin \theta).$$

- Sun-spacecraft distance

$$r_s^2 = (x - \rho \cos \theta)^2 + (y - \rho \sin \theta)^2$$

- Normalized Sun parameters:

- Sun-EM distance $\rho = 3.8881 \cdot 10^2$
- mass of the Sun $m_s = 3.2890 \cdot 10^5$
- angular velocity relative to EM system $\omega_s = -0.9251$

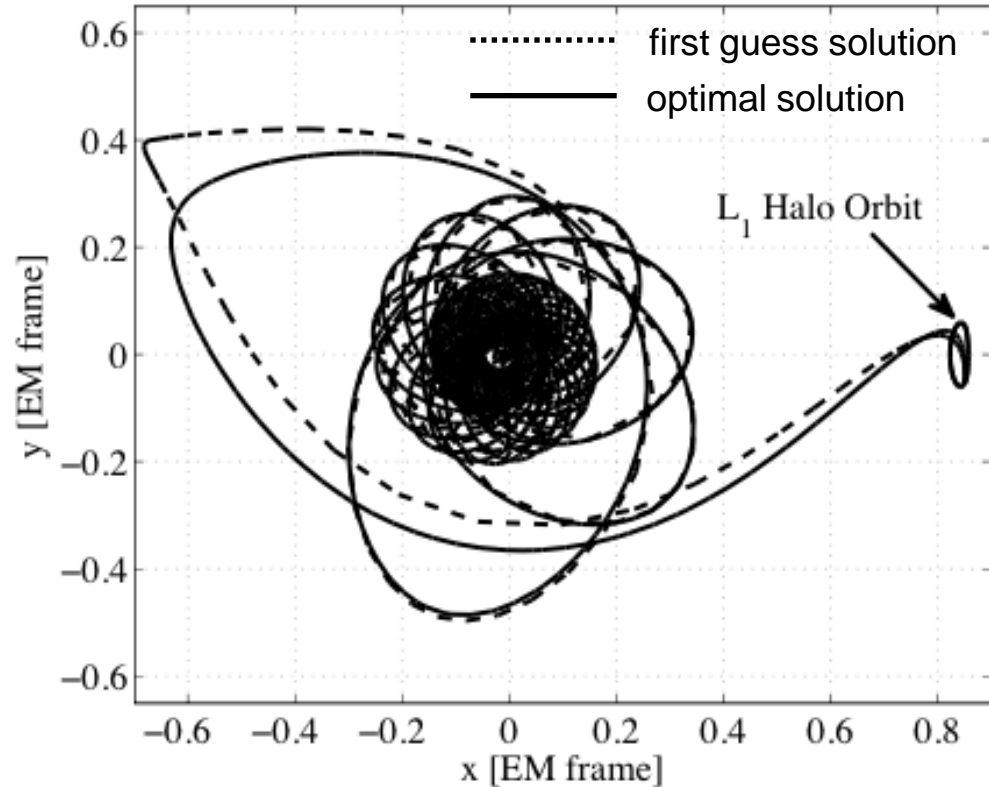


- Objective function (propellant mass minimization): $J = \int_{t_0}^{t_f} \frac{T(\tau)}{I_{sp} g_0} d\tau$
- Path constraints (saturation of thrusters): $T(t) \leq T_{\max}$
- Boundary conditions:
 - depart at periapsis of parking orbit
 - perigee and point on final halo orbit $x_f = x^W, \quad x^W \in W_{\gamma_i}^s$
- Solution by **Direct Transcription and Multiple Shooting**:
 - time domain split into N-1 intervals
 - each interval is split into M-1 segments
 - control discretized over the M nodes, using spline interpolation
 - nonuniform time grid



First Guess vs Optimal Solution

Transfer to halo orbit

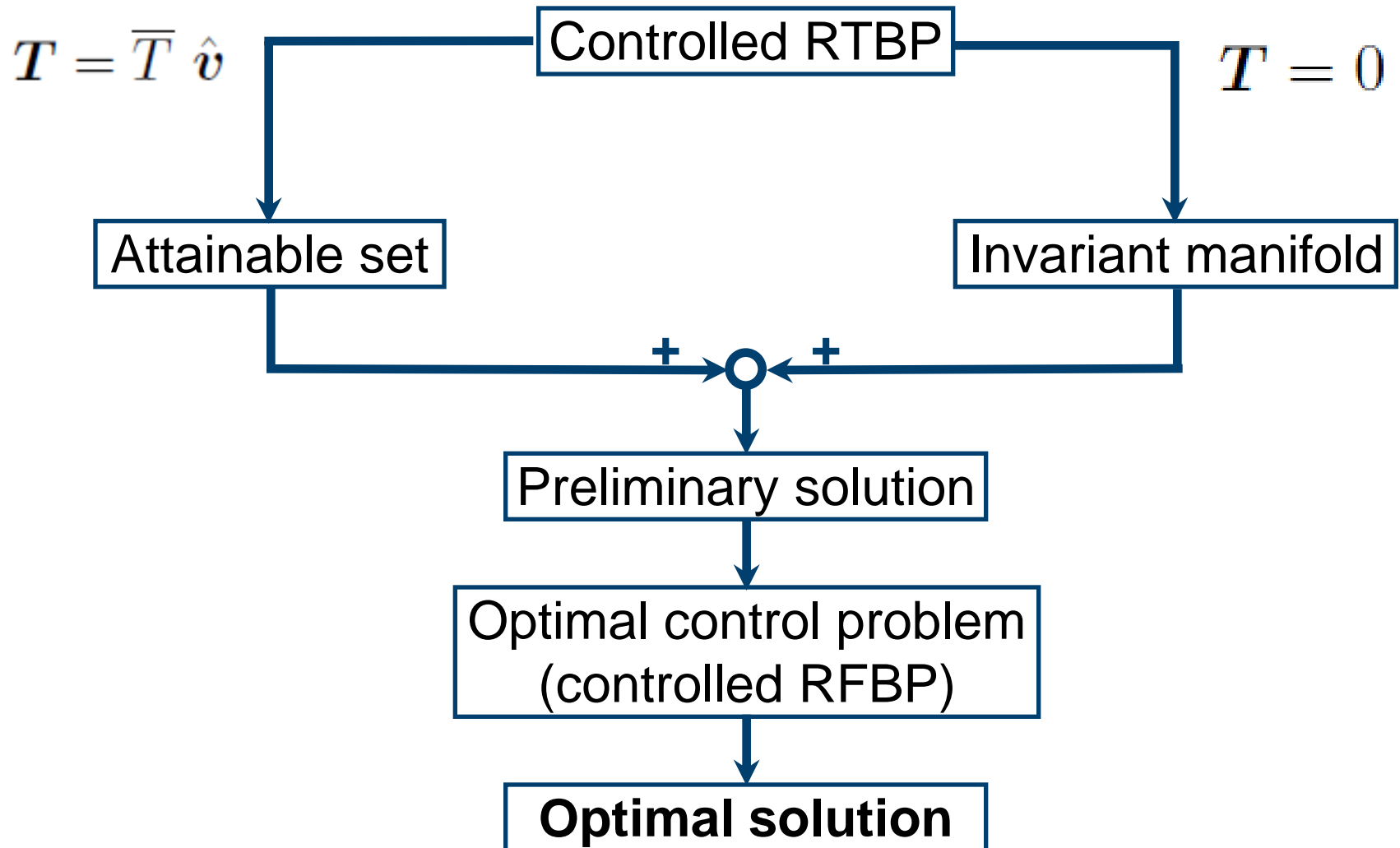


Direct transcription and multiple shooting (RKF78 + cubic spline)

- end-to-end trajectory transcription
- non-uniform grid implementation



Approach Summary





Results

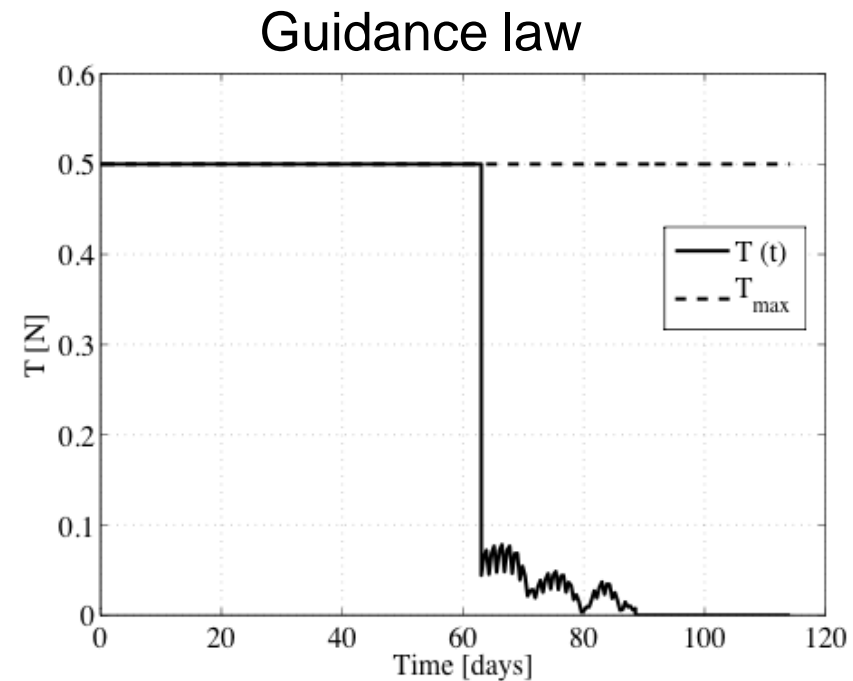
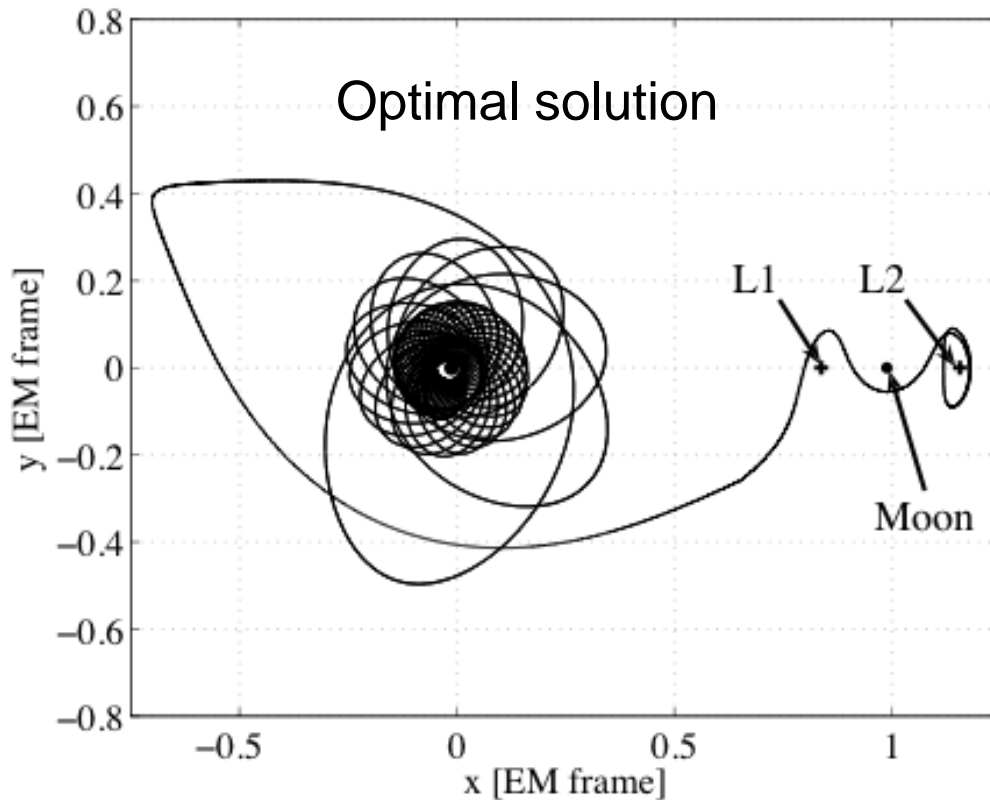
- Departure orbit
- Arrival orbit
- Propellant mass fraction
- Transfer time

Comparisons

- Optimal low-thrust, invariant-manifold solutions vs
 - Solutions achieved with impulsive maneuvers
 - Existing solutions in literature



Low-Thrust, Stable-Manifold Transfers to Halos



- ~ 300 multiple shooting segments (i.e., ~5100 var., 3300 constr.)
- The **signature** of the preliminary solution is clearly visible



Low-Thrust, Stable-Manifold Transfers to Halos

LTSM: Low-thrust, stable-manifold

HTSM: High-thrust, stable-manifold (impulsive maneuvers)

Type	Initial Orbit		Final Orbit		I_{sp} [s]	m_p/m_0	Δt [days]
	h_p^E [km]	e^E	A_z [km]	L_i			
LTSM#1	200	0	8000	L_1	3000	0.171	178
LTSM#2	400	0.72	8000	L_1	3000	0.090	91
LTSM#3	200	0	8000	L_2	3000	0.183	195
LTSM#4	400	0.72	8000	L_2	3000	0.091	114
HTSM#1	200	0	8000	L_1	300	0.711	67
HTSM#2	200	0	8000	L_2	300	0.713	65

- LEO: $h = 200$ km, $e = 0$
- GTO: $h_p = 400$, $e = 0.72$

- $m_0 = 1000$ kg



- A method to **include low-thrust** orbit pieces in the “standard” invariant manifolds frame has been derived
- Low-thrust **attainable sets** are manipulated in the same fashion as invariant manifolds
- Preliminary solutions are **optimized** in a four-body frame (end-to-end optimization)
- Solutions **reduce** the propellant mass (due to higher I_{sp})
- Presented solutions are **not achievable** neither with patched-conics method nor with standard invariant manifolds technique



Suggested readings

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- Topputo, F., Low-Thrust Non-Keplerian Orbits: Analysis, Design, and Control, Ph.D. thesis, Politecnico di Milano, Italy, 2007.
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Incorporating low-thrust propulsion into the invariant manifolds technique for space trajectories design

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