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Minimum-fuel trajectories for a hypersonic passenger aircraft under a temperature restriction for the thermal protection system

Hans Josef Pesch

University of Bayreuth, Germany

**Hyper Aircraft: Kurt Chudej, Markus Wächter,
Gottfried Sachs, Florent le Bras**

Hyper Car: Armin Rund, Wolf von Wahl & Stefan Wendl

**Workshop on Aerospace Applications of Control and Optimization
Kick-off Meeting of the Initial Training Network SADCO,
Paris, France, March. 2-4, 2011**



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mathematik**



- **Introduction/Motivation**
- **The hypersonic trajectory optimization problem**
- **The instationary heat constraint**
- **Numerical results**
- **The hypersonic rocket car problems**
- **Theoretical results**
- **New necessary conditions**
- **Numerical results**
- **Conclusion**



- **Introduction/Motivation**

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Introduction: Supersonic Aircraft



Concorde

first flight: March 21, 1969 (Charger, Korolkov) 1976; 20 planes
first flight: Mach 2.03 (2400 km/h, 8100 ft altitude) 1975; 16 planes
Mach 2.35 (2500 km/h, 13000 ft altitude)

Crash after take-off: July 25, 2000,
Paris, Charles de Gaule



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From Times Online
February 5, 2008

Hypersonic jet could offer day trips Down Under

Day trips to Australia came a step closer to reality today when a British firm unveiled plans for a hypersonic passenger jet which could fly from Europe to Sydney in less than five hours.

Reaction Engines of Oxfordshire says that its A2 plane could be in service within 25 years, carrying 300 passengers at a top speed of almost 4,000 mph, five times the speed of sound and twice the speed of Concorde.

The jet may be more environmentally friendly than other planes because it uses liquid hydrogen for power rather than fossil fuel.



Introduction: Hypersonic Passenger Jets



http://www.reactionengines.co.uk/lapcat_anim.html



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Project LAPCAT
Reading Engines, UK

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Motivation: Hypersonic Passenger Jets

quasilinear PDEs
non-linear boundary conditions
coupled with ODE

PDE

ODE

$$T \leq T_{\max}$$



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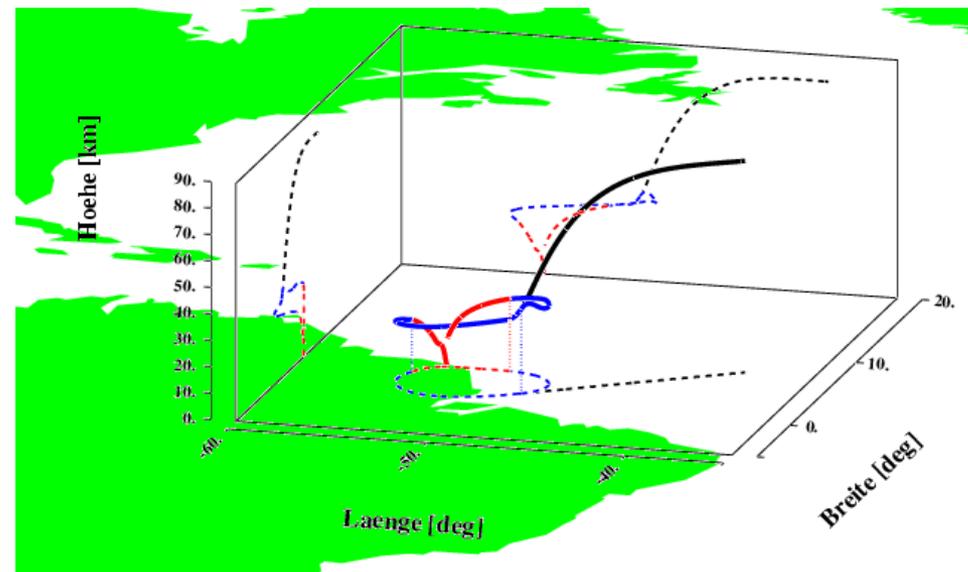


Introduction: The German Sanger II Project



Collaborative Research Center,
Munich: 1989 - 2003

air-breathing
turbojet - ramjet / scramjet



Chudej: 1989; Schnepfer: 1999

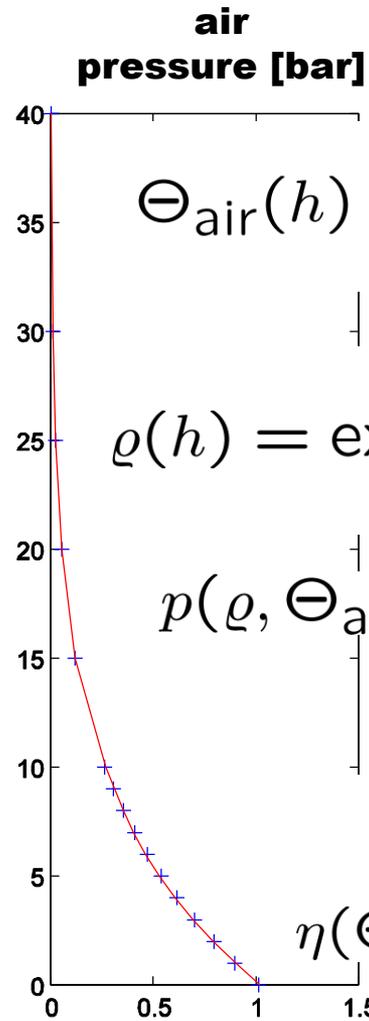
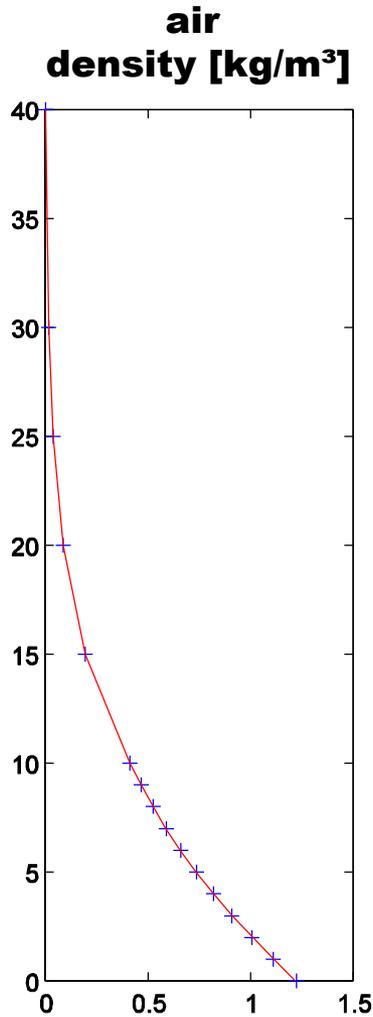
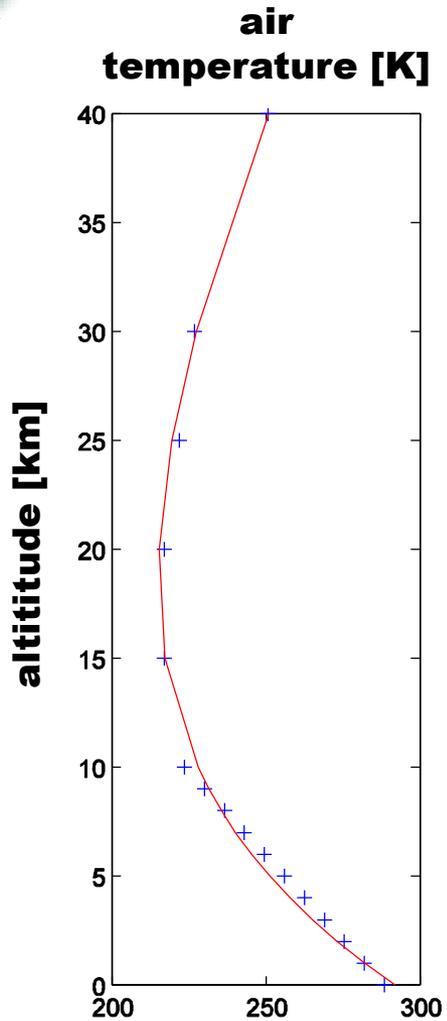


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Model: Atmosphere



$$\Theta_{\text{air}}(h) = \Theta_0 \sum_{i=0}^5 c_i h^i$$

$$\rho(h) = \exp\left(\frac{\sum_{i=0}^5 a_i h^i}{\sum_{j=0}^3 b_j h^j}\right)$$

$$p(\rho, \Theta_{\text{air}}) = \rho R \Theta_{\text{air}}$$

↓

$$a(h) \quad s(\Theta_{\text{air}})$$

$$\eta(\Theta_{\text{air}}) \quad Pr(\Theta_{\text{air}}, p)$$



Heat conductivity

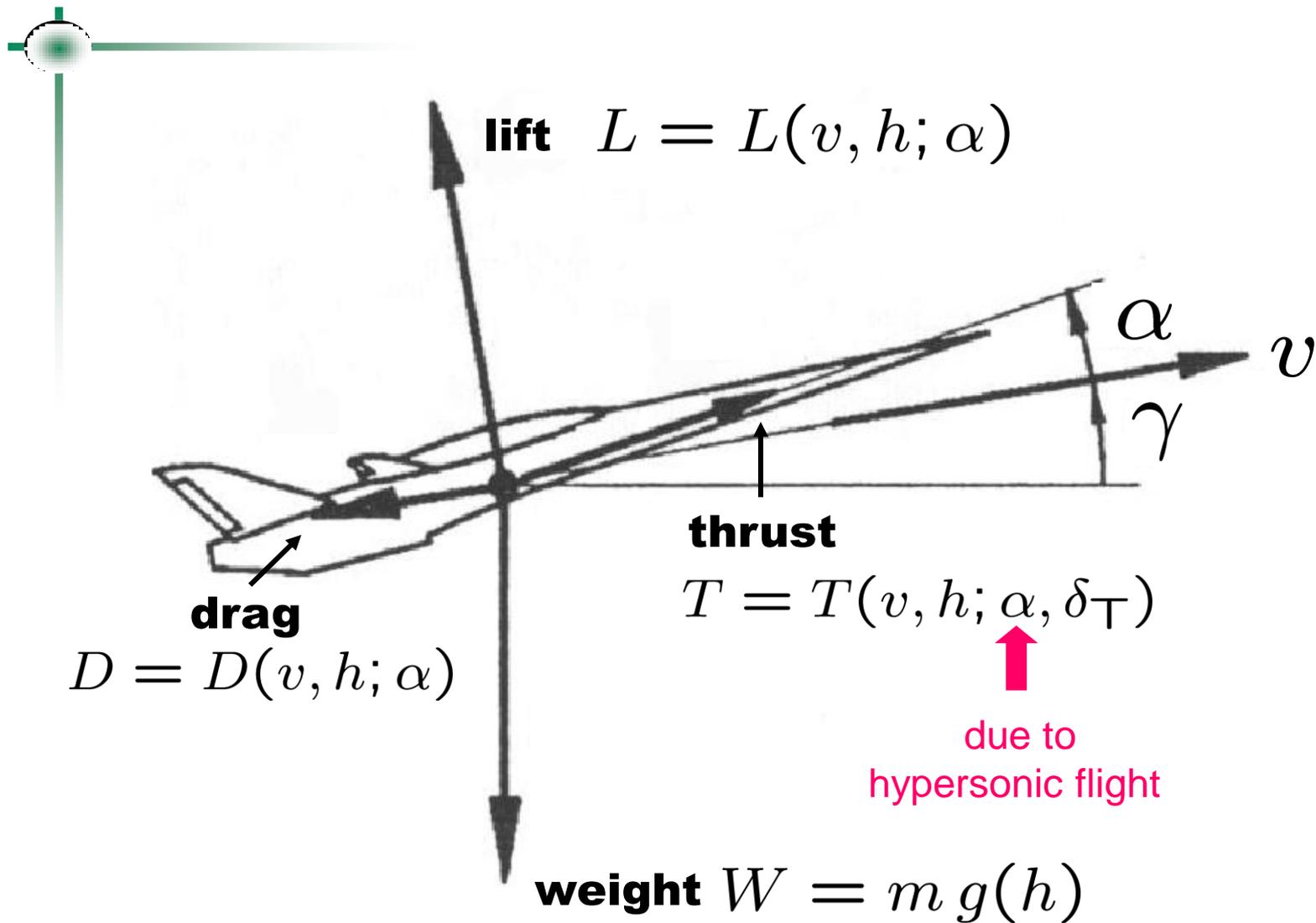
$$\lambda(\Theta_{\text{air}}, p) = \begin{cases} \sum_{i=0}^3 g_i \Theta_{\text{air}}^i & \text{if } \Theta_{\text{air}} \leq 1400 \\ \xi_1 \sum_{i=0}^3 g_i \Theta_{\text{air}}^i + \xi_2 P_\lambda(\Theta_{\text{air}}, p) & \text{if } 1400 < \Theta_{\text{air}} < 1600 \\ P_\lambda(\Theta_{\text{air}}, p) & \text{if } 1600 \leq \Theta_{\text{air}} \end{cases}$$

Heat capacity

$$c_p(\Theta_{\text{air}}) = R \frac{\kappa}{\kappa - 1} \quad \text{with} \quad \kappa = \frac{\sum_{i=0}^5 d_i \Theta_{\text{air}}^i}{\sum_{j=0}^1 e_j \Theta_{\text{air}}^j}$$

Θ_{air} : air temperature





Two-dimensional flight over a great circle of a rotational Earth

$$\dot{v} = \frac{1}{m} \left[T(v, h; \alpha, \delta_T) \cos(\alpha + \sigma_T) - D(v, h; \alpha) \right] - g(h) \sin \gamma + \omega_E^2 r(h) \sin \gamma$$

$$\dot{\gamma} = \frac{1}{m v} \left[T(v, h; \alpha, \delta_T) \sin(\alpha + \sigma_T) + L(v, h; \alpha) \right] + \cos \gamma \left[\frac{v}{r(h)} - \frac{g(h)}{v} + \frac{\omega_E^2 r(h)}{v} \right] + 2 \omega_E$$

$$\dot{h} = v \sin \gamma$$

$$\dot{\zeta} = v \cos \gamma$$

$$\dot{m} = -\beta_T(v, h; \alpha, \delta_T) \quad \text{or} \quad \dot{m} = -\max \{ \beta_C(v, h; \alpha, \delta_T), \beta_I(v, h; \alpha, \delta_T) \}$$

homotopy parameter $0 \leq \epsilon \leq 1$



instantaneous
fuel consumption
for thrust



instantaneous
fuel consumption
for active cooling



Model: Dynamics: Boundary Conditions

	v [m/s]	γ [rad]	h [m]	ζ [km]	m [to]
$t = 0$	150	0	500	0	244
t_f free	150	0	500	9 000	—

Houston – Rome: 9163 km = 5693 m
Munich – Houston: 8714 km = 5414 m



Objective function

$$\max_{\alpha, \delta_T} m(t_f)$$

Constraints

$$-1.5^\circ \leq \alpha \leq 20^\circ \quad 0 \leq \delta_T \leq 1$$

angle of attack : box constraints : throttle setting

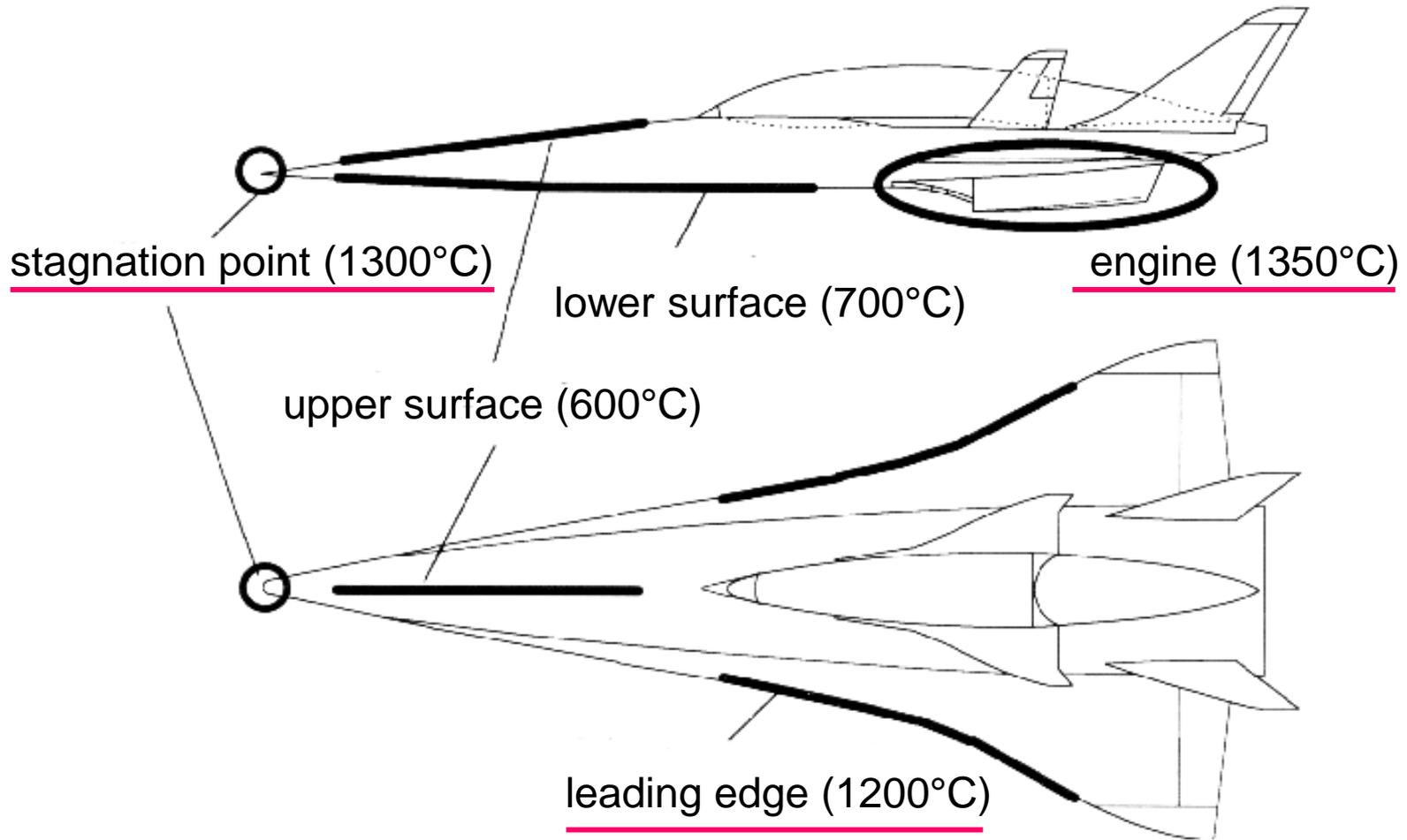
$$0 \leq n(v, h, m; \alpha) \leq 2 \quad 10 \leq \bar{q}(v, h) \leq 50 \text{ [kPa]}$$

control-state constr.: load factor

state constr.: dynamic pressure



Model: Active Engine Cooling



Model: Active Engine Cooling

$$\beta_C = \sum_{i=0}^8 \beta_i(v, h; \alpha; Re, \phi)$$

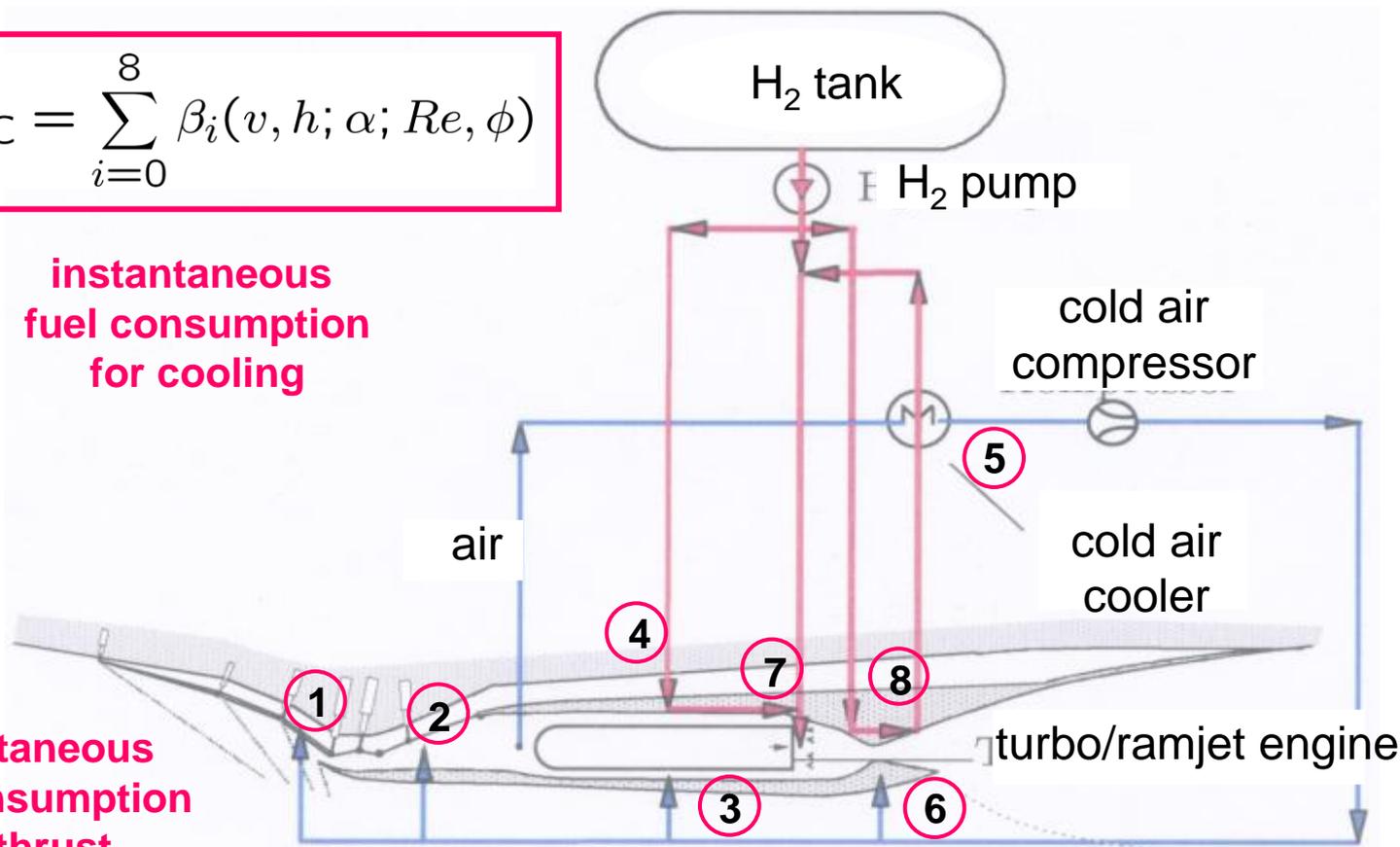
instantaneous
fuel consumption
for cooling

instantaneous
fuel consumption
for thrust

$$\dot{m} = -\beta_T(v, h; \alpha, \delta_T)$$

$$\beta_C \leq \beta_T$$

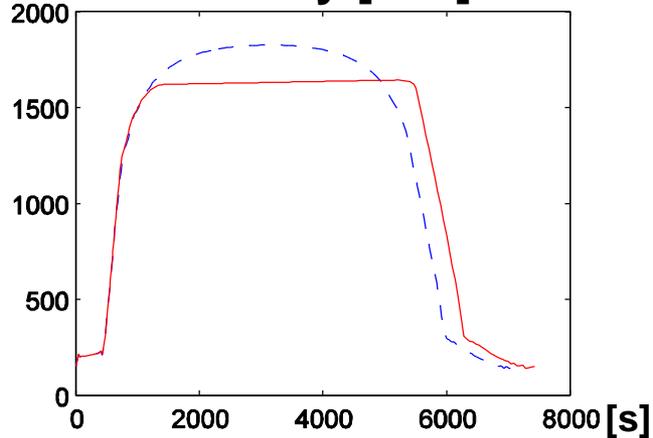
control-state constraint
fuel is reused
for thrust



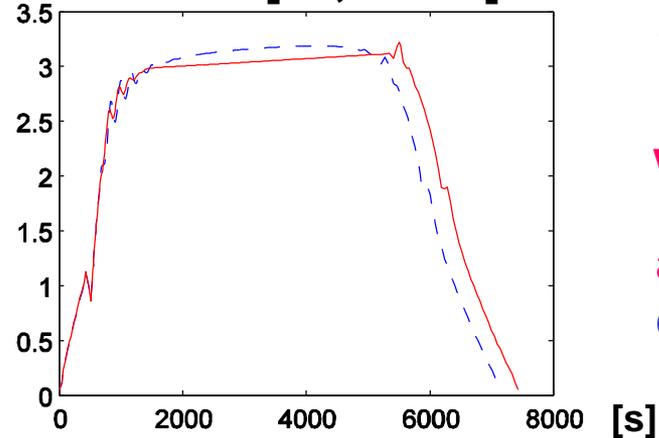
Numerical Results: State Variables



velocity [m/s]

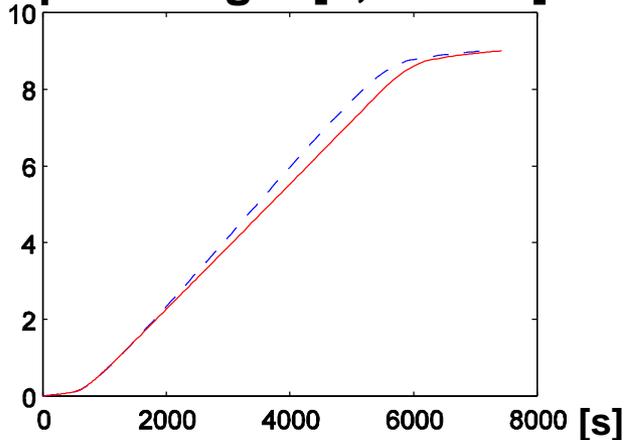


altitude [10,000 m]

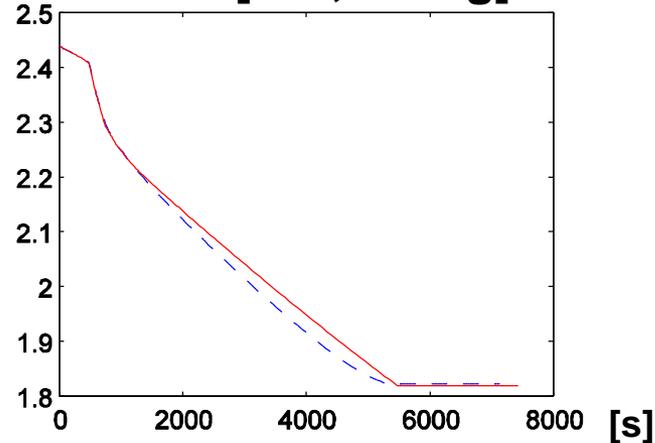


without
with
active
cooling

path length [1,000 km]



mass [100,000 kg]



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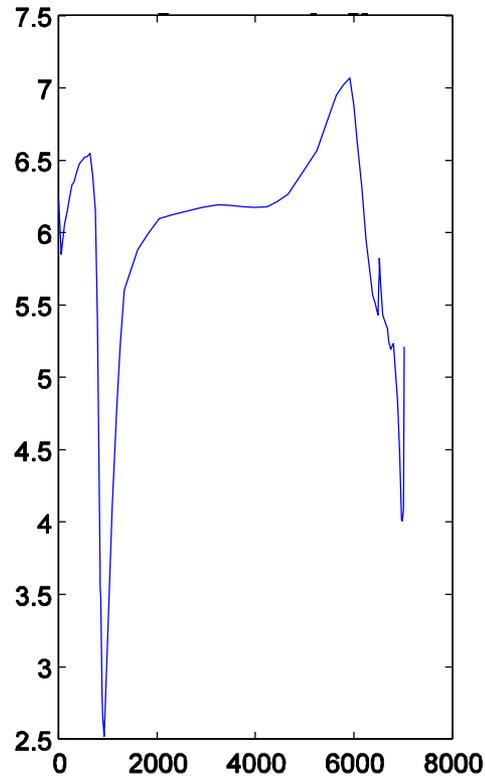
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Florent Le Bras

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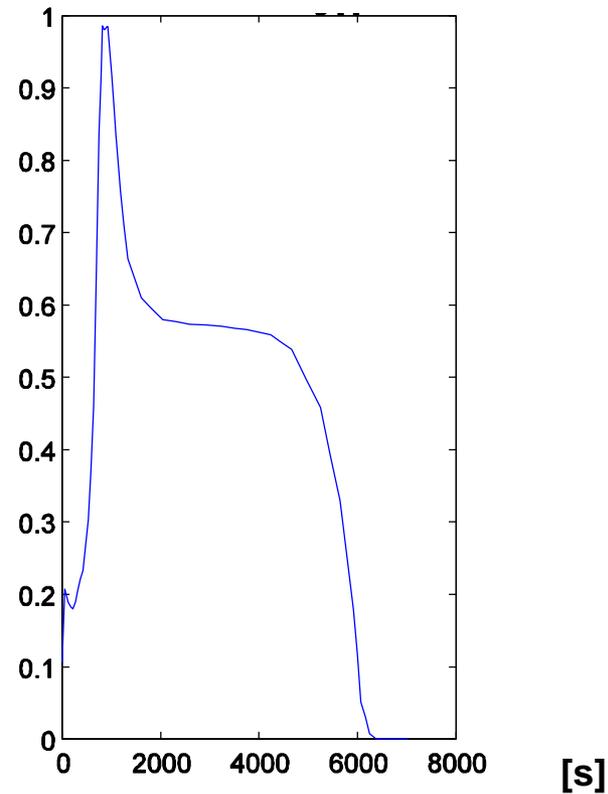


Numerical Results: Control Variables

angle of attack [deg]



throttle setting



$t_f: 7\,023 \text{ [s]} \implies 7\,500 \text{ [s]}$



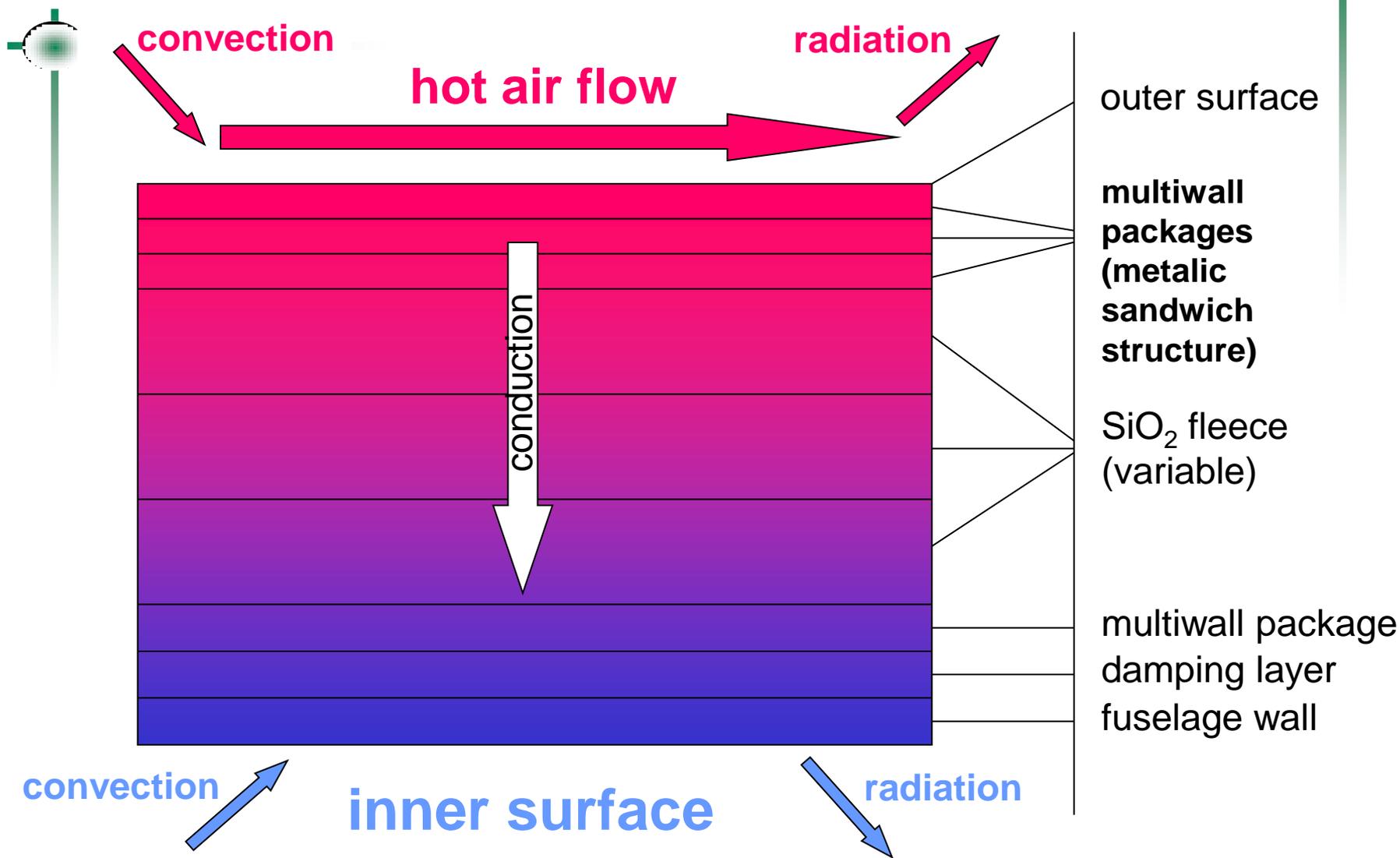
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- Conclusion
- Appendix: more applications



Model: Instationary Heat Constraint: Thermal Protection System



affine linear functions

$$\rho_{\text{TPS}} c_p(\Theta) \frac{\partial \Theta}{\partial t} - \vec{\nabla} \cdot [\lambda(\Theta) \vec{\nabla} \Theta] = 0$$

in $(\vec{x}, t) \in \Omega \times (0, t_f)$

$$\Theta(\vec{x}, 0) = \Theta_0(\vec{x}) = 300 \text{ [K]} \quad \text{in } \vec{x} \in \Omega$$

nonlinear boundary conditions

$$\frac{\partial \Theta}{\partial \vec{n}}(\vec{x}, t) = q_{\text{conv}}(v, h, \alpha) - q_{\text{rad}}(\Theta, \Theta_{\text{air}}(h))$$

in $(\vec{x}, t) \in \partial\Omega \times (0, t_f)$

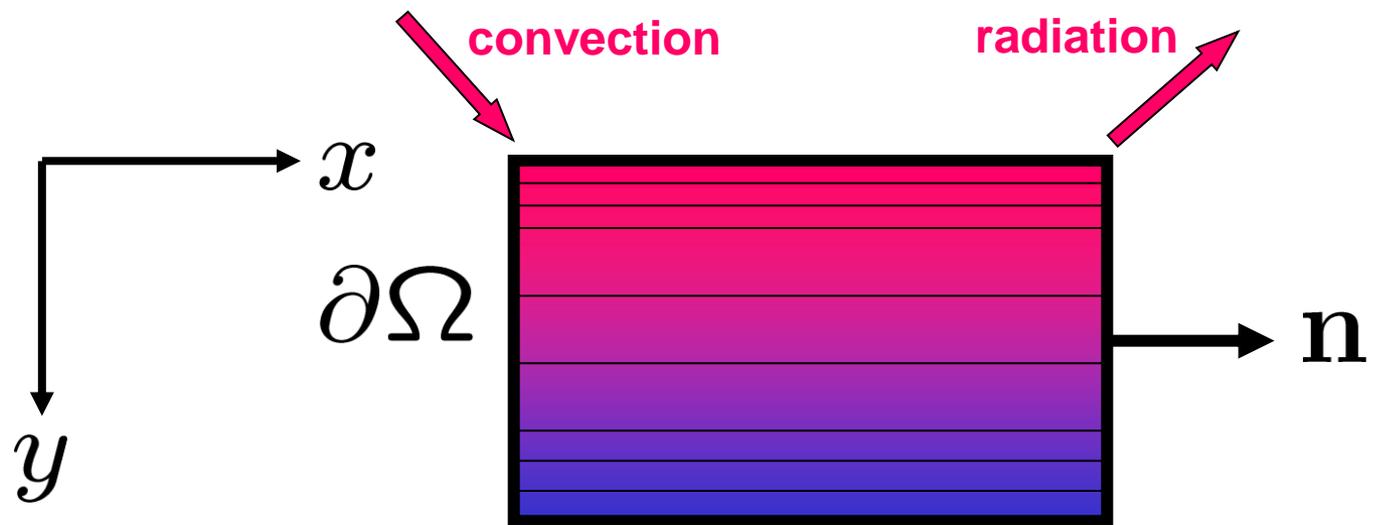
**quasi-linear parabolic initial-boundary value problem
with nonlinear boundary conditions**



Model: Instationary Heat Constraint: Boundary Conditions (1)

$$\frac{\partial \Theta}{\partial \vec{n}}(\vec{x}, t) = q_{\text{conv}} - q_{\text{rad}} \quad \text{in} \quad (\vec{x}, t) \in \partial\Omega \times (0, t_f)$$

$$q_{\text{conv}} = q_{\text{air}}(v, h, \Theta_{\text{air}}(h); \alpha; x_L, Q) \quad q_{\text{rad}} = \varepsilon \sigma (\Theta^4 - \Theta_{\text{air}}^4(h))$$



$$q_{\text{conv}} = \alpha_q(\Theta_{\text{int}}) (\Theta - \Theta_{\text{int}})$$

$$q_{\text{rad}} = \varepsilon \sigma (\Theta^4 - \Theta_{\text{int}}^4)$$



State-constraint for the temperature:

$$\Theta(\vec{x}, t) \leq \Theta_{\max}(\vec{x}, t) \quad \text{for all} \quad (\vec{x}, t) \in \Omega \times (0, t_f)$$



ODE-PDE state-constrained optimal control problem

PDE: quasilinear parabolic with nonlinear bound. conds.

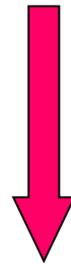
CONTROL: boundary controls indirectly
via ODE states and controls

CONSTRAINT: state constraint



State-constraint for the temperature:

$$\Theta(\vec{x}, t) \leq \Theta_{\max}(\vec{x}, t) \quad \text{for all} \quad (\vec{x}, t) \in \Omega \times (0, t_f)$$



$$\begin{aligned} \tilde{\Omega} &\subset \Omega \\ [t_{\text{in}}, t_{\text{out}}] \end{aligned}$$

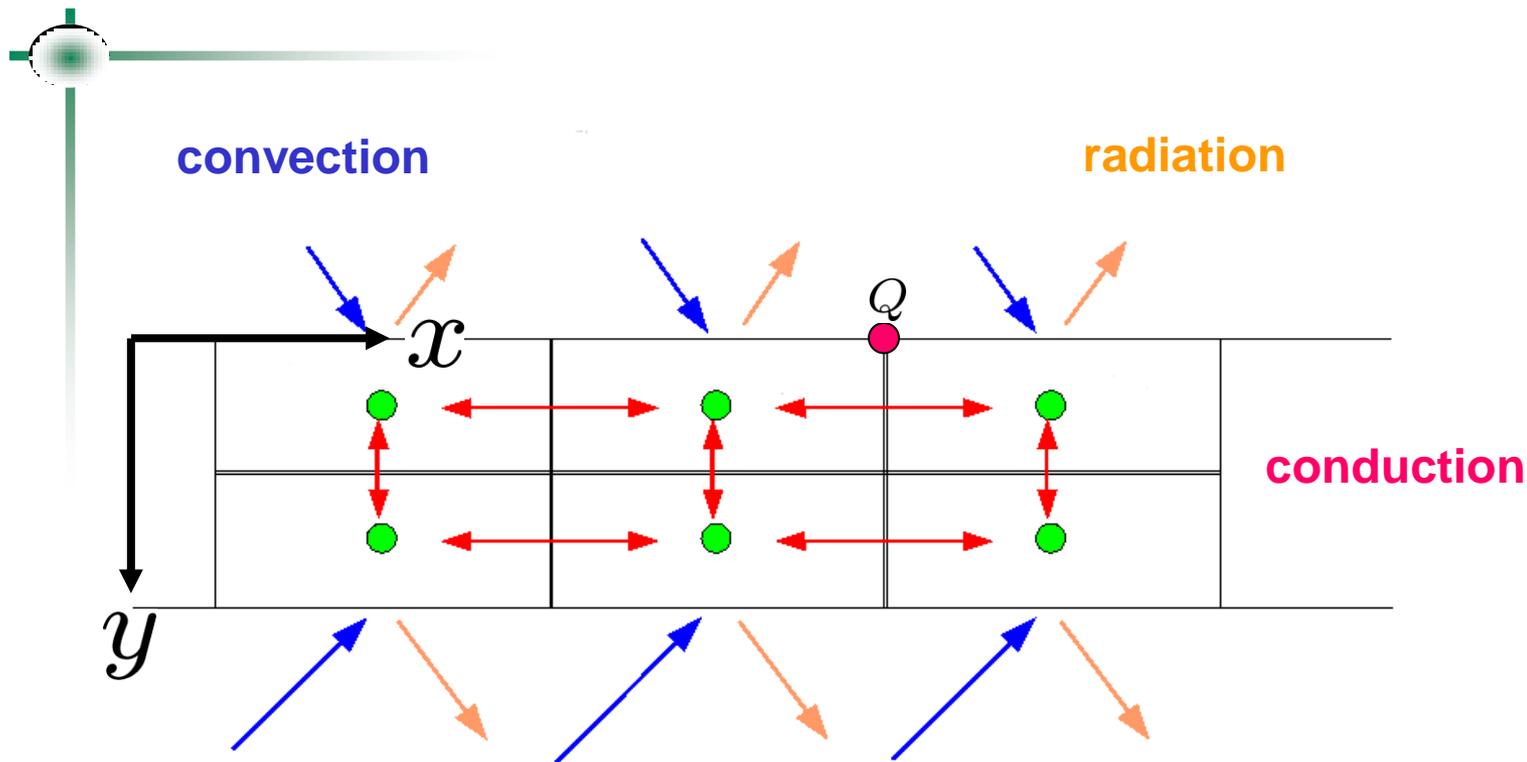
ODE-PDE state-constrained optimal control problem



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- **Numerical results**
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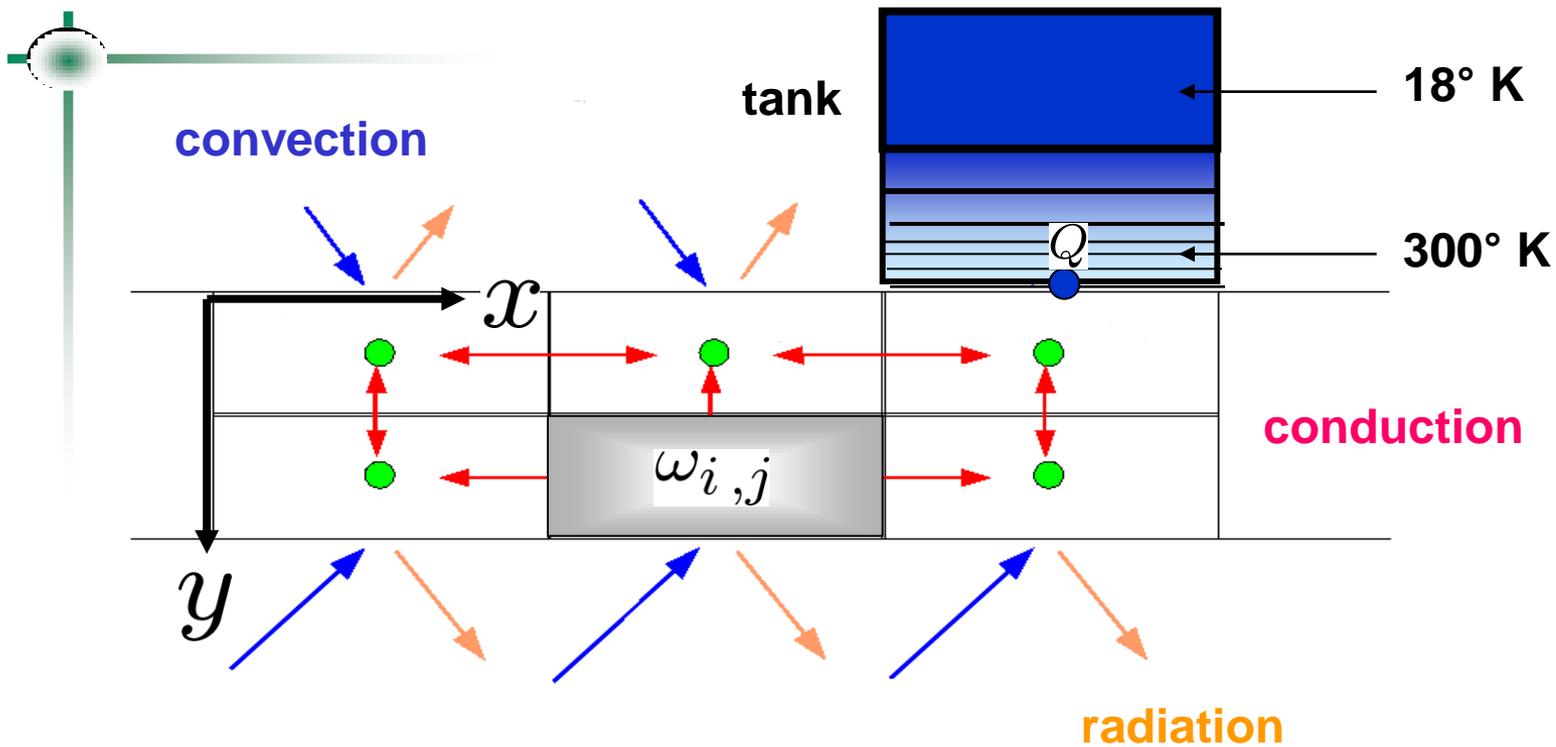
Numerical Method: Semi-Discretization in Space



**Finite Volume Method: locally and globally conservative
second order convergent**



Numerical Method: Semi-Discretization in Space



conservation law in each volume

$$\frac{d}{dt} \int_{\omega_{i,j}} \dots d\vec{x} = - \int_{\partial\omega_{i,j}} \dots d\vec{s}$$

second order
convergent
FVM



$$\rho_{\text{TPS}} c_p(\Theta) \frac{\partial \Theta}{\partial t} - \vec{\nabla} \circ [\lambda(\Theta) \vec{\nabla} \Theta] = 0$$

$$\rho_{\text{TPS}} c_p(\Theta_i) \dot{\Theta}_i$$

$$-\frac{1}{\Delta x} \left[\lambda \left(\frac{\Theta_{i-1} + \Theta_i}{2} \right) \frac{\Theta_{i-1} - \Theta_i}{\Delta x} - \lambda \left(\frac{\Theta_i + \Theta_{i+1}}{2} \right) \frac{\Theta_i - \Theta_{i+1}}{\Delta x} \right] = 0$$

1D case

$$\Theta_i := \Theta(x_i, t)$$



$$\rho_{\text{TPS}} c_p(\Theta_i) \dot{\Theta}_i$$

$$-\frac{1}{\Delta x} \left[\lambda \left(\frac{\Theta_{i-1} + \Theta_i}{2} \right) \frac{\Theta_{i-1} - \Theta_i}{\Delta x} - \lambda \left(\frac{\Theta_i + \Theta_{i+1}}{2} \right) \frac{\Theta_i - \Theta_{i+1}}{\Delta x} \right] = 0$$

$i = 1$: **b.c. towards air**

↑
coupled with ODE.

b.c. towards interior: $i = n - 1$



large scale multiply constrained ODE optimal control problem



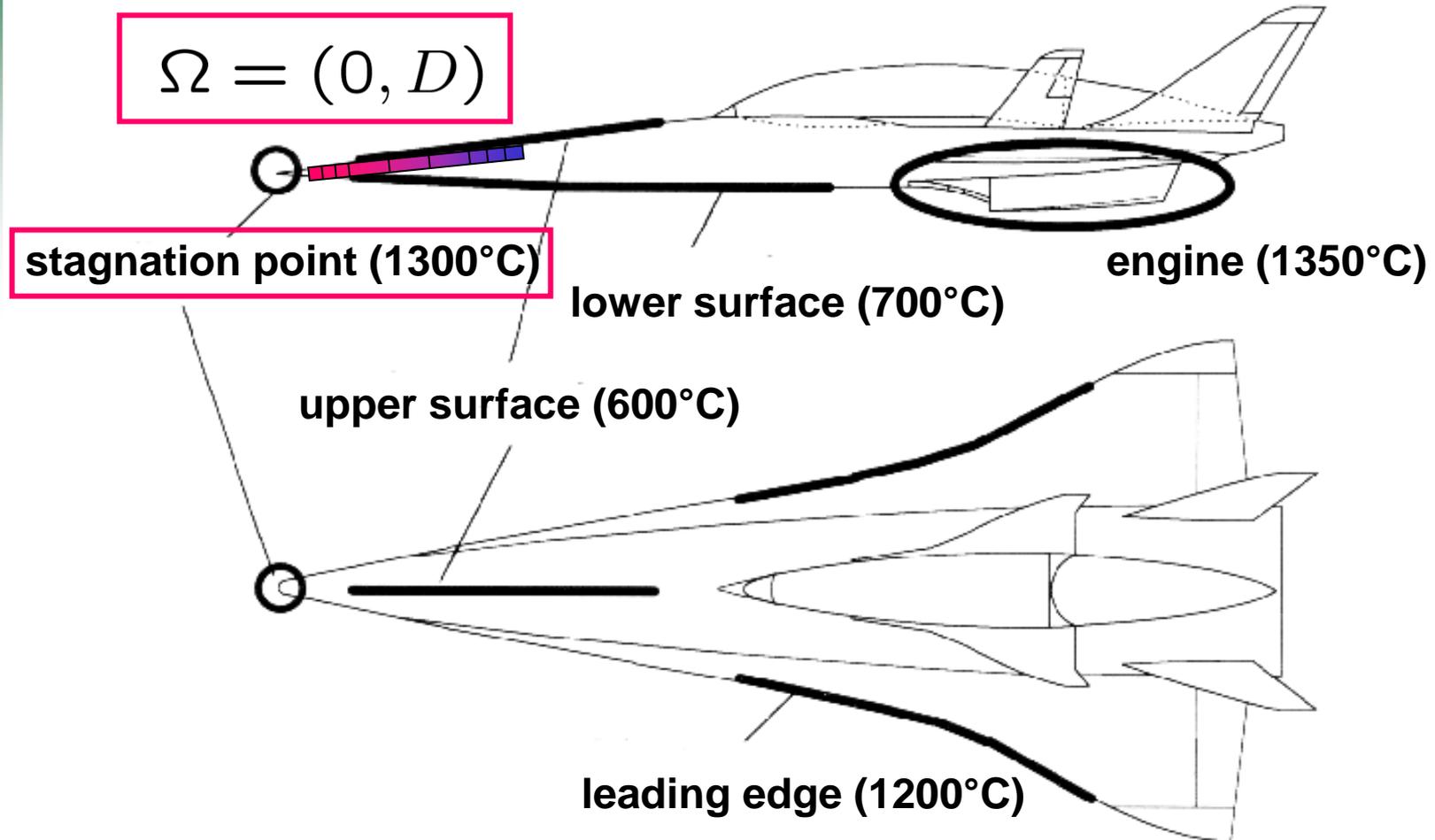
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DIRCOL (O. v. Stryk) with SNOPT (P. Gill)
alternatively: NUOCCCS (C. Büskens)
IPOPT (A. Wächter) with AMPL
WORHP (Büskens, Gerds)

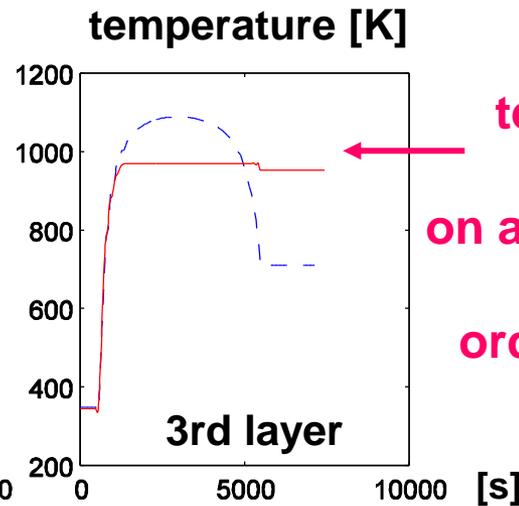
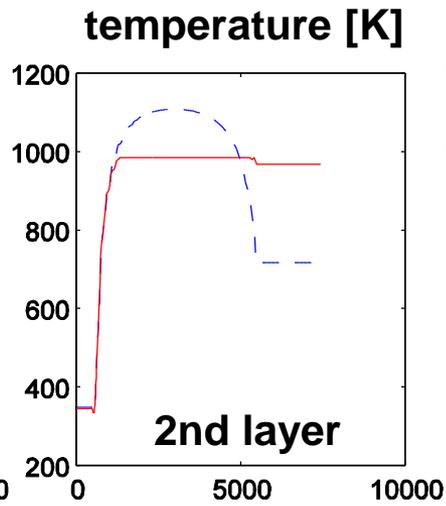
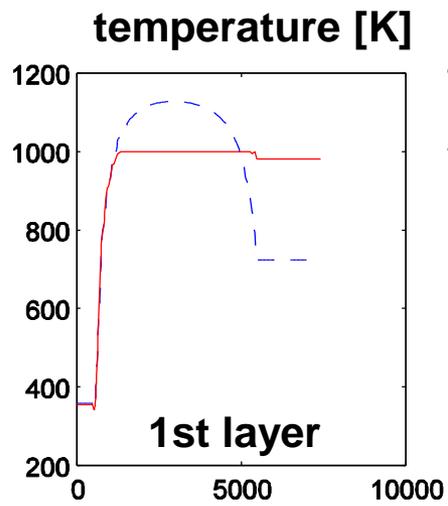
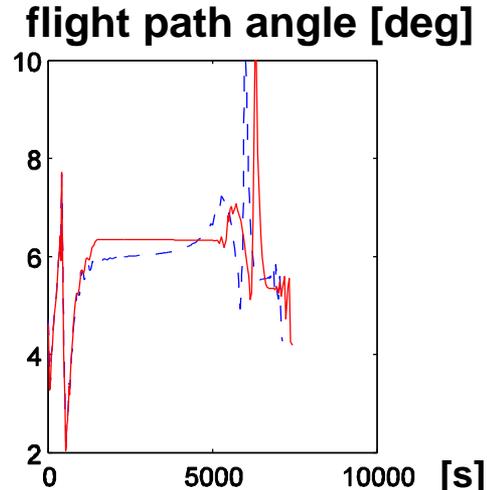
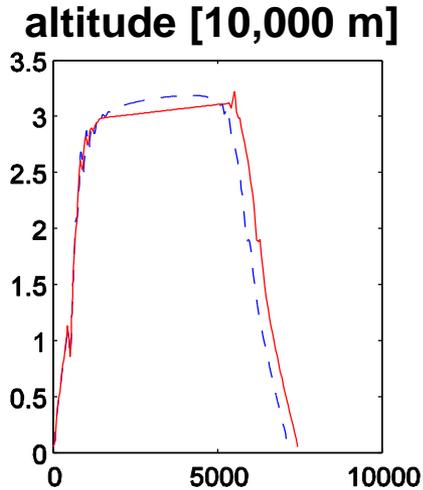
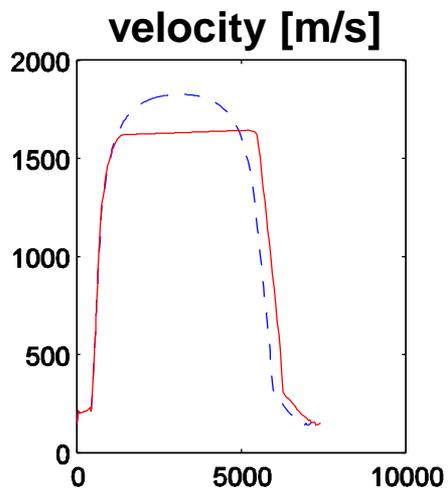
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Numerical Results: Stagnation Point (1D)



Numerical Results: Stagnation Point: States, Heat Loads



**limit
temperature
1000 K
on a boundary arc
order concept?**



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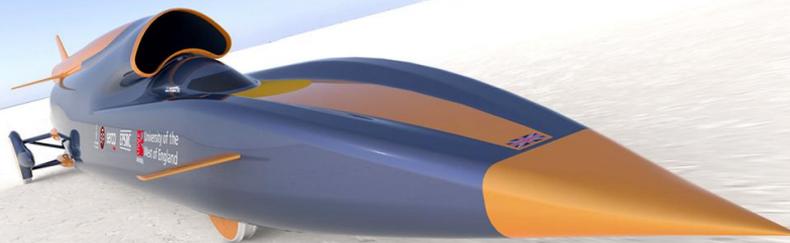


have nothing to do with the supersonic

Bloodhound-SSC-Projekt

1997, Oct. 15: Thrust SSC
officially 1.228 km/h
less than $Ma\ 1=1.234,8\text{ km/h}$
at 20°C

Aim: 1.000 m/h,
hence 1.609 km/h,
faster than a speeding bullet.



we are going with *hypersonic* speed

BLOODHOUND SSC by CURVENTA



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<http://www.bloodhoundssc.com/>

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$$\min_{u \in U} \left\{ t_f + \frac{1}{2} \lambda \int_0^{t_f} u^2 dt \right\}, \quad \lambda \geq 0$$

minimum time control costs

subject to

$$\ddot{w}(t) = u(t) \quad \text{in } (0, t_f)$$

$$w(0) = w_0, \quad \dot{w}(0) = \dot{w}_0$$

$$w(t_f) = 0, \quad \dot{w}(t_f) = 0$$

$$U := \left\{ u \in L^2(0, t_f) : |u(t)| \leq u_{\max} \text{ a. e. in } [0, t_f] \right\}$$





Problem 1

$$\frac{\partial T}{\partial t}(x, t) - \frac{\partial^2 T}{\partial x^2}(x, t) = g(\dot{w}(t))$$

friction term
control via ODE state

in $(0, l) \times (0, t_f)$

$$T(x, 0) = T_0 = 0 \text{ on } (0, l)$$

$$-\frac{\partial T}{\partial x}(0, t) = -(T(0, t) - T_0)$$

$$\frac{\partial T}{\partial x}(l, t) = -(T(l, t) - T_0) \text{ on } [0, t_f]$$

instationary heating of the entire vehicle



$$\frac{\partial T}{\partial t}(x, t) - \frac{\partial^2 T}{\partial x^2}(x, t) = 0$$

Problem 2

$$\text{in } (0, l) \times (0, t_f),$$

$$T(x, 0) = T_0 = 0 \text{ on } (0, l),$$

$$-\frac{\partial T}{\partial x}(0, t) = -(T(0, t) - T_0),$$

$$\frac{\partial T}{\partial x}(l, t) = -(T(l, t) - T_0) + h(\dot{w}(t)) \text{ on } [0, t_f]$$

Transformation
to homogeneous
Robin type b.c.

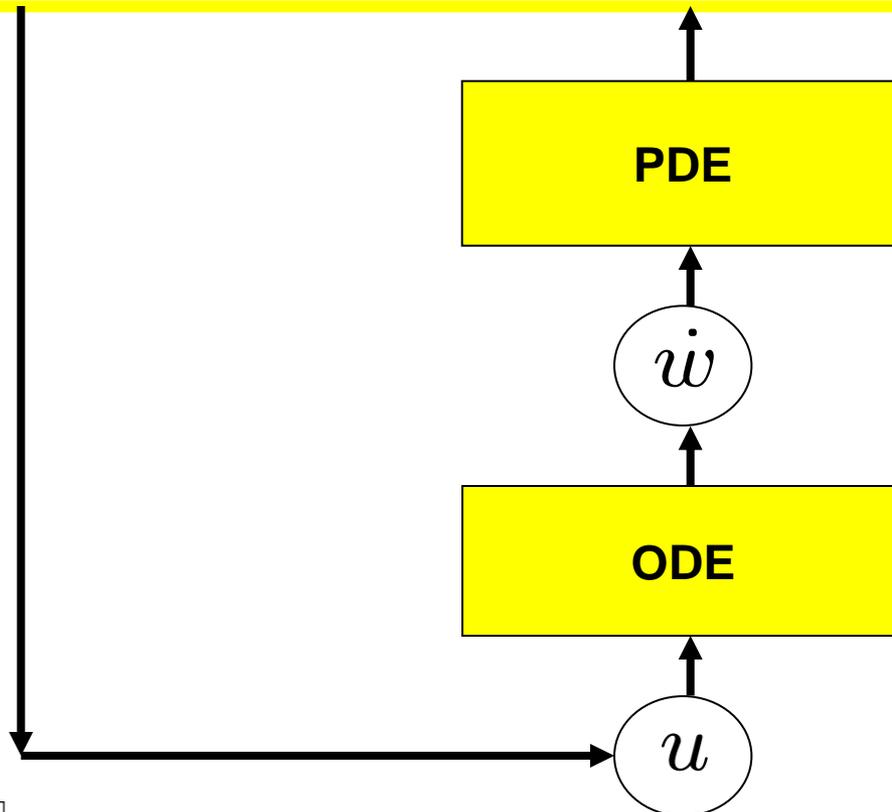
friction term
control via
ODE state

instationary heating at the stagnation point



The State Constraint

$$T(x, t) \leq T_{\max} \text{ a. e. in } (0, l) \times (0, t_f)$$



The state constraint
regenerates
the PDE with the ODE

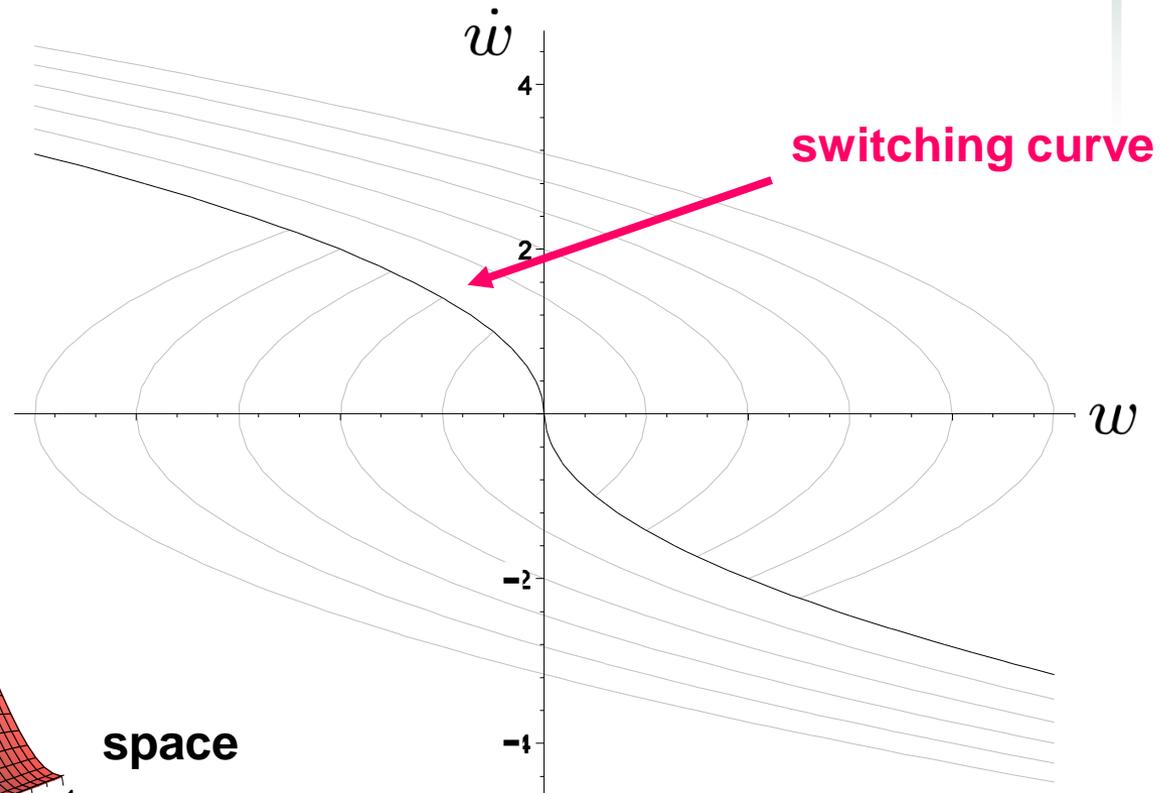
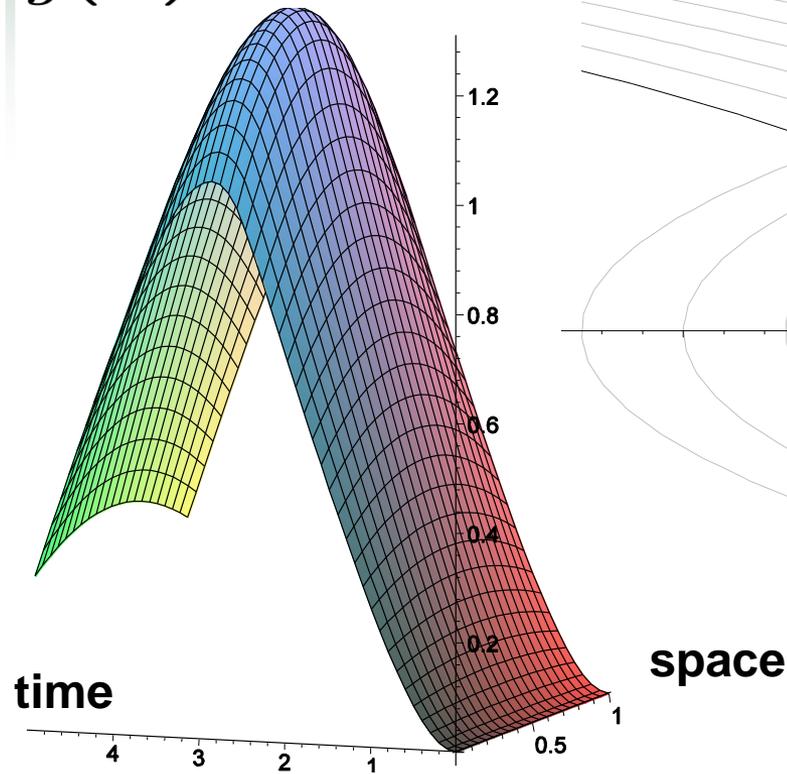


The Optimal Trajectories (Non-regularized, Minimum Time)

Problem 1

$$\min_{u \in U} \left\{ t_f + \frac{1}{2} \lambda \int_0^{t_f} u^2 dt \right\}, \quad \lambda = 0, \quad u_{\max} = 1$$

$$g(\dot{w}) = \dot{w}^2$$



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distributed case
state unconstrained

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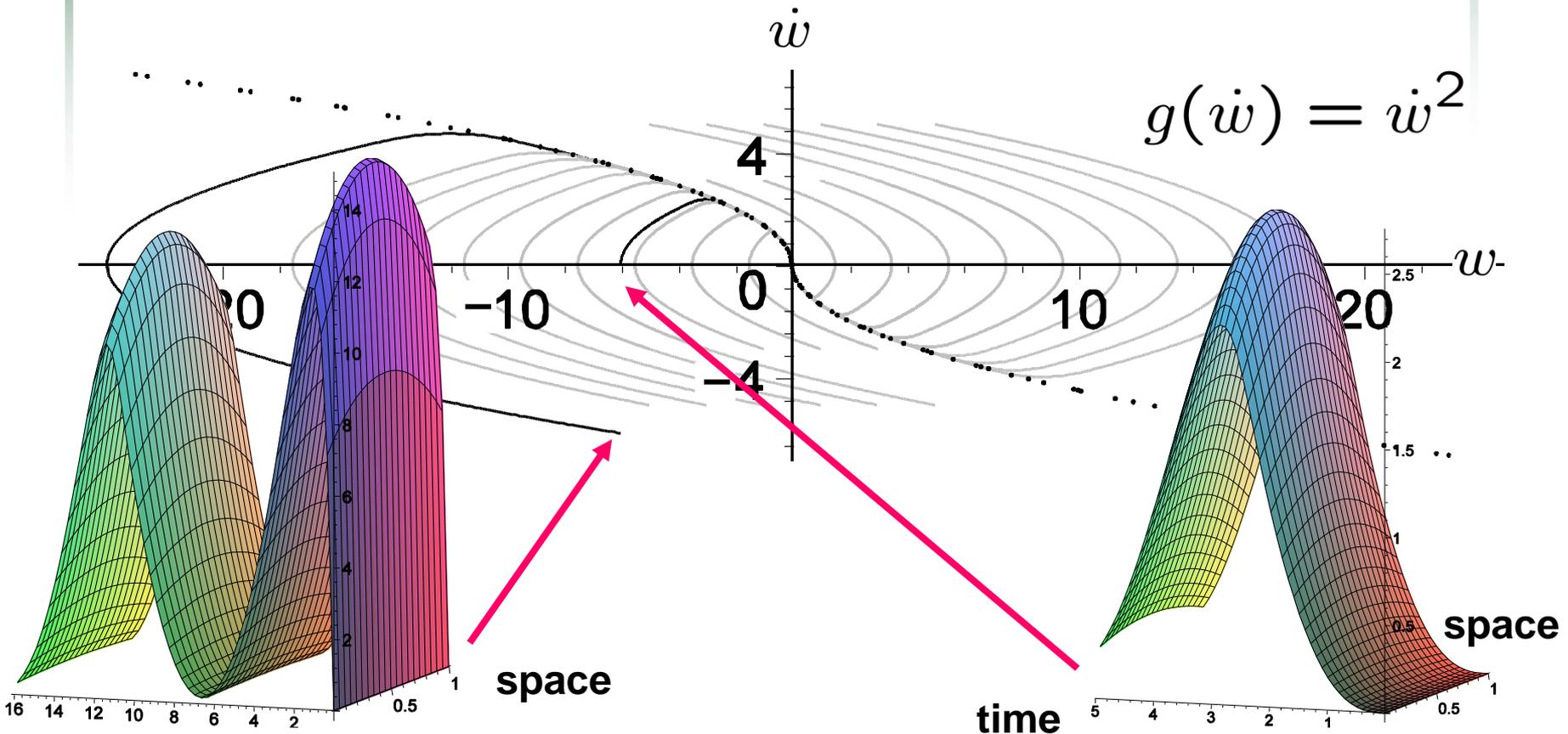


The Optimal Trajectories (Regularized, Control Constrained)

Problem 1

$$\min_{u \in U} \left\{ t_f + \frac{1}{2} \lambda \int_0^{t_f} u^2 dt \right\}, \quad \lambda = \frac{1}{10}, \quad u_{\max} = 1$$

$$g(\dot{w}) = \dot{w}^2$$

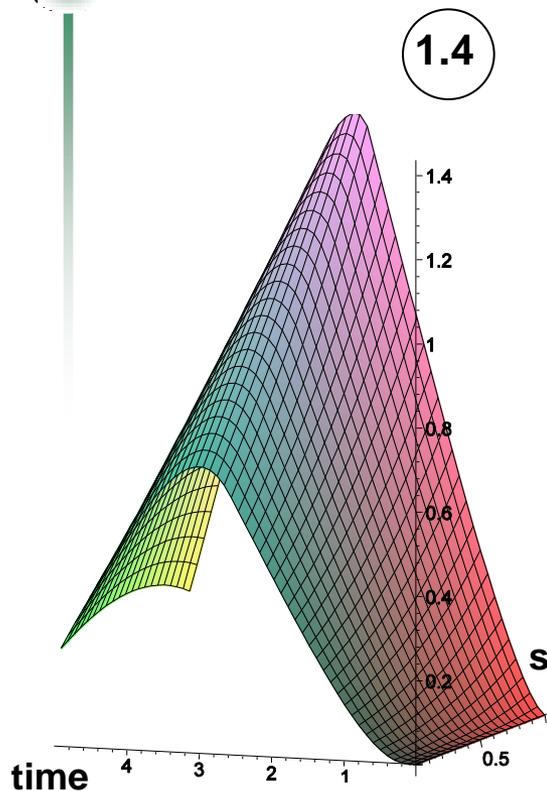


**distributed case
state unconstrained**



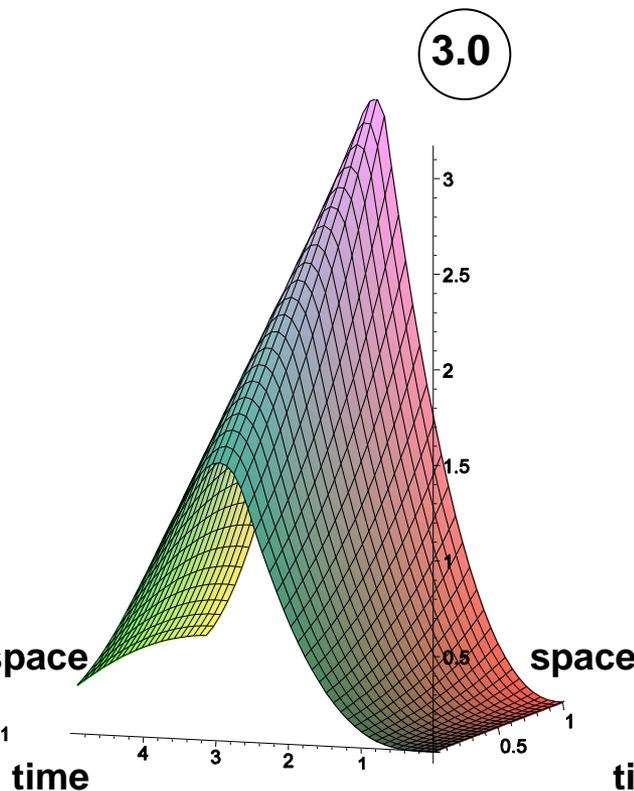
Boundary Control Case

Problem 2



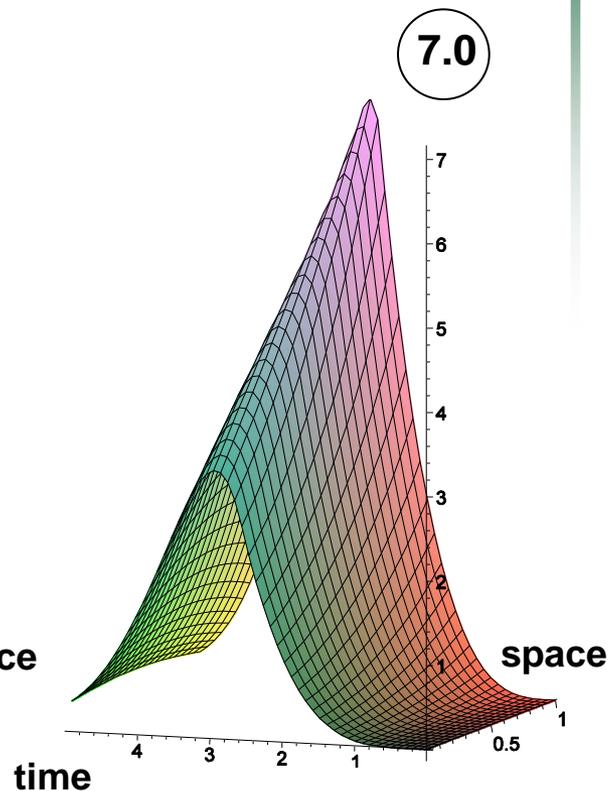
$$h(z) = |z|$$

Coulomb



$$h(z) = z^2$$

Stokes



$$h(z) = |z|^3$$

Newton



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Theoretical results for Problem 1

- Existence, uniqueness, and continuous dependence on data

$$(w, T) \in W_2^1(0, t_f) \times (W_2^{1,0}(Q) \cap C([0, t_f], L^2(0, l)))$$

- Non-negativity of T

- Symmetry $T(x, t) = T(l - x, t)$

- Strong maximum in $x = \frac{l}{2}$

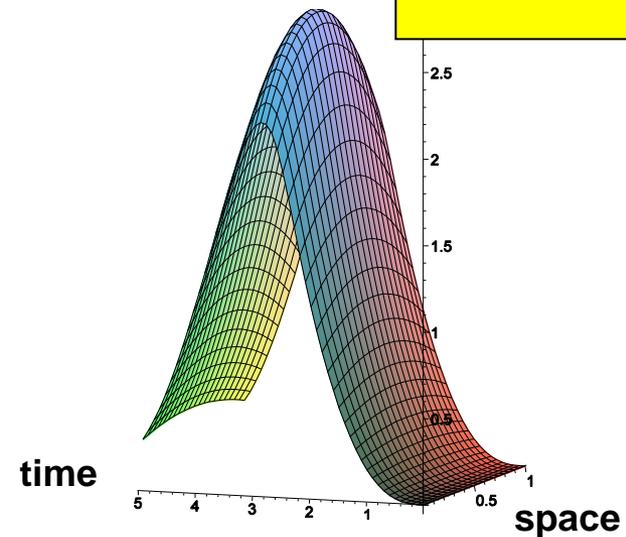
- Classical solution $\frac{d}{dt}T + \mathcal{L}T = g(\dot{w})$

- Maximum regularity

$$T_{tt} \in L^r(\varepsilon, t_f; L^2(0, l)) \quad r \geq 2$$

$$\partial_x^4 T \in L^r(\varepsilon, t_f; L^2(0, l)) \quad \varepsilon > 0$$

Problem 1



Theoretical results (two formulations)

Problem 1: Two equivalent formulations

1) as ODE optimal control problem

$$T\left(\frac{l}{2}, t\right) = \sum_{n=1}^{\infty} \tilde{\gamma}_n \left[\int_0^t g(\dot{w}(s)) e^{-k_n^2(t-s)} ds \right] \leq T_{\max}$$

loss of convergence
if differentiated

non-local, resp. integro-state constraint

2) as PDE optimal control problem

$$T_t + T_{xx} = g\left(\dot{w}_0 + \int_0^t u(s) ds\right)$$

$$T(x, t) \leq T_{\max}$$

non-standard

plus two **isoperimetric constraints** on \mathcal{U} due two ODE boundary conds.



Integro-state constraint

$$S(t, \mathbf{w}([0, t])) := T\left(\frac{l}{2}, t; w_2([0, t])\right) - T_{\max} \leq 0$$

Transformation

$$w_3 := T\left(\frac{l}{2}, t; w_2([0, t])\right)$$

$$\dot{w}_3(t) = \frac{d}{dt} T\left(\frac{l}{2}, t; w_2([0, t])\right)$$

$$w_3(0) = 0$$

$$w_3(t_{\text{on}}) = w_3(t_{\text{off}}) = T_{\max}$$

$$w_3(t) \leq T_{\max}$$

Integro-ODE

**corresponds to
Maurer's intermediate
adjoining approach**

pointwise



Necessary conditions: optimal control law

$$\begin{aligned}\mathcal{L}(\mathbf{w}, u, \mathbf{p}, t_{\text{on}}, t_{\text{off}}, t_f) &:= \int_0^{t_f} 1 + \frac{\lambda}{2} u^2 dt - \int_0^{t_f} (\dot{w}_1 - w_2) p_1 dt - \int_0^{t_f} (\dot{w}_2 - u) p_2 dt \\ &\quad - \int_0^{t_f} \left[\dot{w}_3 - \frac{d}{dt} T \left(\frac{l}{2}, t; w_2([0, t]) \right) \right] p_3 dt \\ &\quad + \int_0^{t_f} (w_3 - T_{\text{max}}) d\mu \\ &\quad + (w_3(t_{\text{on}}) - T_{\text{max}}) \sigma_{\text{on}} + (w_3(t_{\text{off}}) - T_{\text{max}}) \sigma_{\text{off}}\end{aligned}$$

\implies

$$D_u \mathcal{L}[*] (u - u^*) = (\lambda u^* + p_2) (u - u^*) \geq 0 \quad \forall u \in [-u_{\text{max}}, u_{\text{max}}]$$

\implies

$$u^*(t) = P_{[-u_{\text{max}}, +u_{\text{max}}]} \left(-\frac{1}{\lambda} p_2 \right)$$



Necessary conditions: adjoint equations

Retrograde integro-ODE for the adjoint velocity

$$\dot{p}_2 = -p_1 - g'(w_2(t)) \left(p_3(t) - \int_t^{t_f} \sum_{n=1}^{\infty} k_n^2 \tilde{\gamma}_n e^{-k_n^2(s-t)} p_3(s) ds \right)$$

Usual jump conditions for adjoint auxiliary state

discontinuities

$$\dot{p}_3 = \begin{cases} 0 & 0 \leq t < t_{\text{on}} \\ -\mu & t_{\text{on}} < t < t_{\text{off}} \\ 0 & t_{\text{off}} < t \leq t_f \end{cases} \quad \begin{aligned} p_3(t_{\text{on/off}}^-) &= p_3(t_{\text{on/off}}^+) - \sigma_{\text{on/off}} \\ p_3(t_f) &= 0, \quad \sigma_{\text{on/off}} \geq 0 \end{aligned}$$

$$\mu(w_3 - T_{\text{max}}) = 0, \quad \mu \geq 0$$

complementarity condition

difficult to solve
no standard software



Problem 1

$$\frac{\partial T}{\partial t}(x, t) - \frac{\partial^2 T}{\partial x^2}(x, t) = g \left(\dot{w}_0 + \int_0^t u(s) \, ds \right)$$

$$\text{in } (0, l) \times (0, t_f)$$

$$T(x, 0) = 0 \quad \text{on } (0, l)$$

$$-\frac{\partial T}{\partial x}(0, t) = -T(0, t) \quad \frac{\partial T}{\partial x}(l, t) = -T(l, t) \quad \text{on } [0, t_f]$$

$$\int_0^{t_f} u(t) \, dt = -\dot{w}_0 \quad \int_0^{t_f} \int_0^t u(s) \, ds \, dt = -w_0 - \dot{w}_0 t_f$$

$$T(x, t) \leq T_{\max} \quad \text{in } (0, l) \times (0, t_f)$$



By the obtained continuously differentiable solution operator

$$S: f(u, t_f) := \int_0^{t_f} 1 + \frac{\lambda}{2} u^2(t) dt \stackrel{!}{=} \min_{(u, t_f) \in C}$$

subject to

$$G_1(u, t_f) := \begin{pmatrix} \int_0^{t_f} u(t) dt + \dot{w}_0 \\ \int_0^{t_f} \int_0^{t_f} u(s) ds dt + w_0 + \dot{w}_0 t_f \end{pmatrix} = 0$$

$$G_2(u, t_f) := (S(u, t_f) - T_{\max}) \leq_K 0$$

with the convex cone

$$K := \left\{ T \in C([0, l] \times [0, t_f]), \quad T - T_{\max} \leq 0 \text{ a.e.} \right\}$$



Theorem: Let (u^*, t_f^*) be a local solution of the optimization problem.

Let be $(\tilde{u}, \tilde{t}_f) \in \text{int } C(u^*, t_f^*)$ such that

$$G'_1(u^*, t_f^*)(\tilde{u}, \tilde{t}_f) = 0$$

$$G_2(u^*, t_f^*) + G'_2(u^*, t_f^*)(\tilde{u}, \tilde{t}_f) <_K 0.$$

Then there exists a Lagrange multiplier $\bar{\mu} \in \mathcal{M}([0, l] \times [0, t_f])$ associated with (u^*, t_f^*) .

The set of all multipliers belonging to $(u^*, t_f^*) \in \mathcal{M}$ is bounded.



Necessary conditions: adjoint equations

$$-q_t(x, t) - q_{xx}(x, t) = \bar{\mu}$$

$$-q_x(0, t) = -q(0, t), \quad -q_x(l, t) = -q(l, t)$$

$$q(x, t_f^*) = 0$$

so far all seems to be standard, but

Necessary condition: integro optimal control law

$$u^*(t) = P_{[-u_{\max}, u_{\max}]} \left\{ -\frac{1}{\lambda} \left[\nu_1 + \nu_2 t + \int_t^{t_f} g' \left(\dot{w}_0 + \int_0^t u^*(r) dr \right) \cdot \left(\int_0^l q(x, s) dx \right) ds \right] \right\} \quad \boxed{p2}$$

extremely difficult to solve
no standard software



By comparing the two optimal control laws

$$p_1(t) \hat{=} -\nu_2$$

$$p_2(t) \hat{=} \nu_1 + \nu_2 t + \int_t^{t_f^*} g'(v^*(s)) \cdot \left(\int_0^l q(x, s) dx \right) ds$$

discont. deriv. / jump

$$p_3(t) - \int_t^{t_f^*} \sum_{n=1}^{\infty} k_n^2 \tilde{\gamma}_n e^{-k_n^2 (s-t)} p_3(s) ds \hat{=} \int_0^l q(x, t) dx$$

jump



The two formulations allow the comparison of the necessary conditions:

ODE version • known theory

- local jump conditions for ODE formulation
- multipoint boundary value problem for integro-ODEs



PDE version • existence of Lagrange parameter

- non-local jump conditions for PDE formulation
- projection formula for control with integro-terms



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- The instationary heat constraint
- Numerical results

- The hypersonic rocket car problems
- Theoretical results
- **New necessary conditions**
- Numerical results

- Conclusion



Jump condition in the direction of t (equivalent to ODE case):

$$\left[\int_0^l q(x, t) dx \right]_{t_s^-}^{t_s^+} < 0$$

Jump condition in the direction of x (no counterpart in ODE case):

$$\left[\int_0^{t_f} q_x(x, t) dt \right]_{\frac{l}{2}^-}^{\frac{l}{2}^+} < 0$$



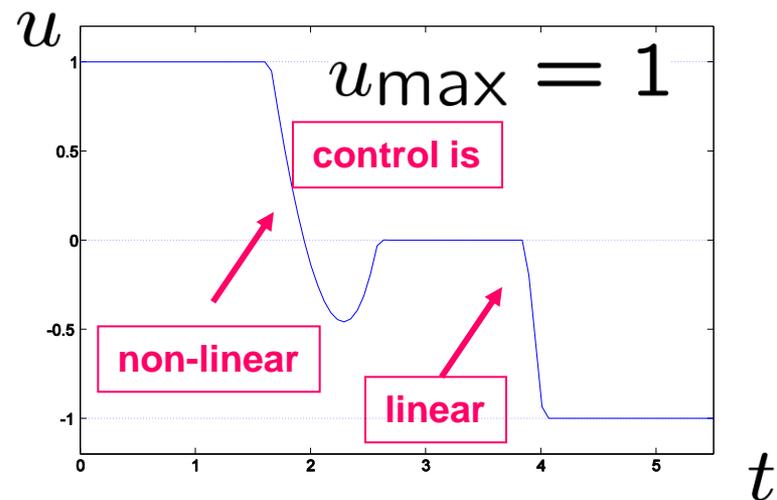
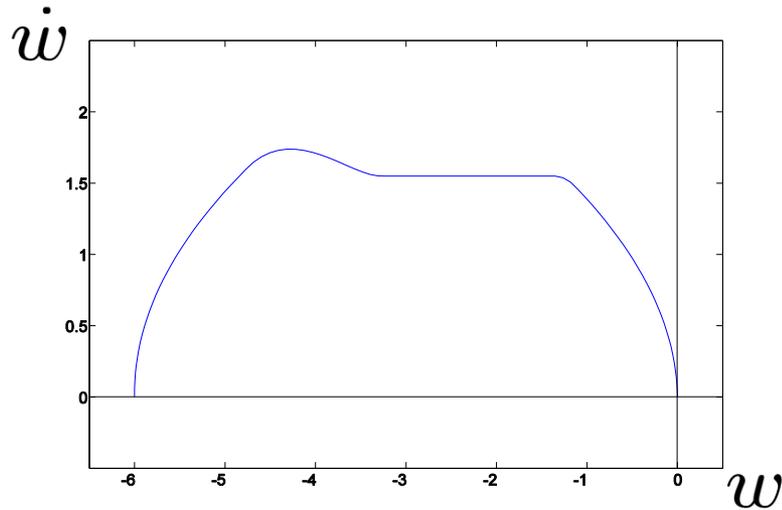
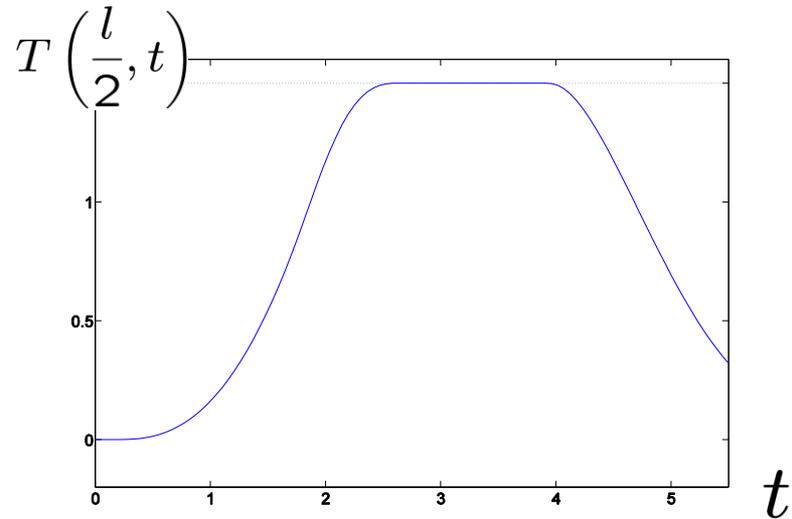
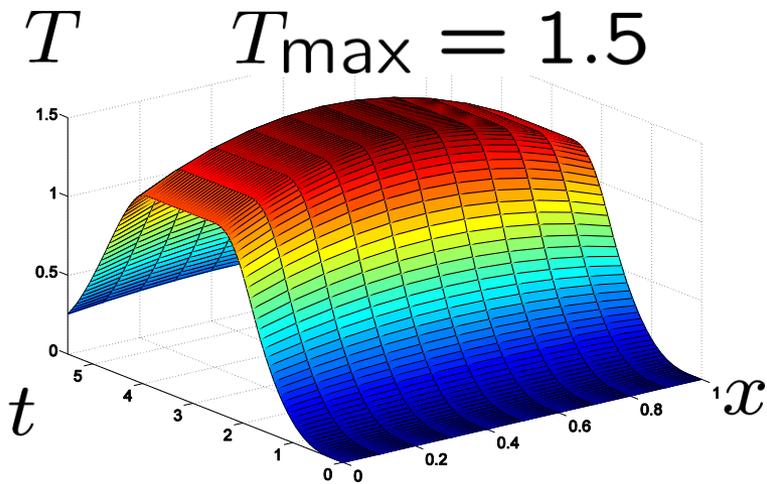
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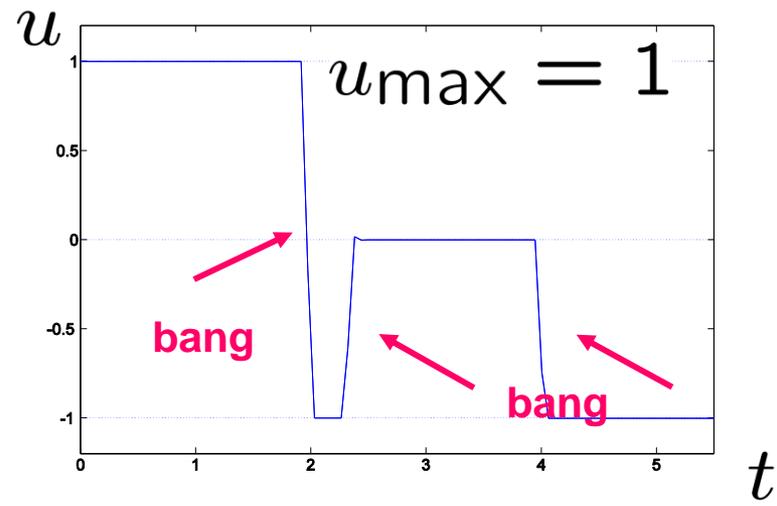
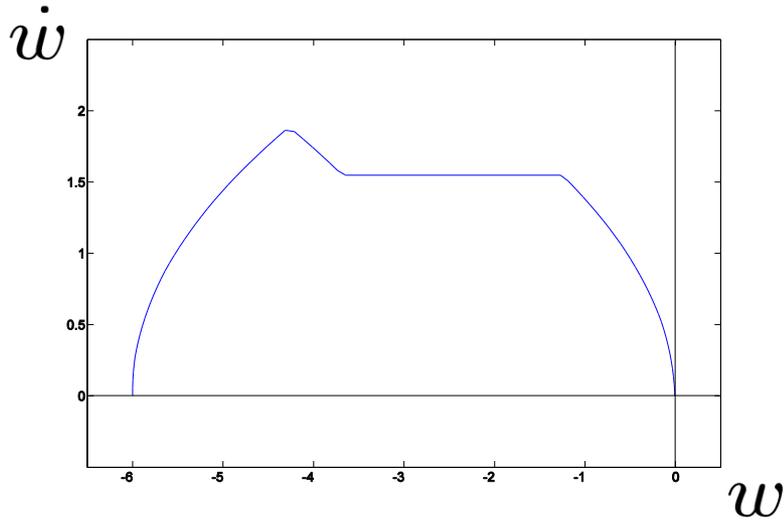
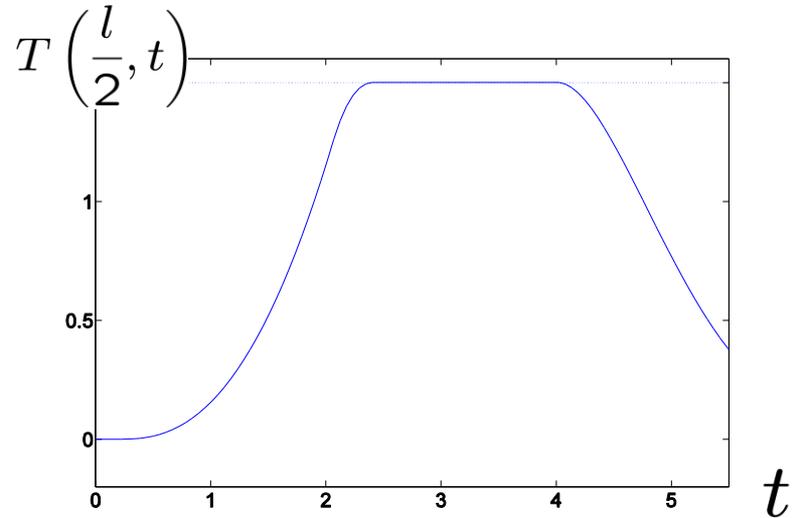
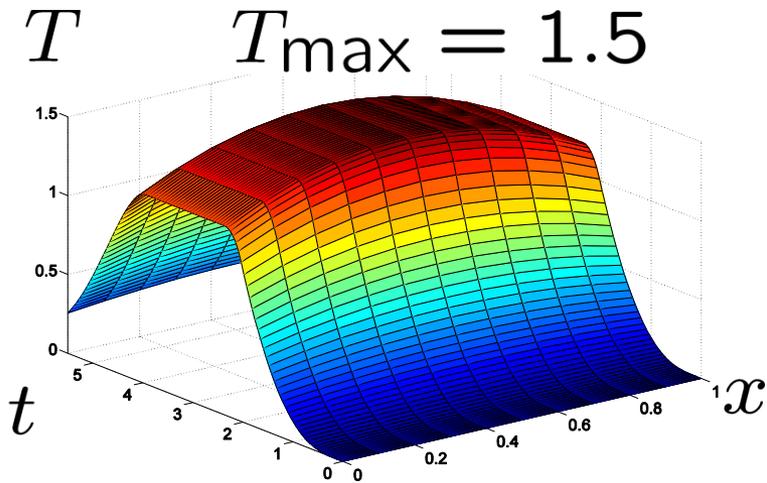
- Conclusion



Numerical results (Type: Problem 1)

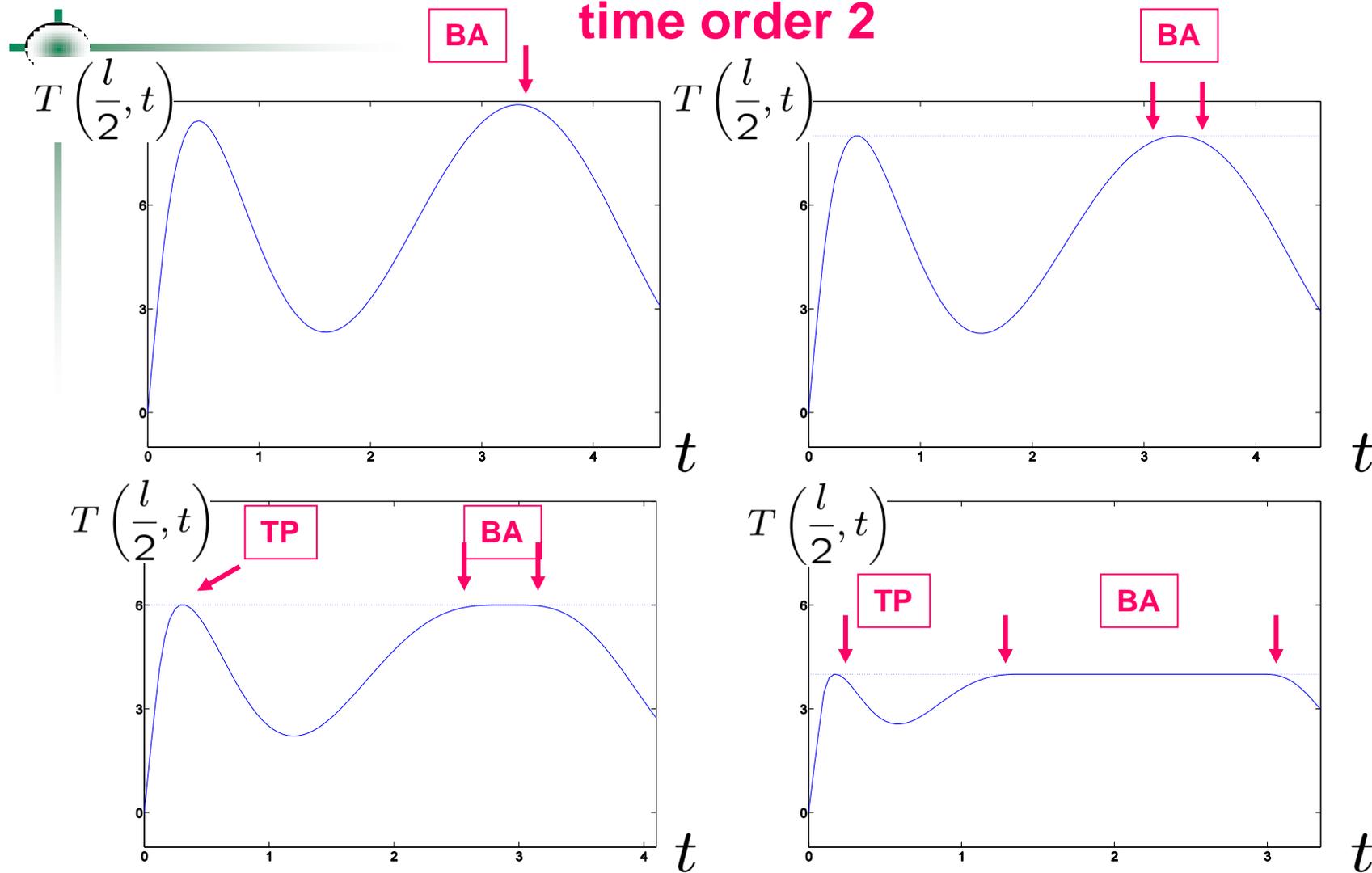


Numerical results (Type: Problem 1)



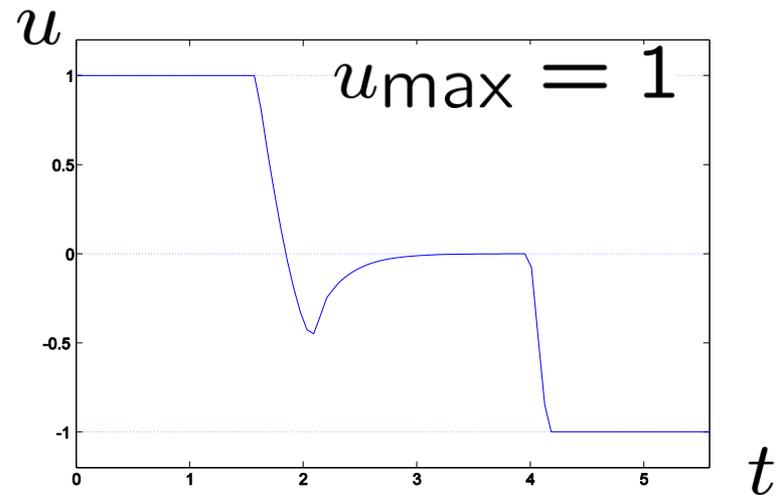
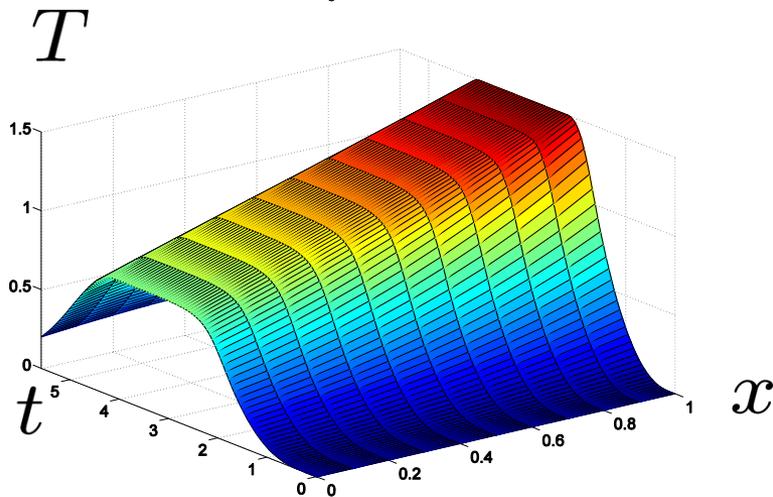
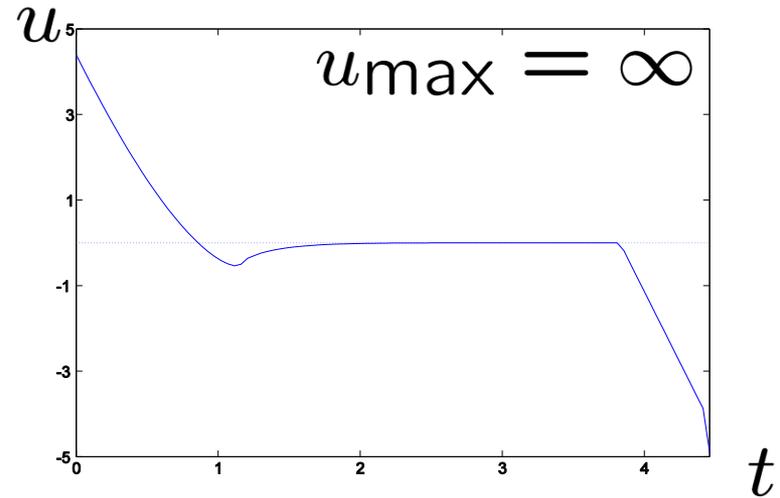
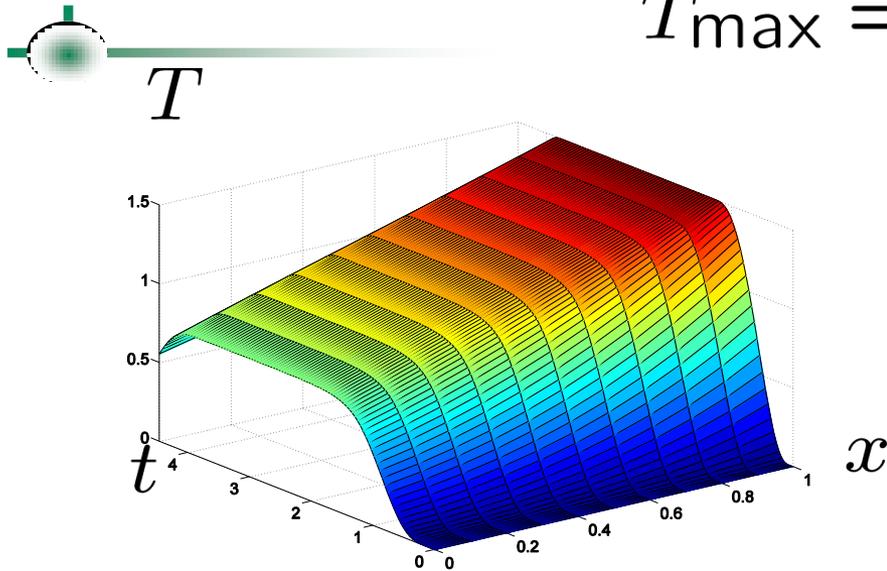
Numerical results (Type: Problem 1)

time order 2



Numerical results (Type: Problem 2)

$$T_{\max} = 1.5$$

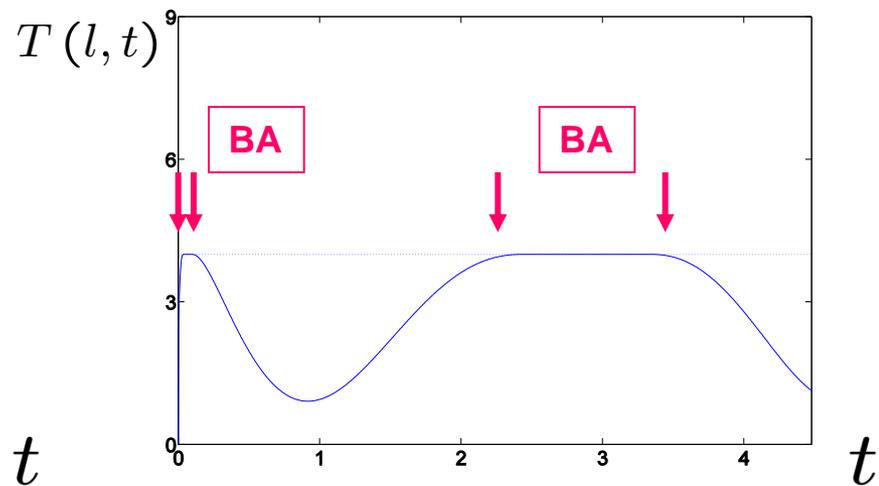
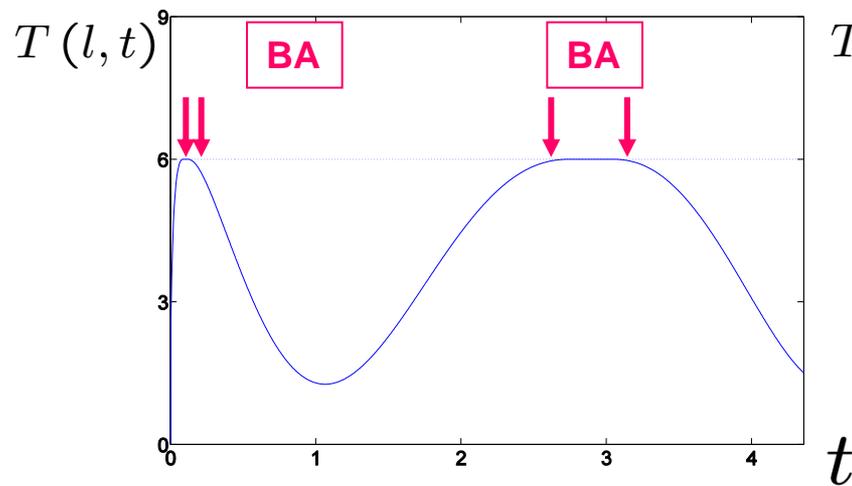
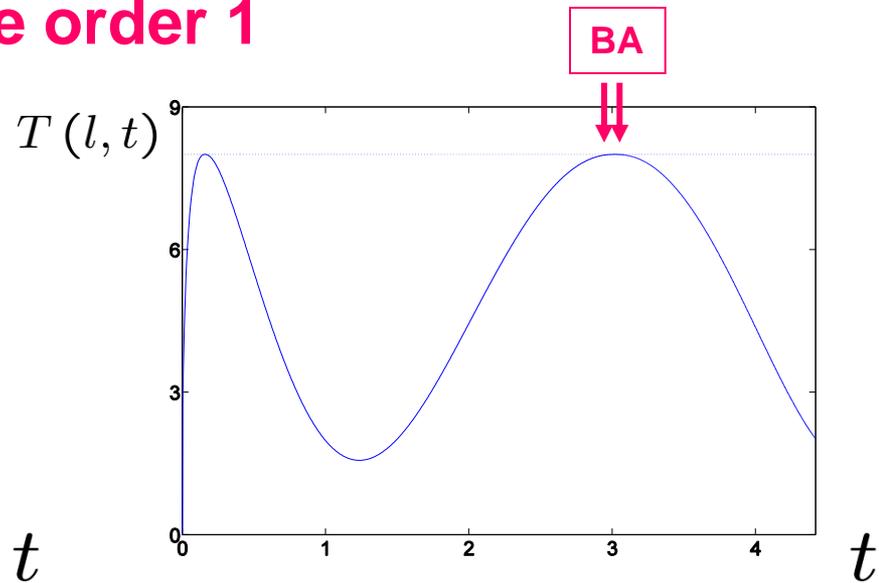
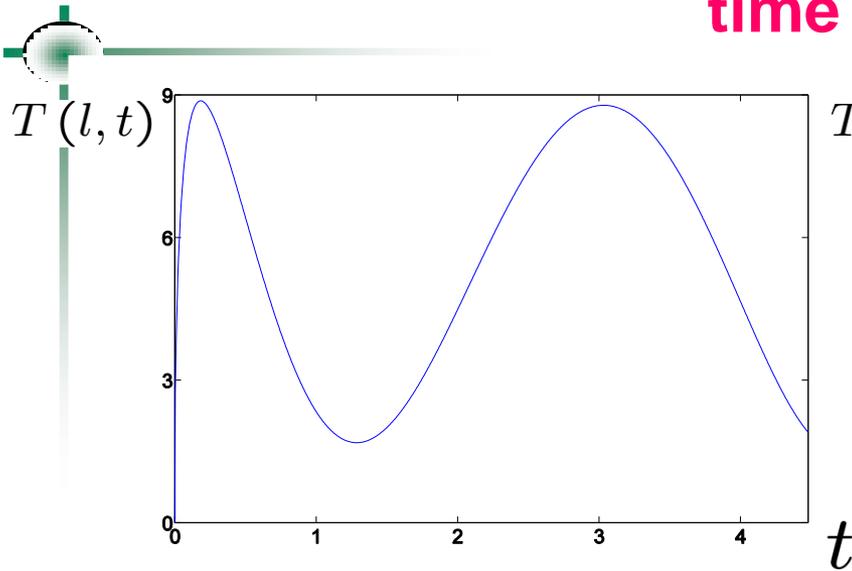


$$\lambda = 10^{-1}$$

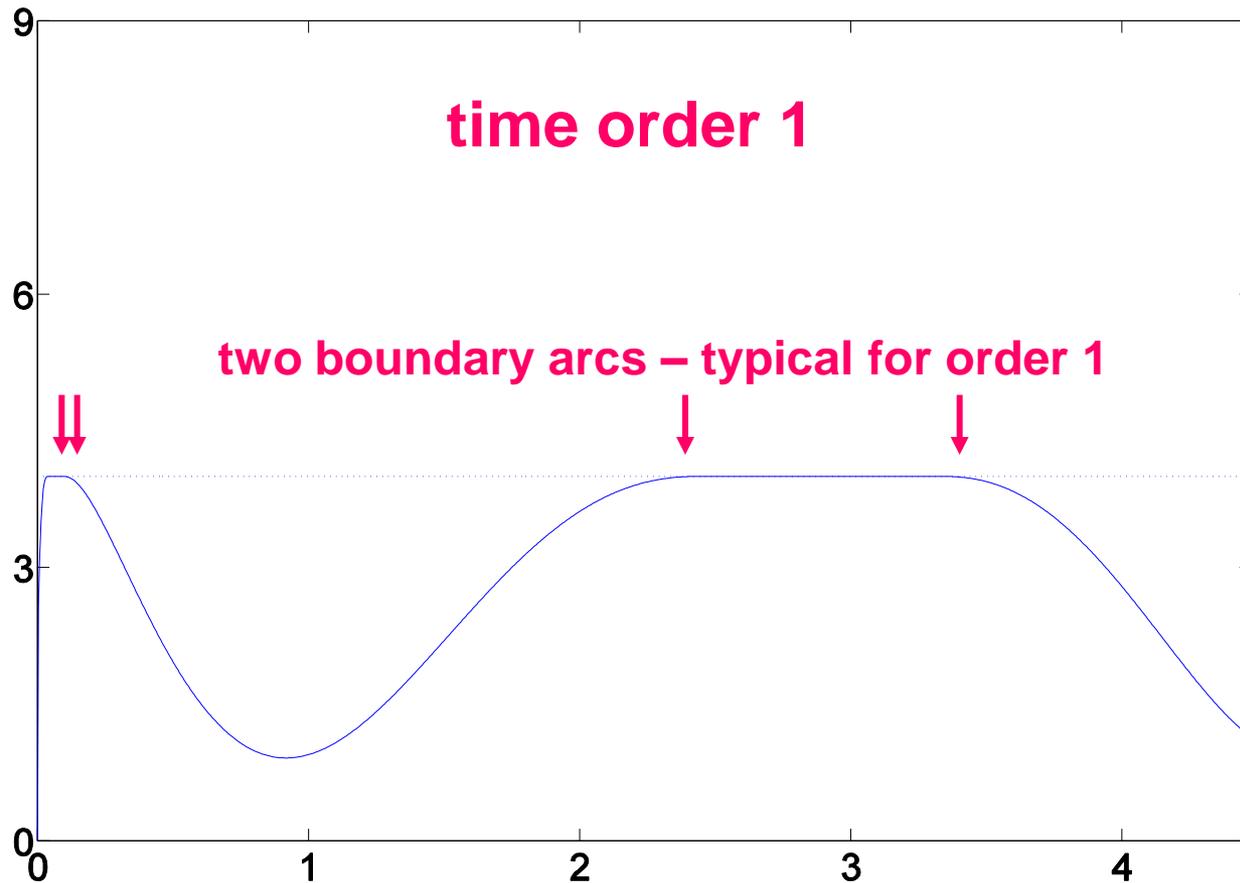


Numerical results (Type: Problem 2)

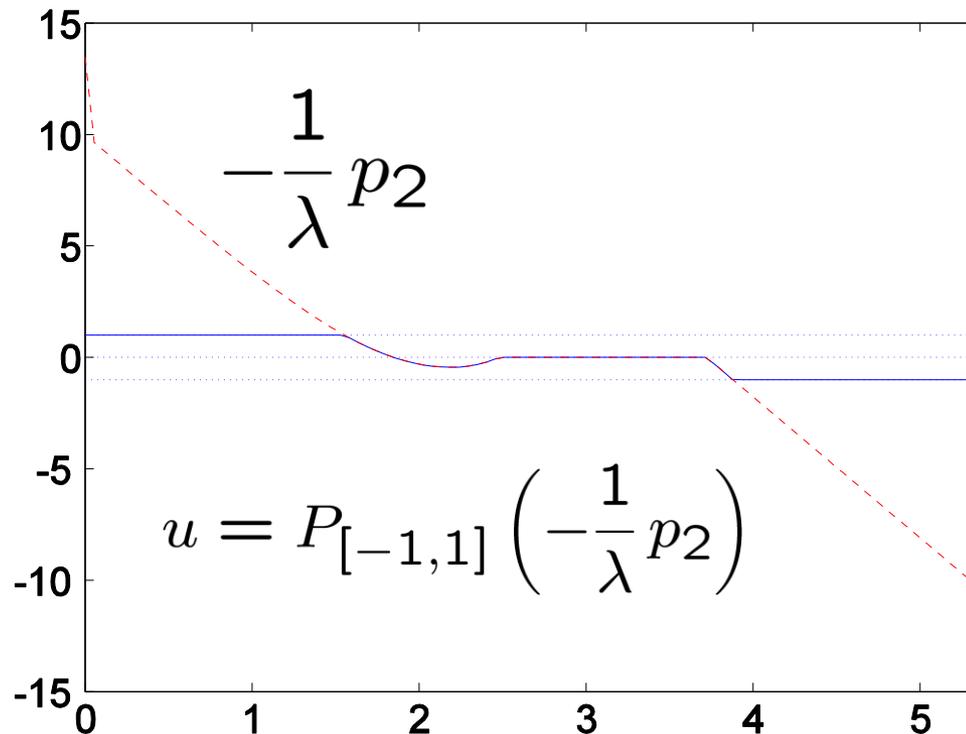
time order 1



Numerical results (Type: Problem 2)



A posteriori verification of optimality conditions: projection formula (ODE)



Method:
Ampl + IPOPT

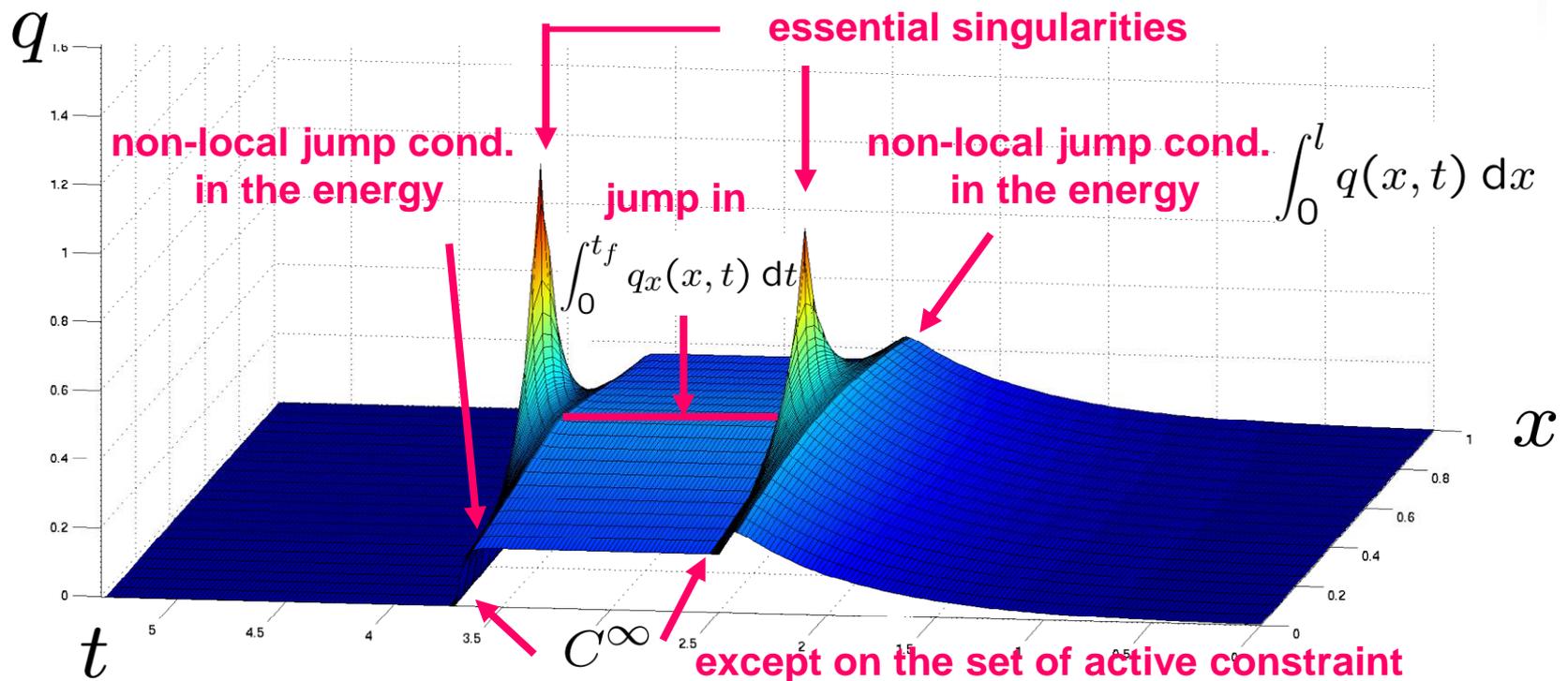
Ref.: IPOPT
Andreas Wächter
2002



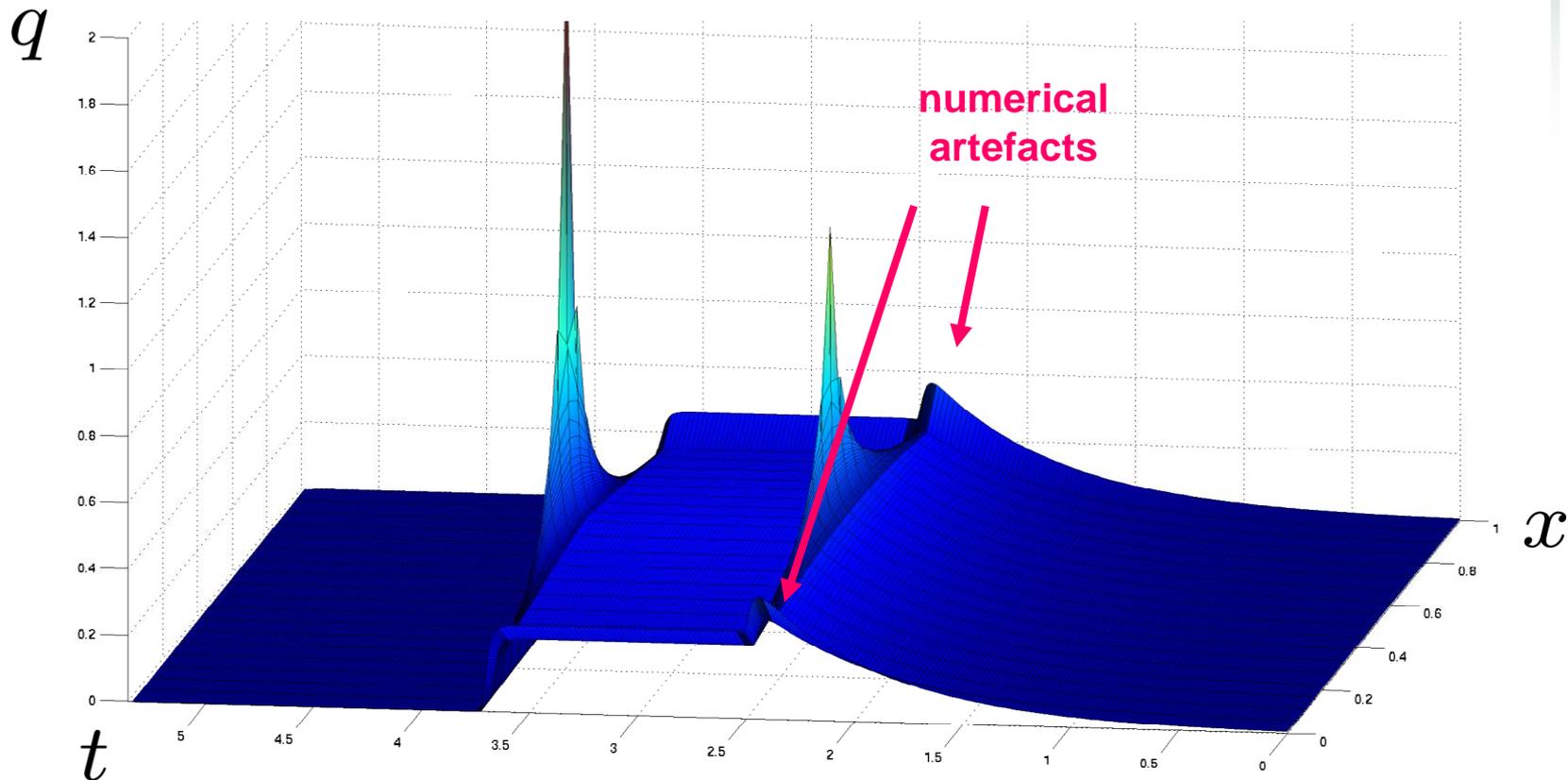
Numerical results (FOTD vs. FDTO)

**A posteriori verification of optimality conditions:
The PDE formulation: adjoint temperature**

$$-q_t - q_{xx} = \varrho_1 \delta\left(x - \frac{l}{2}\right) \otimes \left[\varrho_2 \delta(t - t_{\text{on}}) + \nu(t) + \varrho_3 \delta(t - t_{\text{off}}) \right]$$



A posteriori verification of optimality conditions: The PDE formulation: adjoint temperature



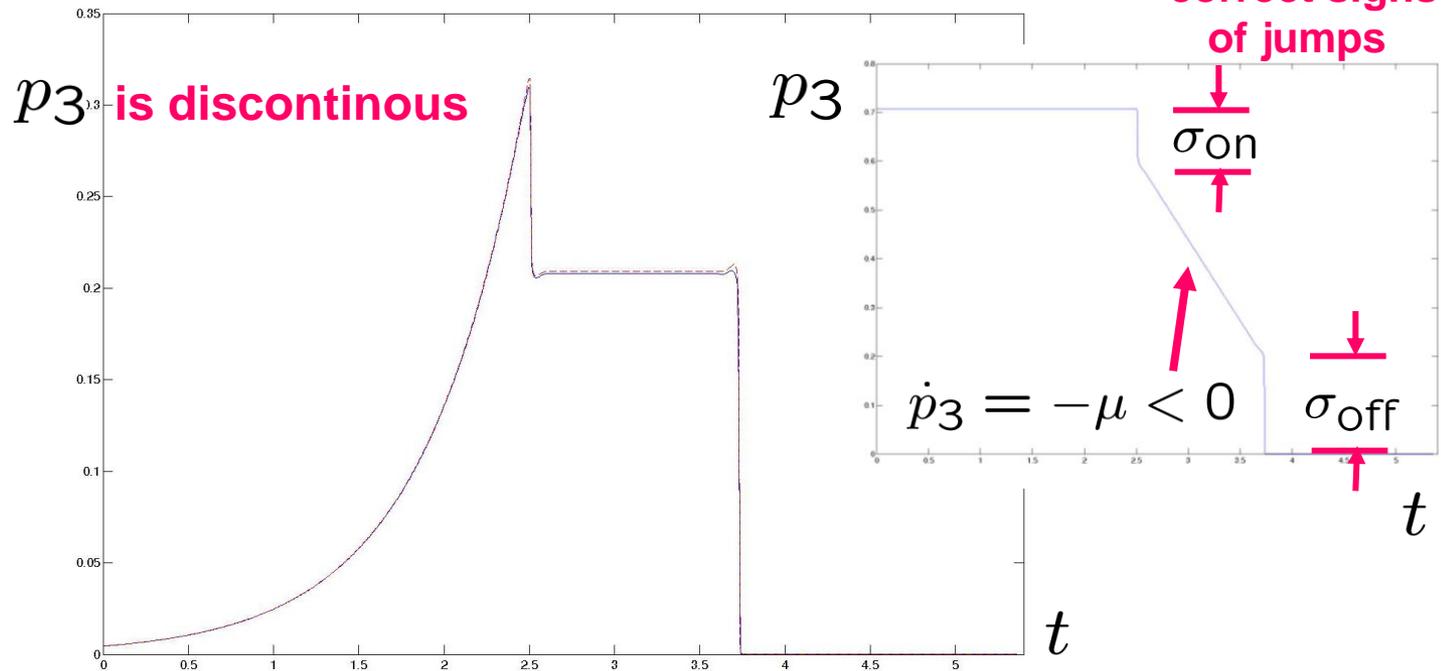
A posteriori verification of optimality conditions: comparison of adjoints (ODE + PDE)



$$p_2(t) \cong \underbrace{\nu_1 + \nu_2 t}_{\text{blue}} + \underbrace{\int_t^{t_f^*} g'(v^*(s)) \cdot \left(\int_0^l q(x, s) dx \right) ds}_{\text{red}}$$



A posteriori verification of optimality conditions: comparison of adjoints/jump conditions (ODE + PDE)



$$p_3(t) - \int_t^{t_f^*} \sum_{n=1}^{\infty} k_n^2 \tilde{\gamma}_n e^{-k_n^2(s-t)} p_3(s) ds \cong \int_0^l q(x, t) dx$$



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Pros:

- Detailed model of high complexity for instationary heating
- Reduction of temperature of TPS due to optimal control
- Challenging problem: ODE - PDE state - constrained optimal control

Cons:

- Lack of theory: nonlinear, state - constrained, ODE-PDE coupling
- Method of lines and SQP methods seem to be at their limits
- Adjoint based methods desirable, but almost impossible to handle



- **Staggered optimal control problems with state constraints**
- **Structural analysis w.r.t. switching structure**
- **Unexpectedly complicated necessary conditions**
- **Problems with free terminal time for PDE problems**
- **Jump conditions in Integro-ODE and PDE optimal control**
- **First optimize, then discretize hardly applicable**
- **First discretize, then optimize with reliable verification of necessary conditions, but with limitations in time and storage**
- **Motivation from hypersonic flight path optimization**

