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# Minimum-fuel trajectories for a hypersonic passenger aircraft under a temperature restriction for the thermal protection system

Hans-Josef Pesch

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# **Minimum-fuel trajectories for a hypersonic passenger aircraft under a temperature restriction for the thermal protection system**

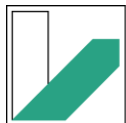
**Hans Josef Pesch**

**University of Bayreuth, Germany**

**Hyper Aircraft: Kurt Chudej, Markus Wächter,  
Gottfried Sachs, Florent le Bras**

**Hyper Car: Armin Rund, Wolf von Wahl & Stefan Wendl**

**Workshop on Aerospace Applications of Control and Optimization  
Kick-off Meeting of the Initial Training Network SADCO,  
Paris, France, March. 2-4, 2011**



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- **Introduction/Motivation**
- **The hypersonic trajectory optimization problem**
- **The instationary heat constraint**
- **Numerical results**
  
- **The hypersonic rocket car problems**
- **Theoretical results**
- **New necessary conditions**
- **Numerical results**
  
- **Conclusion**

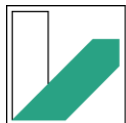


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- **Conclusion**



# Introduction: Supersonic Aircraft



Concorde

first flight: March 21, 1969 (Cherbourg, France) 1976; 20 planes  
first flight: December 21, 1968 (Tbilisi, Georgia) 1975; 16 planes  
Mach 2.35 (2500 km/h, 1554 mph) (altitude 1975)  
Mach 2.35 (2500 km/h, 1554 mph) (altitude)

Crash after take-off: July 25, 2000,  
Paris, Charles de Gaulle



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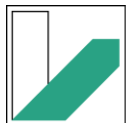
From Times Online  
February 5, 2008

## **Hypersonic jet could offer day trips Down Under**

**Day trips to Australia came a step closer to reality today when a British firm unveiled plans for a hypersonic passenger jet which could fly from Europe to Sydney in less than five hours.**

**Reaction Engines of Oxfordshire says that its A2 plane could be in service within 25 years, carrying 300 passengers at a top speed of almost 4,000 mph, five times the speed of sound and twice the speed of Concorde.**

**The jet may be more environmentally friendly than other planes because it uses liquid hydrogen for power rather than fossil fuel.**



# Introduction: Hypersonic Passenger Jets



[http://www.reactionengines.co.uk/lapcat\\_anim.html](http://www.reactionengines.co.uk/lapcat_anim.html)



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Project LAPCAT  
Reading Engines, UK

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# Motivation: Hypersonic Passenger Jets

quasilinear PDEs  
non-linear boundary conditions  
coupled with ODE

PDE

ODE

$$T \leq T_{\max}$$



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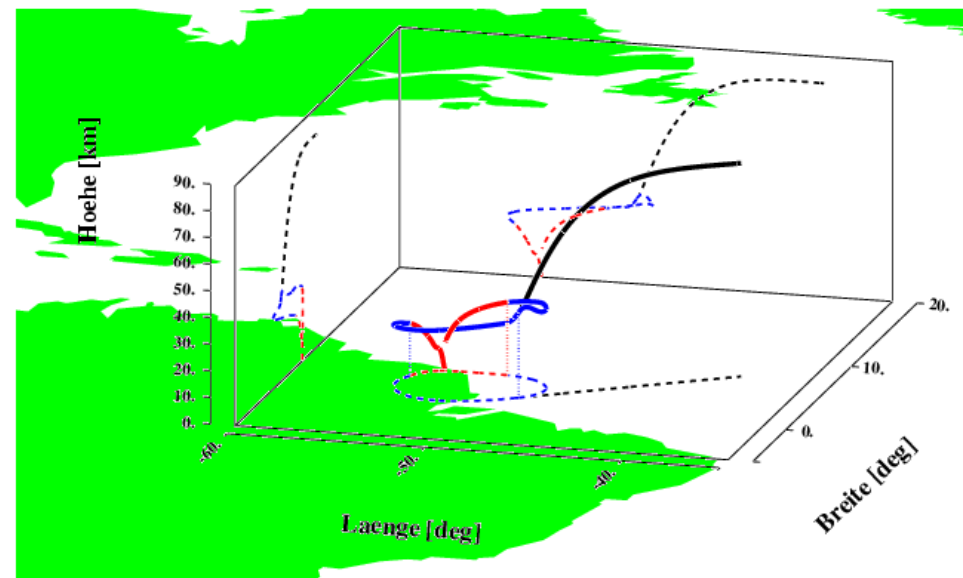


# Introduction: The German Sanger II Project

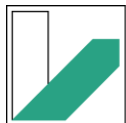


Collaborative Research Center,  
Munich: 1989 - 2003

air-breathing  
turbojet - ramjet / scramjet



Chudej: 1989; Schnepfer: 1999

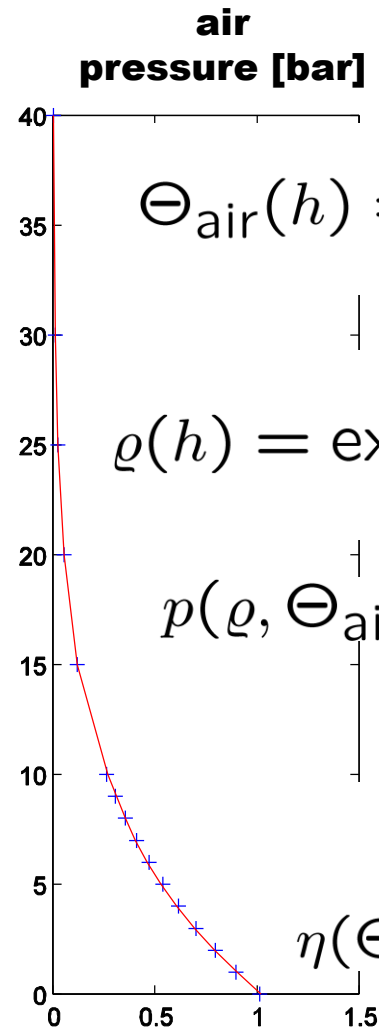
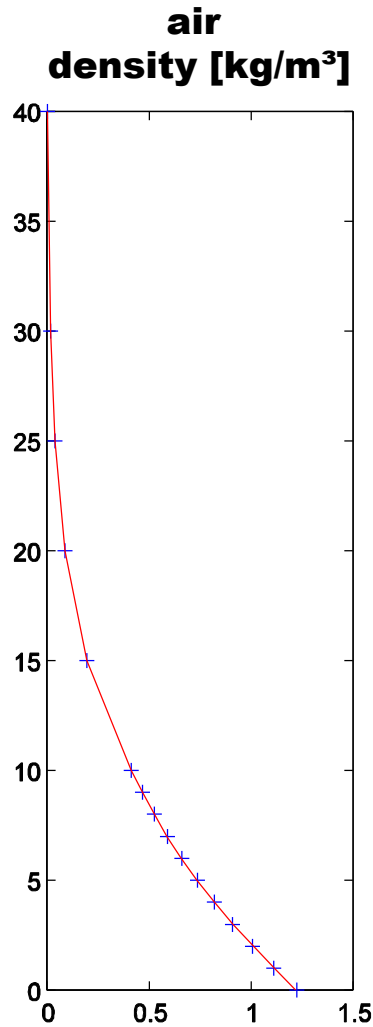
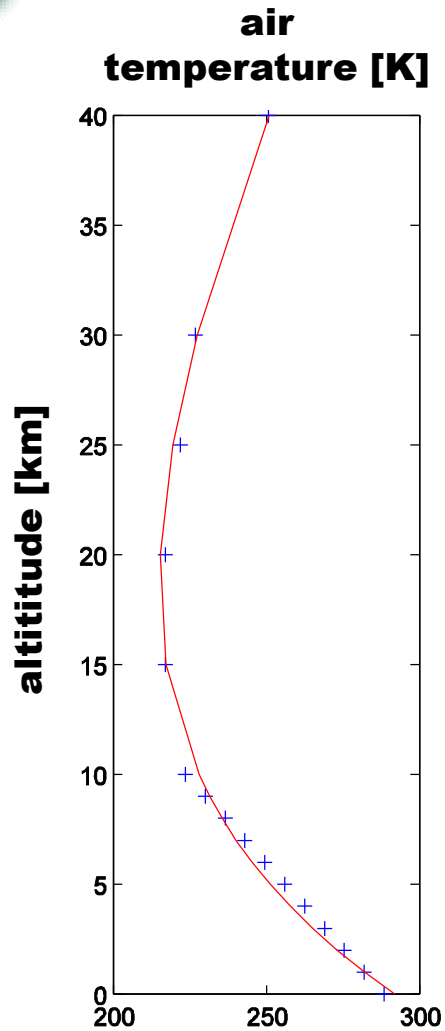


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# Model: Atmosphere



$$\Theta_{\text{air}}(h) = \Theta_0 \sum_{i=0}^5 c_i h^i$$

$$\rho(h) = \exp\left(\frac{\sum_{i=0}^5 a_i h^i}{\sum_{j=0}^3 b_j h^j}\right)$$

$$p(\rho, \Theta_{\text{air}}) = \rho R \Theta_{\text{air}}$$

↓

$$a(h) \quad s(\Theta_{\text{air}})$$

$$\eta(\Theta_{\text{air}}) \quad Pr(\Theta_{\text{air}}, p)$$



## Heat conductivity

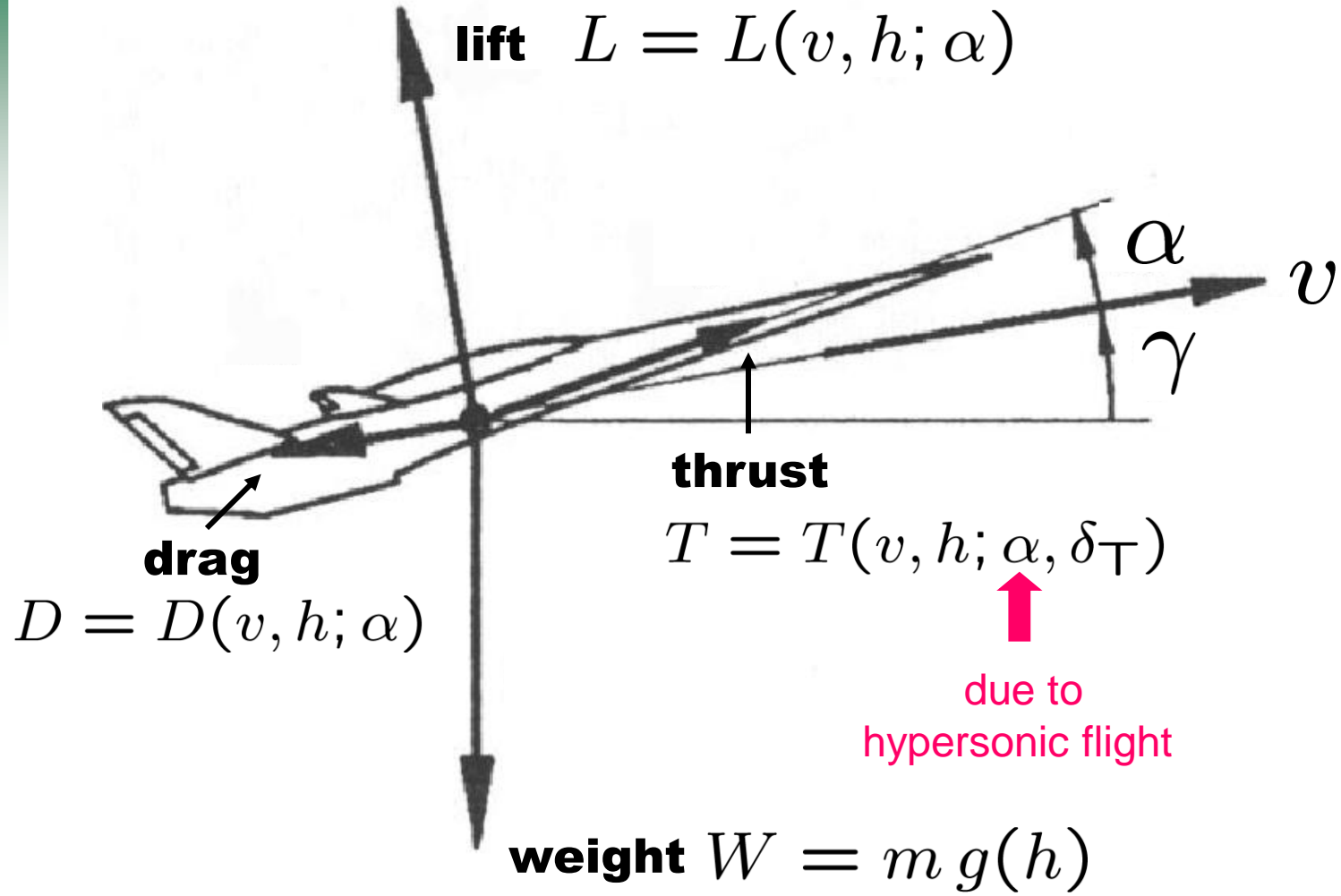
$$\lambda(\Theta_{\text{air}}, p) = \begin{cases} \sum_{i=0}^3 g_i \Theta_{\text{air}}^i & \text{if } \Theta_{\text{air}} \leq 1400 \\ \xi_1 \sum_{i=0}^3 g_i \Theta_{\text{air}}^i + \xi_2 P_\lambda(\Theta_{\text{air}}, p) & \text{if } 1400 < \Theta_{\text{air}} < 1600 \\ P_\lambda(\Theta_{\text{air}}, p) & \text{if } 1600 \leq \Theta_{\text{air}} \end{cases}$$

## Heat capacity

$$c_p(\Theta_{\text{air}}) = R \frac{\kappa}{\kappa - 1} \quad \text{with} \quad \kappa = \frac{\sum_{i=0}^5 d_i \Theta_{\text{air}}^i}{\sum_{j=0}^1 e_j \Theta_{\text{air}}^j}$$

$\Theta_{\text{air}}$ : air temperature





## Two-dimensional flight over a great circle of a rotational Earth

$$\dot{v} = \frac{1}{m} \left[ T(v, h; \alpha, \delta_T) \cos(\alpha + \sigma_T) - D(v, h; \alpha) \right] - g(h) \sin \gamma + \omega_E^2 r(h) \sin \gamma$$

$$\dot{\gamma} = \frac{1}{m v} \left[ T(v, h; \alpha, \delta_T) \sin(\alpha + \sigma_T) + L(v, h; \alpha) \right] + \cos \gamma \left[ \frac{v}{r(h)} - \frac{g(h)}{v} + \frac{\omega_E^2 r(h)}{v} \right] + 2 \omega_E$$

$$\dot{h} = v \sin \gamma$$

$$\dot{\zeta} = v \cos \gamma$$

$$\dot{m} = -\beta_T(v, h; \alpha, \delta_T) \quad \text{or} \quad \dot{m} = -\max \{ \beta_C(v, h; \alpha, \delta_T), \beta_I(v, h; \alpha, \delta_T) \}$$

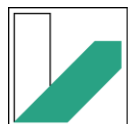
homotopy parameter  $0 \leq \epsilon \leq 1$



instantaneous  
fuel consumption  
for thrust



instantaneous  
fuel consumption  
for active cooling



# Model: Dynamics: Boundary Conditions

	$v$ [m/s]	$\gamma$ [rad]	$h$ [m]	$\zeta$ [km]	$m$ [to]
$t = 0$	150	0	500	0	244
$t_f$ free	150	0	500	9 000	—

Houston – Rome: 9163 km = 5693 m  
Munich – Houston: 8714 km = 5414 m



## Objective function

$$\max_{\alpha, \delta_T} m(t_f)$$

## Constraints

$$-1.5^\circ \leq \alpha \leq 20^\circ \quad 0 \leq \delta_T \leq 1$$

angle of attack : box constraints : throttle setting

$$0 \leq n(v, h, m; \alpha) \leq 2 \quad 10 \leq \bar{q}(v, h) \leq 50 \text{ [kPa]}$$

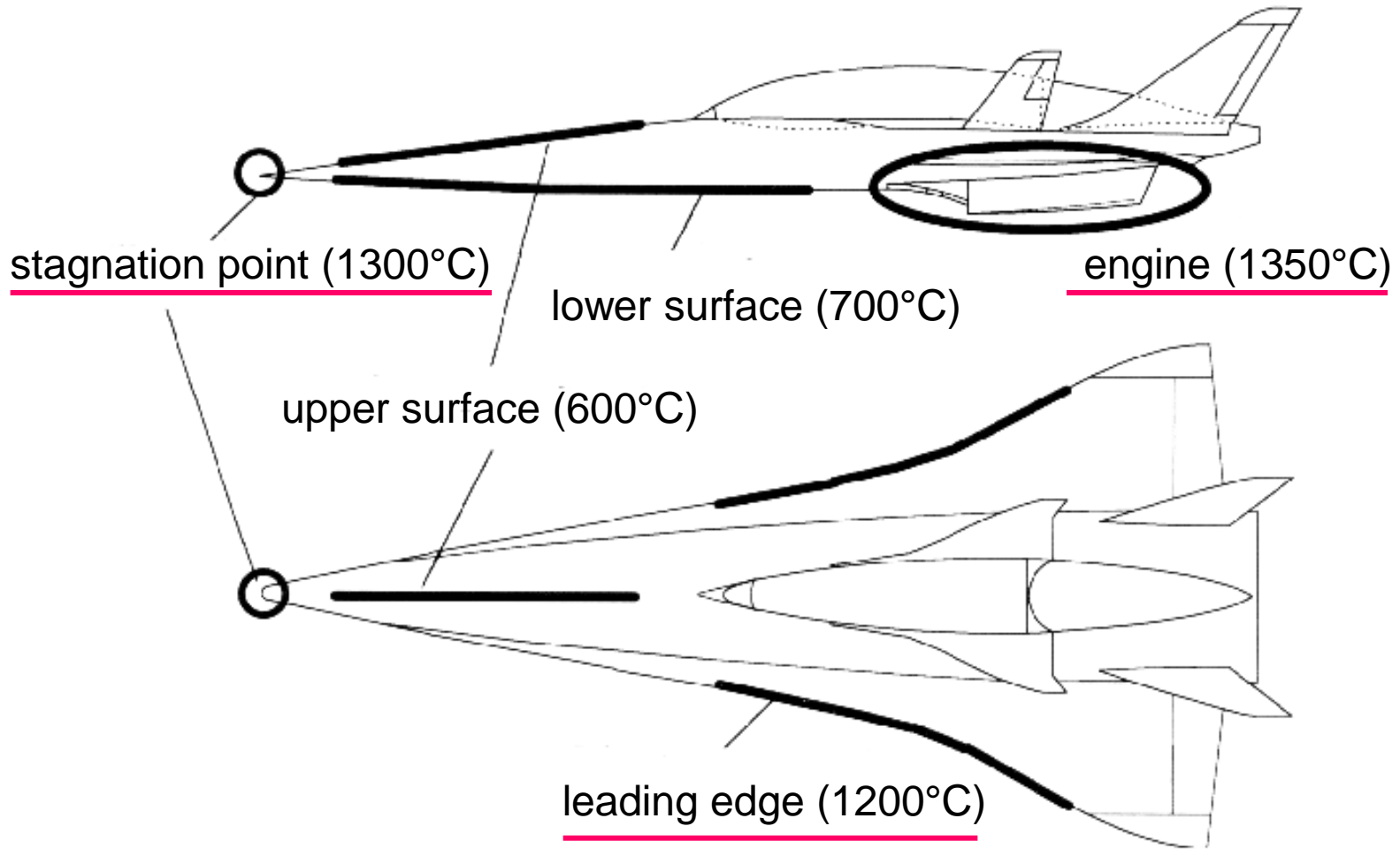
control-state constr.: load factor

state constr.: dynamic pressure





# Model: Active Engine Cooling



# Model: Active Engine Cooling

$$\beta_C = \sum_{i=0}^8 \beta_i(v, h; \alpha; Re, \phi)$$

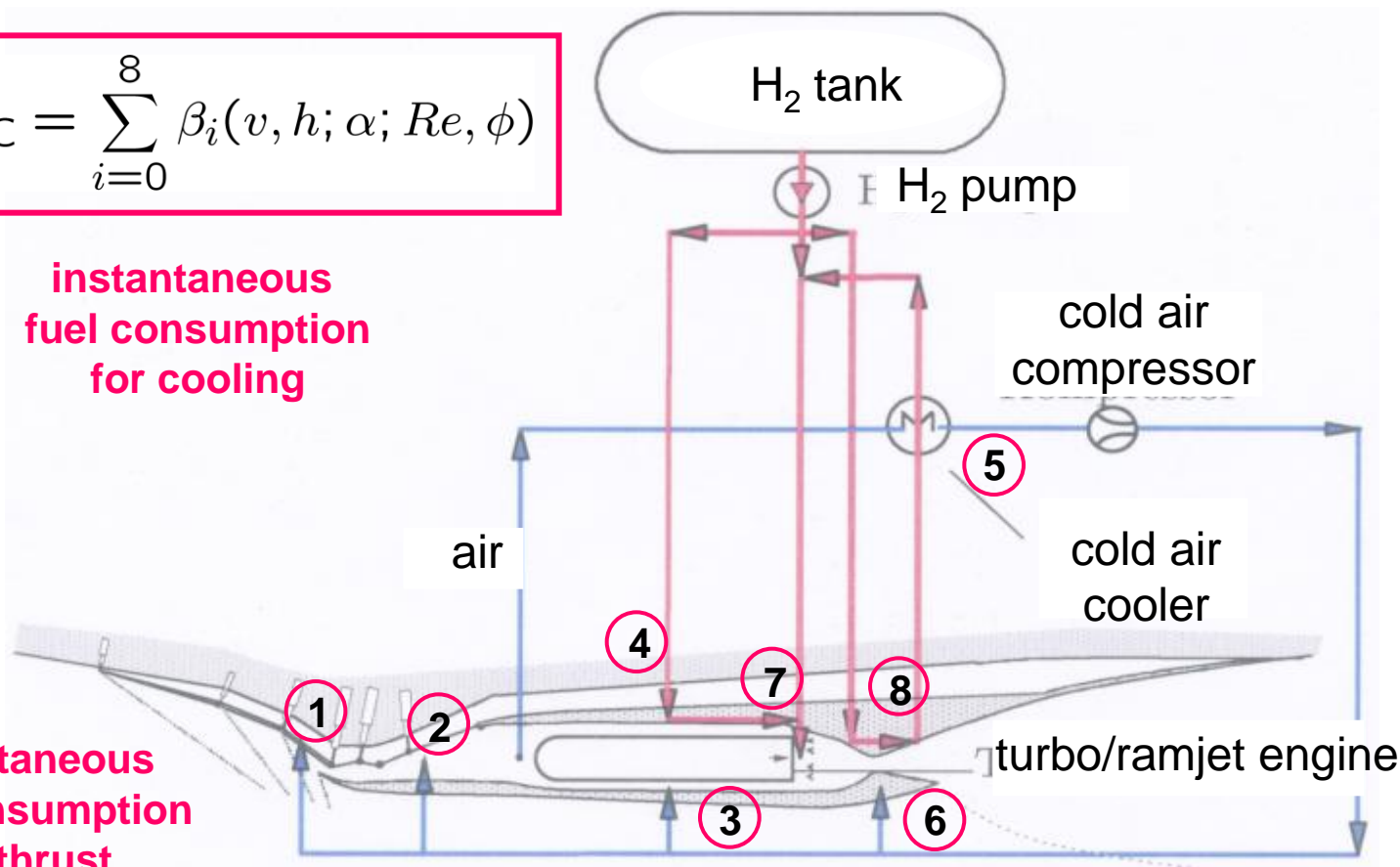
instantaneous  
fuel consumption  
for cooling

instantaneous  
fuel consumption  
for thrust

$$\dot{m} = -\beta_T(v, h; \alpha, \delta_T)$$

$$\beta_C \leq \beta_T$$

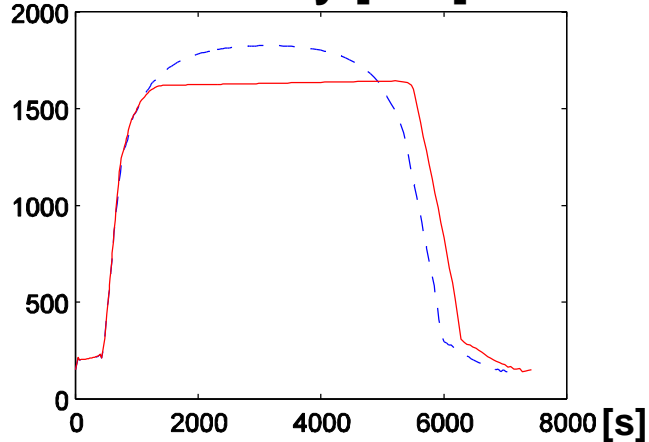
control-state constraint  
fuel is reused  
for thrust



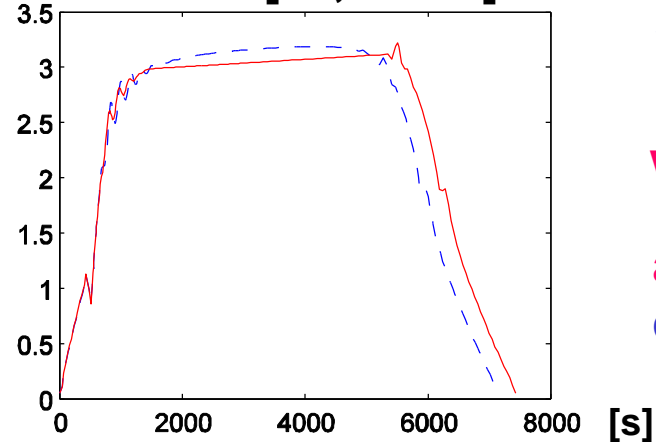
# Numerical Results: State Variables



### velocity [m/s]

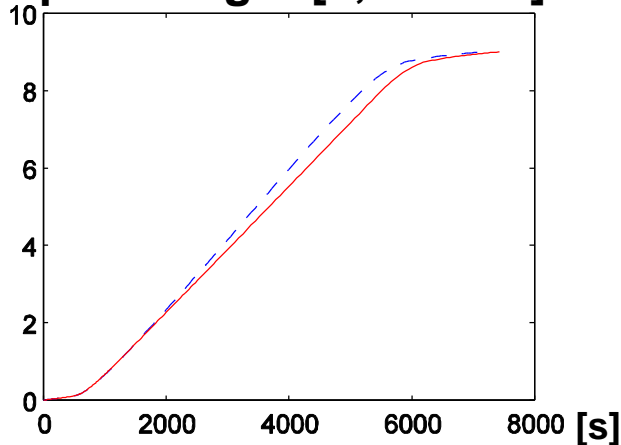


### altitude [10,000 m]

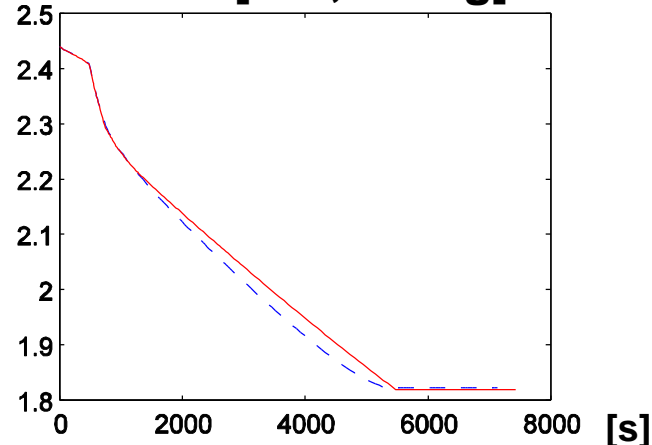


without  
with  
active  
cooling

### path length [1,000 km]



### mass [100,000 kg]



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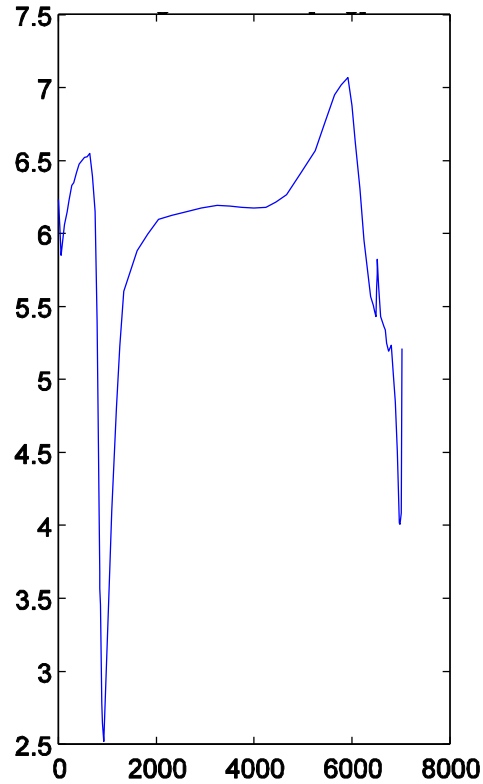
Markus Wächter, Kurt Chudej  
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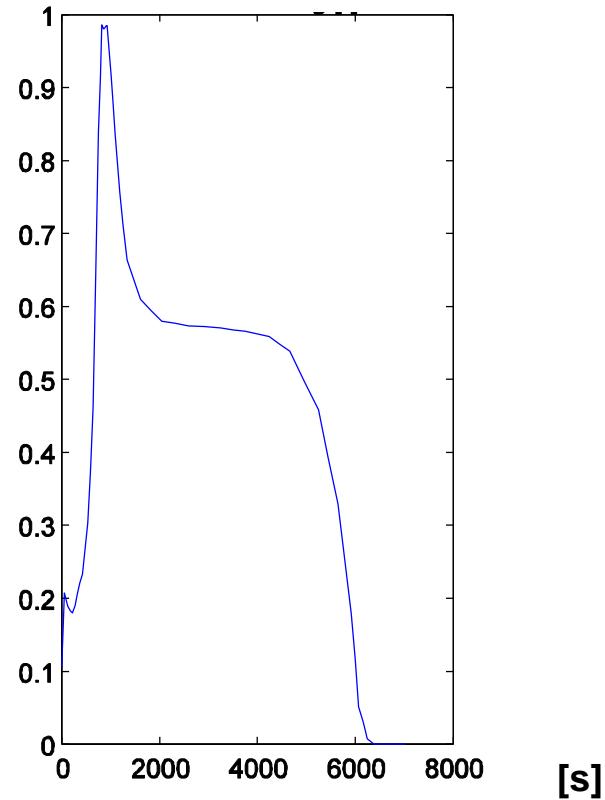


# Numerical Results: Control Variables

angle of attack [deg]



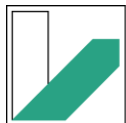
throttle setting



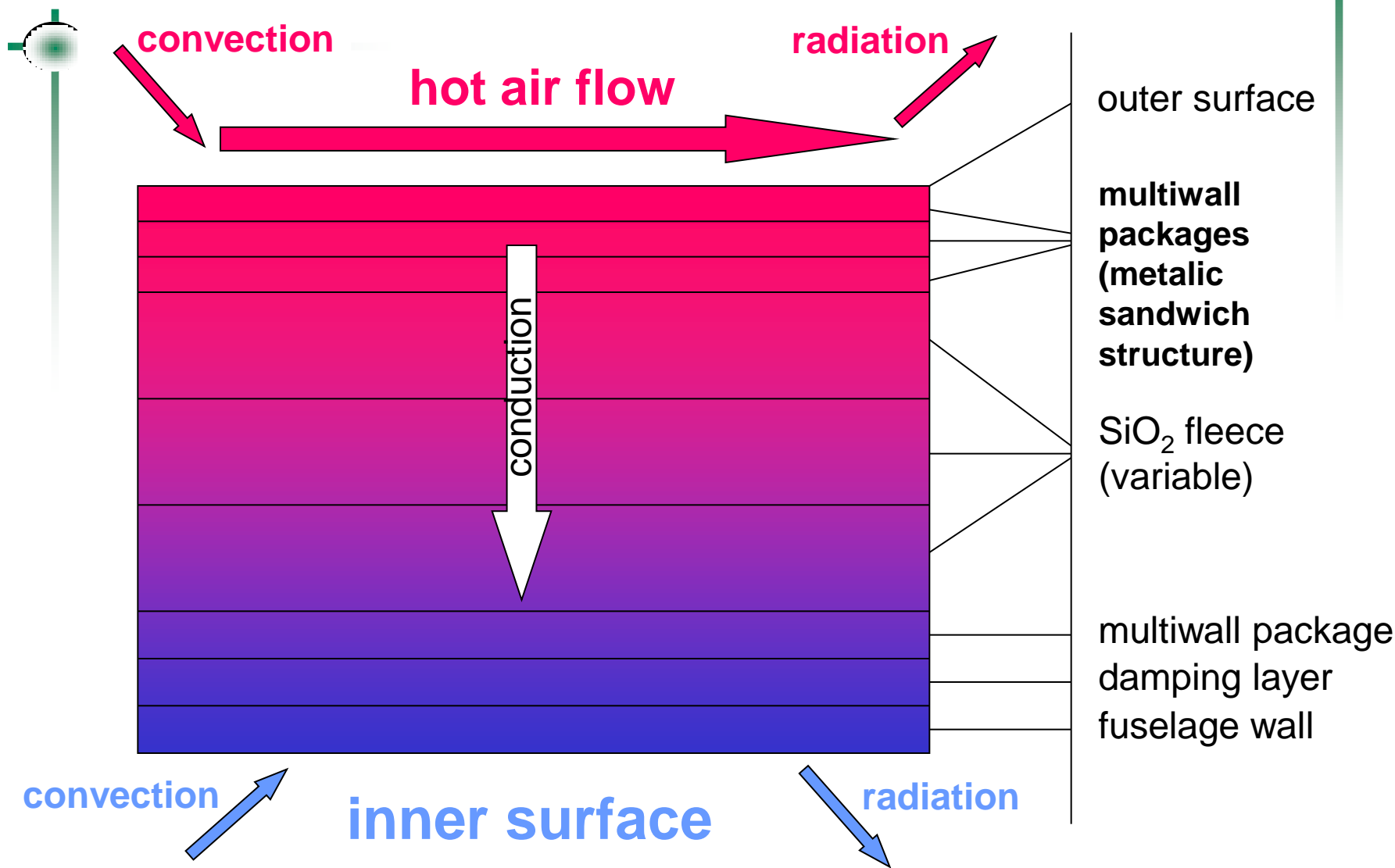
$t_f: 7\,023 \text{ [s]} \implies 7\,500 \text{ [s]}$



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- Appendix: more applications



# Model: Instationary Heat Constraint: Thermal Protection System



affine linear functions

$$\rho_{\text{TPS}} c_p(\Theta) \frac{\partial \Theta}{\partial t} - \vec{\nabla} \cdot [\lambda(\Theta) \vec{\nabla} \Theta] = 0$$

in  $(\vec{x}, t) \in \Omega \times (0, t_f)$

$$\Theta(\vec{x}, 0) = \Theta_0(\vec{x}) = 300 \text{ [K]} \quad \text{in } \vec{x} \in \Omega$$

nonlinear boundary conditions

$$\frac{\partial \Theta}{\partial \vec{n}}(\vec{x}, t) = q_{\text{conv}}(v, h, \alpha) - q_{\text{rad}}(\Theta, \Theta_{\text{air}}(h))$$

in  $(\vec{x}, t) \in \partial\Omega \times (0, t_f)$

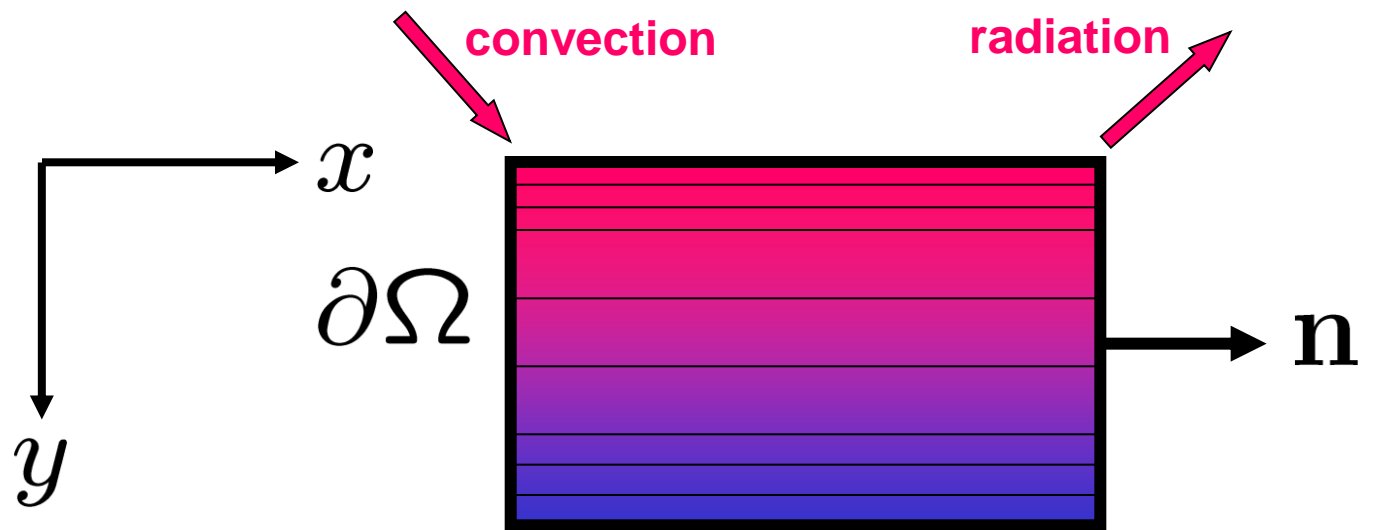
**quasi-linear parabolic initial-boundary value problem  
with nonlinear boundary conditions**



# Model: Instationary Heat Constraint: Boundary Conditions (1)

$$\frac{\partial \Theta}{\partial \vec{n}}(\vec{x}, t) = q_{\text{conv}} - q_{\text{rad}} \quad \text{in} \quad (\vec{x}, t) \in \partial\Omega \times (0, t_f)$$

$$q_{\text{conv}} = q_{\text{air}}(v, h, \Theta_{\text{air}}(h); \alpha; x_L, Q) \quad q_{\text{rad}} = \varepsilon \sigma (\Theta^4 - \Theta_{\text{air}}^4(h))$$



$$q_{\text{conv}} = \alpha_q(\Theta_{\text{int}}) (\Theta - \Theta_{\text{int}})$$

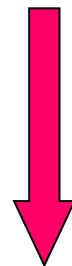
$$q_{\text{rad}} = \varepsilon \sigma (\Theta^4 - \Theta_{\text{int}}^4)$$





**State-constraint for the temperature:**

$$\Theta(\vec{x}, t) \leq \Theta_{\max}(\vec{x}, t) \quad \text{for all} \quad (\vec{x}, t) \in \Omega \times (0, t_f)$$

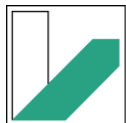


**ODE-PDE state-constrained optimal control problem**

**PDE:** quasilinear parabolic with nonlinear bound. conds.

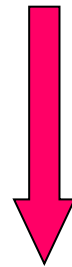
**CONTROL:** boundary controls indirectly  
via ODE states and controls

**CONSTRAINT:** state constraint



**State-constraint for the temperature:**

$$\Theta(\vec{x}, t) \leq \Theta_{\max}(\vec{x}, t) \quad \text{for all} \quad (\vec{x}, t) \in \Omega \times (0, t_f)$$

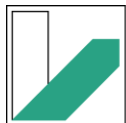


$$\begin{aligned} \tilde{\Omega} &\subset \Omega \\ [t_{\text{in}}, t_{\text{out}}] \end{aligned}$$

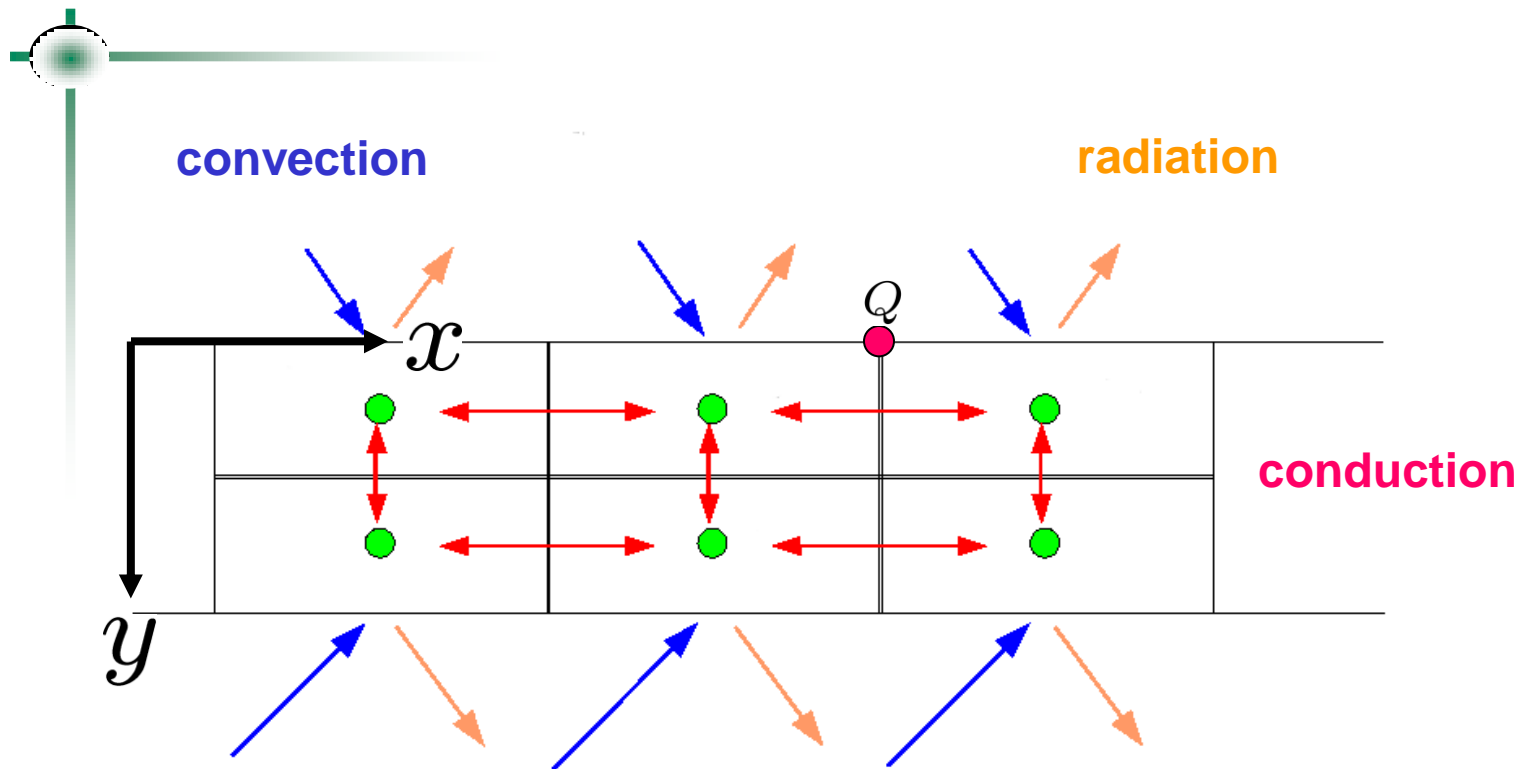
**ODE-PDE state-constrained optimal control problem**



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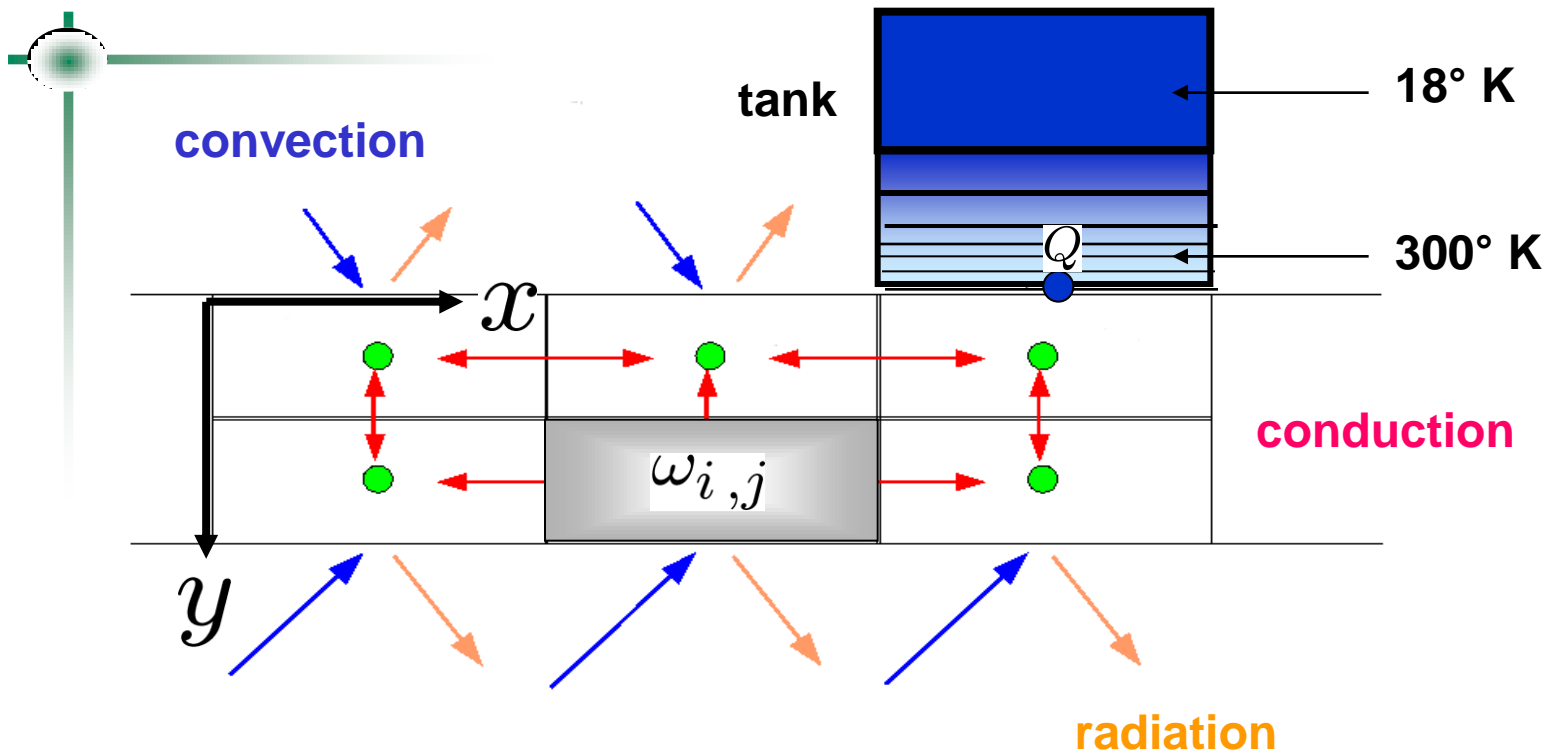
# Numerical Method: Semi-Discretization in Space



**Finite Volume Method: locally and globally conservative  
second order convergent**



# Numerical Method: Semi-Discretization in Space



conservation law in each volume

$$\frac{d}{dt} \int_{\omega_{i,j}} \dots d\vec{x} = - \int_{\partial\omega_{i,j}} \dots d\vec{s}$$

second order  
convergent  
FVM



$$\rho_{\text{TPS}} c_p(\Theta) \frac{\partial \Theta}{\partial t} - \vec{\nabla} \circ [\lambda(\Theta) \vec{\nabla} \Theta] = 0$$

$$\rho_{\text{TPS}} c_p(\Theta_i) \dot{\Theta}_i$$

$$-\frac{1}{\Delta x} \left[ \lambda \left( \frac{\Theta_{i-1} + \Theta_i}{2} \right) \frac{\Theta_{i-1} - \Theta_i}{\Delta x} - \lambda \left( \frac{\Theta_i + \Theta_{i+1}}{2} \right) \frac{\Theta_i - \Theta_{i+1}}{\Delta x} \right] = 0$$

1D case

$$\Theta_i := \Theta(x_i, t)$$



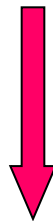
$$\rho_{\text{TPS}} c_p(\Theta_i) \dot{\Theta}_i$$

$$-\frac{1}{\Delta x} \left[ \lambda \left( \frac{\Theta_{i-1} + \Theta_i}{2} \right) \frac{\Theta_{i-1} - \Theta_i}{\Delta x} - \lambda \left( \frac{\Theta_i + \Theta_{i+1}}{2} \right) \frac{\Theta_i - \Theta_{i+1}}{\Delta x} \right] = 0$$

$i = 1$ : **b.c. towards air**

↑  
**coupled with ODE.**

**b.c. towards interior**:  $i = n - 1$



**large scale multiply constrained ODE optimal control problem**



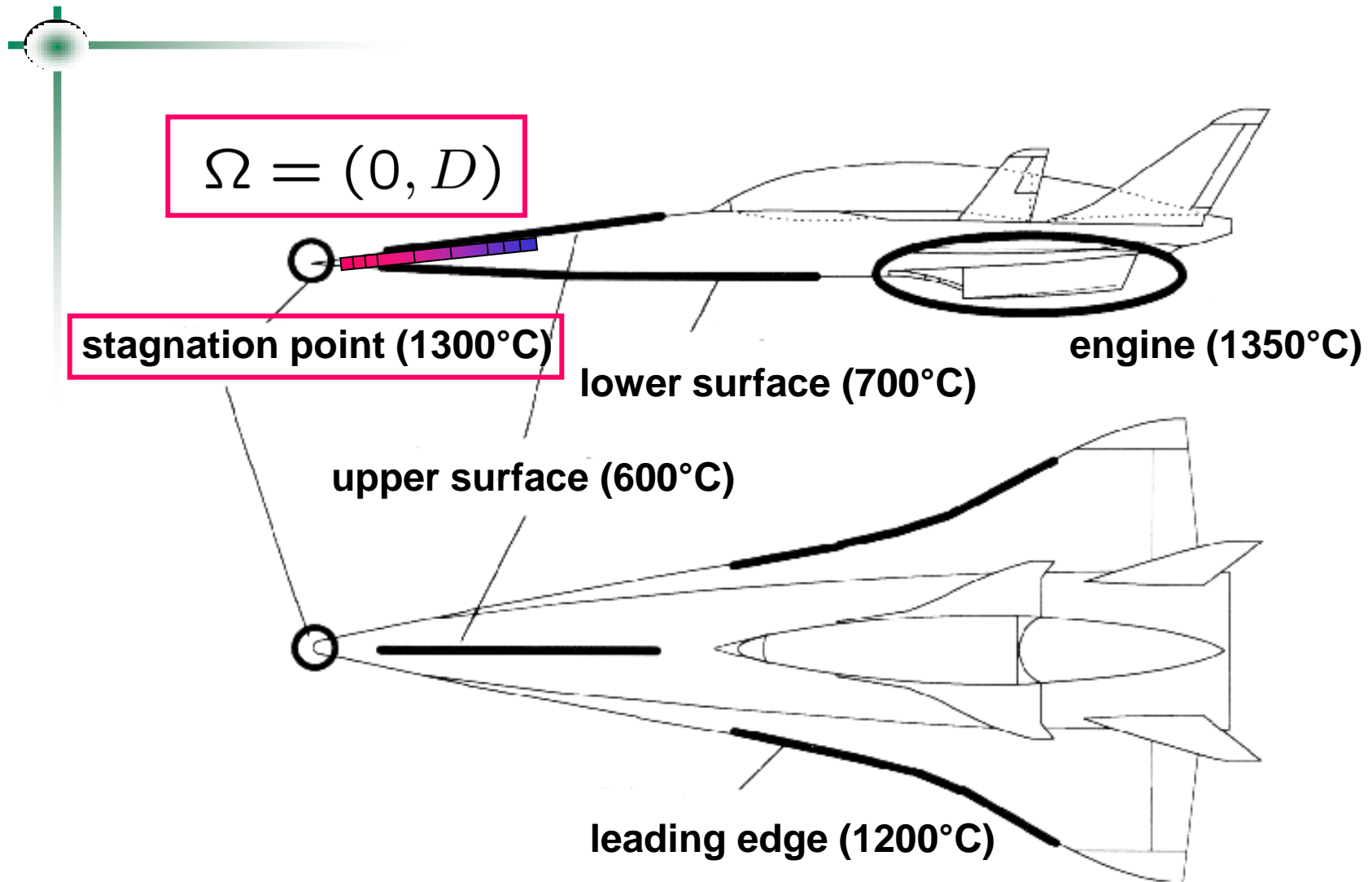
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DIRCOL (O. v. Stryk) with SNOPT (P. Gill)  
alternatively: NUOCCCS (C. Büskens)  
IPOPT (A. Wächter) with AMPL  
WORHP (Büskens, Gerdt)

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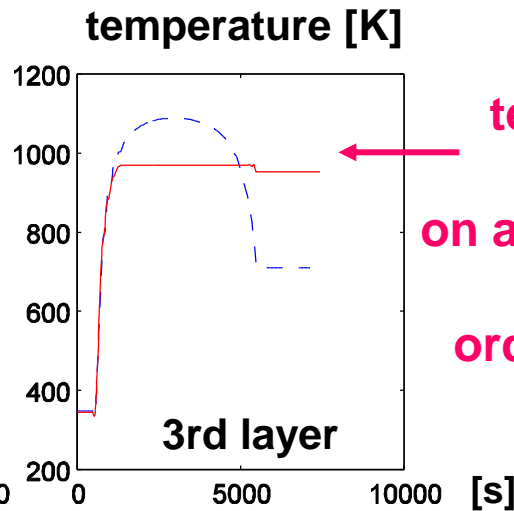
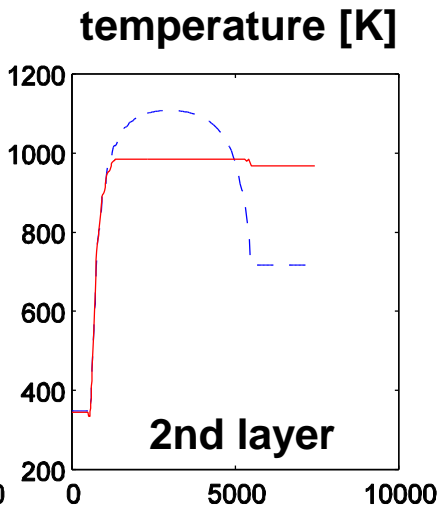
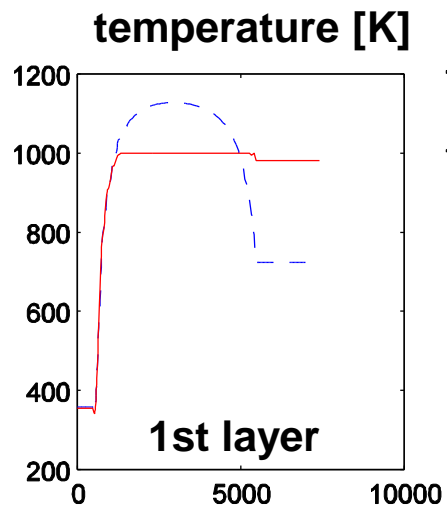
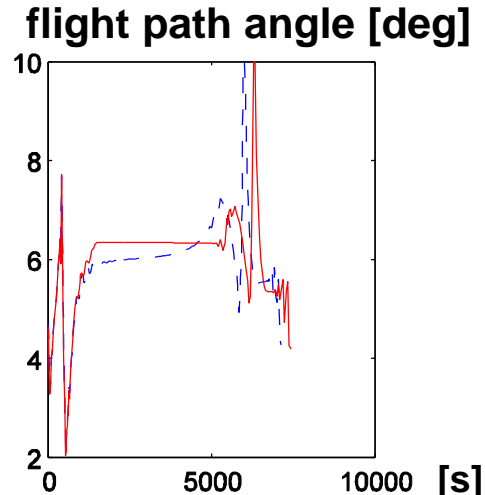
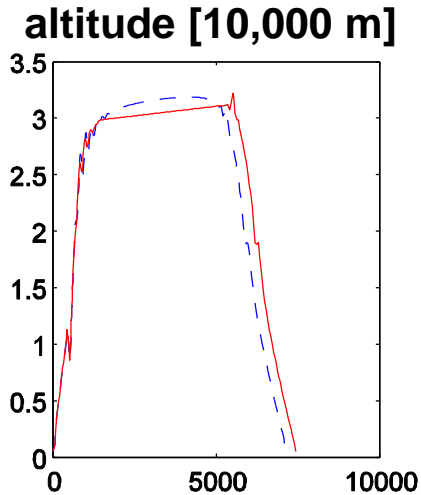
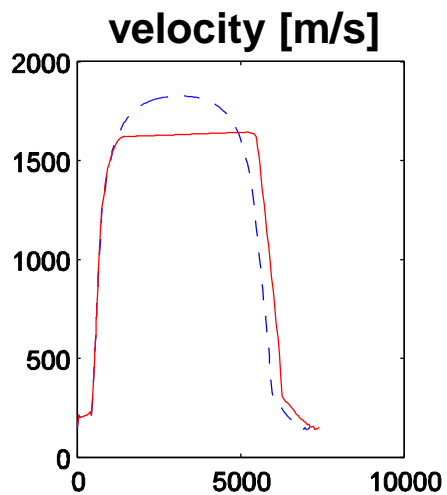
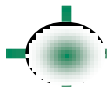


# Numerical Results: Stagnation Point (1D)





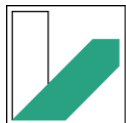
# Numerical Results: Stagnation Point: States, Heat Loads



**limit  
temperature  
1000 K  
on a boundary arc  
order concept?**



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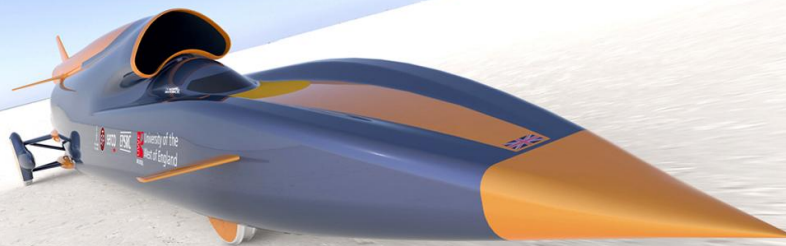


have nothing to do with the supersonic

## Bloodhound-SSC-Projekt

1997, Oct. 15: Thrust SSC  
officially 1.228 km/h  
less than  $Ma\ 1 = 1.234,8\text{ km/h}$   
at  $20^\circ\text{C}$

Aim: 1.000 m/h,  
hence 1.609 km/h,  
faster than a speeding bullet.



**we are going with *hypersonic* speed**

BLOODHOUND SSC by CURVENTA



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<http://www.bloodhoundssc.com/>

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$$\min_{u \in U} \left\{ t_f + \frac{1}{2} \lambda \int_0^{t_f} u^2 dt \right\}, \quad \lambda \geq 0$$

minimum time      control costs

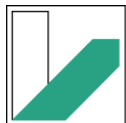
subject to

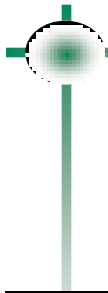
$$\ddot{w}(t) = u(t) \quad \text{in } (0, t_f)$$

$$w(0) = w_0, \quad \dot{w}(0) = \dot{w}_0$$

$$w(t_f) = 0, \quad \dot{w}(t_f) = 0$$

$$U := \left\{ u \in L^2(0, t_f) : |u(t)| \leq u_{\max} \text{ a. e. in } [0, t_f] \right\}$$





## Problem 1

$$\frac{\partial T}{\partial t}(x, t) - \frac{\partial^2 T}{\partial x^2}(x, t) = g(\dot{w}(t))$$

**friction term**  
**control via ODE state**

in  $(0, l) \times (0, t_f)$

$$T(x, 0) = T_0 = 0 \text{ on } (0, l)$$

$$-\frac{\partial T}{\partial x}(0, t) = -(T(0, t) - T_0)$$

$$\frac{\partial T}{\partial x}(l, t) = -(T(l, t) - T_0) \text{ on } [0, t_f]$$

**instationary heating of the entire vehicle**



$$\frac{\partial T}{\partial t}(x, t) - \frac{\partial^2 T}{\partial x^2}(x, t) = 0$$

## Problem 2

$$\text{in } (0, l) \times (0, t_f),$$

$$T(x, 0) = T_0 = 0 \text{ on } (0, l),$$

$$-\frac{\partial T}{\partial x}(0, t) = -(T(0, t) - T_0),$$

$$\frac{\partial T}{\partial x}(l, t) = -(T(l, t) - T_0) + h(\dot{w}(t)) \text{ on } [0, t_f]$$

Transformation  
to homogeneous  
Robin type b.c.

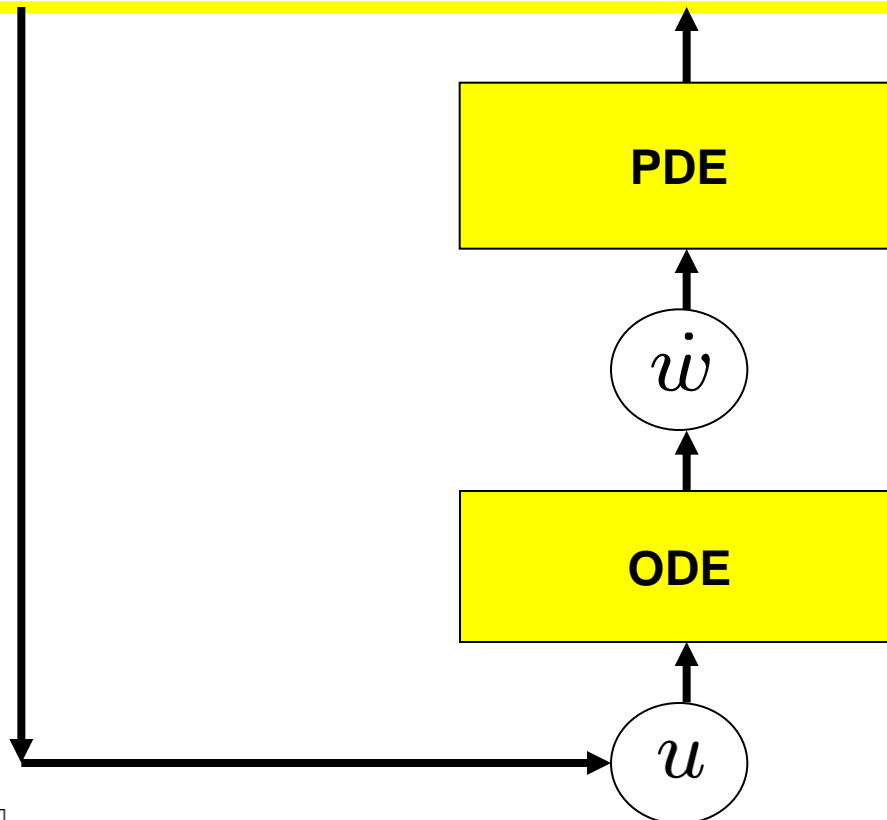
friction term  
control via  
ODE state

instationary heating at the stagnation point



# The State Constraint

$$T(x, t) \leq T_{\max} \text{ a. e. in } (0, l) \times (0, t_f)$$



The state constraint  
regenerates  
the PDE with the ODE

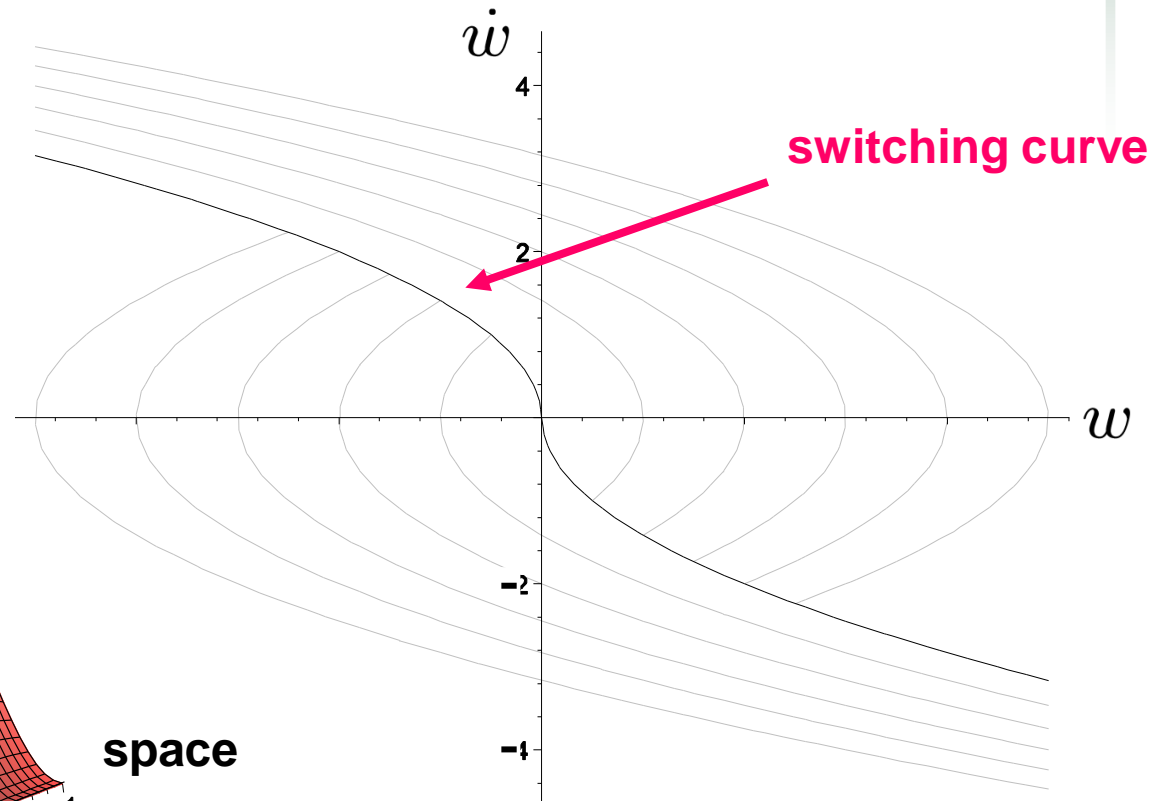
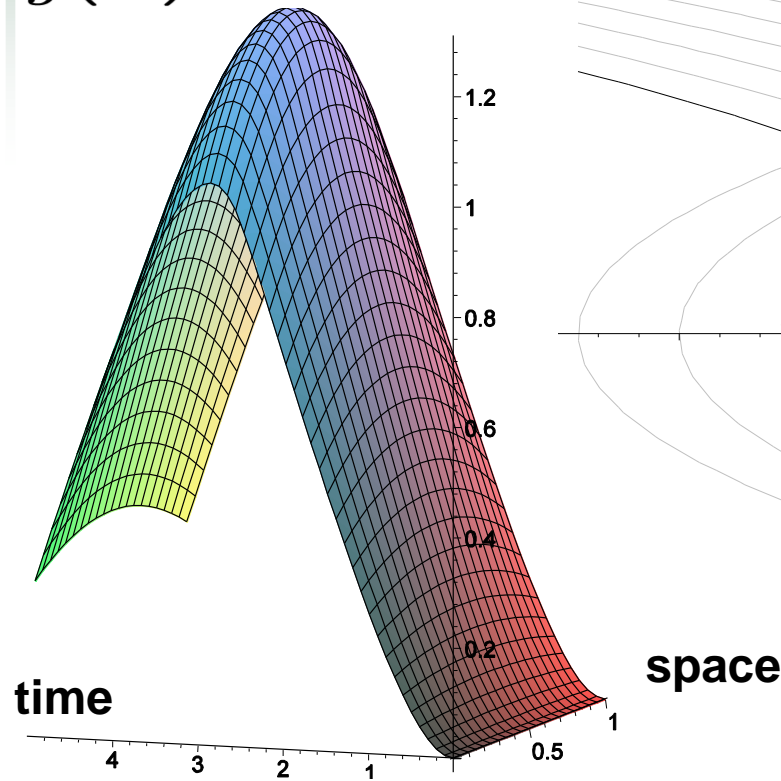


# The Optimal Trajectories (Non-regularized, Minimum Time)

## Problem 1

$$\min_{u \in U} \left\{ t_f + \frac{1}{2} \lambda \int_0^{t_f} u^2 dt \right\}, \quad \lambda = 0, \quad u_{\max} = 1$$

$$g(\dot{w}) = \dot{w}^2$$



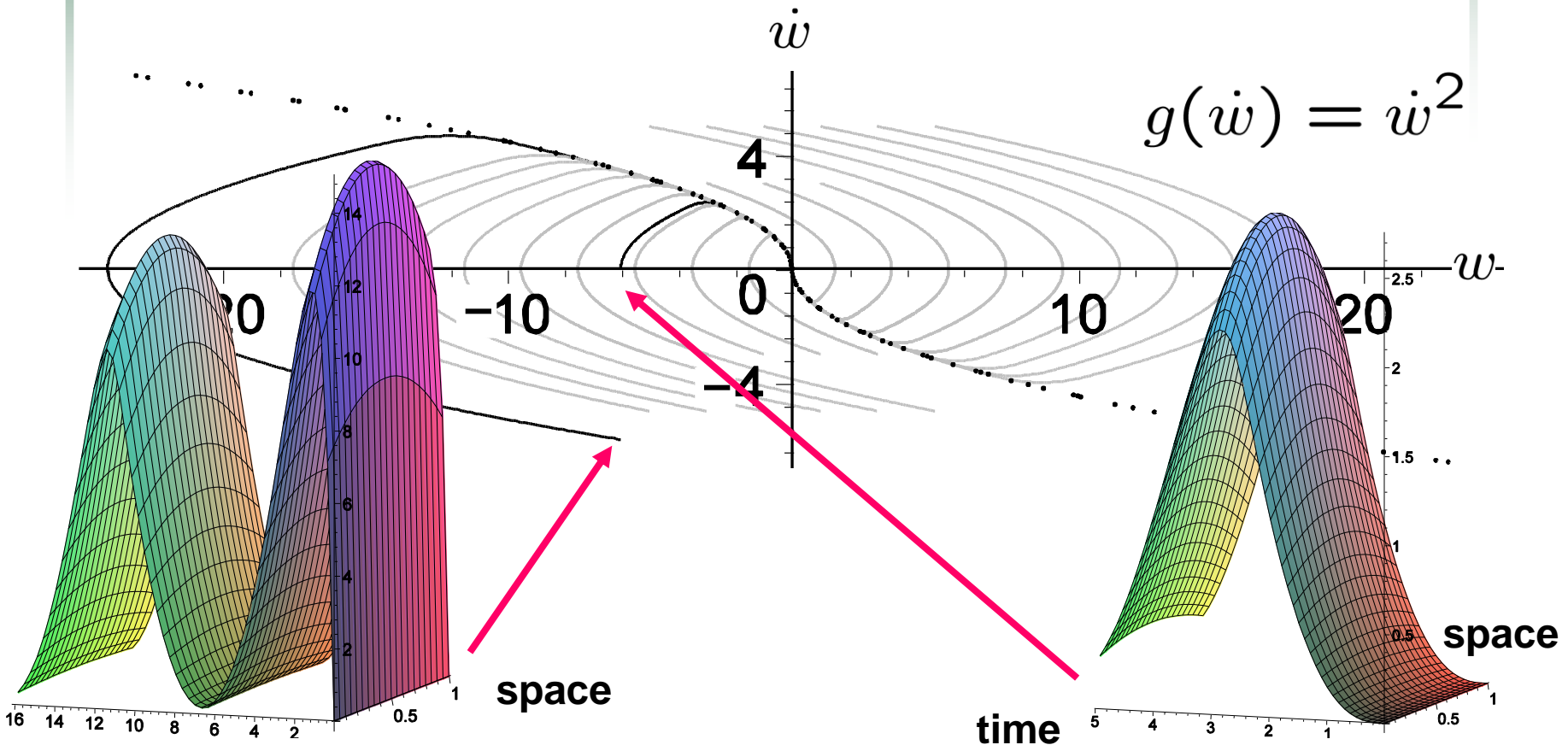


# The Optimal Trajectories (Regularized, Control Constrained)

## Problem 1

$$\min_{u \in U} \left\{ t_f + \frac{1}{2} \lambda \int_0^{t_f} u^2 dt \right\}, \quad \lambda = \frac{1}{10}, \quad u_{\max} = 1$$

$$g(\dot{w}) = \dot{w}^2$$

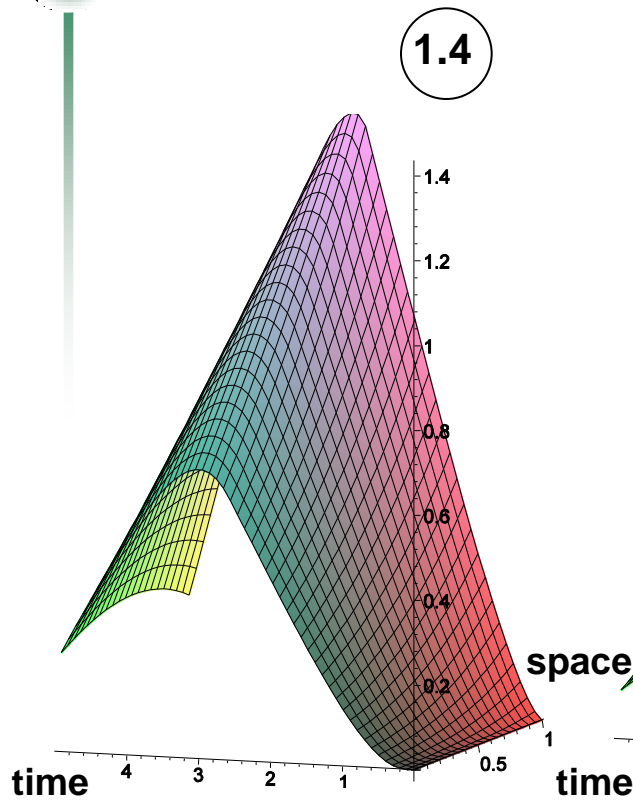


**distributed case  
state unconstrained**



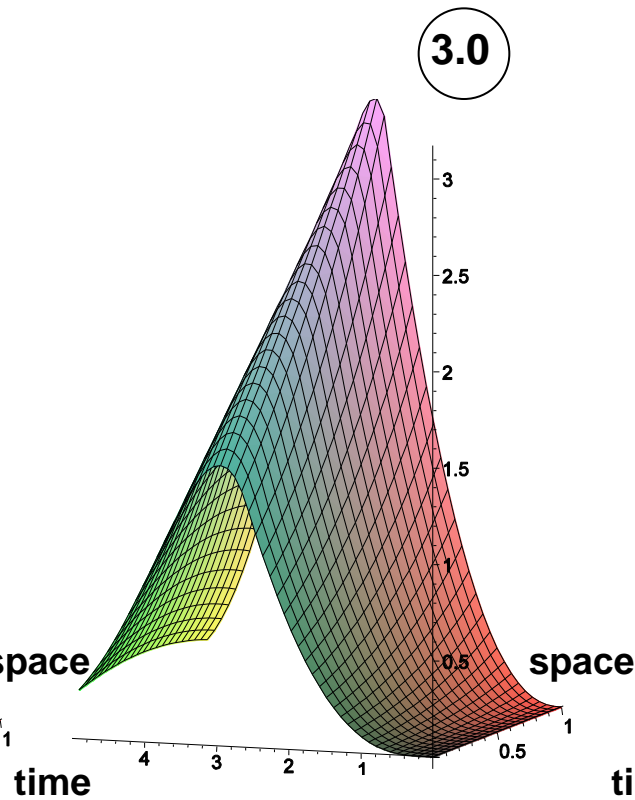
# Boundary Control Case

## Problem 2



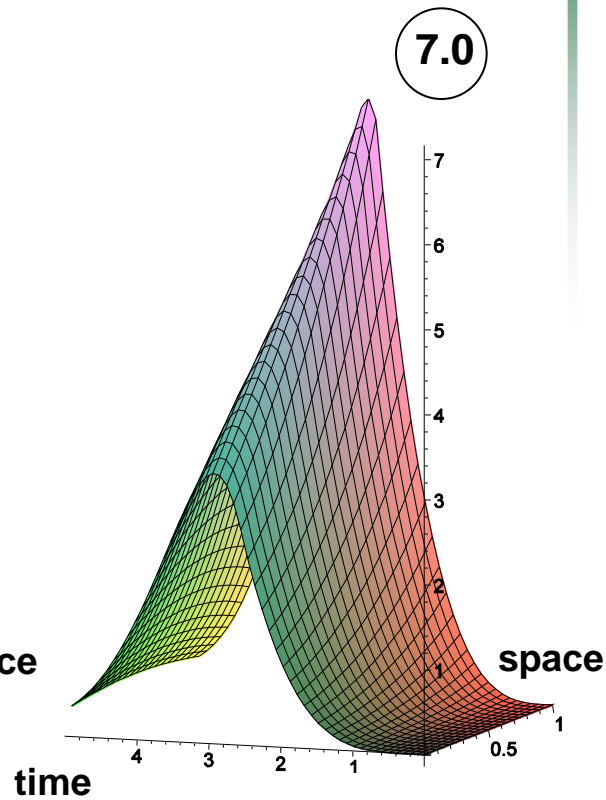
$$h(z) = |z|$$

**Coulomb**



$$h(z) = z^2$$

**Stokes**



$$h(z) = |z|^3$$

**Newton**



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- The hypersonic rocket car problems
- **Theoretical results**
- New necessary conditions
- Numerical results
- Conclusion



# Theoretical results for Problem 1

- Existence, uniqueness, and continuous dependence on data

$$(w, T) \in W_2^1(0, t_f) \times (W_2^{1,0}(Q) \cap C([0, t_f], L^2(0, l)))$$

- Non-negativity of  $T$

- Symmetry  $T(x, t) = T(l - x, t)$

- Strong maximum in  $x = \frac{l}{2}$

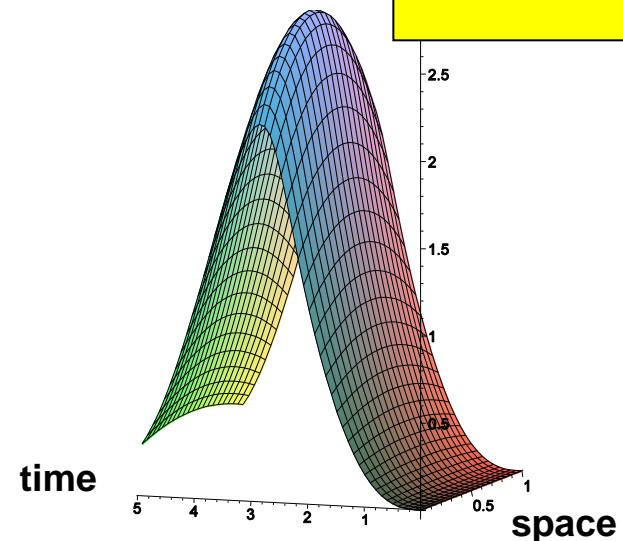
- Classical solution  $\frac{d}{dt}T + \mathcal{L}T = g(\dot{w})$

- Maximum regularity

$$T_{tt} \in L^r(\varepsilon, t_f; L^2(0, l)) \quad r \geq 2$$

$$\partial_x^4 T \in L^r(\varepsilon, t_f; L^2(0, l)) \quad \varepsilon > 0$$

Problem 1



# Theoretical results (two formulations)

## Problem 1: Two equivalent formulations

### 1) as ODE optimal control problem

$$T\left(\frac{l}{2}, t\right) = \sum_{n=1}^{\infty} \tilde{\gamma}_n \left[ \int_0^t g(\dot{w}(s)) e^{-k_n^2(t-s)} ds \right] \leq T_{\max}$$

loss of convergence  
if differentiated

non-local, resp. integro-state constraint

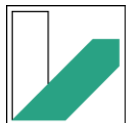
### 2) as PDE optimal control problem

$$T_t + T_{xx} = g\left(\dot{w}_0 + \int_0^t u(s) ds\right)$$

$$T(x, t) \leq T_{\max}$$

non-standard

plus two **isoperimetric constraints** on  $\mathcal{U}$  due two ODE boundary conds.



### Integro-state constraint

$$S(t, \mathbf{w}([0, t])) := T\left(\frac{l}{2}, t; w_2([0, t])\right) - T_{\max} \leq 0$$

### Transformation

$$w_3 := T\left(\frac{l}{2}, t; w_2([0, t])\right)$$

$$\dot{w}_3(t) = \frac{d}{dt} T\left(\frac{l}{2}, t; w_2([0, t])\right)$$

$$w_3(0) = 0$$

$$w_3(t_{\text{on}}) = w_3(t_{\text{off}}) = T_{\max}$$

$$w_3(t) \leq T_{\max}$$

**Integro-ODE**

**corresponds to  
Maurer's intermediate  
adjoining approach**

**pointwise**



## Necessary conditions: optimal control law

$$\begin{aligned}\mathcal{L}(\mathbf{w}, u, \mathbf{p}, t_{\text{on}}, t_{\text{off}}, t_f) &:= \int_0^{t_f} 1 + \frac{\lambda}{2} u^2 dt - \int_0^{t_f} (\dot{w}_1 - w_2) p_1 dt - \int_0^{t_f} (\dot{w}_2 - u) p_2 dt \\ &\quad - \int_0^{t_f} \left[ \dot{w}_3 - \frac{d}{dt} T \left( \frac{l}{2}, t; w_2([0, t]) \right) \right] p_3 dt \\ &\quad + \int_0^{t_f} (w_3 - T_{\text{max}}) d\mu \\ &\quad + (w_3(t_{\text{on}}) - T_{\text{max}}) \sigma_{\text{on}} + (w_3(t_{\text{off}}) - T_{\text{max}}) \sigma_{\text{off}}\end{aligned}$$

$\implies$

$$D_u \mathcal{L}[*] (u - u^*) = (\lambda u^* + p_2) (u - u^*) \geq 0 \quad \forall u \in [-u_{\text{max}}, u_{\text{max}}]$$

$\implies$

$$u^*(t) = P_{[-u_{\text{max}}, +u_{\text{max}}]} \left( -\frac{1}{\lambda} p_2 \right)$$



## Necessary conditions: adjoint equations

### Retrograde integro-ODE for the adjoint velocity

$$\dot{p}_2 = -p_1 - g'(w_2(t)) \left( p_3(t) - \int_t^{t_f} \sum_{n=1}^{\infty} k_n^2 \tilde{\gamma}_n e^{-k_n^2(s-t)} p_3(s) ds \right)$$

### Usual jump conditions for adjoint auxiliary state

discontinuities

$$\dot{p}_3 = \begin{cases} 0 & 0 \leq t < t_{\text{on}} \\ -\mu & t_{\text{on}} < t < t_{\text{off}} \\ 0 & t_{\text{off}} < t \leq t_f \end{cases} \quad \begin{aligned} p_3(t_{\text{on/off}}^-) &= p_3(t_{\text{on/off}}^+) - \sigma_{\text{on/off}} \\ p_3(t_f) &= 0, \quad \sigma_{\text{on/off}} \geq 0 \end{aligned}$$

$$\mu(w_3 - T_{\text{max}}) = 0, \quad \mu \geq 0$$

complementarity condition

difficult to solve  
no standard software





## Problem 1

$$\frac{\partial T}{\partial t}(x, t) - \frac{\partial^2 T}{\partial x^2}(x, t) = g \left( \dot{w}_0 + \int_0^t u(s) \, ds \right)$$

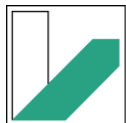
$$\text{in } (0, l) \times (0, t_f)$$

$$T(x, 0) = 0 \quad \text{on } (0, l)$$

$$-\frac{\partial T}{\partial x}(0, t) = -T(0, t) \quad \frac{\partial T}{\partial x}(l, t) = -T(l, t) \quad \text{on } [0, t_f]$$

$$\int_0^{t_f} u(t) \, dt = -\dot{w}_0 \quad \int_0^{t_f} \int_0^t u(s) \, ds \, dt = -w_0 - \dot{w}_0 t_f$$

$$T(x, t) \leq T_{\max} \quad \text{in } (0, l) \times (0, t_f)$$



By the obtained continuously differentiable solution operator

$$S: f(u, t_f) := \int_0^{t_f} 1 + \frac{\lambda}{2} u^2(t) dt \stackrel{!}{=} \min_{(u, t_f) \in C}$$

subject to

$$G_1(u, t_f) := \begin{pmatrix} \int_0^{t_f} u(t) dt + \dot{w}_0 \\ \int_0^{t_f} \int_0^{t_f} u(s) ds dt + w_0 + \dot{w}_0 t_f \end{pmatrix} = 0$$

$$G_2(u, t_f) := ( S(u, t_f) - T_{\max} ) \leq_K 0$$

with the convex cone

$$K := \left\{ T \in C([0, l] \times [0, t_f]), \quad T - T_{\max} \leq 0 \text{ a.e.} \right\}$$



**Theorem:** Let  $(u^*, t_f^*)$  be a local solution of the optimization problem.

Let be  $(\tilde{u}, \tilde{t}_f) \in \text{int } C(u^*, t_f^*)$  such that

$$G'_1(u^*, t_f^*)(\tilde{u}, \tilde{t}_f) = 0$$

$$G_2(u^*, t_f^*) + G'_2(u^*, t_f^*)(\tilde{u}, \tilde{t}_f) <_K 0.$$

Then there exists a Lagrange multiplier  $\bar{\mu} \in \mathcal{M}([0, l] \times [0, t_f])$  associated with  $(u^*, t_f^*)$ .

The set of all multipliers belonging to  $(u^*, t_f^*) \in \mathcal{M}$  is bounded.



## Necessary conditions: adjoint equations

$$-q_t(x, t) - q_{xx}(x, t) = \bar{\mu}$$

$$-q_x(0, t) = -q(0, t), \quad -q_x(l, t) = -q(l, t)$$

$$q(x, t_f^*) = 0$$

so far all seems to be standard, but

## Necessary condition: integro optimal control law

$$u^*(t) = P_{[-u_{\max}, u_{\max}]} \left\{ -\frac{1}{\lambda} \left[ \nu_1 + \nu_2 t \quad \boxed{p_2} \right. \right. \\ \left. \left. + \int_t^{t_f} g' \left( \dot{w}_0 + \int_0^t u^*(r) dr \right) \cdot \left( \int_0^l q(x, s) dx \right) ds \right] \right\}$$

**extremely difficult to solve**  
**no standard software**



By comparing the two optimal control laws

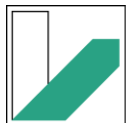
$$p_1(t) \hat{=} -\nu_2$$

$$p_2(t) \hat{=} \nu_1 + \nu_2 t + \int_t^{t_f^*} g'(v^*(s)) \cdot \left( \int_0^l q(x, s) dx \right) ds$$

discont. deriv. / jump

$$p_3(t) - \int_t^{t_f^*} \sum_{n=1}^{\infty} k_n^2 \tilde{\gamma}_n e^{-k_n^2 (s-t)} p_3(s) ds \hat{=} \int_0^l q(x, t) dx$$

jump



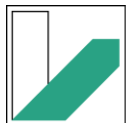
The two formulations allow the comparison of the necessary conditions:

**ODE version** • known theory

- local jump conditions for ODE formulation
- multipoint boundary value problem for integro-ODEs ←

**PDE version** • existence of Lagrange parameter

- non-local jump conditions for PDE formulation
- projection formula for control with integro-terms ←



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**Jump condition in the direction of  $t$  (equivalent to ODE case):**

$$\left[ \int_0^l q(x, t) dx \right]_{t_s^-}^{t_s^+} < 0$$

**Jump condition in the direction of  $x$  (no counterpart in ODE case):**

$$\left[ \int_0^{t_f} q_x(x, t) dt \right]_{\frac{l}{2}^-}^{\frac{l}{2}^+} < 0$$

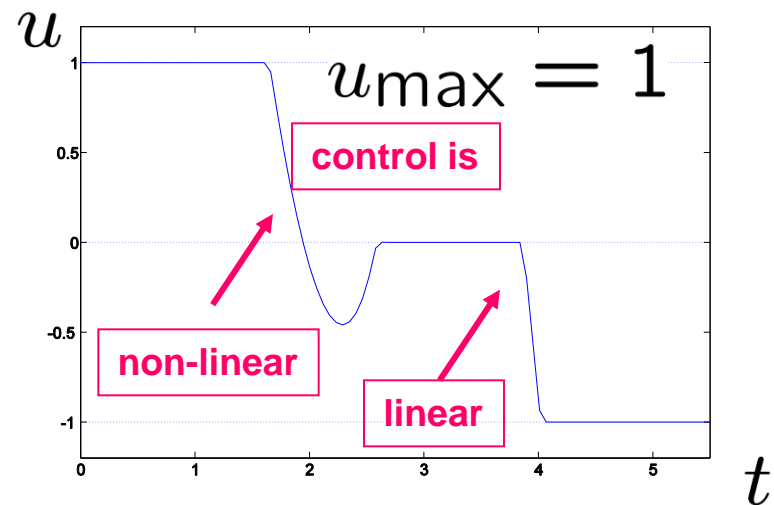
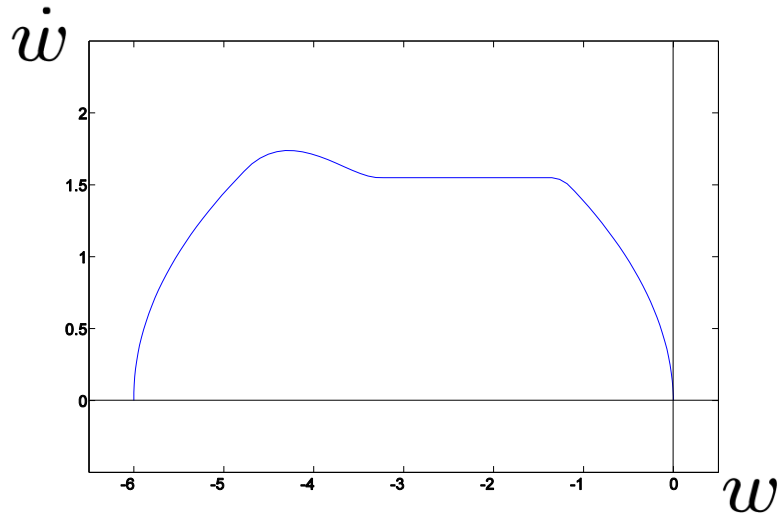
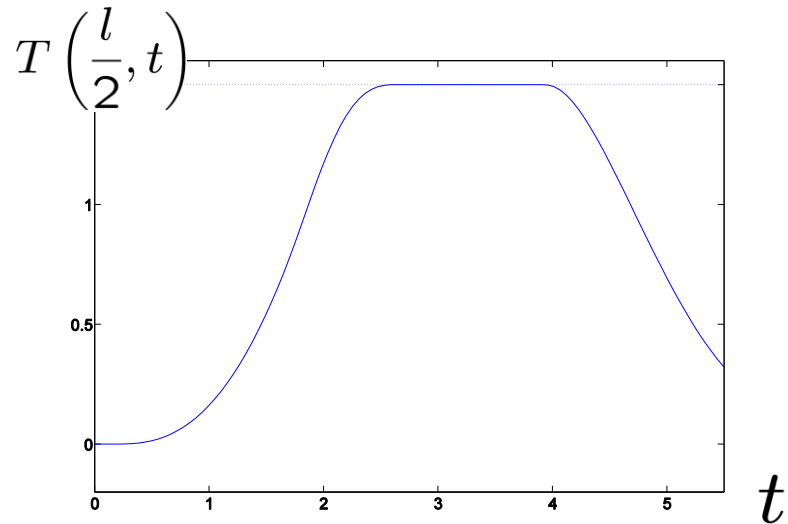
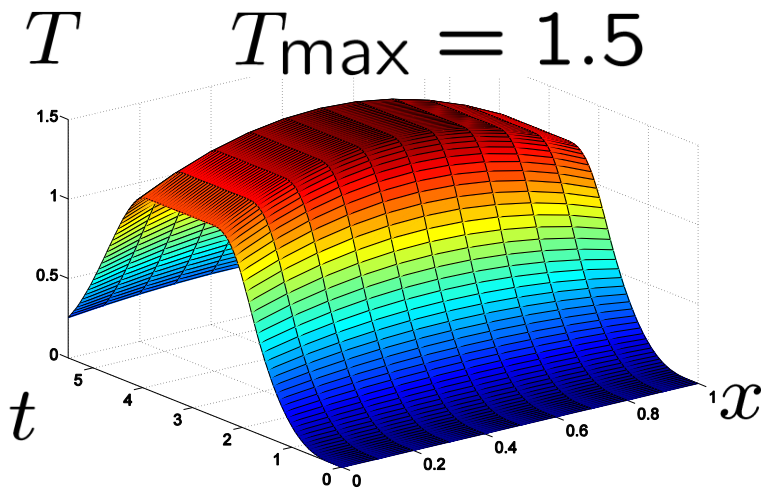




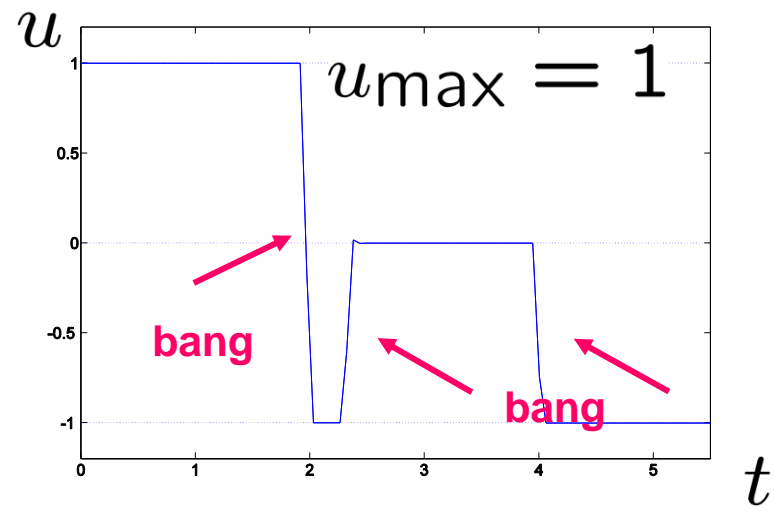
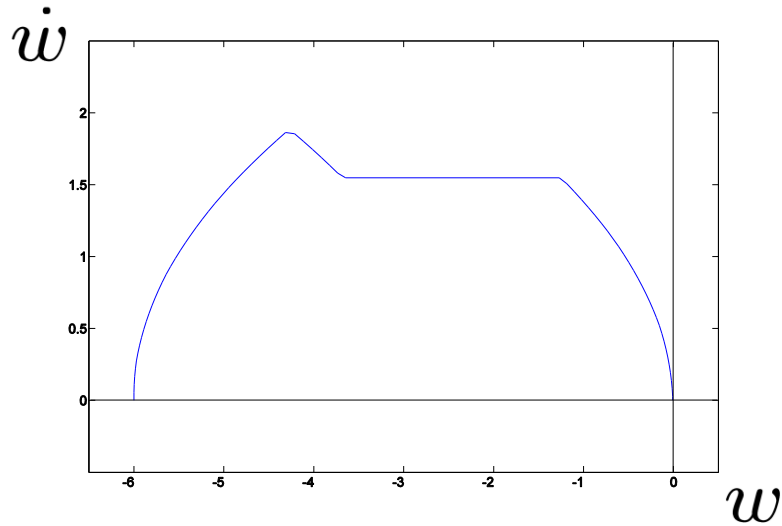
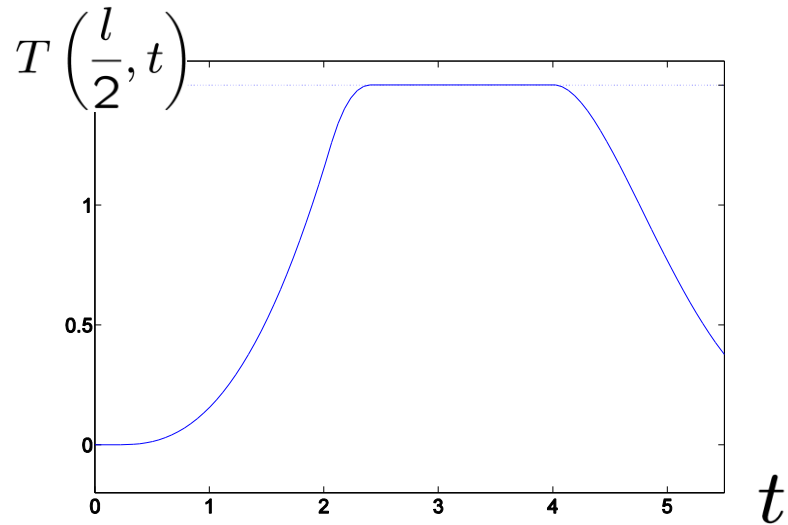
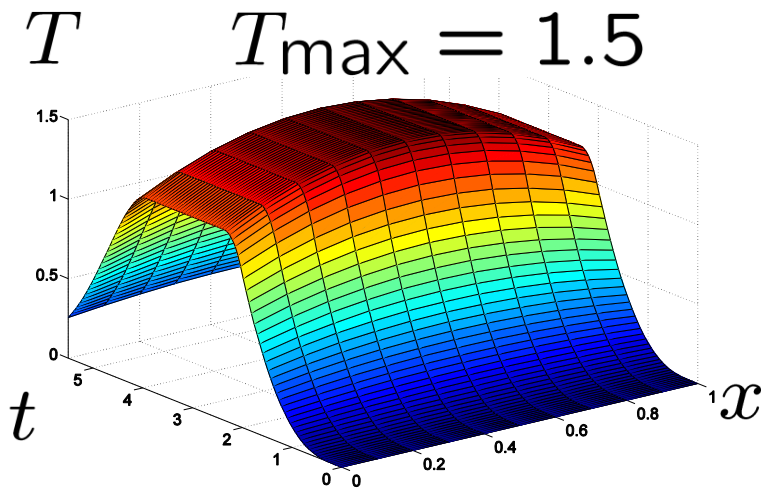
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# Numerical results (Type: Problem 1)

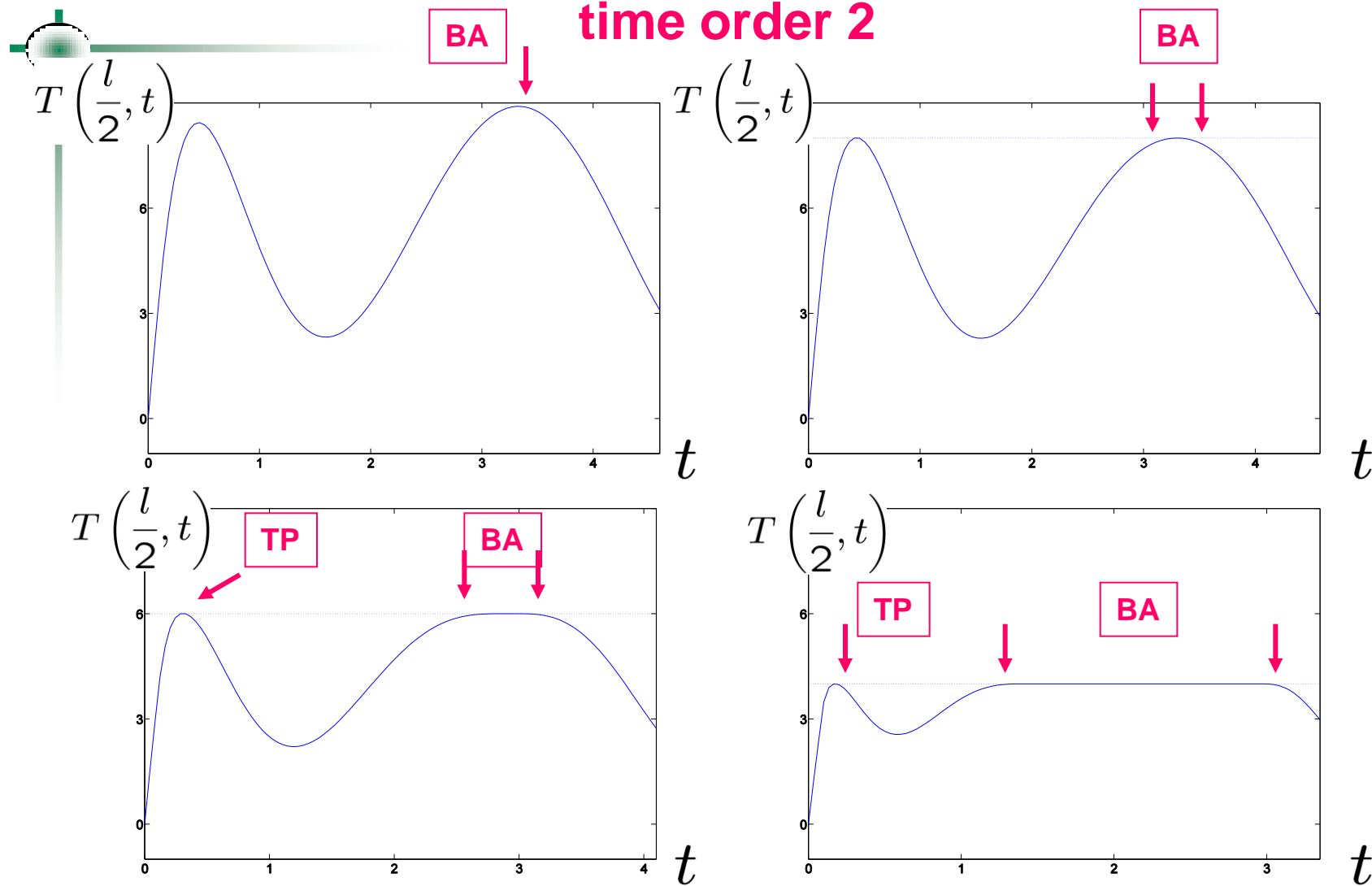


# Numerical results (Type: Problem 1)



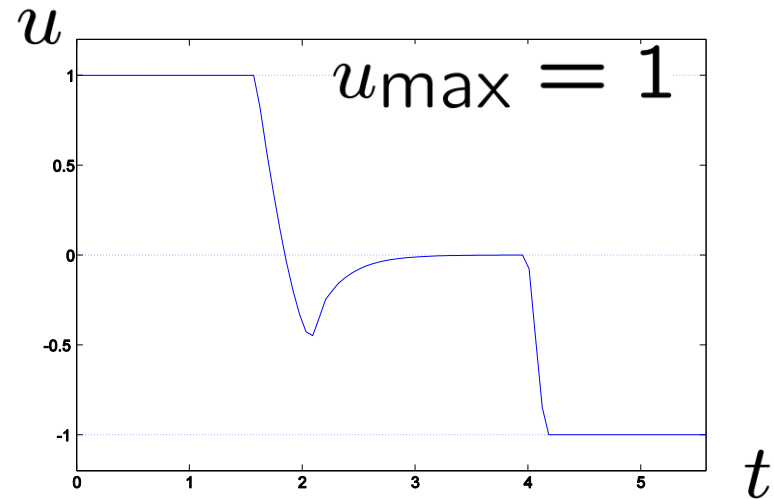
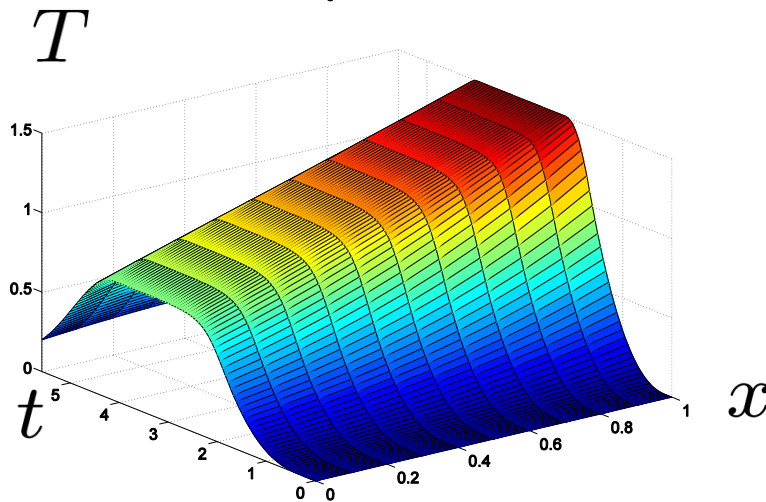
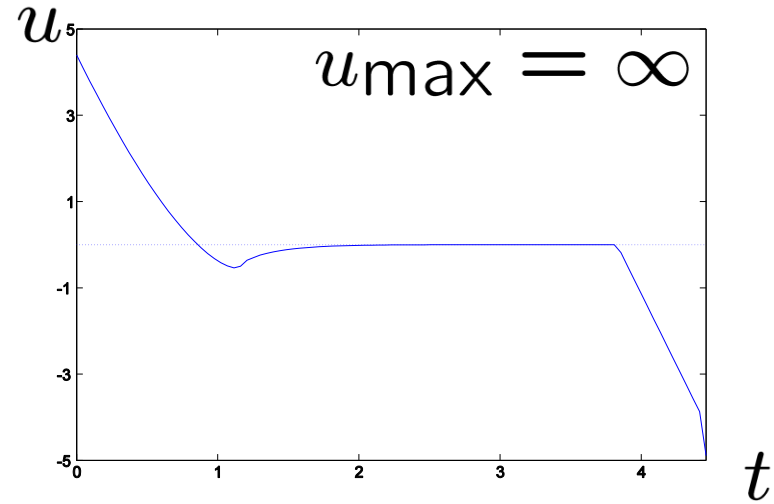
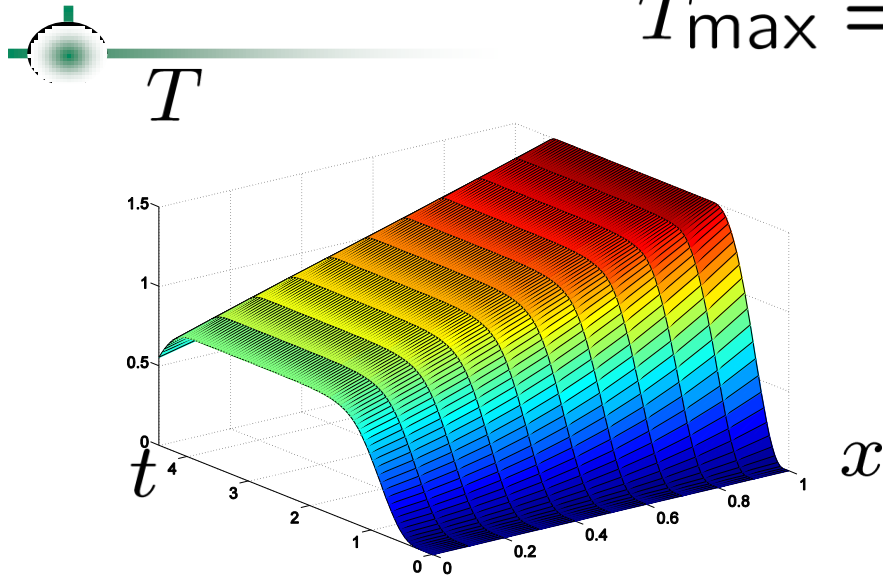
# Numerical results (Type: Problem 1)

time order 2



# Numerical results (Type: Problem 2)

$$T_{\max} = 1.5$$



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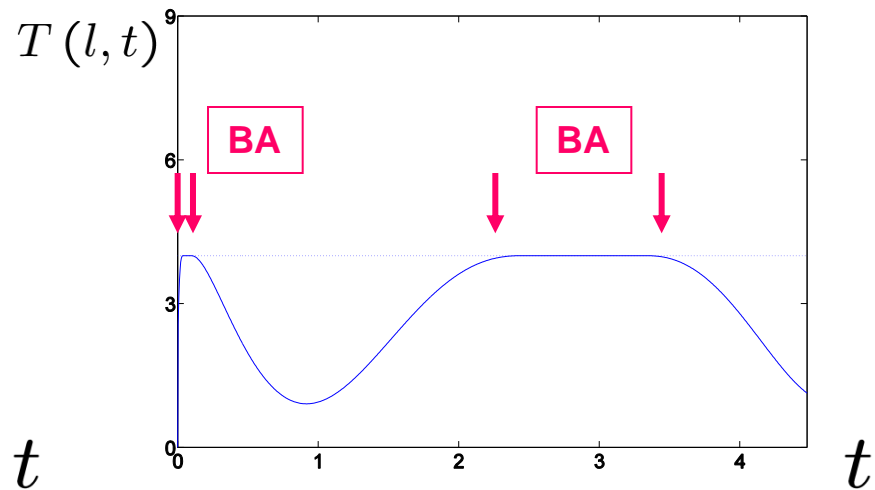
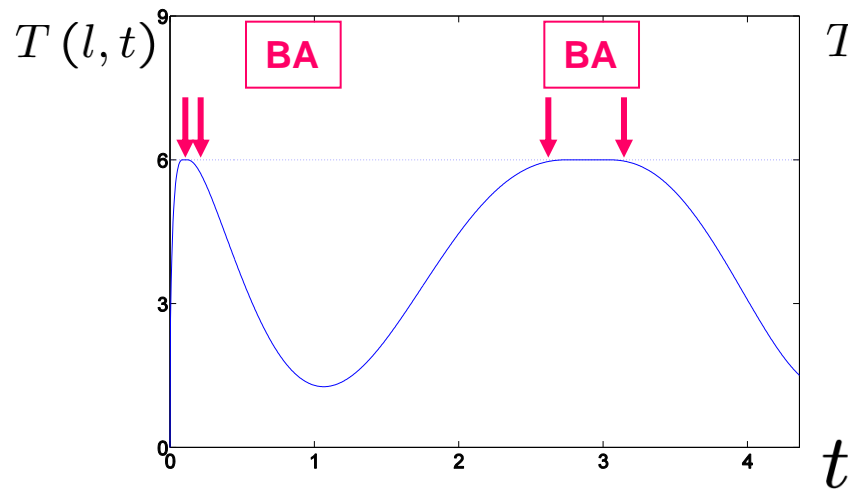
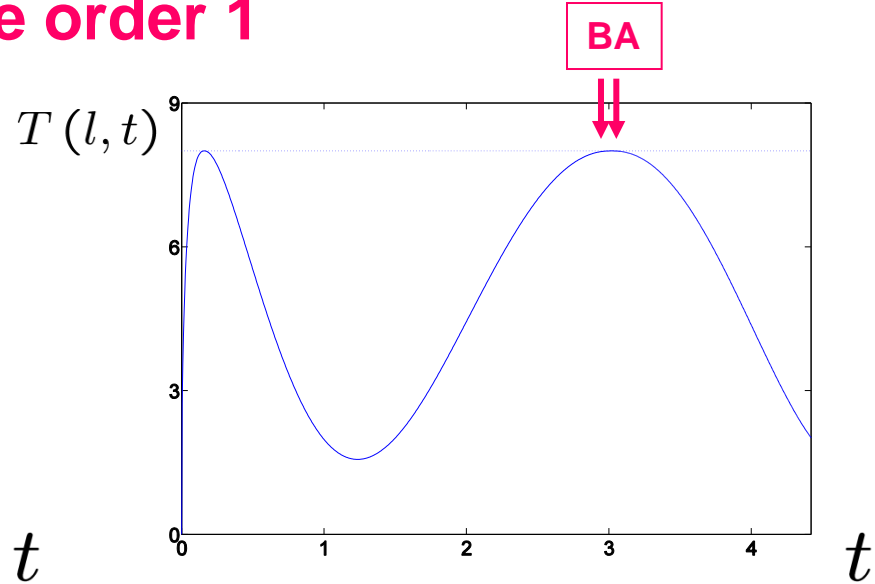
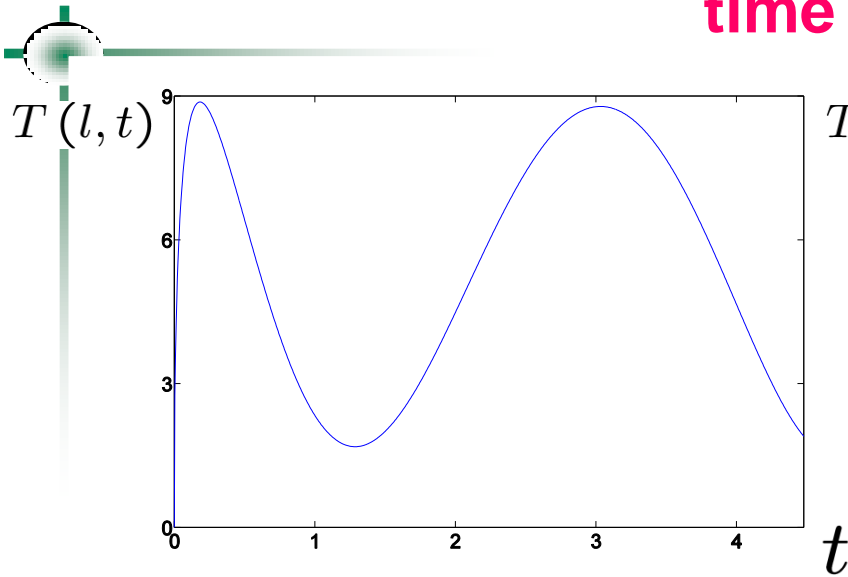
$$\lambda = 10^{-1}$$

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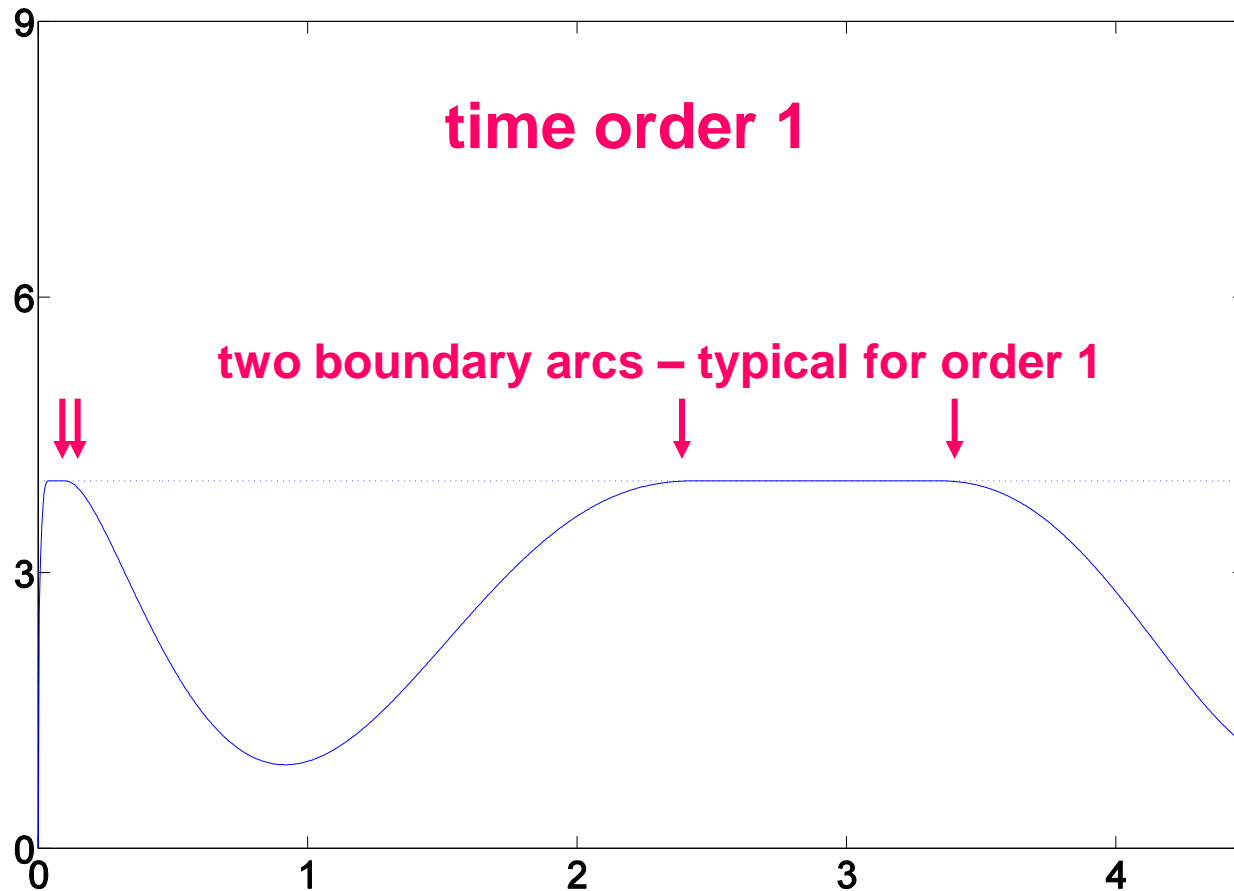


# Numerical results (Type: Problem 2)

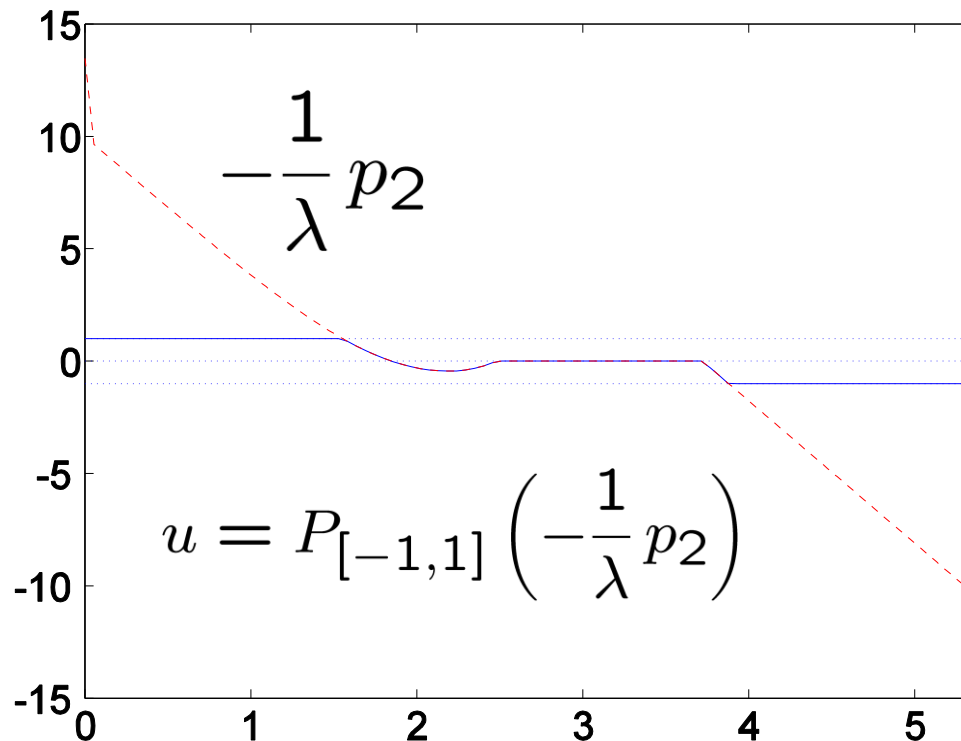
time order 1



# Numerical results (Type: Problem 2)



## A posteriori verification of optimality conditions: projection formula (ODE)



**Method:**  
**Ampl + IPOPT**

**Ref.: IPOPT**  
**Andreas Wächter**  
**2002**

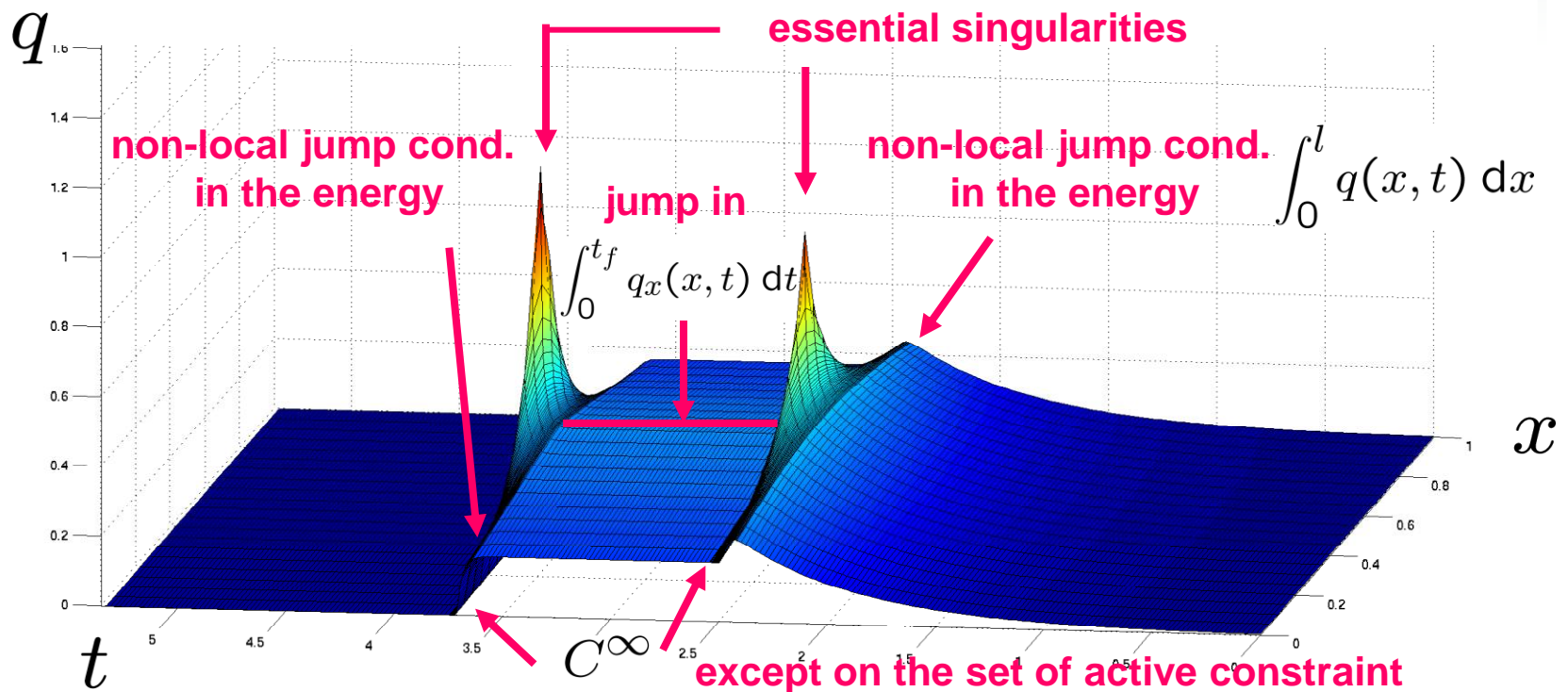




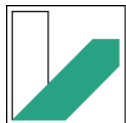
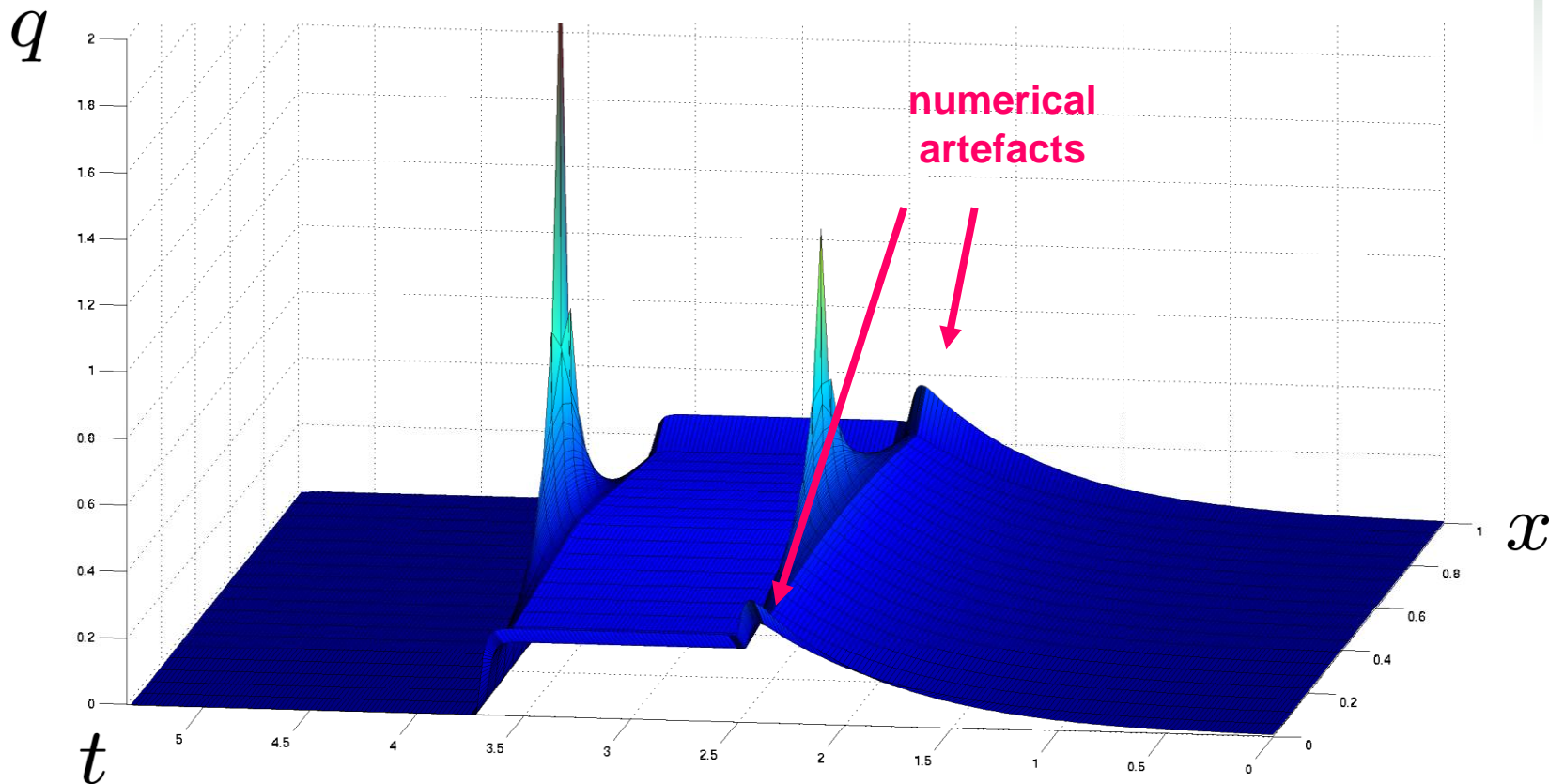
# Numerical results (FOTD vs. FDTO)

**A posteriori verification of optimality conditions:  
The PDE formulation: adjoint temperature**

$$-q_t - q_{xx} = \varrho_1 \delta\left(x - \frac{l}{2}\right) \otimes \left[ \varrho_2 \delta(t - t_{\text{on}}) + \nu(t) + \varrho_3 \delta(t - t_{\text{off}}) \right]$$



## A posteriori verification of optimality conditions: The PDE formulation: adjoint temperature



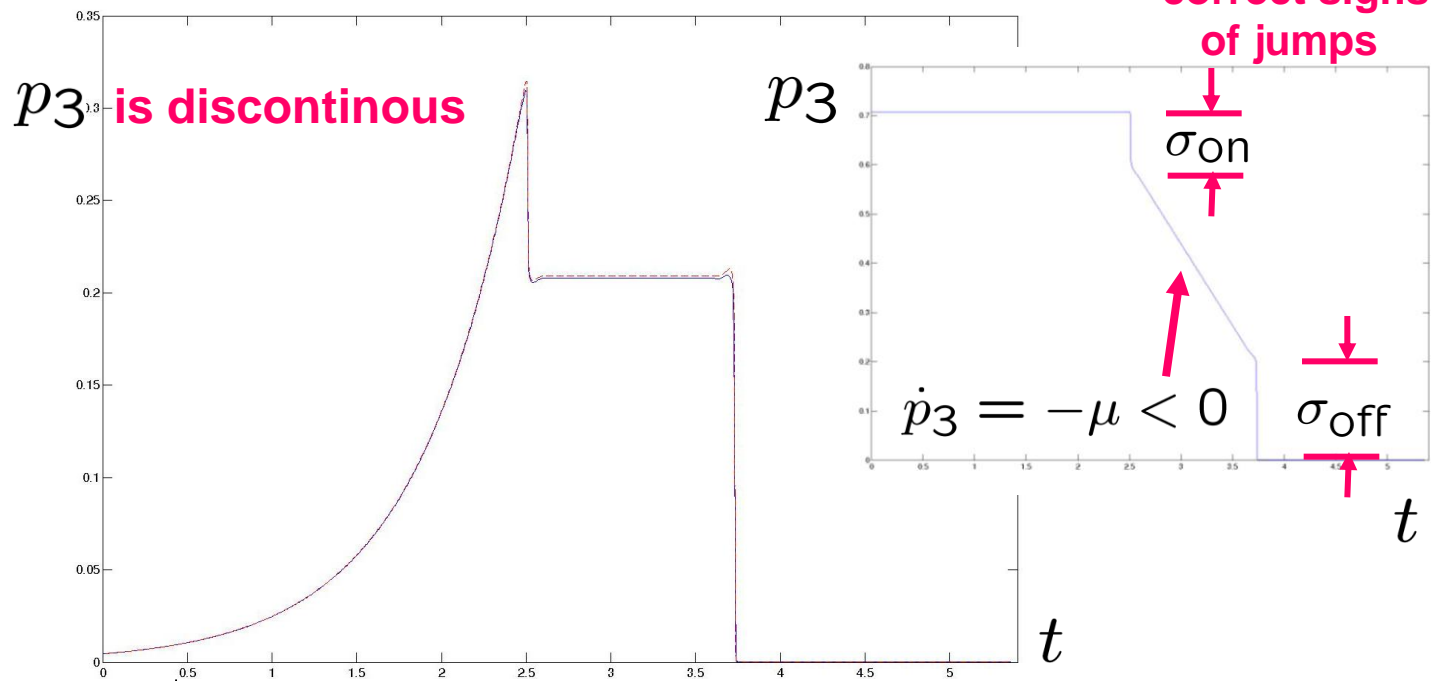
## A posteriori verification of optimality conditions: comparison of adjoints (ODE + PDE)



$$p_2(t) \cong \underbrace{\nu_1 + \nu_2 t}_{\text{blue}} + \underbrace{\int_t^{t_f^*} g'(v^*(s)) \cdot \left( \int_0^l q(x, s) dx \right) ds}_{\text{red}}$$



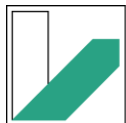
## A posteriori verification of optimality conditions: comparison of adjoints/jump conditions (ODE + PDE)



$$p_3(t) - \int_t^{t_f^*} \sum_{n=1}^{\infty} k_n^2 \tilde{\gamma}_n e^{-k_n^2(s-t)} p_3(s) ds \cong \int_0^l q(x, t) dx$$



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## Pros:

- Detailed model of high complexity for instationary heating
- Reduction of temperature of TPS due to optimal control
- Challenging problem: ODE - PDE state - constrained optimal control

## Cons:

- Lack of theory: nonlinear, state - constrained, ODE-PDE coupling
- Method of lines and SQP methods seem to be at their limits
- Adjoint based methods desirable, but almost impossible to handle



- **Staggered optimal control problems with state constraints**
- **Structural analysis w.r.t. switching structure**
- **Unexpectedly complicated necessary conditions**
- **Problems with free terminal time for PDE problems**
- **Jump conditions in Integro-ODE and PDE optimal control**
- **First optimize, then discretize hardly applicable**
- **First discretize, then optimize with reliable verification of necessary conditions, but with limitations in time and storage**
- **Motivation from hypersonic flight path optimization**

