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Franco Bernelli-Zazzera

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## **Incorporating low-thrust propulsion into the invariant manifolds technique for space trajectories design**

**F. Bernelli-Zazzera, F. Topputo**

**Politecnico di Milano, Dipartimento di Ingegneria Aerospaziale**

**G. Mingotti**

**Universität Paderborn, Institut für Mathematik**



## Context

- Space trajectory design using non-Keplerian orbits
  - Transfers to libration point orbits

## Aims

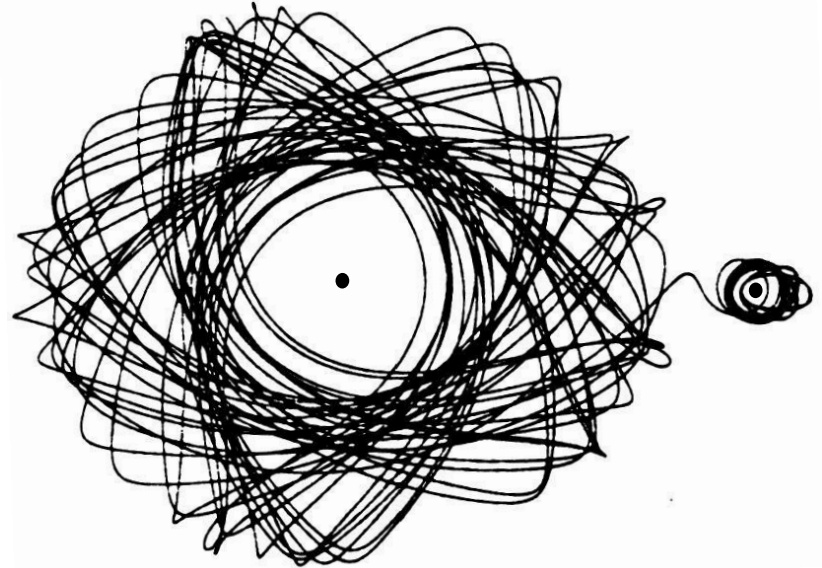
- Improve existing methods
  - Modify “invariant manifolds technique” to include low-thrust propulsion



1. Introduction to **non-Keplerian** orbits
2. Introduction to the **restricted three body problem**
3. Introduction to **equilibrium points, Halo orbits and invariant manifolds**
4. Dynamical system theory in space trajectory design
  - transfers to **Lagrange point orbits**
5. Introduction of **low-thrust** propulsion: controlled RTBP
6. Definition and manipulation of **attainable sets**
7. End-to-end trajectory **optimization**: controlled RFBP
8. Results and comparisons
  - **low-thrust, stable-manifold** transfers to halo orbits
9. Final remarks



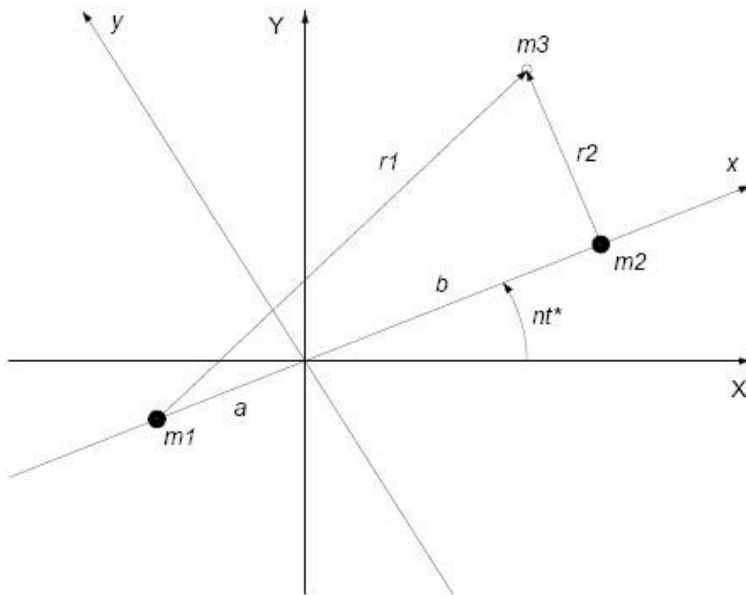
- Solution of  $n$ -body problems,  $n \geq 3$ 
    - Unique features
    - Less propellant
    - Accurate solutions
  - Non-integrable systems
  - Chaotic dynamics
  - Long transfer times
  - No systematic approach
- 
- How can we design non-Keplerian orbits for space missions?
    - ➔ Exploit knowledge of the **phase space** (e.g., equilibrium points, periodic orbits, invariant manifolds, etc.)





# The restricted three-body problem (1)

- Two *primaries* act on a third body (of negligible mass)
- The third body does not affect the motion of the primaries
- The two primaries move in circular orbits around their center of mass (circular restricted three-body problem)



The motion of the primaries is *given*:

$$\begin{aligned} X_1 &= -a \cos nt^*, & X_2 &= b \cos nt^*, \\ Y_1 &= -a \sin nt^*, & Y_2 &= b \sin nt^*. \end{aligned}$$



## The restricted three-body problem (2)



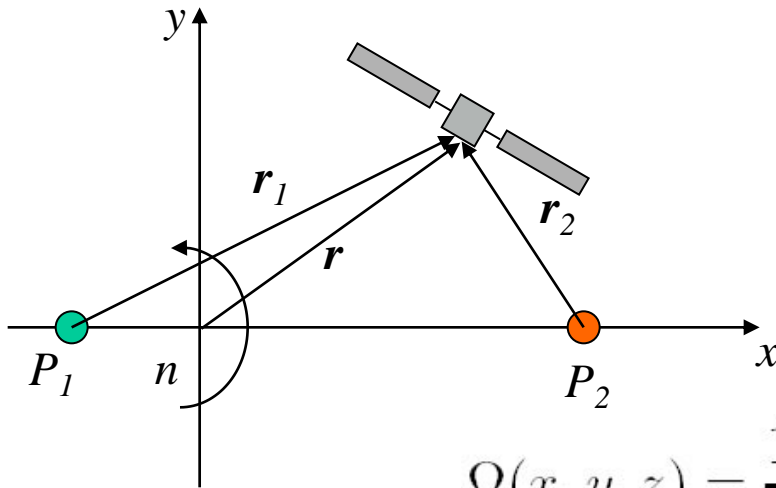
- the problem is *restricted* to the motion of the third body
- Newton equation written in *inertial* frame:

$$\left\{ \begin{array}{l} \frac{d^2 X}{dt^{*2}} \\ \frac{d^2 Y}{dt^{*2}} \\ \frac{d^2 Z}{dt^{*2}} \end{array} \right. = -G \left[ \begin{array}{l} \frac{m_1(X + a \cos nt^*)}{R_1^3} + \frac{m_2(X - b \cos nt^*)}{R_2^3} \\ \frac{m_1(Y + a \sin nt^*)}{R_1^3} + \frac{m_2(Y - b \sin nt^*)}{R_2^3} \\ \frac{m_1 Z}{R_1^3} + \frac{m_2 Z}{R_2^3} \end{array} \right],$$



# The restricted three-body problem (3)

- Equations written in a *dimensionless* rotating (*synodic*) frame make the dynamical system autonomous



$$\ddot{x} - 2\dot{y} = \Omega_x,$$

$$\ddot{y} + 2\dot{x} = \Omega_y,$$

$$\ddot{z} = \Omega_z,$$

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1 - \mu)$$

$$r_1^2 = (x + \mu)^2 + y^2 + z^2, \quad r_2^2 = (x - 1 + \mu)^2 + y^2 + z^2$$

Conventions:

$$G = 1$$

$$n = 1$$

$$m_1 = 1 - \mu$$

$$x_1 = -\mu$$

$$m_2 = \mu$$

$$x_2 = 1 - \mu$$

$$\mu = m_2 / (m_1 + m_2)$$

**Mass parameter of the R3BP**

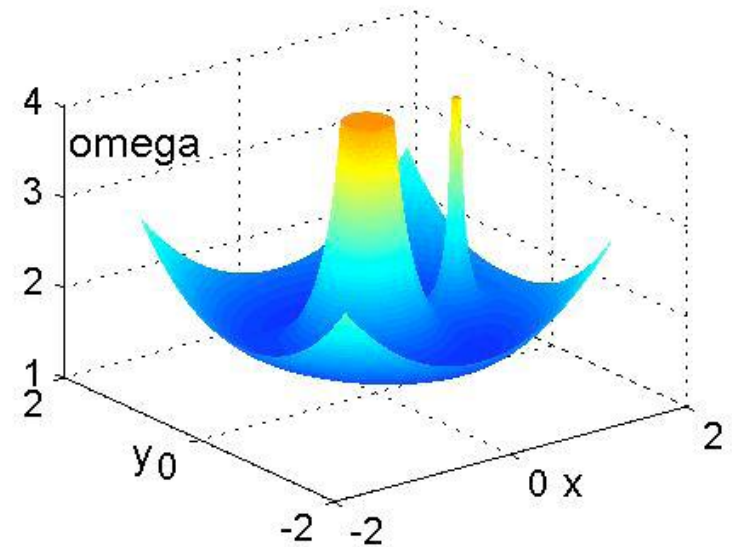




## The restricted three-body problem (4)

- The R3BP has *only one* integral of motion: the Jacobi integral
- The problem is not integrable! Qualitative studies only
- Allowed and forbidden regions are established (Hill's curves)

$$C = 2\Omega(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

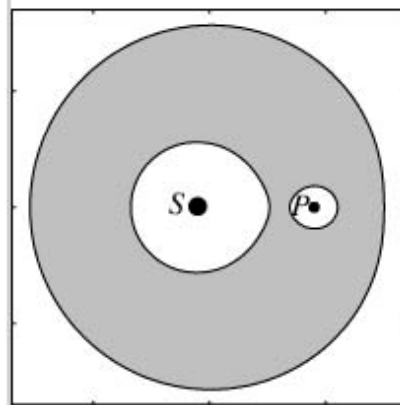


- Given  $x_0$ , then  $C_0 = C(x_0)$  and the allowed region will be  $(x, y, z)$  such that  $\Omega(x, y, z) \geq C_0/2$

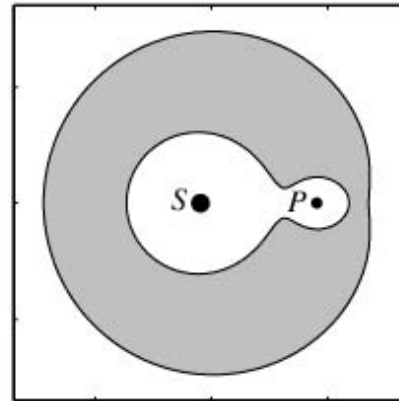


# The restricted three-body problem (5)

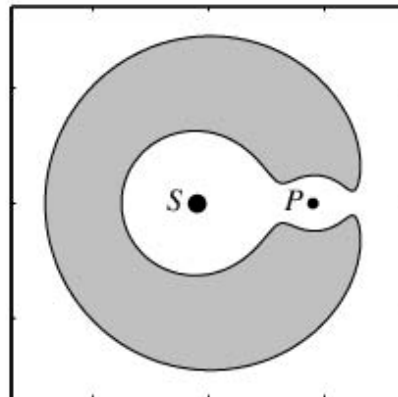
- $C = -2E$
- The higher  $C$ , the lower energy!
- For  $C < 3$ , the S/C is allowed to move in the whole configuration space



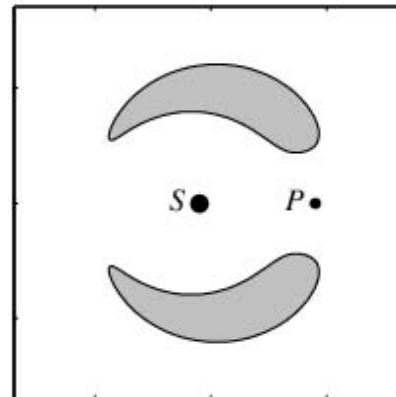
(a) Case 1 :  $E < E_1$



(b) Case 2 :  $E_1 < E < E_2$



(c) Case 3 :  $E_2 < E < E_3$



(d) Case 4 :  $E_3 < E < E_4 = E_5$



# Equilibrium points (1)

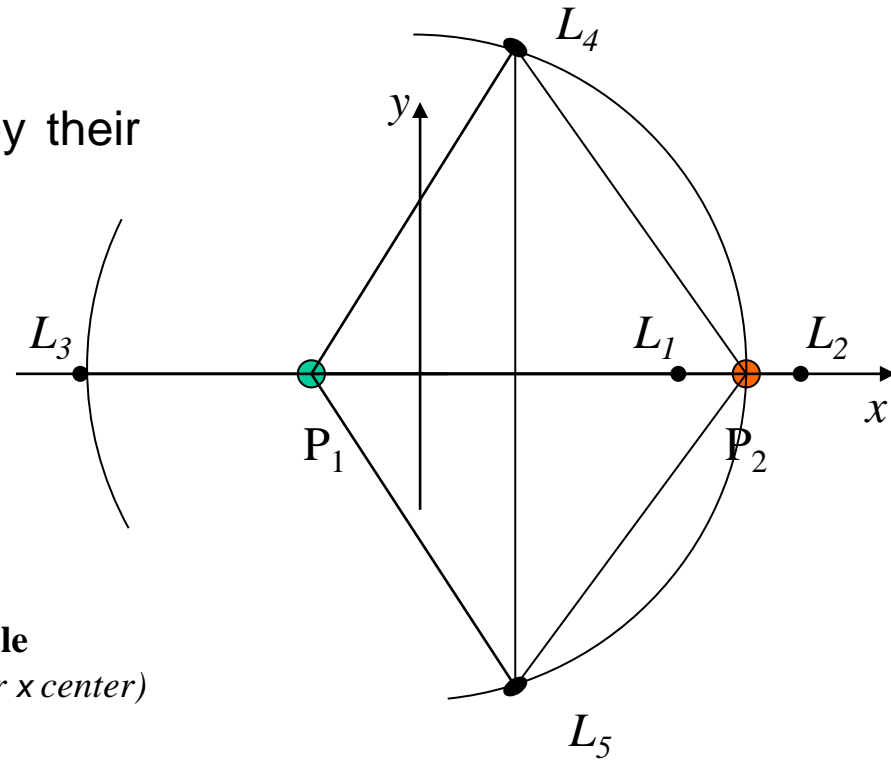
- The differential system has 5 equilibrium points (libration or Lagrangian points).
- The “nature” of the points is given by their linear behaviour.

$$\dot{x} = f(x) \longrightarrow \dot{x} = Ax \longrightarrow \lambda_i$$

$$L1, L2, L3: \begin{cases} \lambda_1 = -\lambda_2 \\ \lambda_{3/4} = \pm i\alpha_1 \longrightarrow \text{Unstable} \\ \lambda_{5/6} = \pm i\alpha_2 \quad (\text{saddle } \times \text{ center } \times \text{ center}) \end{cases}$$

$$L4, L5: \begin{cases} \lambda_{1/2} = \pm i\beta_1 \\ \lambda_{3/4} = \pm i\beta_2 \longrightarrow \text{Stable} \\ \lambda_{5/6} = \pm i\gamma \end{cases} \quad (\text{center } \times \text{ center } \times \text{ center})$$

if  $\mu < \mu_0 = 0.0385$



**Note:** equilibrium is relative to the synodic system!



## Equilibrium points (2)

- Collinear points  $L1$  and  $L2$  are very important for space applications
  - associated to *low energy* values (high values of  $C$ )
  - have “fast” stable and unstable dynamics (strong instability)
- *Saddle* part (stable and unstable eigenvalues)
  - Hyperbolic orbit near  $L1$  and  $L2$
- *Center* part (complex conjugate eigenvalues)
  - Periodic orbits around  $L1$  and  $L2$
- *Invariant manifolds* of the points
  - $W^s_{L_i}$ , stable manifold
  - $W^u_{L_i}$ , unstable manifold
- The invariant manifolds of the points have interesting dynamical properties that can be exploited for space applications



# Invariant manifolds of equilibrium points (1)

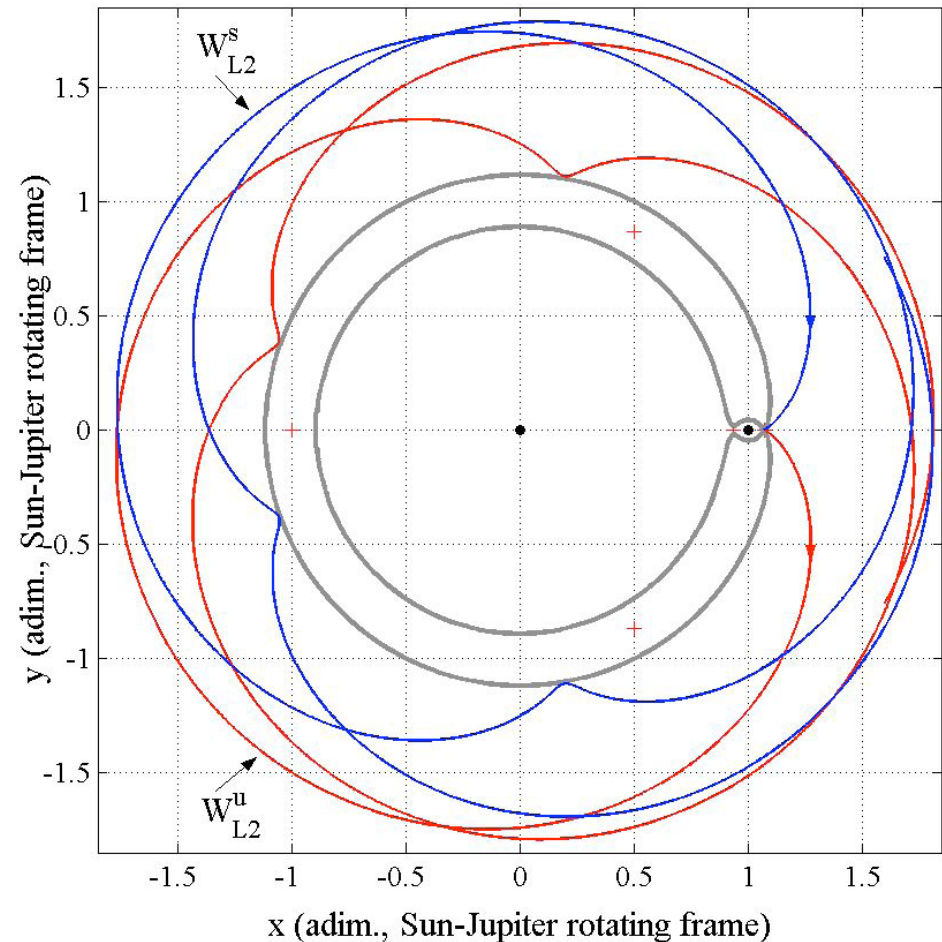
- *Nonlinear* invariant manifolds are defined in the phase space as *1D* invariant subsets such that

$$W_{L_i}^s \equiv \{x \in \mathbb{R}^6 / x(t) \rightarrow L_i, t \rightarrow +\infty\}$$

$$W_{L_i}^u \equiv \{x \in \mathbb{R}^6 / x(t) \rightarrow L_i, t \rightarrow -\infty\}$$

## Sun-Jupiter system

- $W_{L_2}^s$  (blue)
- $W_{L_2}^u$  (red)
- The two *1D* manifolds extend for more than *3 AU* far from the equilibrium point.

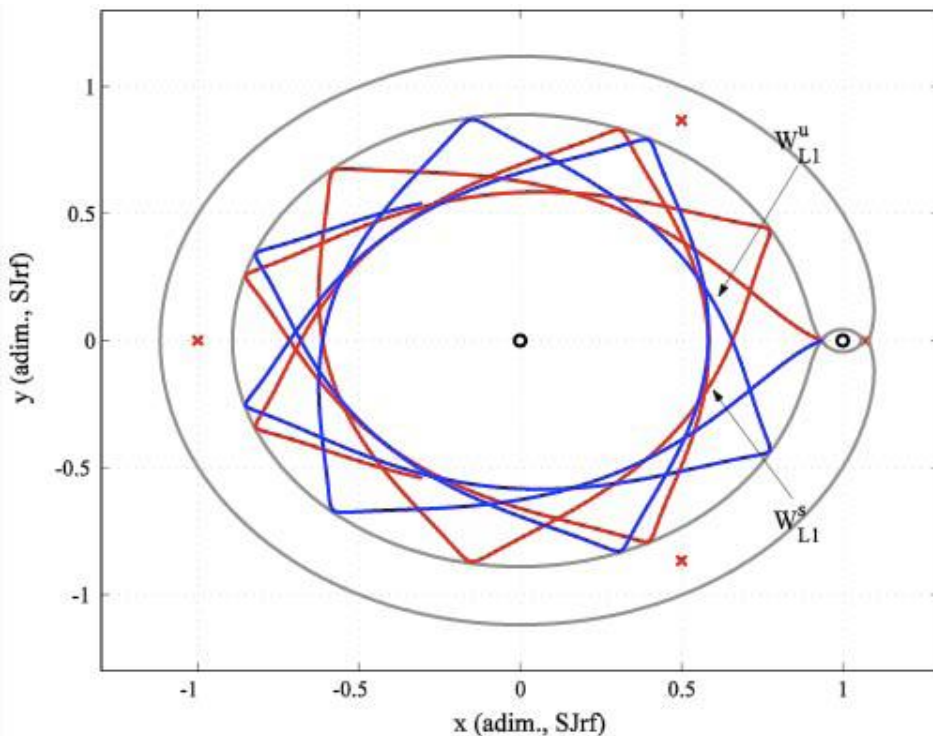




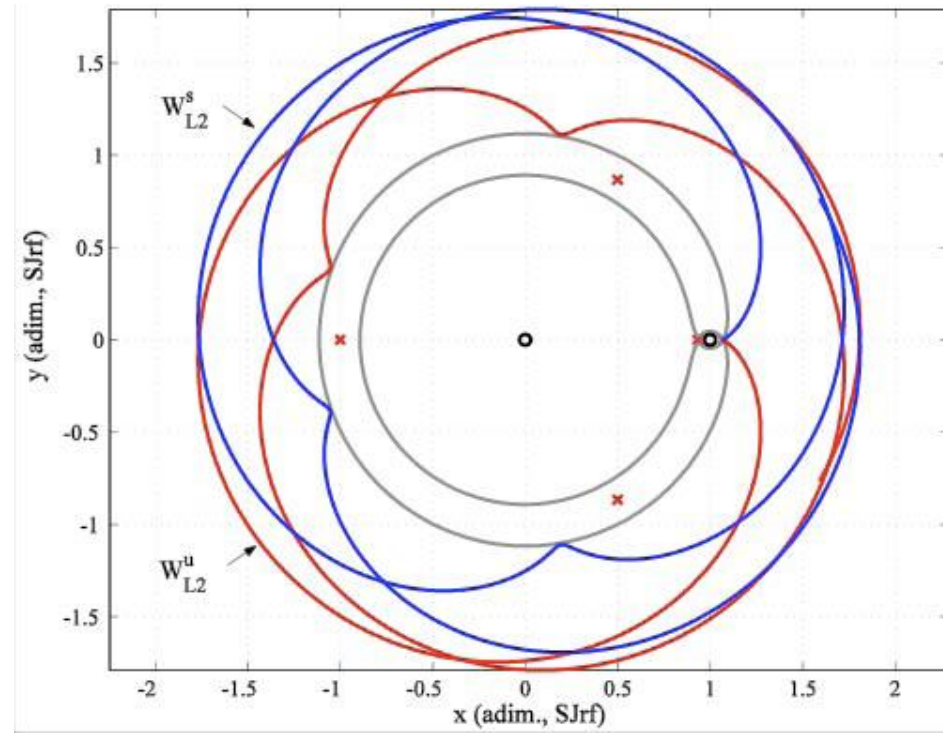
## Invariant manifolds of equilibrium points (2)

- If a point lies on the stable (unstable) manifold, in the phase space sense, it will move toward (away) the libration point
- Possibility to have free transfers in the solar system

$W^{s,u}_{L1}$



$W^{s,u}_{L2}$

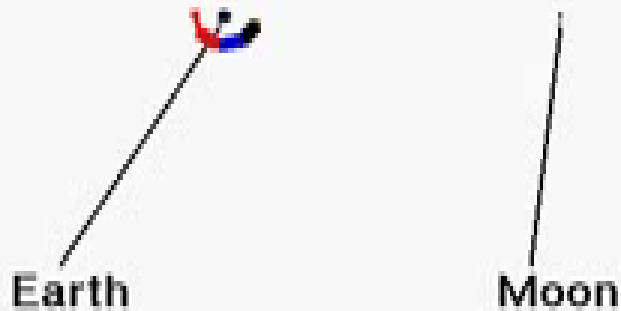




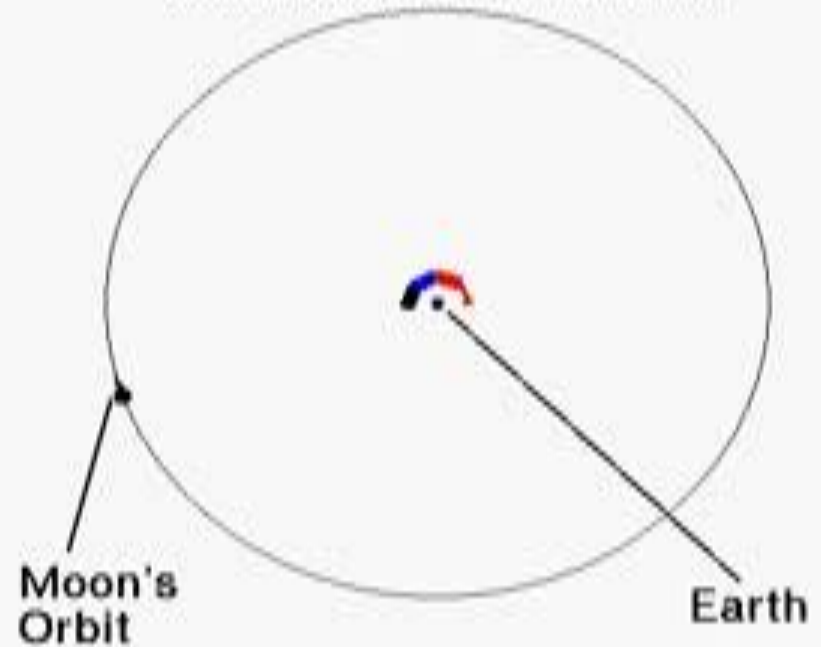
# Example – video by Shane Ross, Virginia Tech



**GEO to Moon Orbit Transfer  
Seen in Lunar Rotating Frame**



**GEO to Moon Orbit Transfer  
Seen in Geocentric Inertial Frame**

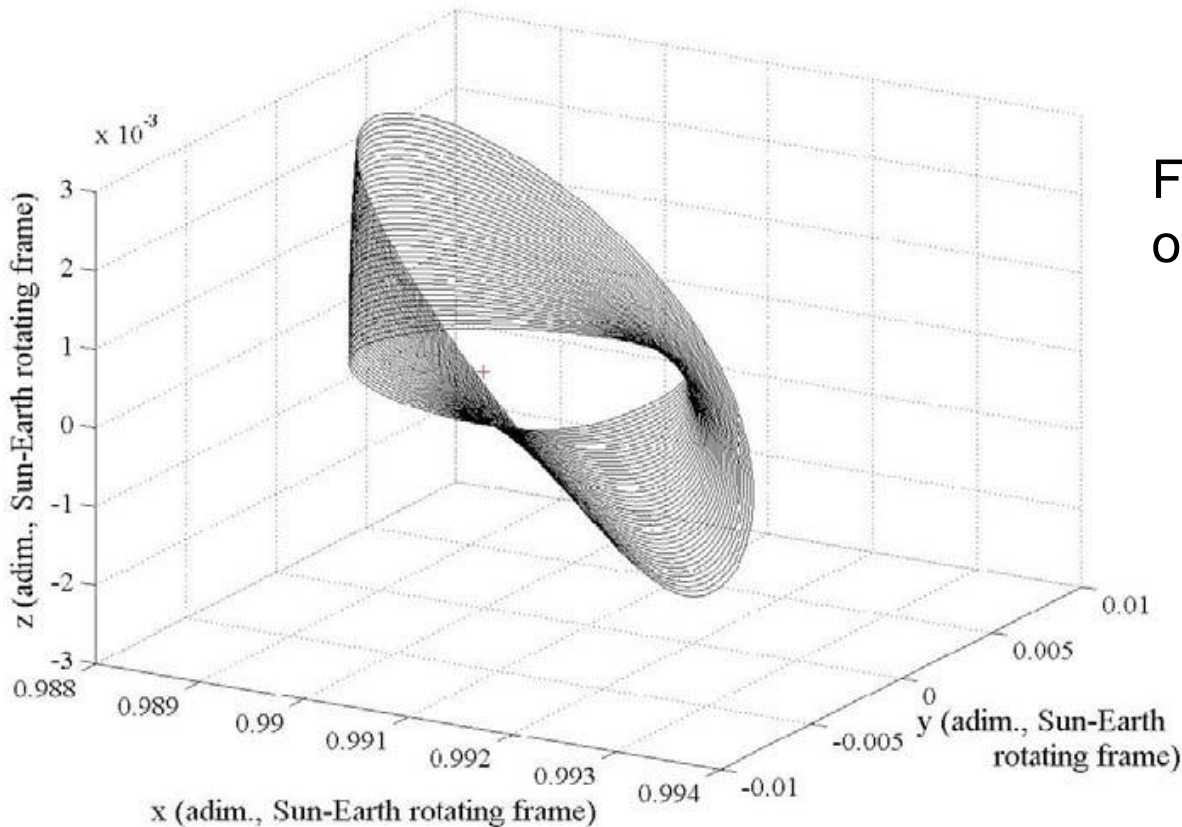






# Halo orbits (1)

- Large, 3D periodic orbits around collinear Lagrangian points
- $\omega = \nu$ , forced by the higher order (nonlinear) terms
- Suitable for many missions (large out-of-plane excursions)
- Obtained through semi-analytical methods



Family of halos around  $L_1$  of the Sun-Earth system





*Richardson's method* to construct halo orbits:

1. Find a third-order *analytical* first guess solution
2. *Differential correction* under the nonlinear dynamics

The R3BP has the following symmetry

$$S : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \rightarrow (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t)$$

- a trajectory that crosses perpendicularly the  $y=0$  plane twice is a periodic orbit

Given the analytic initial guess, it is corrected through

$$\Phi(\mathbf{x}_0 + \Delta\mathbf{x}, T_{1/2} + \Delta t) = \Phi(\mathbf{x}_0, T_{1/2}) + \left[ \frac{\partial\Phi(\mathbf{x}_0, T_{1/2})}{\partial\mathbf{x}} \right] \cdot \Delta\mathbf{x} + \frac{\partial\Phi(\mathbf{x}_0, T_{1/2})}{\partial t} \cdot \Delta t$$



# Invariant manifolds of Halo orbits (1)

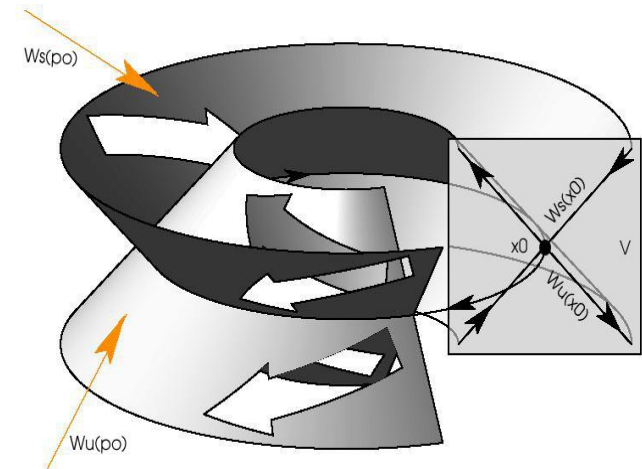
- In a neighborhood of the equilibrium points there are also *invariant manifolds* associated to the periodic orbits!
- These manifolds are more complicated “objects” than the simple manifolds associated to the equilibrium points
- They are due to the *saddle x center x center* part
- They are 2-dimensional subsets!
- Stable manifold (definition)  
$$W^s_{Li,p.o.} \equiv \{x \in \mathbb{R}^6 / x(t) \rightarrow p.o. , t \rightarrow +\infty\}$$
- Unstable manifold (definition)  
$$W^u_{Li,p.o.} \equiv \{x \in \mathbb{R}^6 / x(t) \rightarrow p.o. , t \rightarrow -\infty\}$$



## Invariant manifolds of Halo orbits (2)

- Periodic orbits have stable and unstable manifolds too!
- Viewed as equilibrium points on a Poincaré section

$$\mathbf{x} \mapsto \mathbf{x}_0 + M(\mathbf{x} - \mathbf{x}_0)$$



- They are given by the stable and unstable eigenvalues and eigenvectors of the monodromy matrix  $M$  associated to the periodic orbit



$$\mathbf{x}_0$$

- initial condition to generate a periodic orbit (given by the Richardson's method)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad , \quad \mathbf{x}(0) = \mathbf{x}_0$$

- integrate the dynamical system (of order 42) for  $0 \leq t \leq T$  to obtain  $\mathbf{x} = \mathbf{x}(t)$  and  $M = M(t)$

$$\dot{M} = \left[ \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right] M \quad , \quad M(0) = \mathbf{I}$$

$$\mathbf{x}^i = \mathbf{x}(t_i), \quad M^i = M(t_i)$$

- generic point of the orbit

$$\mathbf{v}_s^i$$

- stable eigenvector of  $M$

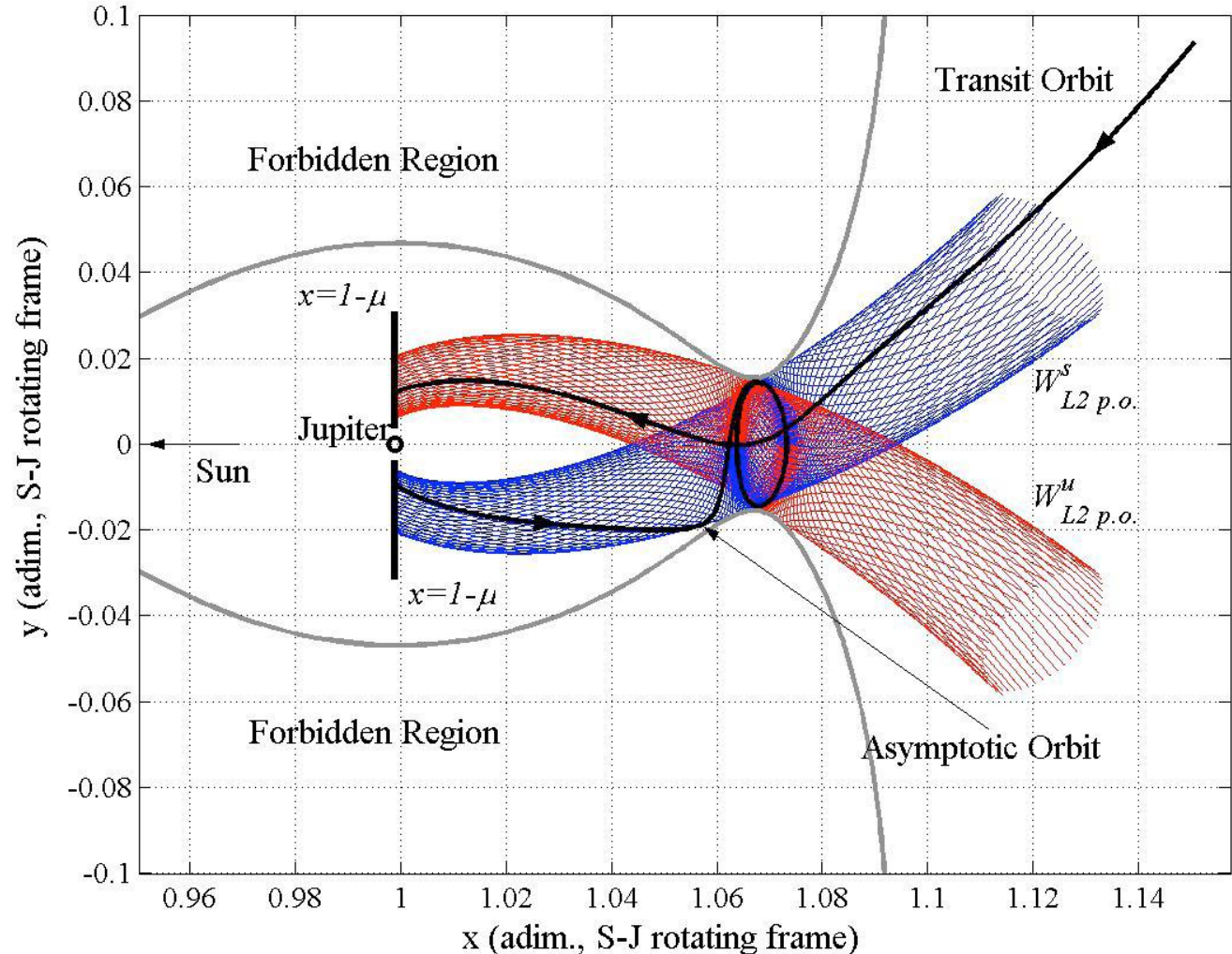
$$\mathbf{x}_{0,s}^i = \mathbf{x}^i \pm \varepsilon \mathbf{v}_s^i$$

- initial condition to integrate backward to get the stable manifold of  $\mathbf{x}_i$



# Invariant manifolds of Halo orbits (4)

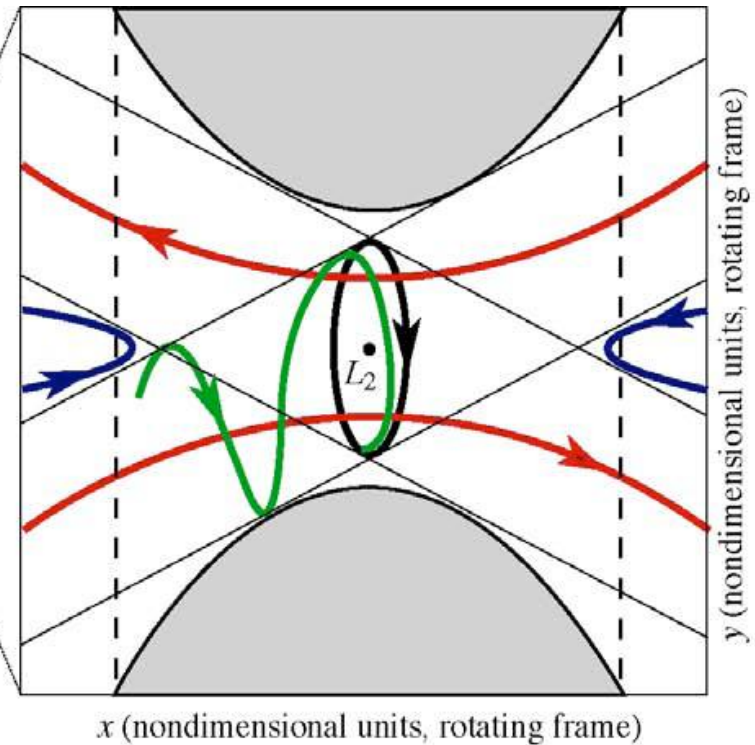
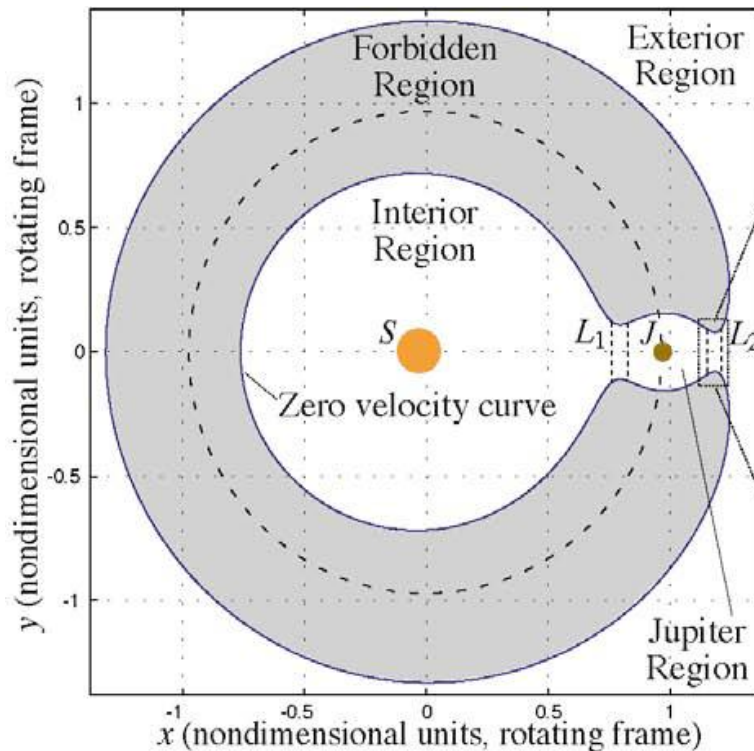
- Example of invariant manifold associated to a periodic orbit around L2 in the Sun-Jupiter system



blue:  $W^s_{L2 p.o.}$   
red:  $W^u_{L2 p.o.}$



- Complete phase portrait
  - asymptotic orbits
  - periodic orbits
  - *transit orbits* (inside the tubes)
  - non-transit orbits

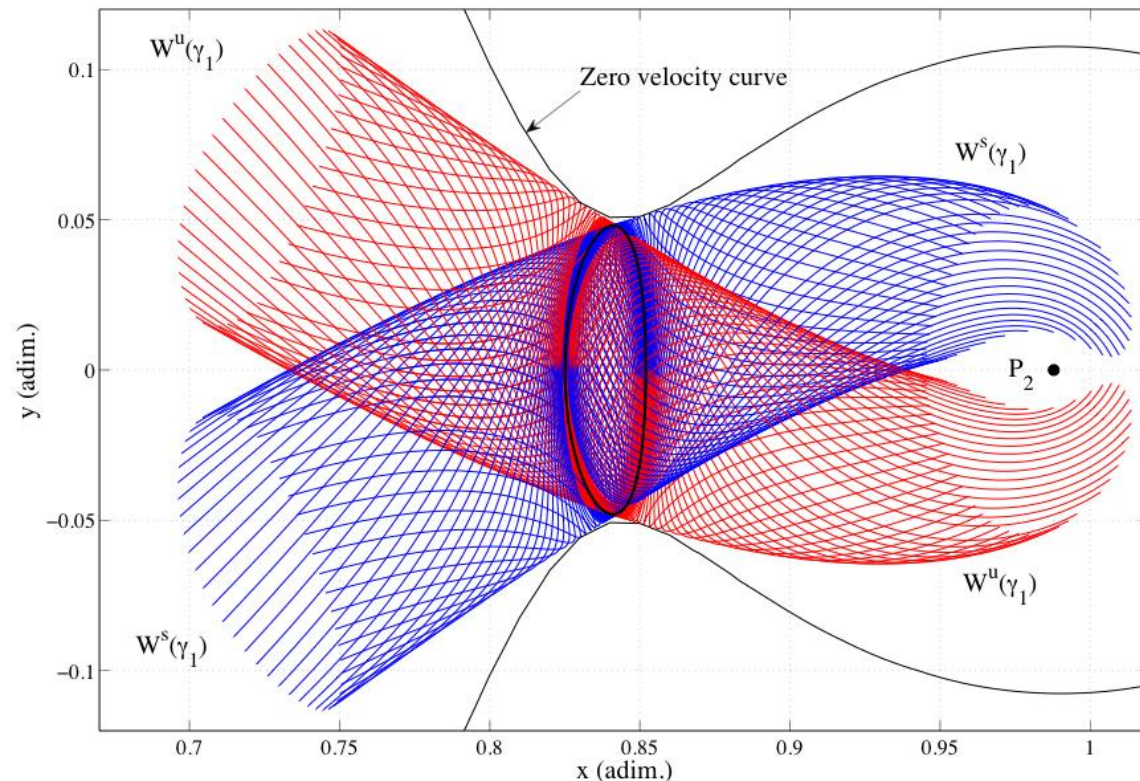


➤ Asymptotic orbits are useful for the trajectory design of libration point missions





- The phase space structure is exploited in a “natural” way.
- Equilibrium points, periodic orbits, invariant manifolds, etc. are used in place of orbital elements.



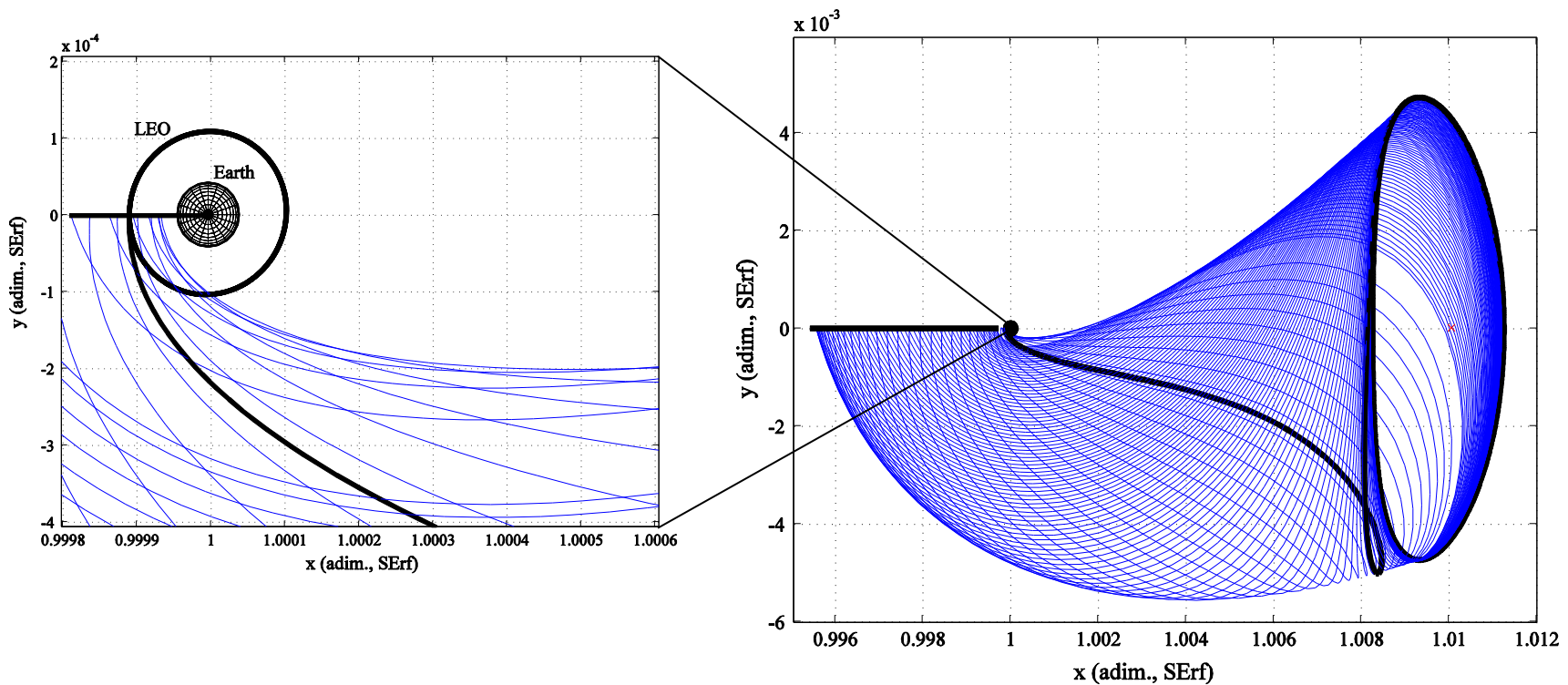
## Typical case studies

- Transfers to halo orbits
- Low energy transfers to the Moon



# Transfer to Halo orbits (1)

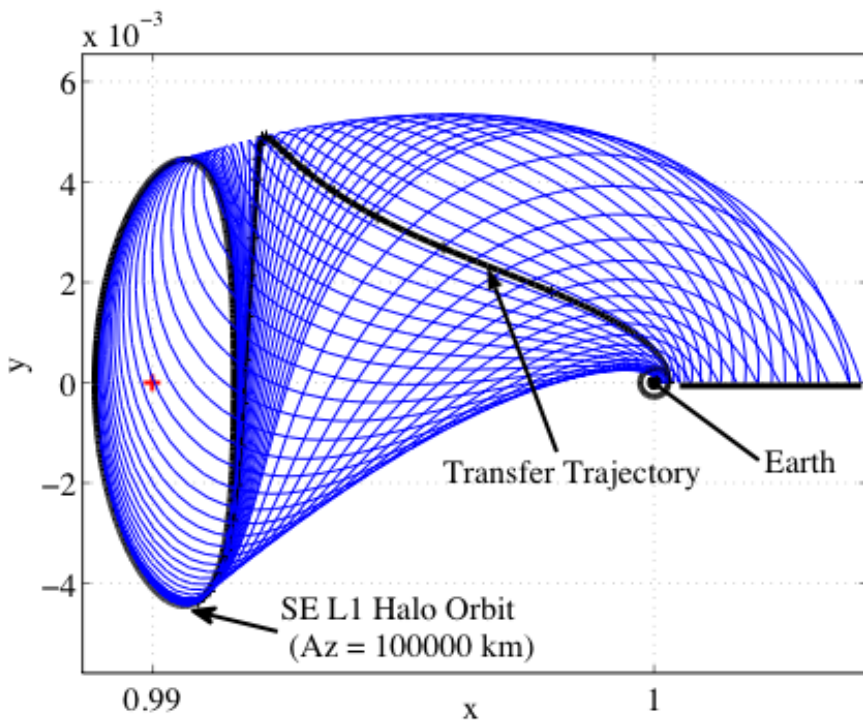
- The concept of **stable manifold** is exploited for a transfer to Halo orbits
  - place the S/C on the stable manifold associated to that orbit
  - *Wait* (the S/C wraps on the final orbit)
  - perform an orbit injection maneuver to stabilize in a halo





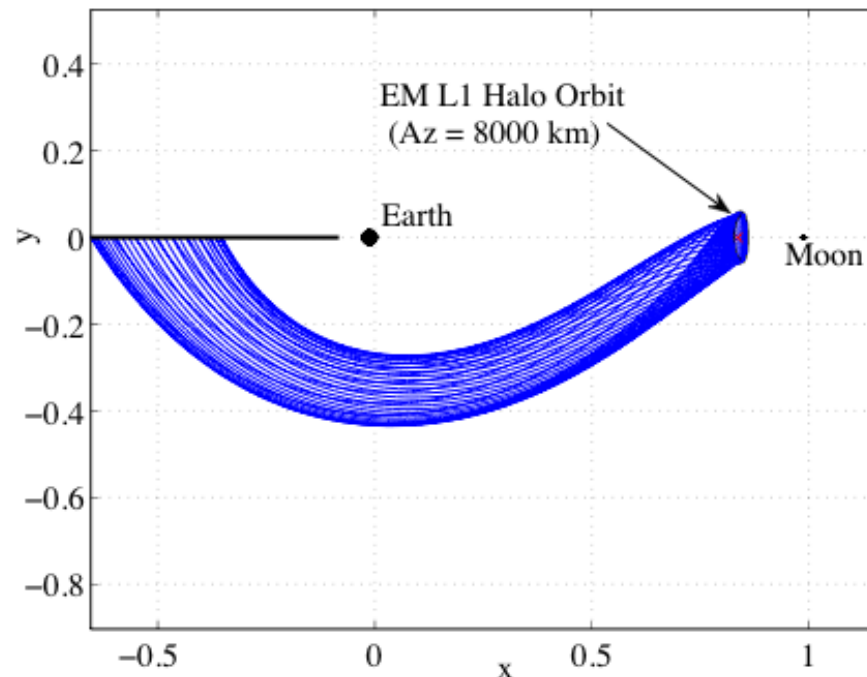


## Transfer to Halo orbits (2)



### Sun-Earth system

- $\Delta v \approx 3200$  m/s (from LEO)
- Single-impulse transfer



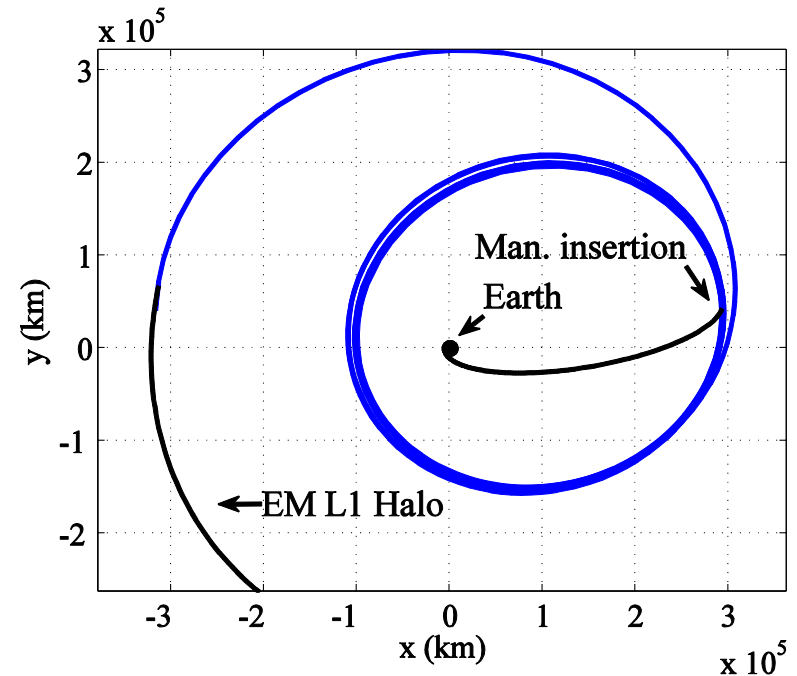
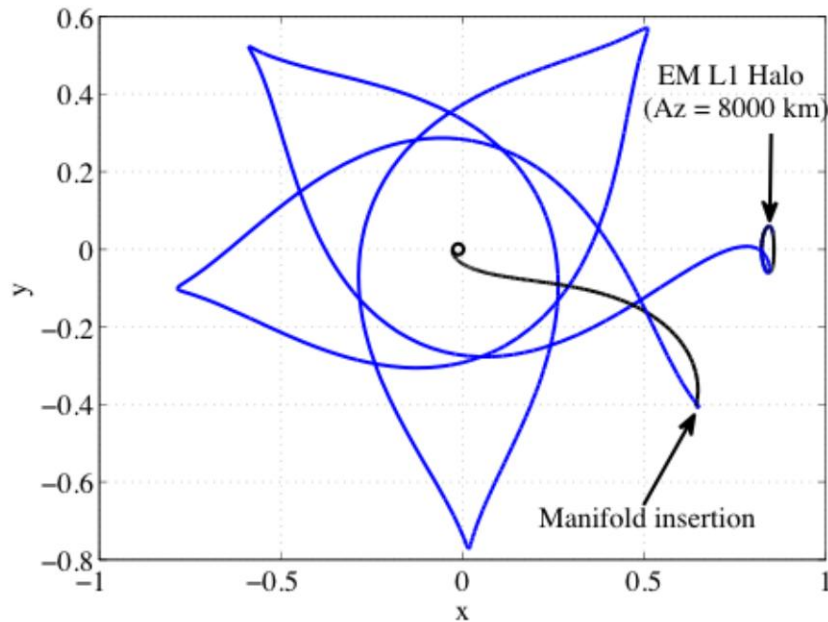
### Earth-Moon system

- $\Delta v = ?$
- No single-impulse transfer



## Transfer to Halo orbits (3)

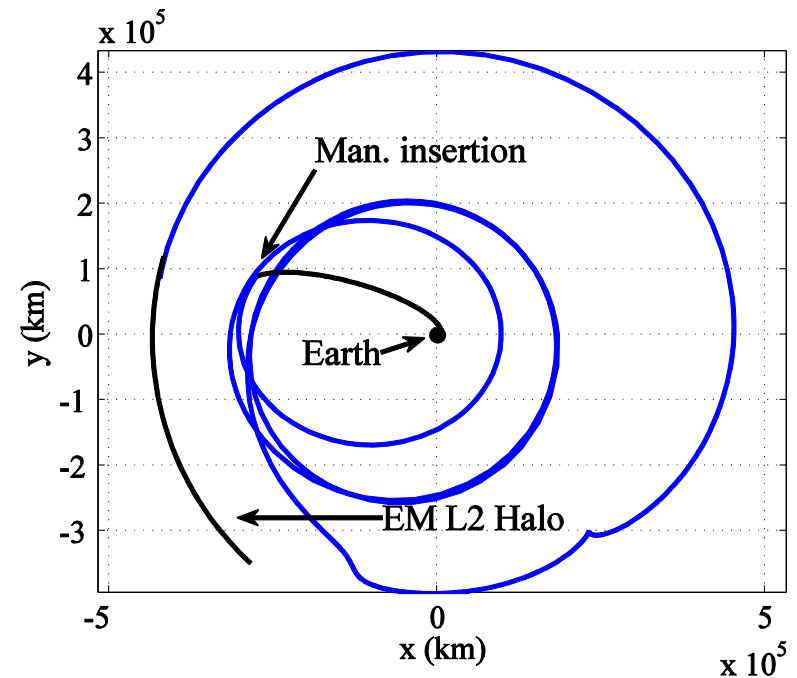
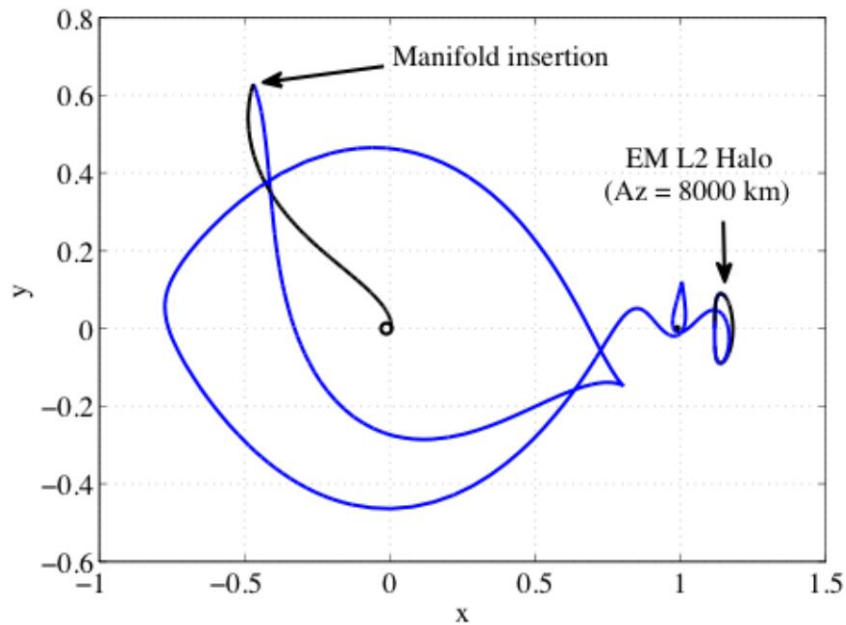
- A point on the stable manifold has to be targeted (two-impulse strategy)
- Transfer to  $L_1$





## Transfer to Halo orbits (4)

- A point on the **stable manifold** has to be targeted (two-impulse strategy)
- Transfer to  $L_2$





## Transfer to Halo orbits (5)



- In summary

Departing from circular, 200 km LEO

Type	$A_z$ (km)	$\Delta t$ (days)	$\Delta v$ (m/s)	$I_{sp}$ (s)	$m_p/m_0$
$L_1$	8000	67.3	3659	300	0.711
$L_2$	8000	65.0	3676	300	0.713



# Introduction of Low-Thrust Propulsion (1)

Dynamical system theory allows us to define solutions with

- Unique features (e.g., libration point orbits)
- Less propellant mass (e.g., low energy transfers)

Rocket equation:

$$\frac{m_p}{m_0} = 1 - e^{-\frac{\Delta v}{I_{sp}g_0}}$$

Invariant manifolds technique  
is used to reduce  $\Delta v$

Can we further reduce the propellant mass  
using **low-thrust**, **high- $I_{sp}$**  systems?

Can we further **improve** low-energy solutions?



## Introduction of Low-Thrust Propulsion (2)



The effect of low-thrust systems is **no longer** modeled as instantaneous velocity changes !



Need to **re-formulate** existing “invariant manifolds technique” to incorporate **low-thrust propulsion** ...



# Controlled Restricted Three-Body Problem

Equations of motion including propulsion

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega_3}{\partial x} + \frac{T_x}{m}, \quad \ddot{y} + 2\dot{x} = \frac{\partial \Omega_3}{\partial y} + \frac{T_y}{m}, \quad \ddot{z} = \frac{\partial \Omega_3}{\partial z} + \frac{T_z}{m}, \quad \dot{m} = -\frac{T}{I_{sp} g_0}$$

Present thrust magnitude:  $T = \sqrt{T_x^2 + T_y^2 + T_z^2}$

Variations of spacecraft mass,  $m$ , are considered through last eq.

The system is singular when  $m \rightarrow 0$



$$\mathbf{T}(t) = (T_x(t), T_y(t), T_z(t)), \quad t \in [t_0, t_f] \quad ?$$

The **guidance law** is not given, but rather it is an **unknown** of the optimal control problem ...



$$\mathbf{x}_0 = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, m_0) \quad \text{initial condition}$$

$$\phi_{\mathbf{T}(\tau)}(\mathbf{x}_0, t_0; t) \quad \text{with} \quad \mathbf{T}(\tau), \tau \in [t_0, t] \quad \text{controlled flow}$$

$$\mathbf{x}(t) = \phi_{\bar{\mathbf{T}}}(\mathbf{x}_0, t_0; t) \quad \text{with} \quad \mathbf{T} = \bar{\mathbf{T}} \hat{\mathbf{v}} \quad \text{tangential thrust flow}$$

$$\gamma_{\bar{\mathbf{T}}}(\mathbf{x}_0, t) = \{\phi_{\bar{\mathbf{T}}}(\mathbf{x}_0, t_0; \tau) \mid \tau < t\} \quad \text{tangential thrust orbit}$$

$$\boxed{\mathcal{A}_{\bar{\mathbf{T}}}(t) = \bigcup_{\mathbf{x}_0 \in \mathcal{X}} \gamma_{\bar{\mathbf{T}}}(\mathbf{x}_0, t)} \quad \text{Attainable set}$$



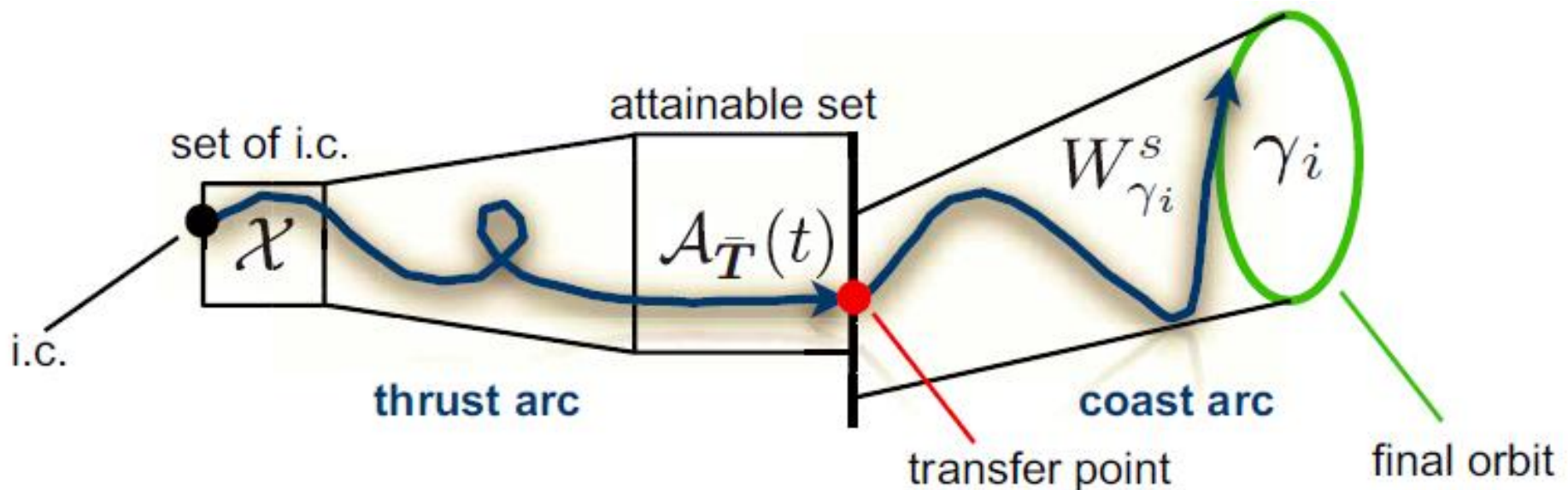
$\mathcal{A}_{\bar{\mathbf{T}}}$  is associated to an admissible set of initial conditions  $\mathcal{X}$





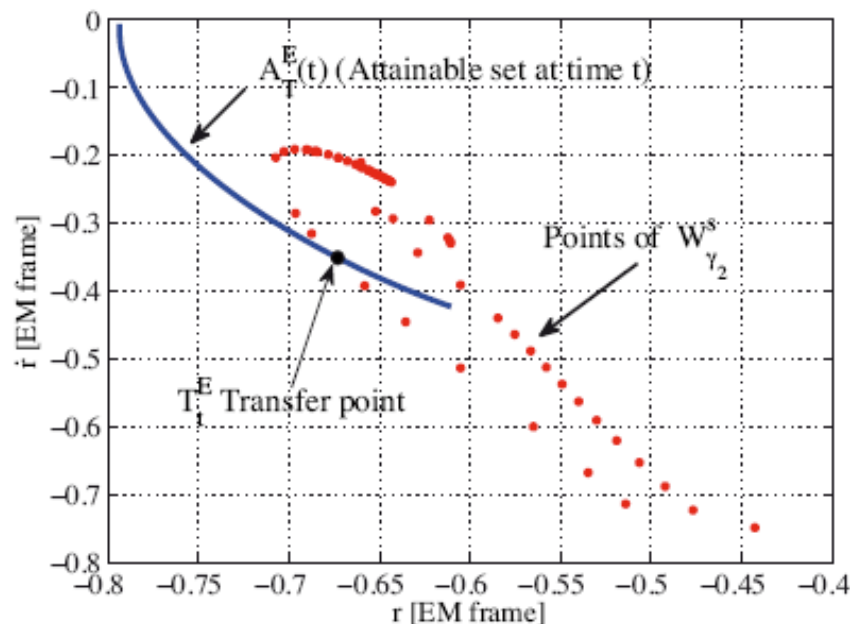
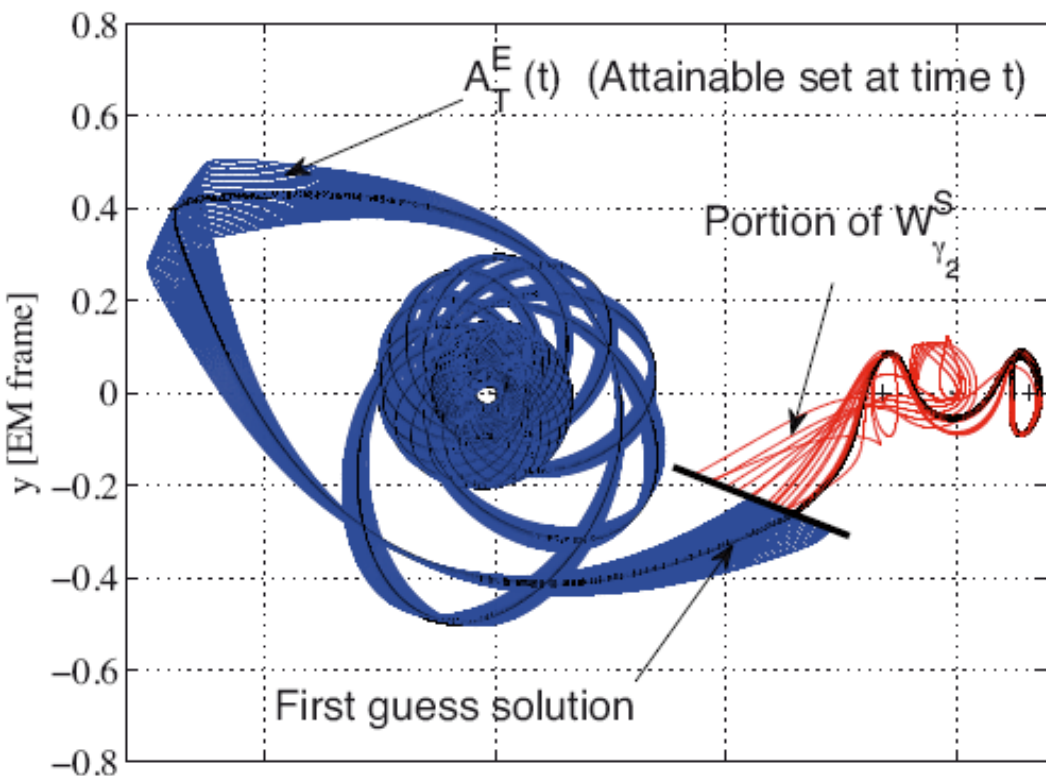
## Attainable sets

- Contain a set of **tangential low-thrust** orbits associated to admissible initial conditions
- **Tangential thrust** is considered in this stage (preliminary solution later used as first guess of optimal control problem)
- **Attainable sets** are manipulated in the same fashion as **invariant manifolds**





# Attainable Sets for Transfers to Halo Orbits



$T = 0.5$  N;  $A_z = 8000$  km;  $t = 14.8$

- Departure: GTO
- Arrival: L<sub>2</sub> halo

same as before !

- Preliminary solution found simply by:  $\mathcal{T}_t^E = \mathcal{A}_T^E(t) \cap W_{\gamma_i}^s$
- Small discontinuities are admitted, i.e.,  $\|\mathbf{x}^A - \mathbf{x}^W\| \leq \varepsilon$



- Improve preliminary solutions found with attainable sets
- Eliminate possible discontinuities
- More refined dynamical model (controlled, bicircular RFBP):

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega_4}{\partial x} + \frac{T_x}{m}, \quad \ddot{y} + 2\dot{x} = \frac{\partial \Omega_4}{\partial y} + \frac{T_y}{m}, \quad \ddot{z} = \frac{\partial \Omega_4}{\partial z} + \frac{T_z}{m}, \quad \dot{\theta} = \omega_S, \quad \dot{m} = -\frac{T}{I_{sp} g_0}$$

- Includes perturbation of the Sun in the EM model

$$\Omega_4(x, y) = \Omega_3(x, y, \mu_{EM}) + \frac{m_s}{r_s} - \frac{m_s}{\rho^2} (x \cos \theta + y \sin \theta).$$

- Sun-spacecraft distance

$$r_s^2 = (x - \rho \cos \theta)^2 + (y - \rho \sin \theta)^2$$

- Normalized Sun parameters:

- Sun-EM distance  $\rho = 3.8881 \cdot 10^2$
- mass of the Sun  $m_s = 3.2890 \cdot 10^5$
- angular velocity relative to EM system  $\omega_s = -0.9251$

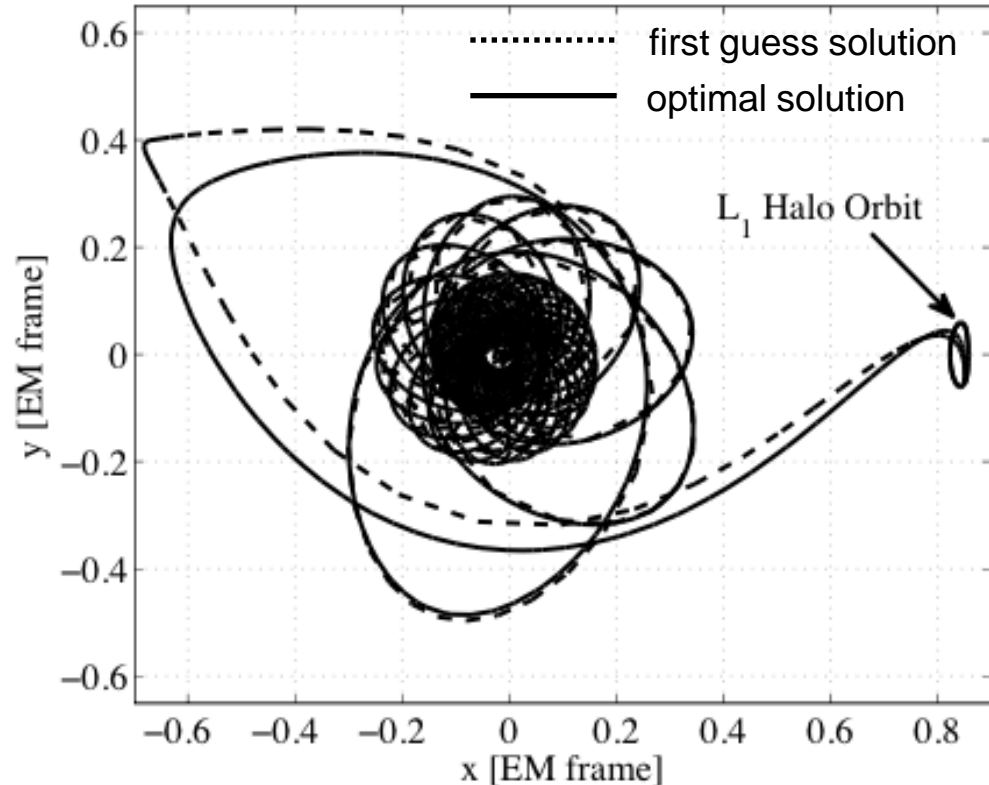


- Objective function (propellant mass minimization):  $J = \int_{t_0}^{t_f} \frac{T(\tau)}{I_{sp} g_0} d\tau$
- Path constraints (saturation of thrusters):  $T(t) \leq T_{\max}$
- Boundary conditions:
  - depart at periapsis of parking orbit
  - perigee and point on final halo orbit  $x_f = x^W, \quad x^W \in W_{\gamma_i}^s$
- Solution by **Direct Transcription and Multiple Shooting**:
  - time domain split into N-1 intervals
  - each interval is split into M-1 segments
  - control discretized over the M nodes, using spline interpolation
  - nonuniform time grid



# First Guess vs Optimal Solution

Transfer to halo orbit

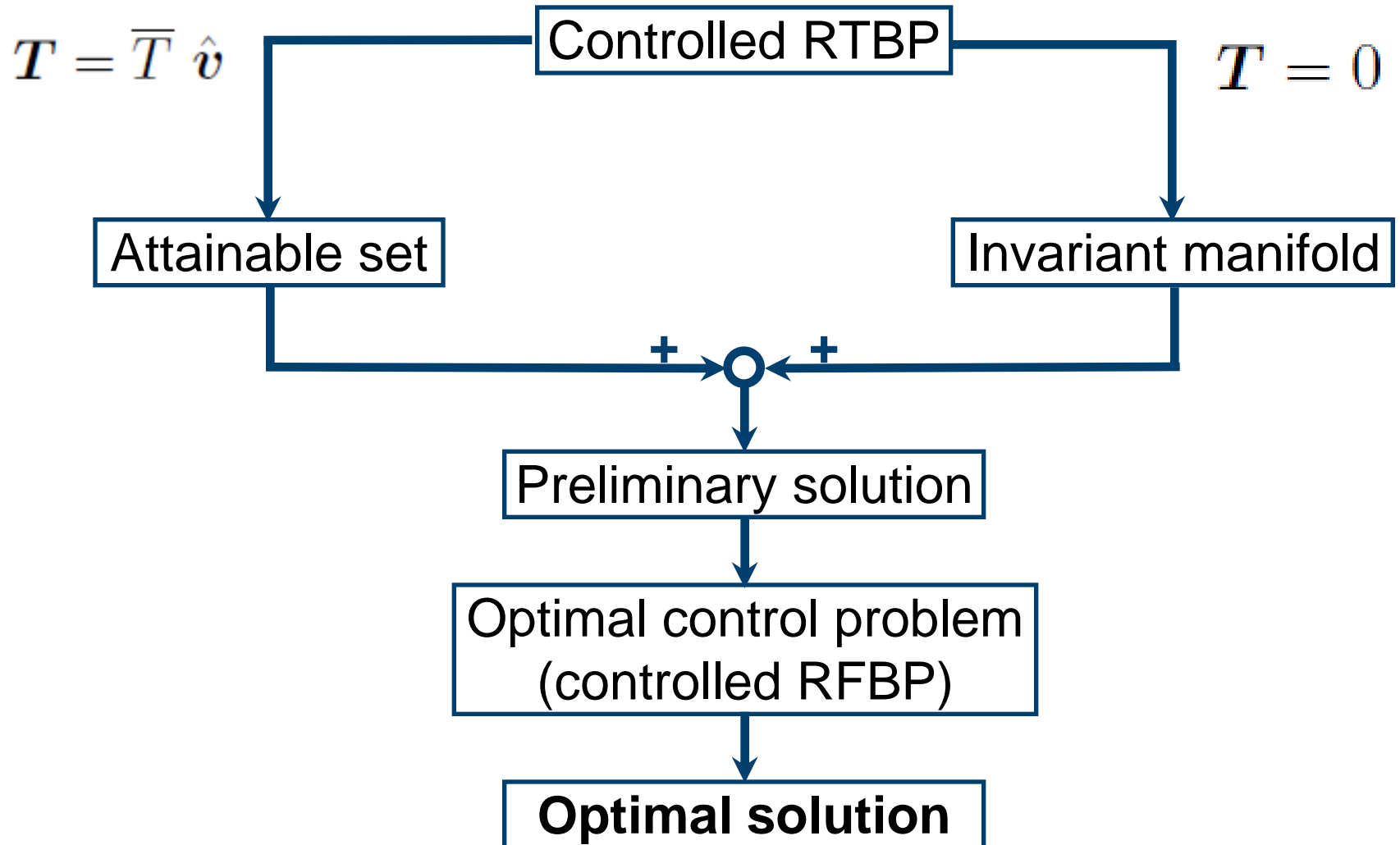


Direct transcription and multiple shooting (RKF78 + cubic spline)

- end-to-end trajectory transcription
- non-uniform grid implementation



# Approach Summary





## Results

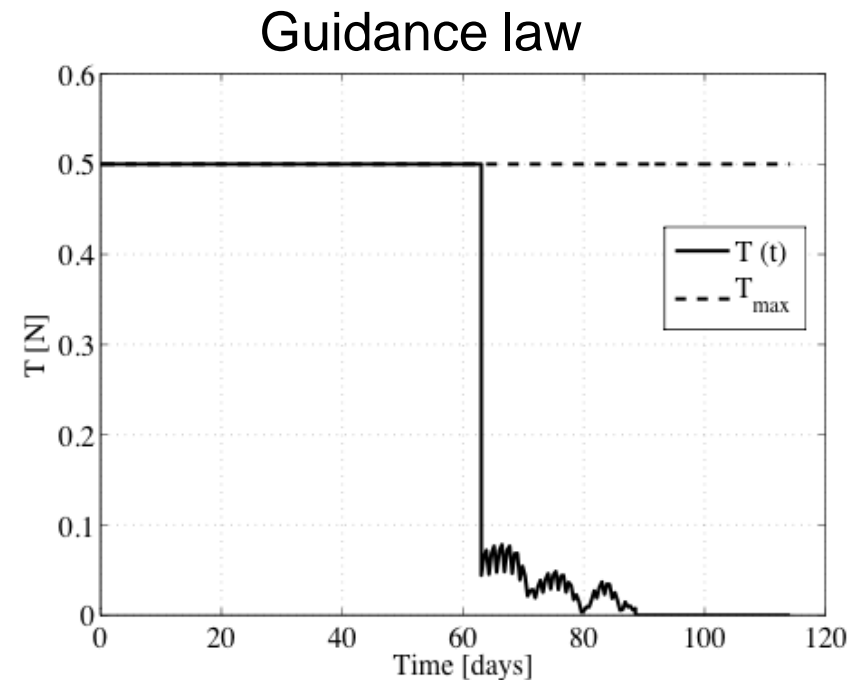
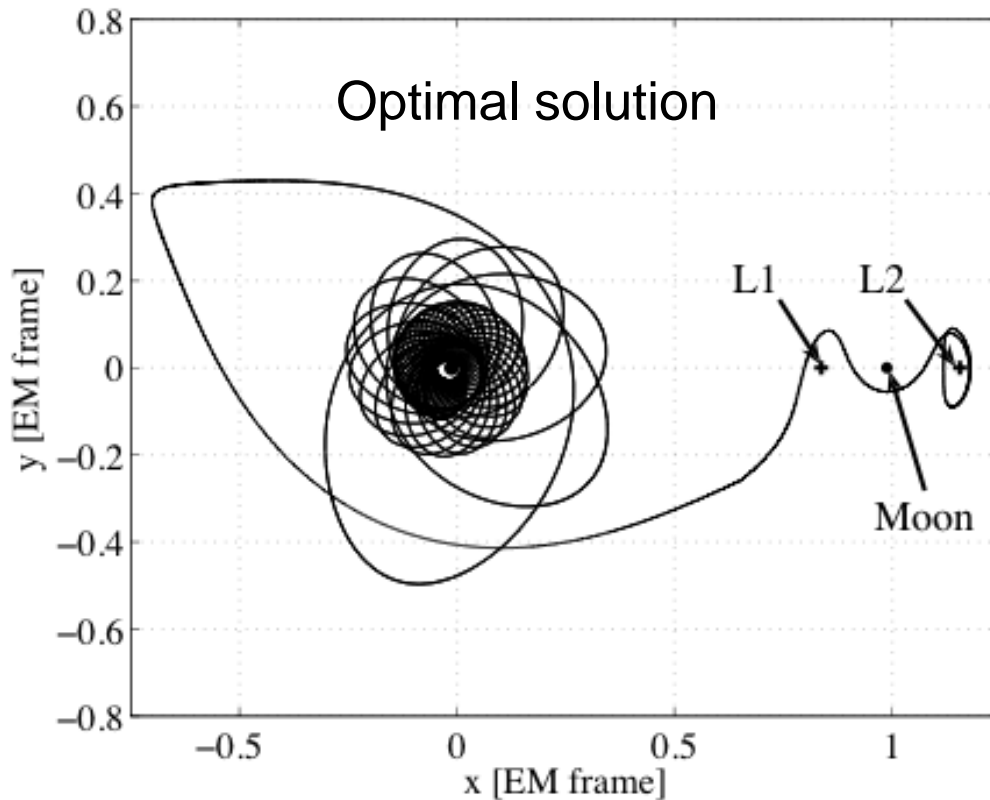
- Departure orbit
- Arrival orbit
- Propellant mass fraction
- Transfer time

## Comparisons

- Optimal low-thrust, invariant-manifold solutions vs
  - Solutions achieved with impulsive maneuvers
  - Existing solutions in literature



# Low-Thrust, Stable-Manifold Transfers to Halos



- ~ 300 multiple shooting segments (i.e., ~5100 var., 3300 constr.)
- The **signature** of the preliminary solution is clearly visible





# Low-Thrust, Stable-Manifold Transfers to Halos

LTSM: Low-thrust, stable-manifold

HTSM: High-thrust, stable-manifold (impulsive maneuvers)

Type	Initial Orbit		Final Orbit		$I_{sp}$ [s]	$m_p/m_0$	$\Delta t$ [days]
	$h_p^E$ [km]	$e^E$	$A_z$ [km]	$L_i$			
LTSM#1	200	0	8000	$L_1$	3000	0.171	178
LTSM#2	400	0.72	8000	$L_1$	3000	0.090	91
LTSM#3	200	0	8000	$L_2$	3000	0.183	195
LTSM#4	400	0.72	8000	$L_2$	3000	0.091	114
HTSM#1	200	0	8000	$L_1$	300	0.711	67
HTSM#2	200	0	8000	$L_2$	300	0.713	65

- LEO:  $h = 200$  km,  $e = 0$
- GTO:  $h_p = 400$ ,  $e = 0.72$

- $m_0 = 1000$  kg



- A method to **include low-thrust** orbit pieces in the “standard” invariant manifolds frame has been derived
- Low-thrust **attainable sets** are manipulated in the same fashion as invariant manifolds
- Preliminary solutions are **optimized** in a four-body frame (end-to-end optimization)
- Solutions **reduce** the propellant mass (due to higher  $I_{sp}$ )
- Presented solutions are **not achievable** neither with patched-conics method nor with standard invariant manifolds technique



## Suggested readings

- Szebehely, V., Theory of Orbits: The Restricted Problem of Three Bodies, Academic Press Inc., New York, 1967, pp. 16–22
- Farquhar, R., “Lunar Communications with Libration-Point Satellites,” Journal of Spacecraft and Rockets, Vol. 4, 1967, pp. 1383–1384
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- Gomez, G., Simo, C., Masdemont, J., and Jorba, A., Dynamics and Mission Design near Libration Points: Vol. III Advanced Methods for Collinear Points, World Scientific, London, 2001, pp. 69–92
- Gomez, G., Koon, W., Lo, M., Marsden, J., Masdemont, J., and Ross, S., “Invariant Manifolds, the Spatial Three-Body Problem and Space Mission Design,” Advances in the Astronautical Sciences, Vol. 109, 2001, pp. 3–22
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- Topputo, F., Low-Thrust Non-Keplerian Orbits: Analysis, Design, and Control, Ph.D. thesis, Politecnico di Milano, Italy, 2007.
- Mingotti, G., Trajectory Design and Optimization in Highly Nonlinear Astrodynamics, Ph.D. thesis, Politecnico di Milano, Italy, 2010



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## **Incorporating low-thrust propulsion into the invariant manifolds technique for space trajectories design**

**F. Bernelli-Zazzera, F. Topputo**

**Politecnico di Milano, Dipartimento di Ingegneria Aerospaziale**

**G. Mingotti**

**Universität Paderborn, Institut für Mathematik**