



HAL
open science

Quantitative version of Kipnis-Varadhan's theorem and Monte-Carlo approximation of homogenized coefficients

Antoine Gloria, Jean-Christophe Mourrat

► **To cite this version:**

Antoine Gloria, Jean-Christophe Mourrat. Quantitative version of Kipnis-Varadhan's theorem and Monte-Carlo approximation of homogenized coefficients. [Research Report] 2011, pp.26. inria-00579424v2

HAL Id: inria-00579424

<https://inria.hal.science/inria-00579424v2>

Submitted on 31 Aug 2011 (v2), last revised 24 May 2012 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

QUANTITATIVE VERSION OF KIPNIS-VARADHAN'S THEOREM AND MONTE-CARLO APPROXIMATION OF HOMOGENIZED COEFFICIENTS

ANTOINE GLORIA & JEAN-CHRISTOPHE MOURRAT

ABSTRACT. This article is devoted to the analysis of a Monte-Carlo method to approximate effective coefficients in stochastic homogenization of discrete elliptic equations. We consider the case of independent and identically distributed coefficients, and adopt the point of view of the random walk in a random environment. Given some final time $t > 0$, a natural approximation of the homogenized coefficients is given by the empirical average of the final squared positions rescaled by t of n independent random walks in n independent environments. Relying on a new quantitative version of Kipnis-Varadhan's theorem (which is of independent interest) combined with estimates of spectral exponents obtained by pde arguments, we first give a sharp estimate of the error between the homogenized coefficients and the expectation of the rescaled final position of the random walk in terms of t . We then complete the error analysis by quantifying the fluctuations of the empirical average in terms of n and t , and prove a large-deviation estimate. Our estimates are optimal, up to a logarithmic correction in dimension 2.

Keywords: random walk, random environment, stochastic homogenization, effective coefficients, Monte-Carlo method, quantitative estimates.

2010 Mathematics Subject Classification: 35B27, 60K37, 60H25, 65C05, 60H35, 60G50.

1. MAIN RESULT AND STRUCTURE OF THE PROOF

1.1. Main result. We consider the discrete elliptic operator $-\nabla^* \cdot A \nabla$, where ∇^* and ∇ are the discrete backward divergence and forward gradient, respectively. For all $x \in \mathbb{Z}^d$, $A(x)$ is the diagonal matrix whose entries are the conductances $\omega_{x, x+e_i}$ of the edges $(x, x+e_i)$ starting at x , where $(e_i)_{i \in \{1, \dots, d\}}$ denotes the canonical basis of \mathbb{Z}^d . Let \mathbb{B} denote the set of edges of \mathbb{Z}^d . We call the family of conductances $\omega = (\omega_e)_{e \in \mathbb{B}}$ the *environment*. The environment ω is random, and we write \mathbb{P} for its distribution (with corresponding expectation \mathbb{E}). We make the following assumptions :

- (H1) the measure \mathbb{P} is invariant under translations,
- (H2) the conductances are i. i. d.¹,
- (H3) there exists $0 < \alpha < \beta$ such that $\alpha \leq \omega_e \leq \beta$ almost surely.

Under these conditions, standard homogenization results ensure that there exists some *deterministic* symmetric matrix A_{hom} such that the solution operator of the deterministic continuous differential operator $-\nabla \cdot A_{\text{hom}} \nabla$ describes the large scale behavior of the solution operator of the random discrete differential operator $-\nabla^* \cdot A \nabla$ almost surely (for this statement, (H2) can in fact be replaced by the weaker assumption that the measure \mathbb{P} is ergodic with respect to the group of translations, see [Kü83]).

¹(H2) obviously implies (H1) in the present form. Yet for most qualitative (and some quantitative) results (H2) can be weakened and may not imply (H1) any longer.

The operator $-\nabla^* \cdot A \nabla$ is the infinitesimal generator of a stochastic process $(X(t))_{t \in \mathbb{R}_+}$ which can be defined as follows. Given an environment ω , it is the Markov process whose jump rate from a site $x \in \mathbb{Z}^d$ to a neighbouring site y is given by $\omega_{x,y}$. We write \mathbf{P}_x^ω for the law of this process starting from $x \in \mathbb{Z}^d$.

It is proved in [KV86] that under the averaged measure $\mathbb{P}\mathbf{P}_0^\omega$, the rescaled process $\sqrt{\varepsilon}X(\varepsilon^{-1}t)$ converges in law, as ε tends to 0, to a Brownian motion whose infinitesimal generator is $-\nabla \cdot A_{\text{hom}} \nabla$, or in other words, a Brownian motion with covariance matrix $2A_{\text{hom}}$ (see also [AKS82, Kü83, Ko85] for prior results). We will use this fact to construct computable approximations of A_{hom} . As proved in [DFGW89], this invariance principle holds as soon as (H1) is true, (H2) is replaced by the ergodicity of the measure \mathbb{P} , and (H3) by the integrability of the conductances. Under the assumptions (H1-H3), [SS04] strengthens this result in another direction, showing that for almost every environment, $\sqrt{\varepsilon}X(\varepsilon^{-1}t)$ converges in law under \mathbf{P}_0^ω to a Brownian motion with covariance matrix $2A_{\text{hom}}$. This has been itself extended to environments which do not satisfy the uniform ellipticity condition (H3), see [BB07, MP07, BP07, Ma08, BD10].

Let $(Y(t))_{t \in \mathbb{N}}$ denote the sequence of consecutive sites visited by the random walk $(X(t))_{t \in \mathbb{R}_+}$ (note that the “times” are different in nature for $X(t)$ and $Y(t)$). This sequence is itself a Markov chain that satisfies for any two neighbours $x, y \in \mathbb{Z}^d$:

$$\mathbf{P}_x^\omega[Y(1) = y] = \frac{\omega_{x,y}}{p_\omega(x)},$$

where $p_\omega(x) = \sum_{|z|=1} \omega_{x,x+z}$. We simply write $p(\omega)$ for $p_\omega(0)$. Let us introduce a “tilted” version of the law \mathbb{P} on the environments, that we write $\tilde{\mathbb{P}}$ and define by

$$(1.1) \quad d\tilde{\mathbb{P}}(\omega) = \frac{p(\omega)}{\mathbb{E}[p]} d\mathbb{P}(\omega).$$

The reason why this measure is natural to consider is that it makes the environment seen from the position of the random walk Y a stationary process (see (3.2) for a definition of this process).

Interpolating between two integers by a straight line, we can think of Y as a continuous function on \mathbb{R}_+ . With this in mind, it is also true that there exists a matrix $A_{\text{hom}}^{\text{disc}}$ such that, as ε tends to 0, the rescaled process $\sqrt{\varepsilon}Y(\varepsilon^{-1}t)$ converges in law under $\tilde{\mathbb{P}}\mathbf{P}_0^\omega$ to a Brownian motion with covariance matrix $2A_{\text{hom}}^{\text{disc}}$. Moreover, $A_{\text{hom}}^{\text{disc}}$ and A_{hom} are related by (see [DFGW89, Theorem 4.5 (ii)]):

$$(1.2) \quad A_{\text{hom}} = \mathbb{E}[p] A_{\text{hom}}^{\text{disc}}.$$

Given that the numerical simulation of Y saves some operations compared to the simulation of X (there is no waiting time to compute, and the running time is equal to the number of steps), we will focus on approximating $A_{\text{hom}}^{\text{disc}}$. More precisely, we fix once and for all some $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, and define

$$(1.3) \quad \sigma_t^2 = t^{-1} \tilde{\mathbb{E}}\mathbf{E}_0^\omega[(\xi \cdot Y(t))^2],$$

$$(1.4) \quad \sigma^2 = 2\xi \cdot A_{\text{hom}}^{\text{disc}} \xi = \frac{2\xi \cdot A_{\text{hom}} \xi}{\mathbb{E}[p]}.$$

It follows from results of [KV86] (or [DFGW89, Theorem 2.1]) that σ_t^2 tends to σ^2 as t tends to infinity. We now describe a Monte-Carlo method to approximate σ_t^2 . Using the definition of the tilted measure (1.1), one can see that

$$(1.5) \quad \sigma_t^2 = \frac{\tilde{\mathbb{E}}\mathbf{E}_0^\omega[(\xi \cdot Y(t))^2]}{t} = \frac{\mathbb{E}\mathbf{E}_0^\omega[p(\omega)(\xi \cdot Y(t))^2]}{t\mathbb{E}[p]}.$$

Assuming that we have easier access to the measure \mathbb{P} than to the tilted $\tilde{\mathbb{P}}$, we prefer to base our Monte-Carlo procedure on the r. h. s. of the second identity in

(1.5). Let $Y^{(1)}, Y^{(2)}, \dots$ be independent random walks evolving in the environments $\omega^{(1)}, \omega^{(2)}, \dots$ respectively. We write $\mathbf{P}_0^{\bar{\omega}}$ for their joint distribution, all random walks starting from 0, where $\bar{\omega}$ stands for $(\omega^{(1)}, \omega^{(2)}, \dots)$. The family of environments $\bar{\omega}$ is itself random, and we let \mathbb{P}^{\otimes} be the product distribution with marginal \mathbb{P} . In other words, under \mathbb{P}^{\otimes} , the environments $\omega^{(1)}, \omega^{(2)}, \dots$ are independent and distributed according to \mathbb{P} . Our computable approximation of σ_t^2 is defined by

$$(1.6) \quad \hat{A}_n(t) = \frac{p(\omega^{(1)})(\xi \cdot Y^{(1)}(t))^2 + \dots + p(\omega^{(n)})(\xi \cdot Y^{(n)}(t))^2}{t(p(\omega^{(1)}) + \dots + p(\omega^{(n)}))}.$$

The main result of this paper is the following optimal bounds on the distribution of the error $|\hat{A}_n(t) - \sigma^2|$.

Theorem 1.1. *Under the assumptions (H1-H3), there exist $C, c > 0$ such that, for any $n \in \mathbb{N}^*$, any $\varepsilon > 0$ and any t large enough,*

$$(1.7) \quad \mathbb{P}^{\otimes} \mathbf{P}_0^{\bar{\omega}} \left[|\hat{A}_n(t) - \sigma^2| \geq (C\mu_d(t) + \varepsilon)/t \right] \leq \exp\left(-\frac{n\varepsilon^2}{ct^2}\right),$$

where σ^2 and $\hat{A}_n(t)$ are defined respectively in (1.4) and (1.6), and

$$\mu_d(t) = \begin{cases} \ln^q t & (d = 2) \\ 1 & (d > 2) \end{cases}$$

for some $q > 0$ depending only on α and β .

This result precisely quantifies the convergence rate of a method proposed by Papanicolaou in [Pap83] in the beginning of the eighties to approximate the homogenized coefficients A_{hom} numerically.

Let us quickly discuss the sharpness of these results. If A was a periodic matrix (or even a constant matrix) we would get the same estimate as in Theorem 1.1, except in dimension 2 for which no logarithmic correction would be needed (note however that in this i. i. d. case the additional logarithm seems unavoidable for $d = 2$, as discussed in the introduction of [GO10a] — although we fail to identify the precise exponent q , which we conjecture to be equal to 2). Numerical tests illustrating (1.7) for $d = 2$ are reported and commented on in the last section of this article.

1.2. Structure of the proof. Although the result of Theorem 1.1 is purely probabilistic (we estimate a distribution) its proof involves both nontrivial probabilistic arguments (martingale decomposition and Kipnis-Varadhan theory, large deviation estimates) and nontrivial arguments of elliptic theory (Harnack inequality, De Giorgi-Nash-Moser theory, and L^p -theory). What allows to combine these arguments is spectral theory. This makes the overall structure of the proof interesting and rather unusual.

The starting point of the proof is the rewriting of the l. h. s. of (1.7) in the form

$$\begin{aligned} & \mathbb{P}^{\otimes} \mathbf{P}_0^{\bar{\omega}} \left[|\hat{A}_n(t) - \sigma^2| \geq (C\mu_d(t) + \varepsilon)/t \right] \\ &= \mathbb{P}^{\otimes} \mathbf{P}_0^{\bar{\omega}} \left[|\hat{A}_n(t) - \sigma_t^2 + \sigma_t^2 - \sigma^2| \geq (C\mu_d(t) + \varepsilon)/t \right] \\ &\leq \mathbb{P}^{\otimes} \mathbf{P}_0^{\bar{\omega}} \left[|\hat{A}_n(t) - \sigma_t^2| \geq (C\mu_d(t) + \varepsilon)/t - |\sigma_t^2 - \sigma^2| \right]. \end{aligned}$$

The result then follows from the following two estimates:

$$(1.8) \quad |\sigma_t^2 - \sigma^2| \leq C \frac{\mu_d(t)}{t},$$

$$(1.9) \quad \mathbb{P}^{\otimes} \mathbf{P}_0^{\bar{\omega}} \left[|\hat{A}_n(t) - \sigma_t^2| \geq \varepsilon/t \right] \leq \exp\left(-\frac{n\varepsilon^2}{ct^2}\right),$$

see Theorems 3.1 and 4.1. The second estimate is a large deviation estimate. Its proof is standard once we are given sharp upper bounds on the transition probabilities of the random walk in the random environment — which are also by now standard under assumption (H3). The proof is given in Section 4 for completeness.

The core of this article is the estimate (1.8). We call its l. h. s. the systematic error. As proved in the celebrated paper [KV86] by Kipnis and Varadhan (see also [DFGW89]), the systematic error vanishes as t goes to infinity as soon as the measure \mathbb{P} is ergodic under translations. The strategy to prove this result is to find a decomposition of $Y(t) \cdot \xi$ into a martingale plus a remainder, in such a way that the remainder term becomes negligible in the limit, and conclude using the orthogonality of the increments of the martingale and ergodicity. The approach taken up by [KV86] is based on the spectral analysis of the (self-adjoint) operator of the environment viewed by the particle. More precisely, it is shown that in order for this decomposition with negligible remainder to exist, it suffices that the spectral measure of this operator, once projected on the “local drift” \mathfrak{d} (see (3.4)), satisfies some integrability condition (IC) at the edge of the spectrum. Condition (IC) is then seen to be equivalent to asking \mathfrak{d} to belong to the function space H^{-1} , a fact which is automatically true due to certain symmetry considerations that were systematized in [DFGW89].

Our proof of (1.8) consists in two steps. We first make the argument of Kipnis and Varadhan quantitative in Section 2. That is, we show that stronger integrability conditions than (IC) on the spectral measure can be turned into quantitative estimates on the systematic error — this is a general result of independent interest.

In the second step we prove that indeed, condition (IC) can be strengthened to higher integrability properties, provided ergodicity is replaced by the stronger assumption that the conductances are i. i. d., the hypothesis (H2). This is where elliptic theory comes into the picture. In particular, following our approach in [GM10], we shall use in Section 3 that the boundedness of moments of solutions to the so-called corrector equation in homogenization theory yields estimates on the spectral measure, and conversely. The moments of these solutions are estimated using tools developed by Otto and the first author in [GO10a, GO10b], based on a spectral gap estimate (in the forms of a variance and a covariance estimate) and on sharp bounds on Green’s functions (obtained by elliptic theory). These tools have been recognised to give nontrivial estimates on the spectral measure by the second author in [Mo11]. Using these tools and a subtle interplay between the bounds on the moments of solutions to the corrector equation and the estimates on the spectral measure, we prove in Section 3 that the assumptions of our quantitative version of Kipnis-Varadhan’s result are indeed satisfied in the case of i. i. d. conductances, which implies the desired decay (1.8) of the systematic error.

Taken independently from one another, each of the following three sections (quantification of Kipnis-Varadhan’s theorem, quantitative estimates on the spectral measure, large deviation estimates) may appear as technical adaptations of arguments which have already appeared in the literature. What makes Theorem 1.1 surprisingly difficult to prove and its proof interesting is the variety of the mathematics involved.

Let us point out that although the results of this paper are proved under assumptions (H1-H3), the assumption (H2) on the statistics of ω is only used to obtain the variance estimate of [GO10a, Lemma 2.3]. In particular, (H2) can be weakened as follows:

- the distribution of $\omega_{(z, z+\mathbf{e}_i)}$ may in addition depend on \mathbf{e}_i ,

- independence can be replaced by finite correlation length $C_L > 0$, that is for all $e, e' \in \mathbb{B}$, ω_e and $\omega_{e'}$ are independent if $|e - e'| \geq C_L$.

Notation. So far we have already introduced the probability measures \mathbf{P}_0^ω (distribution of Y), $\mathbf{P}_0^{\overline{\omega}}$ (distribution of $Y^{(1)}, Y^{(2)}, \dots$), \mathbb{P} (i.i.d. distribution for $\omega = (\omega_e)_{e \in \mathbb{B}}$), $\tilde{\mathbb{P}}$ (tilted measure defined in (1.1)) and \mathbb{P}^\otimes (product distribution of $\overline{\omega}$ with marginal \mathbb{P}). It will be convenient to define $\tilde{\mathbb{P}}^\otimes$ the product distribution of $\overline{\omega}$ with marginal $\tilde{\mathbb{P}}$. For convenience, we write \mathbb{P}_0 as a short-hand notation for $\mathbb{P}\mathbf{P}_0^\omega$, $\tilde{\mathbb{P}}_0$ for $\tilde{\mathbb{P}}\mathbf{P}_0^\omega$, \mathbb{P}_0^\otimes for $\mathbb{P}^\otimes\mathbf{P}_0^\omega$, and $\tilde{\mathbb{P}}_0^\otimes$ for $\tilde{\mathbb{P}}^\otimes\mathbf{P}_0^\omega$. The corresponding expectations are written accordingly, replacing “P” by “E” with the appropriate typography. Finally, we write $|\cdot|$ for the Euclidian norm of \mathbb{R}^d .

2. QUANTITATIVE VERSION OF KIPNIS-VARADHAN’S THEOREM

Kipnis-Varadhan’s theorem [KV86] concerns additive functionals of reversible Markov processes. It gives conditions for such additive functionals to satisfy an invariance principle. The proof of the result relies on a decomposition of the additive functional as the sum of a martingale term plus a remainder term, the latter being shown to be negligible. In this section, which can be read independently of the rest of the paper, we give conditions that enable to obtain some quantitative bounds on this remainder term.

We consider discrete and continuous times simultaneously. Let $(\eta_t)_{t \geq 0}$ be a Markov process defined on some measurable state space \aleph (here, $t \geq 0$ stands either for $t \in \mathbb{N}$ or for $t \in \mathbb{R}_+$). We denote by P_x the distribution of the process started from $x \in \aleph$, and by E_x the associated expectation. We assume that this Markov process is reversible and ergodic with respect to some probability measure ν . We write P_ν for the law of the process started from the distribution ν , and E_ν for the associated expectation.

To the Markov process is naturally associated a semi-group $(P_t)_{t \geq 0}$ defined, for any $f \in \mathbb{L}_2(\nu)$, by

$$P_t f(x) = E_x[f(\eta_t)].$$

Each P_t is a self-adjoint contraction of $\mathbb{L}^2(\nu)$. In the continuous-time case, we assume further that the semi-group is strongly continuous, that is to say, that $P_t f$ converges to f in $\mathbb{L}^2(\nu)$ as t tends to 0, for any $f \in \mathbb{L}^2(\nu)$. We let L be the $\mathbb{L}^2(\nu)$ -infinitesimal generator of the semi-group. It is self-adjoint in $\mathbb{L}^2(\nu)$, and we fix the sign convention so that it is a positive operator (i.e., $P_t = e^{-tL}$).

Note that in general, one can see using spectral analysis that there exists a projection \overline{P} such that $P_t f$ converges to $\overline{P}f$ as t tends to ∞ , $t > 0$. Changing $\mathbb{L}^2(\nu)$ to the image of the projection \overline{P} , and P_0 for \overline{P} , one recovers a strongly continuous semigroup of contractions, and one can still carry the analysis below replacing $\mathbb{L}^2(\nu)$ by the image of \overline{P} when necessary.

In discrete time, we set $L = \text{Id} - P_1$. Again, L is a positive self-adjoint operator on $\mathbb{L}^2(\nu)$. Note that we slightly depart from the custom of defining the generator as P_1 in order to match more closely the continuous time situation.

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{L}^2(\nu)$. For any function $f \in \mathbb{L}^2(\nu)$ we define the *spectral measure* of L projected on the function f as the measure e_f on \mathbb{R}_+ that satisfies, for any bounded continuous $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}$, the relation

$$(2.1) \quad \langle f, \Psi(L)f \rangle = \int \Psi(\lambda) \, de_f(\lambda).$$

The Dirichlet form associated to L is given by

$$(2.2) \quad \|f\|_1^2 = \int \lambda \, de_f(\lambda).$$

We denote by H^1 the completion of the space $\{f \in \mathbb{L}^2(\nu) : \|f\|_1 < +\infty\}$ with respect to this $\|\cdot\|_1$ norm, taken modulo functions of zero $\|\cdot\|_1$ norm. This turns $(H^1, \|\cdot\|_1)$ into a Hilbert space, and we let H^{-1} denote its dual. One can identify H^{-1} with the completion of the space $\{f \in \mathbb{L}^2(\nu) : \|f\|_{-1} < +\infty\}$ with respect to the norm $\|\cdot\|_{-1}$ defined by

$$\|f\|_{-1}^2 = \int \lambda^{-1} \, de_f(\lambda).$$

Indeed, for all $f \in \mathbb{L}^2(\nu)$, the linear form

$$\begin{cases} (\mathbb{L}^2(\nu) \cap H^1, \|\cdot\|_1) & \rightarrow \mathbb{R} \\ \phi & \mapsto \langle f, \phi \rangle \end{cases}$$

has norm $\|f\|_{-1}$, and thus defines an element of H^{-1} (with norm $\|f\|_{-1}$) iff $\|f\|_{-1}$ is finite. The notion of spectral measure introduced in (2.1) for functions of $\mathbb{L}^2(\nu)$ can be extended to elements of H^{-1} . Indeed, let $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a continuous function such that $\Psi(\lambda) = O(\lambda^{-1})$ as $\lambda \rightarrow +\infty$. One can check that the map

$$\begin{cases} (\mathbb{L}^2(\nu) \cap H^{-1}, \|\cdot\|_{-1}) & \rightarrow H^1 \\ f & \mapsto \Psi(L)f \end{cases}$$

extends to a bounded linear map on H^{-1} . One can then define the spectral measure of L projected on the function f as the measure e_f such that for any continuous Ψ with $\Psi(\lambda) = O(\lambda^{-1})$, (2.1) holds. With a slight abuse of notation, for all $f \in H^{-1}$ and $g \in H^1$, we write $\langle f, g \rangle$ for the $H^{-1} - H^1$ duality product between f and g .

For any $f \in H^{-1}$, we define $(Z_f(t))_{t \geq 0}$ as

$$(2.3) \quad Z_f(t) = \int_0^t f(\eta_s) \, ds \quad \text{or} \quad Z_f(t) = \sum_{s=0}^{t-1} f(\eta_s),$$

according to whether we consider the continuous or the discrete time cases. In the continuous case, the meaning of (2.3) is unclear a priori. Yet it is proved in [DFGW89, Lemma 2.4] that for any $t \geq 0$ the map

$$\begin{cases} \mathbb{L}^2(\nu) \cap H^{-1} & \rightarrow \mathbb{L}^2(P_\nu) \\ f & \mapsto Z_f(t) \end{cases}$$

can be extended by continuity to a bounded linear map on H^{-1} , and moreover, that (2.3) coincides with the usual integral as soon as $f \in \mathbb{L}^1(\nu)$. The following theorem is due to [DFGW89], building on previous work of [KV86].

Theorem 2.1. (i) For all $f \in H^{-1}$, there exists $(M_t)_{t \geq 0}$, $(\xi_t)_{t \geq 0}$ such that $Z_f(t)$ defined in (2.3) satisfies the identity $Z_f(t) = M_t + \xi_t$, where (M_t) is a square-integrable martingale with stationary increments under P_ν (and the natural filtration), and (ξ_t) is such that :

$$(2.4) \quad t^{-1} E_\nu[(\xi_t)^2] \xrightarrow[t \rightarrow +\infty]{} 0.$$

As a consequence, $t^{-1/2} Z_f(t)$ converges in law under P_ν to a Gaussian random variable of variance $\sigma^2(f) < +\infty$ as t goes to infinity, and

$$(2.5) \quad t^{-1} E_\nu[(Z_f(t))^2] \xrightarrow[t \rightarrow +\infty]{} \sigma^2(f).$$

(ii) If, moreover, $f \in \mathbb{L}^1(\nu)$ and, for some $\bar{t} > 0$, $\sup_{0 \leq t \leq \bar{t}} |Z_f(t)|$ is in $\mathbb{L}^2(\nu)$, then the process $t \mapsto \sqrt{\varepsilon} Z_f(\varepsilon^{-1}t)$ converges in law under P_ν to a Brownian motion of variance $\sigma^2(f)$ as ε goes to 0.

Remarks. The additional conditions appearing in statement (ii) are automatically satisfied in discrete time, due to the fact that $H^{-1} \subseteq \mathbb{L}^2(\nu)$ in this case. In the continuous-time setting and when $f \in \mathbb{L}^1(\nu)$, the process $t \mapsto Z_f(t)$ is almost surely continuous, and $\sup_{0 \leq t \leq \bar{t}} |Z_f(t)|$ is indeed a well-defined random variable.

Under some additional information on the spectral measure of f , we can estimate the rates of convergence in the limits (2.4) and (2.5). For any $\gamma > 1$ and $q \geq 0$, we say that the spectral exponents of a function $f \in H^{-1}$ are at least $(\gamma, -q)$ if

$$(2.6) \quad \int_0^\mu de_f(\lambda) = O(\mu^\gamma \ln^q(\mu^{-1})) \quad (\mu \rightarrow 0).$$

Note that the phrasing is consistent, since if $(\gamma', -q') \leq (\gamma, -q)$ for the lexicographical order, and if the spectral exponents of f are at least $(\gamma, -q)$, then they are at least $(\gamma', -q')$. In [Mo11], it was found more convenient to consider, instead of (2.6), a condition of the following form :

$$(2.7) \quad \int_0^\mu \lambda^{-1} de_f(\lambda) = O(\mu^{\gamma-1} \ln^q(\mu^{-1})) \quad (\mu \rightarrow 0).$$

One can easily check that conditions (2.6) and (2.7) are equivalent. Indeed, on the one hand, one has the obvious inequality

$$\int_0^\mu de_f(\lambda) \leq \mu \int_0^\mu \lambda^{-1} de_f(\lambda),$$

which shows that (2.7) implies (2.6). On the other hand, one may perform a kind of integration by parts, use Fubini's theorem :

$$\begin{aligned} \int_0^\mu \lambda^{-1} de_f(\lambda) &= \int_0^\mu \int_{\delta=\lambda}^{+\infty} \delta^{-2} d\delta de_f(\lambda) \\ &= \int_{\delta=0}^{+\infty} \delta^{-2} \int_{\lambda=0}^{\delta \wedge \mu} de_f(\lambda) d\delta, \end{aligned}$$

and obtain the converse implication by examining separately the integration over δ in $[0, \mu)$ and in $[\mu, +\infty)$.

For all $\gamma > 1$ and $q \geq 0$, we set

$$(2.8) \quad \psi_{\gamma,q}(t) = \begin{cases} t^{1-\gamma} \ln^q(t) & \text{if } \gamma < 2, \\ t^{-1} \ln^{q+1}(t) & \text{if } \gamma = 2, \\ t^{-1} & \text{if } \gamma > 2. \end{cases}$$

The quantitative version of Theorem 2.1 is as follows.

Theorem 2.2. *If the spectral exponents of $f \in H^{-1}$ are at least $(\gamma, -q)$, then the decomposition $Z_f(t) = M_t + \xi_t$ of Theorem 2.1 holds with the additional property that*

$$t^{-1} E_\nu[(\xi_t)^2] = O(\psi_{\gamma,q}(t)) \quad (t \rightarrow +\infty).$$

Moreover,

$$\sigma^2(f) - \frac{E_\nu[Z_f(t)^2]}{t} = O(\psi_{\gamma,q}(t)) \quad (t \rightarrow +\infty).$$

Proof. In the continuous-time setting, the argument for the first estimate is very similar to the one of [Mo11, Proposition 8.2], and we do not repeat the details here. It is based on the observation that

$$(2.9) \quad \frac{1}{t} E_\nu[(\xi_t)^2] = 2 \int \frac{1 - e^{-\lambda t}}{\lambda^2 t} de_f(\lambda).$$

One needs to take into account the possible logarithmic terms that appear in (2.7) and which are not considered in [Mo11]. Some care is also needed because we do not assume that $f \in \mathbb{L}^2(\nu)$. Yet one can easily replace the bound involving the

$\mathbb{L}^2(\nu)$ norm of f by its H^{-1} norm. The second part of the statement is given by [Mo11, Proposition 8.3].

We now turn to the discrete time setting. In this context, identity (2.9) should be replaced by

$$\frac{1}{t} E_\nu[(\xi_t)^2] = 2 \int \frac{1 - (1 - \lambda)^t}{\lambda^{2t}} de_f(\lambda).$$

By definition, $L = \text{Id} - P_1$, where P_1 is the semi-group at time 1. Hence the spectrum of L is contained in $[0, 2]$. One can then follow the same computations as before to prove the first part of Theorem 2.2.

Somewhat surprisingly, the second part of the statement requires additional attention in the discrete time setting. Indeed, in the continuous case, the argument of [Mo11, Proposition 8.3] (which already appears in [DFGW89]) is that $Z_f(t)$ and $\xi(t)$ are orthogonal in $\mathbb{L}^2(P_\nu)$, a fact obtained using the invariance under time symmetry. This orthogonality is only approximately valid in the discrete-time setting. Indeed, let us recall that $Z_f(t)$ is given by (2.3), while ξ_t is obtained as the limit in $\mathbb{L}^2(P_\nu)$ of

$$-u_\varepsilon(\eta_t) + u_\varepsilon(\eta_0),$$

where $u_\varepsilon = (\varepsilon + L)^{-1}f$. Using time symmetry, what we obtain is that ξ_t is orthogonal to $(Z_f(t) + f(\eta_t))$. As a consequence, the cross-product $E_\nu[Z_f(t)\xi_t]$, which is equal to 0 in the proof of [Mo11, Proposition 8.3], is in the present case equal to $-E_\nu[f(\eta_t)\xi_t]$. Yet spectral analysis ensures that this term is equal to

$$\int \frac{1 - (1 - \lambda)^t}{\lambda} de_f(\lambda) = O(1) \quad (t \rightarrow +\infty),$$

which is what we need to obtain the second claim of the theorem. \square

3. THE SYSTEMATIC ERROR

We now come back to the analysis of the Monte-Carlo approximation of the homogenized coefficients within assumptions (H1)-(H3). The aim of this section is to estimate the difference between σ_t^2 and the quantity σ^2 we wish to approximate (both being defined in (1.3)). This difference, that we refer to as the *systematic error* after [GO10a], is shown to be of order $1/t$ as t tends to infinity, up to a logarithmic correction in dimension 2.

Theorem 3.1. *Under assumptions (H1)-(H3), there exists $q \geq 0$ such that, as t tends to infinity,*

$$(3.1) \quad \sigma_t^2 - \sigma^2 = \begin{cases} O(t^{-1} \ln^q(t)) & \text{if } d = 2, \\ O(t^{-1}) & \text{if } d > 2. \end{cases}$$

Theorem 3.1 is a discrete-time version of [Mo11, Corollary 2.6]. Its proof makes use of an auxiliary process that we now introduce.

Let $(\theta_x)_{x \in \mathbb{Z}^d}$ be the translation group that acts on the set of environments as follows: for any pair of neighbours $y, z \in \mathbb{Z}^d$, $(\theta_x \omega)_{y,z} = \omega_{x+y, x+z}$. The *environment viewed by the particle* is the process defined by

$$(3.2) \quad \omega(t) = \theta_{Y(t)} \omega.$$

One can check that $(\omega(t))_{t \in \mathbb{N}}$ is a Markov chain, whose generator is given by

$$(3.3) \quad -\mathcal{L}f(\omega) = \frac{1}{p(\omega)} \sum_{|z|=1} \omega_{0,z} (f(\theta_z \omega) - f(\omega)),$$

so that $\mathbf{E}_0^\omega[f(\omega(1))] = (I - \mathcal{L})f(\omega)$. Moreover, the measure $\tilde{\mathbb{P}}$ defined in (1.1) is reversible and ergodic for this process [DFGW89, Lemma 4.3 (i)]. As a consequence, the operator \mathcal{L} is (positive and) self-adjoint in $\mathbb{L}^2(\tilde{\mathbb{P}})$.

The proof of Theorem 3.1 relies on spectral analysis. For any function $f \in L^2(\tilde{\mathbb{P}})$, let e_f be the spectral measure of \mathcal{L} projected on the function f . This measure is such that, for any positive continuous function $\Psi : [0, +\infty) \rightarrow \mathbb{R}_+$, one has

$$\tilde{\mathbb{E}}[f \Psi(\mathcal{L})f] = \int \Psi(\lambda) de_f(\lambda).$$

For any $\gamma > 1$ and $q \geq 0$, we recall that we say that the spectral exponents of a function f are at least $(\gamma, -q)$ if (2.6) holds.

Let us define the local drift \mathfrak{d} in direction ξ as

$$(3.4) \quad \mathfrak{d}(\omega) = \mathbf{E}_0^\omega[\xi \cdot Y(1)] = \frac{1}{p(\omega)} \sum_{|z|=1} \omega_{0,z} \xi \cdot z.$$

As we shall prove at the end of this section, we have the following bounds on the spectral exponents of \mathfrak{d} .

Proposition 3.2. *Under assumptions (H1)-(H3), there exists $q \geq 0$ such that the spectral exponents of the function \mathfrak{d} are at least*

$$(3.5) \quad \begin{cases} (2, -q) & \text{if } d = 2, \\ (d/2 + 1, 0) & \text{if } 3 \leq d \leq 5, \\ (4, -1) & \text{if } d = 6, \\ (4, 0) & \text{if } d \geq 7. \end{cases}$$

Let us see how this result implies Theorem 3.1. In order to do so, we also need the following information, that is a consequence of Proposition 3.2.

Corollary 3.3. *Let*

$$(3.6) \quad \mathfrak{d}_t(\omega) = \mathbf{E}_0^\omega[\mathfrak{d}(\omega(t))]$$

be the image of \mathfrak{d} by the semi-group at time t associated with the Markov chain $(\omega(t))_{t \in \mathbb{N}}$. There exists $q \geq 0$ such that

$$\tilde{\mathbb{E}}[(\mathfrak{d}_t)^2] = \begin{cases} O(t^{-2} \ln^q(t)) & \text{if } d = 2, \\ O(t^{-(d/2+1)}) & \text{if } 3 \leq d \leq 5, \\ O(t^{-4} \ln(t)) & \text{if } d = 6, \\ O(t^{-4}) & \text{if } d \geq 7. \end{cases}$$

Proof. This result is the discrete-time analog of [GM10, Corollary 1]. It is obtained the same way, noting that

$$\tilde{\mathbb{E}}[(\mathfrak{d}_t)^2] = \int (1 - \lambda)^{2t} de_f(\lambda),$$

and that the support of the measure e_f is contained in $[0, 2]$. \square

We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. The proof has the same structure as for the continuous-time case of [Mo11, Proposition 8.4]. Note that [DFGW89, Theorem 2.1] ensures that

$$(3.7) \quad \lim_{t \rightarrow \infty} \sigma_t^2 \stackrel{(\text{def})}{=} \lim_{t \rightarrow \infty} t^{-1} \mathbb{E}_0[(\xi \cdot Y(t))^2] = \sigma^2.$$

The starting point is the observation that, under $\tilde{\mathbb{P}}_0$, the process defined by

$$(3.8) \quad N_t = \xi \cdot Y(t) - \sum_{s=0}^{t-1} \mathfrak{d}(\omega(s))$$

is a square-integrable martingale with stationary increments. On the one hand, following (2.3), we denote by $Z_{\mathfrak{d}}(t)$ the sum appearing in the r. h. s. of (3.8). From Proposition 3.2 and Theorem 2.2, we learn that there exist $\bar{\sigma}$ and $q \geq 0$ such that

$$(3.9) \quad t\bar{\sigma}^2 - \tilde{\mathbb{E}}_0[(Z_{\mathfrak{d}}(t))^2] = \begin{cases} O(\ln^q(t)) & \text{if } d = 2, \\ O(1) & \text{if } d > 2. \end{cases}$$

On the other hand, since N_t is a martingale with stationary increments,

$$(3.10) \quad \tilde{\mathbb{E}}_0[(N_t)^2] = t\tilde{\mathbb{E}}_0[(N_1)^2].$$

As in the proof of Theorem 2.2 in the discrete time case, we then use that $\xi \cdot Y(t)$ is orthogonal to $(Z_{\mathfrak{d}}(t) + \mathfrak{d}(\omega(t)))$ to turn (3.8) into

$$(3.11) \quad t^{-1}\tilde{\mathbb{E}}_0[(N_t)^2] = t^{-1}\tilde{\mathbb{E}}_0[(\xi \cdot Y(t))^2] + t^{-1}\tilde{\mathbb{E}}_0[(Z_{\mathfrak{d}}(t))^2] + 2t^{-1}\tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot Y(t))].$$

We already control the l. h. s. and the second term of the r. h. s. of (3.11). In order to quantify the convergence of $t^{-1}\tilde{\mathbb{E}}_0[(\xi \cdot Y(t))^2]$ it remains to control the last term. In particular, provided we show that

$$(3.12) \quad \tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot Y(t))] = \begin{cases} O(\ln^q(t)) & \text{if } d = 2, \\ O(1) & \text{if } d > 2, \end{cases}$$

(3.11), (3.9), (3.10), and (3.7) imply first that $\sigma^2 = \tilde{\mathbb{E}}_0[(N_1)^2] - \bar{\sigma}^2$, and then the desired quantitative estimate (3.1). We now turn to (3.12) and write

$$\begin{aligned} \tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot Y(t))] &= \sum_{s=0}^{t-1} \tilde{\mathbb{E}}_0[\mathfrak{d}(\omega(t))(\xi \cdot (Y(s+1) - Y(s)))] \\ &= \sum_{s=0}^{t-1} \tilde{\mathbb{E}}_0[\mathfrak{d}_{t-s-1}(\omega(s+1))(\xi \cdot (Y(s+1) - Y(s)))] \end{aligned}$$

where we have used the Markov property at time $s+1$, together with the definition (3.6) of \mathfrak{d}_{t-s-1} . Using Cauchy-Schwarz inequality and the stationarity of the process $(\omega(t))_{t \in \mathbb{N}}$ under $\tilde{\mathbb{E}}_0$, this sum is bounded by

$$|\xi|^2 \sum_{s=0}^{t-1} \tilde{\mathbb{E}}_0[(\mathfrak{d}_{t-s-1})^2]^{1/2}.$$

Estimate (3.12) then follows from Corollary 3.3. This concludes the proof of the theorem. \square

We finally turn to the proof of Proposition 3.2, which is a discrete-time counterpart of [GM10, Theorem 5]. In [GM10, Theorem 5] however, we had proved in addition that the spectral exponents are at least $(d/2 - 2, 0)$, which is sharper than the exponents of Proposition 3.2 for $d > 10$. In particular for $d > 10$ the bounds of [GM10, Theorem 5] follow from results of [Mo11], whose adaptation to the discrete time setting is not straightforward. As shown above, the present statement is sufficient to prove the optimal scaling of the systematic error, and we do not investigate further this issue. The proof of Proposition 3.2 is rather involved and one may wonder whether this is worth the effort in terms of the application we have in mind — namely Theorem 3.1. In order to obtain the optimal convergence rate in Theorem 3.1 we need the spectral exponents to be larger than $(2, 0)$. Proving that the exponents are at least $(2, 0)$ is rather direct using results of [GO10a] (see the first three steps of the proof of Proposition 3.2). Yet proving that they are larger than $(2, 0)$ for $d > 2$ is as involved as proving Proposition 3.2 itself. This is the reason why we display the complete proof of Proposition 3.2 — although the precise values of the spectral exponents are not that important in the context of this paper.

Proof of Proposition 3.2. The present proof is a direct version of the proof of [GM10, Theorem 5]. In particular, the proof of [GM10, Theorem 5] relies on a nontrivial (weaker) estimate of the spectral exponents obtained in [GO10b] using a covariance estimate. Yet if one wants to extend these results to more general statistics of the conductivity function one has to give up the covariance estimate (which we are not able to prove any longer, see an example in [GO11]). With this in mind we only rely on the variance estimate of [GO10a, Lemma 2.3]. As opposed to the strategy used in [GM10, Theorem 5] we directly focus on the spectral exponents rather than on some other quantity like the systematic error itself. Our starting point is the inequality:

$$(3.13) \quad \int_0^{T^{-1}} de_{\mathfrak{d}}(\lambda) \lesssim T^{-4} \int_0^\infty \frac{1}{(T^{-1} + \lambda)^4} de_{\mathfrak{d}}(\lambda),$$

which follows from the fact that for $\lambda \leq T^{-1}$, $\frac{T^{-4}}{(T^{-1} + \lambda)^4} \gtrsim 1$ (here and below \lesssim and \gtrsim stand respectively for \leq and \geq up to multiplicative constants). The variable T^{-1} for T large plays the role of μ in (2.6). In what follows we make the standard identification between stationary functions $(z, \omega) \mapsto f(z, \omega)$ of both the space variable $z \in \mathbb{Z}^d$ and the environment ω and their translated versions at 0 $\omega \mapsto f(0, \theta_z \omega)$ depending on the environment only. We define ϕ_T as the unique stationary solution to

$$(3.14) \quad T^{-1} \phi_T(x) - \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \nabla \phi_T = \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \xi,$$

whose existence and uniqueness follow from Lax-Milgram's theorem in $\mathbb{L}^2(\tilde{\mathbb{P}})$ using the identification between the stationary function ϕ_T and its version defined on the environment only (see a similar argument of [Kü83]). In particular, with the notation $\mathfrak{d} = \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \xi$,

$$\phi_T = (T^{-1} + \mathcal{L})^{-1} \mathfrak{d},$$

where \mathcal{L} is the operator defined in (3.3), and the spectral theorem ensures that

$$\tilde{\mathbb{E}}(\phi_T^2) = \tilde{\mathbb{E}}(\mathfrak{d}(T^{-1} + \mathcal{L})^{-2} \mathfrak{d}) = \int_0^\infty \frac{1}{(T^{-1} + \lambda)^2} de_{\mathfrak{d}}(\lambda)$$

where $e_{\mathfrak{d}}$ is the spectral measure of \mathcal{L} projected on the drift \mathfrak{d} . We also let ψ_T be the unique stationary solution to

$$(3.15) \quad T^{-1} \psi_T(x) - \frac{1}{p_\omega(x)} \nabla^* \cdot A(x) \nabla \psi_T(x) = \phi_T(x),$$

whose existence and uniqueness also follows from Lax-Milgram's theorem in the probability space as well. This time,

$$\psi_T = (T^{-1} + \mathcal{L})^{-2} \mathfrak{d},$$

and the spectral theorem yields

$$\tilde{\mathbb{E}}(\psi_T^2) = \tilde{\mathbb{E}}(\mathfrak{d}(T^{-1} + \mathcal{L})^{-4} \mathfrak{d}) = \int_0^\infty \frac{1}{(T^{-1} + \lambda)^4} de_{\mathfrak{d}}(\lambda)$$

From now on, we shall use the shorthand notation $\langle u \rangle := \tilde{\mathbb{E}}(u)$ and $\text{var}[u] = \langle (u - \langle u \rangle)^2 \rangle$ for all $u \in \mathbb{L}^2(\tilde{\mathbb{P}})$. In particular the identity above turns into

$$(3.16) \quad \int_0^\infty \frac{1}{(T^{-1} + \lambda)^4} de_{\mathfrak{d}}(\lambda) = \text{var}[\psi_T],$$

since $\langle \psi_T \rangle = \frac{1}{\mathbb{E}[p]} \int \psi_T p \, d\mathbb{P} = \frac{T}{\mathbb{E}[p]} \int \phi_T p \, d\mathbb{P} = 0$ using equations (3.15) and (3.14).

The rest of the proof, which is dedicated to the estimate of $\text{var}[\psi_T]$, is divided in six steps. Starting point is the application of the variance estimate of [GO10a, Lemma 2.3] to ψ_T , which requires to estimate the susceptibility of ψ_T with respect to the random coefficients. In view of (3.15) it is not surprising that we will have to estimate not only the susceptibility of ψ_T but also of ϕ_T and of some Green's function with respect to the random coefficients. In the first step, we establish the susceptibility estimate for the Green's function. In Step 2 we turn to the susceptibility estimate for the approximate corrector ϕ_T . We then show in Step 3 that, relying on [GO10a], this implies that the spectral exponents are at least

$$(3.17) \quad \left| \begin{array}{ll} (2, -q) & \text{for } d = 2, \\ (2, 0) & \text{for } d > 2. \end{array} \right.$$

Following step is to estimate the susceptibility of ψ_T . In Step 5 we show that this, combined with the suboptimal estimates

$$(3.18) \quad \langle |\nabla \psi_T|^2 \rangle \lesssim \left| \begin{array}{ll} T \ln^q T & \text{for } d = 2, \\ T & \text{for } d > 2, \end{array} \right. \quad \text{and} \quad \langle \psi_T^2 \rangle \lesssim \left| \begin{array}{ll} T^2 \ln^q T & \text{for } d = 2, \\ T^2 & \text{for } d > 2 \end{array} \right.$$

obtained using (3.17), allow us to improve the spectral exponents to

$$(3.19) \quad \left| \begin{array}{ll} (2, -q) & \text{for } d = 2, \\ (5/2, 0) & \text{for } d = 3, \\ (3, -1) & \text{for } d = 4, \\ (3, 0) & \text{for } d > 4. \end{array} \right.$$

In the last step, we quickly argue that in turn these spectral exponents yield the optimal and suboptimal estimates

$$(3.20) \quad \langle |\nabla \psi_T|^2 \rangle \lesssim \left| \begin{array}{ll} T \ln^q T & \text{for } d = 2, \\ \sqrt{T} & \text{for } d = 3, \\ \ln T & \text{for } d = 4, \\ 1 & \text{for } d > 4, \end{array} \right. \quad \text{and} \quad \langle \psi_T^2 \rangle \lesssim \left| \begin{array}{ll} T^2 \ln^q T & \text{for } d = 2, \\ T^{3/2} & \text{for } d = 3, \\ T \ln T & \text{for } d = 4, \\ T & \text{for } d > 4, \end{array} \right.$$

which finally bootstrap (3.19) to the desired estimates of the spectral exponents, and consequently yield the following optimal estimate of $\text{var}[\psi_T]$:

$$\text{var}[\psi_T] \lesssim \left| \begin{array}{ll} T^2 \ln^q T & \text{for } d = 2, \\ T^{3/2} & \text{for } d = 3, \\ T & \text{for } d = 4, \\ \sqrt{T} & \text{for } d = 5, \\ \ln T & \text{for } d = 6, \\ 1 & \text{for } d > 6. \end{array} \right.$$

We do not need Step 6 to prove Theorem 3.1. Yet this step is interesting in itself since it may be the starting point of a subtle “multi-level” induction to obtain the sharp spectral exponents in any dimension.

Step 1. Susceptibility of the Green's function.

For all $y \in \mathbb{Z}^d$ we define the Green's function $G_T(\cdot, y)$ with singularity at y as the unique solution in $L^2(\mathbb{Z}^d)$ to the equation

$$(3.21) \quad T^{-1} p_\omega(x) G_T(x, y) - \nabla_x^* \cdot A(x) \nabla_x G_T(x, y) = \delta(x - y),$$

using Lax-Milgram's theorem. We shall prove for all $e = (z, z') \in \mathbb{B}$, $z \in \mathbb{Z}^d$, $z' = z + \mathbf{e}_i$,

$$(3.22) \quad \frac{\partial G_T}{\partial \omega_e}(x, y) = -T^{-1} \left(G_T(z, y)G_T(x, z) + G_T(z', y)G_T(x, z') \right) \\ - \nabla_{z_i} G_T(x, z) \nabla_{z_i} G_T(z, y),$$

and

$$(3.23) \quad \sup_{\omega_e} |\nabla_{z_i} G_T(z, y)| \lesssim |\nabla_{z_i} G_T(z, y)| + T^{-1} g_T(y - z), \\ \sup_{\omega_e} |\nabla_{z_i} G_T(y, z)| \lesssim |\nabla_{z_i} G_T(y, z)| + T^{-1} g_T(y - z),$$

where $g_T : \mathbb{Z}^d \rightarrow \mathbb{R}^+$ satisfies for some constant $c > 0$ (depending on α, β, d)

$$(3.24) \quad g_T(x) = (1 + |x|)^{2-d} \exp\left(-c \frac{|x|}{\sqrt{T}}\right)$$

for $d > 2$, and

$$(3.25) \quad g_T(x) = \left| \ln \left(\frac{\sqrt{T}}{1 + |x|} \right) \right| \exp\left(-c \frac{|x|}{\sqrt{T}}\right)$$

for $d = 2$. We define the elliptic operator L_T as

$$(L_T u)(x) = \sum_{x', |x-x'|=1} \omega_{(x,x')} T^{-1} u(x) + \sum_{x', |x-x'|=1} \omega_{(x,x')} (u(x) - u(x')),$$

so that (3.21) takes the form

$$(3.26) \quad (L_T G_T(\cdot, y))(x) = \delta(x - y).$$

Formally differentiating this equation with respect to ω_e yields

$$L_T \left(\frac{\partial G_T}{\partial \omega_e}(\cdot, y) \right)(x) + T^{-1} G_T(x, y) (\delta(x - z) + \delta(x - z')) \\ + (G_T(z, y) - G_T(z', y)) \delta(x - z) + (G_T(z', y) - G_T(z, y)) \delta(x - z') = 0.$$

Using (3.26) this identity turns into

$$L_T \left(\frac{\partial G_T}{\partial \omega_e}(\cdot, y) + T^{-1} (G_T(z, y)G_T(\cdot, z) + G_T(z', y)G_T(\cdot, z')) \right. \\ \left. + \nabla_{z_i} G_T(\cdot, z) \nabla_{z_i} G_T(z, y) \right)(x) = 0.$$

Formally erasing the operator L_T on the l. h. s. then yields the desired identity (3.22). To turn this into a rigorous argument, we may proceed as in [GO10a, Proof of Lemma 2.5], first consider finite differences instead of derivative w. r. t. ω_e , use that L_T is bijective on $L^2(\mathbb{Z}^d)$, and then pass to the limit. We leave the details to the reader and directly turn to (3.23).

From (3.22) we infer that

$$\frac{\partial \nabla_{z_i} G_T(z, y)}{\partial \omega_e} = -\nabla_{z_i} \nabla_{z_i} G_T(z, z) \nabla_{z_i} G_T(z, y) \\ - T^{-1} (G_T(z, y) \nabla_{z_i} G_T(z, z) + G_T(z', y) \nabla_{z_i} G_T(z, z')).$$

Using then the uniform pointwise estimate $|\nabla G_T| \lesssim 1$ (see [GO10a, Corollary 2.3], whose proof holds as well in the present case with a non-constant zero order term since $2^d \alpha T^{-1} \leq T^{-1} p_\omega(x) \leq 2^d \beta T^{-1}$), the uniform pointwise estimate of the Green's function from [Gl10, Lemma 3], and considering this identity as an ODE for $\nabla_{z_i} G_T(z, y)$ in function of ω_e , we obtain (3.23).

Step 2. Susceptibility of the approximate corrector ϕ_T .

In this step we shall prove that for $e = (z, z') \in \mathbb{B}$, $z \in \mathbb{Z}^d$ and $z' = z + \mathbf{e}_i$,

$$(3.27) \quad \frac{\partial \phi_T}{\partial \omega_e}(x) = -(\nabla_i \phi_T(z) + \xi_i) \nabla_{z_i} G_T(x, z) - T^{-1} \phi_T(z) (G_T(x, z) + G_T(x, z')),$$

$$(3.28) \quad \sup_{\omega_e} |\phi_T(x)| \lesssim |\phi_T(x)| + (|\nabla_i \phi_T(z)| + 1) \left(|\nabla_{z_i} G_T(x, z)| + T^{-1/2} g_T(x - z) \right),$$

$$(3.29) \quad \sup_{\omega_e} \left| \frac{\partial \phi_T}{\partial \omega_e}(x) \right| \lesssim (|\nabla_i \phi_T(z)| + 1) \left(|\nabla_{z_i} G_T(x, z)| + T^{-1/2} g_T(x - z) \right),$$

and for all $n \in \mathbb{N}$,

$$(3.30) \quad \sup_{\omega_e} \left| \frac{\partial (\phi_T(x)^{n+1})}{\partial \omega_e} \right| \lesssim (|\nabla_i \phi_T(z)| + 1) \left(|\nabla_{z_i} G_T(x, z)| + T^{-1/2} g_T(x - z) \right) \times \left(|\phi_T(x)| + (|\nabla_i \phi_T(z)| + 1) \left(|\nabla_{z_i} G_T(z, x)| + T^{-1/2} g_T(x - z) \right) \right)^n.$$

As for the Green's function, we rewrite the defining equation for ϕ_T as

$$(3.31) \quad (L_T G_T(\cdot, y))(x) - \nabla^* \cdot A(x) \xi = 0.$$

Formally differentiating (3.31) w. r. t. ω_e yields

$$L_T \frac{\partial \phi_T}{\partial \omega_e}(x) - (\nabla_i \phi_T(z) + \xi_i) (\delta(x - z) - \delta(x - z')) + T^{-1} \phi_T(z) (\delta(x - z) + \delta(x - z')) = 0,$$

which using (3.26) turns into

$$L_T \left(\frac{\partial \phi_T}{\partial \omega_e} - (\nabla_i \phi_T(z) + \xi_i) (G_T(\cdot, z) - G_T(\cdot, z')) + T^{-1} \phi_T(z) (G_T(\cdot, z) + G_T(\cdot, z')) \right)(x) = 0.$$

This (formally) shows (3.27). To turn this into a rigorous argument, we may use the Green representation formula

$$\phi_T(x) = \int_{\mathbb{Z}^d} G_T(x, y) \nabla^* \cdot A(y) \xi \, dy,$$

where $\int_{\mathbb{Z}^d} dy$ denotes the sum over all $y \in \mathbb{Z}^d$, and proceed as in [GO10a, Proof of Lemma 2.4].

We now turn to (3.29). This estimate follows from (3.27), (3.23), and the following two facts:

$$(3.32) \quad |\phi_T| \lesssim \sqrt{T}$$

and

$$(3.33) \quad \sup_{\omega_e} |\nabla_i \phi_T(z)| \lesssim |\nabla_i \phi_T(z)| + 1.$$

Starting point to prove (3.32) is the Green representation formula in the form of

$$\begin{aligned} |\phi_T(x)| &= \left| \int_{\mathbb{Z}^d} G_T(x, y) \nabla^* \cdot A(y) \xi \, dy \right| \\ &= \left| \int_{\mathbb{Z}^d} \nabla_y G_T(x, y) \cdot A(y) \xi \, dy \right| \\ &\lesssim \int_{\mathbb{Z}^d} |\nabla_y G_T(0, y)| \, dy, \end{aligned}$$

from which we deduce the claim by a dyadic decomposition of space combined with Cauchy-Schwarz' inequality and [GO10a, Lemma 2.9] (a similar calculation is detailed in [Gl10, Proof of Lemma 4]). For (3.33), we first note that (3.27) implies

$$\begin{aligned} \frac{\partial \nabla_i \phi_T(z)}{\partial \omega_e} &= -(\nabla_i \phi_T(z) + \xi_i) (\nabla_{z_i} G_T(z', z) - \nabla_{z_i} G_T(z, z)) \\ &\quad + T^{-1} \phi_T(z) (\nabla_{z_i} G_T(z, z) + \nabla_{z_i} G_T(z, z')), \end{aligned}$$

which — seen as an ODE w. r. t. ω_e — yields the claim using the uniform bound $|\nabla G_T| \lesssim 1$ of [GO10a, Corollary 2.3] and (3.32).

Estimate (3.28) is a direct consequence of (3.29), whereas (3.30) follows from the Leibniz' rule combined with (3.27), (3.28), and (3.29).

Step 3. Proof of (3.17).

The estimates (3.17) of the spectral exponents follow from the more general estimates: for all $q > 0$ there exists $\gamma(q) > 0$ such that

$$(3.34) \quad \langle |\phi_T|^q \rangle \lesssim \begin{cases} \ln^{\gamma(q)} T & \text{for } d = 2, \\ 1 & \text{for } d > 2, \end{cases}$$

combined with the fact that

$$\int_0^{T^{-1}} de_\partial(\lambda) \lesssim T^{-2} \int_0^\infty \frac{1}{(T^{-1} + \lambda)^2} de_\partial(\lambda) = T^{-2} \langle \phi_T^2 \rangle.$$

The proof of (3.34) is an adaptation of [GO10a, Proof of Proposition 2.1] which already covers the case of a constant coefficient in the zero order term of L_T , that is for $T^{-1} \phi_T$ instead of $T^{-1} p_\omega \phi_T$ (no randomness in the zero order term).

The first step to apply the variance estimate of [GO10a, Lemma 2.3] is to show that ϕ_T is measurable with respect to cylindrical topology associated with the random variables. This can be proved exactly as in [GO10a, Lemma 2.6]. The estimates (3.23), (3.28), (3.29), and (3.30) of Steps 1 and 2 are like in the auxiliary lemmas of [GO10a] provided we replace the terms $|\nabla_{z_i} G_T(x, z)|$ in [GO10a, Lemmas 2.4 & 2.5] by $|\nabla_{z_i} G_T(x, z)| + T^{-1/2} g_T(x - z)$. A close look at the proof of [GO10a, Proposition 2.1] shows that these terms $|\nabla_{z_i} G_T(x, z)|$ are either estimated by the Green's function itself (in which case the additional term $T^{-1/2} g_T(x - z)$ is of higher order), or they are controlled on dyadic annuli by the Meyers' estimate [GO10a, Lemma 2.9]. This lemma shows in particular that there exists $p > 2$ such that for all $p \geq q \geq 2$, $k >$ and $R \gg 1$,

$$\int_{R \leq |x-z| < 2R} |\nabla_z G_T(x, z)|^q dz \lesssim R^d (R^{1-d})^q \min\{1, \sqrt{T} R^{-1}\}^k.$$

By the properties (3.24) for $d > 2$ and (3.25) for $d = 2$ of the function g_T , it is easy to see that for $d > 2$

$$\int_{R \leq |x-z| < 2R} (T^{-1/2} g_T(x - z))^q dz \lesssim R^d (R^{1-d})^q \min\{1, \sqrt{T} R^{-1}\}^k$$

as well, whereas for $d = 2$

$$\int_{R \leq |x-z| < 2R} (T^{-1/2} g_T(x-z))^q dz \lesssim R^2 (R^{-1})^q \ln^q T \min\{1, \sqrt{T} R^{-1}\}^k.$$

Hence the proof of [GO10a, Proposition 2.1] adapts mutadis mutandis to the present case (with possibly an additional logarithmic correction for $d = 2$), and we have (3.34).

Step 4. Susceptibility of ψ_T .

In this step we shall prove that for $e = (z, z')$, $z \in \mathbb{Z}^d$ and $z' = z + \mathbf{e}_i$,

$$(3.35) \quad \begin{aligned} \frac{\partial \psi_T}{\partial \omega_e}(x) &= -\nabla_{z_i} G_T(x, z) \nabla_i \psi_T(z) - T^{-1} G_T(x, z) \psi_T(z) \\ &\quad - T^{-1} G_T(x, z') \psi_T(z') \\ &\quad - (\nabla_i \phi_T(z) + \xi_i) \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) \nabla_{z_i} G_T(y, z) \, dy \\ &\quad - T^{-1} \phi_T(z) \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) (G_T(y, z) + G_T(y, z')) \, dy \\ &\quad + G_T(x, z) \phi_T(z) + G_T(x, z') \phi_T(z'), \end{aligned}$$

and

$$(3.36) \quad \begin{aligned} &\sup_{\omega_e} \left| \frac{\partial \psi_T}{\partial \omega_e}(0) \right| \\ &\lesssim g_T(z) (|\nabla_i \psi_T(z)| + T^{-1} |\psi_T(z)| + \nu_d(T) (1 + |\phi_T(z)| + |\phi_T(z')|)) \\ &\quad + (1 + |\phi_T(z)| + |\phi_T(z')|) \int_{\mathbb{Z}^d} g_T(y) (|\nabla_{z_i} G_T(y, z)| + T^{-1} g_T(y-z)) \, dy, \end{aligned}$$

where

$$(3.37) \quad \nu_d(T) = \begin{cases} T & \text{for } d = 2, \\ \sqrt{T} & \text{for } d = 3, \\ \ln T & \text{for } d = 4, \\ 1 & \text{for } d > 4. \end{cases}$$

Starting point is again the Green representation formula

$$\psi_T(x) = \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) \phi_T(y) \, dy,$$

associated with (3.15) in the form

$$T^{-1} p \psi_T - \nabla^* \cdot A \nabla \psi_T = p \phi_T.$$

Differentiated w. r. t. ω_e it turns into

$$\begin{aligned} \frac{\partial \psi_T(x)}{\partial \omega_e} &= \int_{\mathbb{Z}^d} \frac{\partial G_T(x, y)}{\partial \omega_e} p_\omega(y) \phi_T(y) \, dy \\ &\quad + \int_{\mathbb{Z}^d} G_T(x, y) \frac{\partial p_\omega(y)}{\partial \omega_e} \phi_T(y) \, dy + \int_{\mathbb{Z}^d} G_T(x, y) p_\omega(y) \frac{\partial \phi_T(y)}{\partial \omega_e} \, dy. \end{aligned}$$

Combined with (3.22), (3.27), and the Green representation formula itself, this shows (3.35).

We now turn to (3.36) and treat each term of the r. h. s. of (3.35) separately. We begin with the third line of (3.35), appeal to (3.33), then bound the gradient of the approximate corrector $|\nabla_i \phi_T(z)|$ by the approximate corrector $|\phi_T(z)| + |\phi_T(z')|$ itself, control the Green's function G_T by g_T , and use (3.23) to estimate the supremum in ω_e of the gradient of the Green's function $|\nabla_{z_i} G_T(y, x)|$. This term is thus controlled by the second term of the r. h. s. of (3.36). The term in the

fourth line of (3.35) is also estimated by the second term of the r. h. s. of (3.36) using (3.28) (and the uniform bounds $|\nabla G_T|, T^{-1/2}G_T \lesssim 1$), whereas the last two terms of (3.35) are bounded by the first term of the r. h. s. of (3.36) using (3.28). The subtle terms are the first three ones, for which we have to estimate the suprema of $|\nabla_i \psi_T(z)$, $|\psi_T(z)|$, and $|\psi_T(z')|$ w. r. t. ω_e .

We begin with the following two estimates

$$(3.38) \quad \sup_{\omega_e} |\psi_T(z)| \lesssim |\psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1)\nu_d(T) \\ + \sup_{\omega_e} |\nabla_i \psi_T(z)|,$$

$$(3.39) \quad \sup_{\omega_e} |\nabla_i \psi_T(z)| \lesssim |\nabla_i \psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1)\nu_d(T) \\ + T^{-1} \sup_{\omega_e} |\psi_T(z)|,$$

which — considered as a system of two coupled ODEs— show that there exists some $T_* > 0$ such that for all $T \geq T_*$,

$$(3.40) \quad \sup_{\omega_e} |\psi_T(z)| \lesssim |\psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1)\nu_d(T) \\ + |\nabla_i \psi_T(z)|,$$

$$(3.41) \quad \sup_{\omega_e} |\nabla_i \psi_T(z)| \lesssim |\nabla_i \psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1)\nu_d(T) \\ + T^{-1} |\psi_T(z)|.$$

To prove (3.38) we consider (3.35) as an ODE on $\psi_T(z)$, bound $\psi_T(z')$ by $\psi_T(z) + |\nabla_i \psi_T(z)|$, and use that the four last terms of the r. h. s. of (3.35) are bounded by the second term of the r. h. s. of (3.36), as discussed above. Hence (3.35) turns into

$$\left| \frac{\partial \psi_T}{\partial \omega_e}(z) \right| \lesssim \sup_{\omega_e} \{ |\nabla_{z_i} G_T(x, z)| |\nabla_i \psi_T(z)| \} + T^{-1} G_T(z, z) |\psi_T(z)| \\ + T^{-1} G_T(z, z') (|\psi_T(z)| + \sup_{\omega_e} |\nabla_i \psi_T(z)|) \\ + (1 + |\phi_T(z)| + |\phi_T(z')|) \\ \times \int_{\mathbb{Z}^d} g_T(y) (|\nabla_{z_i} G_T(y, z)| + T^{-1} g_T(y - z)) dy.$$

Using the uniform bounds $|\nabla G_T|, T^{-1}G_T \lesssim 1$, and replacing the gradient of the Green's function by the Green's function itself in the integral — which we then control by $\nu_d(T)$ —, this estimate yields (3.38) by integrating the ODE. We now turn to (3.39) and infer from (3.35) that

$$\frac{\partial \nabla_i \psi_T(z)}{\partial \omega_e} = -\nabla_{z_i} \nabla_{z_i} G_T(z, z) \nabla_i \psi_T(z) - T^{-1} \nabla_{z_i} G_T(z, z) \psi_T(z) \\ - T^{-1} \nabla_{z_i} G_T(z, z') \psi_T(z') \\ - (\nabla_i \phi_T(z) + \xi_i) \int_{\mathbb{Z}^d} \nabla_{z_i} G_T(z, y) p_\omega(y) \nabla_{z_i} G_T(y, z) dy \\ - T^{-1} \phi_T(z) \int_{\mathbb{Z}^d} \nabla_{z_i} G_T(z, y) p_\omega(y) (G_T(y, z) + G_T(y, z')) dy \\ + \nabla_{z_i} G_T(z, z) \phi_T(z) + \nabla_{z_i} G_T(z, z') \phi_T(z').$$

Proceeding as from (3.35) to (3.38), this implies (3.39), and therefore (3.40) and (3.41). Combining the inequality $|\psi_T(z')| \leq |\psi_T(z)| + |\nabla_i \psi_T(z)|$ with (3.40) and

(3.41) yields the last estimate we need:

$$(3.42) \quad \sup_{\omega_e} |\psi_T(z')| \lesssim |\psi_T(z)| + (|\phi_T(z)| + |\phi_T(z')| + 1)\nu_d(T) + |\nabla_i \psi_T(z)|.$$

We are finally in position to conclude the proof of (3.36): the first three terms of the r. h. s. of (3.35) are estimated by (3.41), (3.40), and (3.42). Replacing then the gradient of the Green's function by the Green's function itself, and the Green's function by g_T , we obtain (3.36), as desired.

Step 5. Proof of (3.19).

This is an application of the variance estimate [GO10a, Lemma 2.3] on ψ_T based on (3.36). In particular,

$$(3.43) \quad \text{var} [\psi_T] \lesssim \sum_{e \in \mathbb{B}} \left\langle \sup_{\omega_e} \left| \frac{\partial \psi_T(0)}{\partial \omega_e} \right|^2 \right\rangle.$$

We distinguish the contributions of the two terms of the r. h. s. of (3.36) in this sum and use the notation

$$\begin{aligned} A_e &:= g_T(z) (|\nabla_i \psi_T(z)| + T^{-1} |\psi_T(z)| + \nu_d(T) (1 + |\phi_T(z)| + |\phi_T(z')|)), \\ B_e &:= (1 + |\phi_T(z)| + |\phi_T(z')|) \int_{\mathbb{Z}^d} g_T(y) (|\nabla_{z_i} G_T(y, z)| + T^{-1} g_T(y - z)) \, dy. \end{aligned}$$

The contribution of the first term is estimated as follows:

$$\begin{aligned} & \sum_{e \in \mathbb{B}} \langle A_e^2 \rangle \\ & \lesssim \sum_{e \in \mathbb{B}} \langle g_T(z)^2 (|\nabla_i \psi_T(z)|^2 + T^{-2} |\psi_T(z)|^2 + \nu_d(T)^2 (1 + |\phi_T(z)|^2 + |\phi_T(z')|^2)) \rangle \\ & \lesssim \left(\sum_{z \in \mathbb{Z}^d} g_T(z)^2 \right) (\langle |\nabla \psi_T|^2 \rangle + T^{-2} \langle \psi_T^2 \rangle + \nu_d(T)^2 (1 + \langle \phi_T^2 \rangle)) \\ & \lesssim \nu_d(T) (\langle |\nabla \psi_T|^2 \rangle + T^{-2} \langle \psi_T^2 \rangle + \nu_d(T)^2 (1 + \langle \phi_T^2 \rangle)), \end{aligned}$$

by stationarity of ϕ_T , ψ_T , and $\nabla \psi_T$. Combined with (3.34) for $q = 2$, and (3.18) (which is proved below), this turns into

$$(3.44) \quad \sum_{e \in \mathbb{B}} \langle A_e^2 \rangle \lesssim \begin{cases} T^3 \ln^q T & \text{for } d = 2, \\ T^{3/2} & \text{for } d = 3, \\ T \ln T & \text{for } d = 4, \\ T & \text{for } d > 4. \end{cases}$$

Estimate (3.18) is a consequence of the estimate (3.17) of the spectral exponents and of the spectral representations

$$\begin{aligned} \langle \psi_T^2 \rangle &= \int_0^\infty \frac{1}{(T^{-1} + \lambda)^4} \, de_\partial(\lambda) \\ &\leq T^2 \int_0^1 \frac{1}{\lambda^2} \, de_\partial(\lambda) + \int_0^\infty \frac{1}{\lambda} \, de_\partial(\lambda) \\ \alpha \langle |\nabla \psi_T|^2 \rangle &\leq \langle \nabla \psi_T \cdot A \nabla \psi_T \rangle \\ &= \int_0^\infty \frac{\lambda}{(T^{-1} + \lambda)^4} \, de_\partial(\lambda) \\ &\leq T \int_0^1 \frac{1}{\lambda^2} \, de_\partial(\lambda) + \int_0^\infty \frac{1}{\lambda} \, de_\partial(\lambda), \end{aligned}$$

since $\int_0^\infty \frac{1}{\lambda} \, de_\partial(\lambda) \lesssim 1$, and by definition (2.6) of the spectral exponents.

We now turn to the second term of the r. h. s. of (3.36), and note that it coincides with the term treated in [GM10, Step 3, Proof of Lemma 5] so that we have

$$(3.45) \quad \sum_{e \in \mathbb{B}} \langle B_e^2 \rangle \lesssim \begin{cases} T^2 \ln^q T & \text{for } d = 2, \\ T^{3/2} & \text{for } d = 3, \\ T & \text{for } d = 4, \\ \sqrt{T} & \text{for } d = 5, \\ \ln T & \text{for } d = 6, \\ 1 & \text{for } d > 6. \end{cases}$$

Hence, (3.43), (3.36), (3.44) and (3.45) yield (3.19) for $d > 2$ using (3.13) and (3.16). Note that (3.44) is not optimal for $d = 2$ in view of the estimate of $\langle \psi_T^2 \rangle$ in (3.18) (which is sharper). This comes from our estimate of $\sup_{\omega_e} \psi_T(z)$ where we have replaced a gradient of the Green's function by the Green's function itself and therefore given up some potential better decay (at least integrated on dyadic annuli). In high dimensions however, there is no loss because g_T is then square-integrable itself. Anyway, the optimal spectral exponents for $d = 2$ have been already obtained in (3.17).

Step 6. Proof of (3.5).

We quickly show how (3.19) allows to get the sharper spectral exponents (3.5). This is the same argument as in Step 5, except that (3.18) can now be bootstrapped to (3.20) using (3.19) and

$$\begin{aligned} \langle \psi_T^2 \rangle &= \int_0^\infty \frac{1}{(T^{-1} + \lambda)^4} \mathrm{d}e_{\mathfrak{d}}(\lambda) \\ &\leq T \int_0^{T_*^{-1}} \frac{1}{\lambda^3} \mathrm{d}e_{\mathfrak{d}}(\lambda) + T_*^3 \int_0^\infty \frac{1}{\lambda} \mathrm{d}e_{\mathfrak{d}}(\lambda) \\ \langle \nabla \psi_T \cdot A \nabla \psi_T \rangle &= \int_0^\infty \frac{\lambda}{(T^{-1} + \lambda)^4} \mathrm{d}e_{\mathfrak{d}}(\lambda) \\ &\leq \int_0^{T_*^{-1}} \frac{1}{\lambda^3} \mathrm{d}e_{\mathfrak{d}}(\lambda) + T_*^2 \int_0^\infty \frac{1}{\lambda} \mathrm{d}e_{\mathfrak{d}}(\lambda), \end{aligned}$$

where $T_* < \infty$ has been defined in Step 4. Proceeding as in the end of Step 5 this yields the spectral exponents (3.5), and concludes the proof of the proposition. \square

4. THE RANDOM FLUCTUATIONS

In this section, we show that the computable quantity $\hat{A}_n(t)$ defined in (1.6) is a good approximation of σ_t^2 , in the sense that its random fluctuations are small as soon as n/t^2 is large. We write \mathbb{N}^* for $\mathbb{N} \setminus \{0\}$.

Theorem 4.1. *There exists $c > 0$ such that, for any $n \in \mathbb{N}^*$, $\varepsilon > 0$ and t large enough,*

$$\mathbb{P}_0^\otimes \left[|\hat{A}_n(t) - \sigma_t^2| \geq \varepsilon/t \right] \leq \exp \left(-\frac{n\varepsilon^2}{ct^2} \right).$$

In order to prove this result, we rewrite $\hat{A}_n(t)$ as $A_n(t)/\hat{p}_n$, where

$$A_n(t) = \frac{p(\omega^{(1)})(\xi \cdot Y^{(1)}(t))^2 + \dots + p(\omega^{(n)})(\xi \cdot Y^{(n)}(t))^2}{nt\mathbb{E}[p]},$$

and

$$\hat{p}_n = \frac{p(\omega^{(1)}) + \dots + p(\omega^{(n)})}{n\mathbb{E}[p]}.$$

Both are sums of independent and identically distributed random variables under \mathbb{P}_0^\otimes , thus enabling us to use standard tools from large deviations theory. We start with the following classical but instructing fact.

Proposition 4.2. *There exists $c_0 > 0$ such that, for any $n \in \mathbb{N}$ and any small enough $\varepsilon > 0$:*

$$\mathbb{P}^\otimes \left[|\hat{p}_n - 1| \geq \varepsilon \right] \leq \exp(-n\varepsilon^2/c_0).$$

Proof. Let $\bar{p}(\omega) = p(\omega) - \mathbb{E}[p]$. It suffices to show that, for some $C > 0$,

$$(4.1) \quad \mathbb{P}^\otimes \left[|\bar{p}(\omega^{(1)}) + \dots + \bar{p}(\omega^{(n)})| \geq n\varepsilon \right] \leq \exp(-n\varepsilon^2/C).$$

Chebyshev's inequality implies that, for any $\lambda \geq 0$,

$$\begin{aligned} \mathbb{P}^\otimes \left[\bar{p}(\omega^{(1)}) + \dots + \bar{p}(\omega^{(n)}) \geq n\varepsilon \right] &\leq e^{-n\varepsilon\lambda} \mathbb{E}^\otimes \left[\exp(\lambda(\bar{p}(\omega^{(1)}) + \dots + \bar{p}(\omega^{(n)})) \right] \\ &\leq e^{-n\varepsilon\lambda} \mathbb{E} \left[\exp(\lambda\bar{p}(\omega))^n \right]. \end{aligned}$$

Using a series expansion of the exponential, one can check that there exists $C > 0$ such that, for any λ small enough,

$$(4.2) \quad \ln \mathbb{E} \left[\exp(\lambda\bar{p}(\omega)) \right] \leq C\lambda^2.$$

As a consequence, for any λ small enough,

$$\mathbb{P}^\otimes \left[\bar{p}(\omega^{(1)}) + \dots + \bar{p}(\omega^{(n)}) \geq \varepsilon \right] \leq \exp(-n(\lambda\varepsilon - C\lambda^2)),$$

and for $\lambda = \varepsilon/(2C)$, the latter term becomes $\exp(-n\varepsilon^2/(4C))$. The event $\bar{p}(\omega^{(1)}) + \dots + \bar{p}(\omega^{(n)}) \leq -n\varepsilon$ can be handled the same way, and we thus obtain (4.1). \square

What makes the proof of Proposition 4.2 work is the observation (4.2) that the log-Laplace transform of $\bar{p} = p - \mathbb{E}[p]$ is quadratic close to the origin. In order to prove the corresponding result for $A_n(t)$, we will need to control the log-Laplace transform of $\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2$ uniformly in t . To that end, we use a sharp upper bound on the transition probabilities of the random walk recalled in the following theorem. We refer the reader to [HS93] or [Wo, Theorem 14.12] for a proof.

Theorem 4.3. *There exists a constant $c_1 > 0$ such that, for any environment ω with conductances in $[\alpha, \beta]$, any $t \in \mathbb{N}^*$ and $x \in \mathbb{Z}^d$,*

$$\mathbf{P}_0^\omega[Y(t) = x] \leq \frac{c_1}{t^{d/2}} \exp\left(-\frac{|x|^2}{c_1 t}\right).$$

From Theorem 4.3 we deduce the following result.

Corollary 4.4. *Let c_1 be given by Theorem 4.3. For all $\lambda < 1/c_1$, one has*

$$\sup_{t \in \mathbb{N}^*} \tilde{\mathbb{E}}_0 \left[\exp\left(\lambda \frac{|Y(t)|^2}{t}\right) \right] < +\infty.$$

Proof. Let $\delta = 1/c_1 - \lambda$. By Theorem 4.3,

$$\mathbf{E}_0^\omega \left[e^{\lambda|Y(t)|^2/t} \right] \leq c_1 t^{-d/2} \sum_{x \in \mathbb{Z}^d} e^{-\delta|x|^2/t}.$$

If the sum ranges over all $x \in (\mathbb{N}^*)^d$, it is easy to bound it by a convergent integral:

$$t^{-d/2} \sum_{x \in (\mathbb{N}^*)^d} e^{-\delta|x|^2/t} \leq t^{-d/2} \int_{\mathbb{R}_+^d} e^{-\delta|x|^2/t} dx = \int_{\mathbb{R}_+^d} e^{-\delta|x|^2} dx.$$

By symmetry, the estimate carries over to the sum over all $x \in (\mathbb{Z}^*)^d$. The same argument applies for the sum over all $x = (x_1, \dots, x_d)$ having exactly one component equal to 0, and so on. \square

The following lemma contains the required uniform control on the log-Laplace transform of $\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2$.

Lemma 4.5. *There exist $\lambda_1 > 0$ and c_2 such that, for any $\lambda < \lambda_1$ and any $t \in \mathbb{N}^*$,*

$$\ln \tilde{\mathbb{E}}_0 \left[\exp \left(\lambda \left(\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right) \right) \right] \leq c_2 \lambda^2.$$

Proof. It is sufficient to prove that there exists c_3 such that, for any λ small enough and any t ,

$$\tilde{\mathbb{E}}_0 \left[\exp \left(\lambda \left(\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right) \right) \right] \leq 1 + c_3 \lambda^2.$$

We use the series expansion of the exponential to rewrite this expectation as

$$\sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \tilde{\mathbb{E}}_0 \left[\left(\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right)^k \right].$$

The term corresponding to $k = 0$ is equal to 1, whereas the term for $k = 1$ vanishes. The remaining sum, for k ranging from 2 to infinity, can be controlled using Corollary 4.4 combined with the bound

$$\tilde{\mathbb{E}}_0 \left| \frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right|^k \leq 2 \tilde{\mathbb{E}}_0 \left[\frac{(\xi \cdot Y(t))^{2k}}{t^k} \right],$$

which follows from the definition of σ_t^2 and Jensen's inequality. \square

We are now in position to prove Theorem 4.1.

Proof of Theorem 4.1. Starting point is the inequality

$$(4.3) \quad \mathbb{P}_0^\otimes \left[\hat{A}_n(t) - \sigma_t^2 \geq 2\varepsilon/t \right] \leq \mathbb{P}_0^\otimes \left[A_n(t) - \sigma_t^2 \geq \varepsilon/t \right] + \mathbb{P}_0^\otimes \left[A_n(t)(\hat{p}_n^{-1} - 1) \geq \varepsilon/t \right].$$

We treat both terms of the r. h. s. separately. For the first one, the key observation is that (recalling the definition of $\tilde{\mathbb{P}}$ given in (1.1))

$$\mathbb{P}_0^\otimes \left[A_n(t) - \sigma_t^2 \geq \varepsilon/t \right] = \tilde{\mathbb{P}}_0^\otimes \left[\frac{(\xi \cdot Y^{(1)}(t))^2 + \dots + (\xi \cdot Y^{(n)}(t))^2}{nt} - \sigma_t^2 \geq \varepsilon/t \right].$$

Let $\lambda > 0$. As in the proof of Proposition 4.2, we bound this term using Chebyshev's inequality:

$$(4.4) \quad \begin{aligned} \mathbb{P}_0^\otimes \left[A_n(t) - \sigma_t^2 \geq \varepsilon/t \right] &\leq \tilde{\mathbb{E}}_0^\otimes \left[\exp \left(\lambda \left(\frac{(\xi \cdot Y^{(1)}(t))^2 + \dots + (\xi \cdot Y^{(n)}(t))^2}{t} - n\sigma_t^2 \right) \right) \right] \exp \left(-\frac{n\lambda\varepsilon}{t} \right) \\ &\leq \tilde{\mathbb{E}}_0 \left[\exp \left(\lambda \left(\frac{(\xi \cdot Y(t))^2}{t} - \sigma_t^2 \right) \right) \right]^n \exp \left(-\frac{n\lambda\varepsilon}{t} \right). \end{aligned}$$

By Lemma 4.5, the r. h. s. of (4.4) is bounded by

$$\exp \left(n \left(c_2 \lambda^2 - \frac{\lambda\varepsilon}{t} \right) \right),$$

for all λ small enough. Choosing $\lambda = \varepsilon/2c_2t$ (which is small enough for t large enough), we obtain

$$(4.5) \quad \mathbb{P}_0^\otimes \left[A_n(t) - \sigma_t^2 \geq \varepsilon/t \right] \leq \exp \left(-\frac{n\varepsilon^2}{4c_2t^2} \right),$$

as needed.

t	10	20	40	80	160	320	640
$K(t)$	10^5	3000	3000	1000	500	100	20
Systematic error	1.27E-01	7.43E-02	4.17E-02	2.46E-02	1.26E-02	6.96E-03	3.72E-03

TABLE 1. Systematic error in function of the final time t for $K(t)t^2$ realizations.

We now turn to the second term of the r. h. s. of (4.3). From inequality (4.5), we infer that there exists $M > 0$ such that

$$\mathbb{P}_0^\otimes [A_n(t) \geq M] \leq \exp\left(-\frac{n\varepsilon^2}{4c_2t^2}\right).$$

Since \hat{p}_n is almost surely bounded by a constant, it is enough to evaluate the probability

$$\mathbb{P}_0^\otimes [\hat{p}_n^{-1} - 1 \geq \varepsilon/(Mt)],$$

which is controlled by Proposition 4.2.

We have thus obtained the required control of the l. h. s. of (4.3). The probability of the symmetric event

$$\mathbb{P}_0^\otimes \left[\sigma_i^2 - \hat{A}_n(t) \geq 2\varepsilon/t \right]$$

can be handled the same way. \square

5. NUMERICAL VALIDATION AND COMMENTS

In this section, we illustrate on a simple two-dimensional example the sharpness of the estimates of the systematic error and of the random fluctuations obtained in Theorems 3.1 and 4.1.

In the numerical tests, each conductivity of \mathbb{B} takes the value $\alpha = 1$ or $\beta = 4$ with probability $1/2$. In this simple case, the homogenized matrix is given by Dykhne's formula, namely $A_{\text{hom}} = \sqrt{\alpha\beta}\text{Id} = 2\text{Id}$ (see for instance [Gl10, Appendix A]). For the simulation of the random walk, we generate — and store — the environment along the trajectory of the walk. In particular, this requires to store up to a constant times t data. In terms of computational cost, the expensive part of the computations is the generation of the randomness. In particular, to compute one realization of $\hat{A}_{t^2}(t)$ costs approximately the generation of $t^2 \times 4t = 4t^3$ random variables. A natural advantage of the method is its full scalability: the t^2 random walks used to calculate a realization of $\hat{A}_{t^2}(t)$ are completely independent.

We first test the estimate of the systematic error: up to a logarithmic correction, the convergence is proved to be linear in time. In view of Theorem 4.1, typical fluctuations of $t(\hat{A}_{n(t)}(t) - \sigma_i^2)$ are of order no greater than $t/\sqrt{n(t)}$, and thus become negligible when compared with the systematic error as soon as the number $n(t)$ of realizations satisfies $n(t) \gg t^2$. We display in Table 1 an estimate of the systematic error obtained with $n(t) = K(t)t^2$ realizations. The systematic error is plotted on Figure 1 in function of the time in logarithmic scale. The apparent convergence rate (linear fitting) is -0.85 , which is consistent with Theorem 3.1, which predicts -1 and a logarithmic correction.

We now turn to the random fluctuations of $\hat{A}_{n(t)}(t)$. Theorem 4.1 gives us a large deviation estimate which essentially says that the fluctuations of $t(\hat{A}_{n(t)}(t) - \sigma_i^2)$ have a Gaussian tail, measured in units of $t/\sqrt{n(t)}$. The Figures 2-5 display the histograms of $t(\hat{A}_{t^2}(t) - \sigma_i^2)$ for $t = 10, 20, 40$ and 80 (with 10000 realizations of $\hat{A}_{t^2}(t)$ in each case). As expected, they look Gaussian.

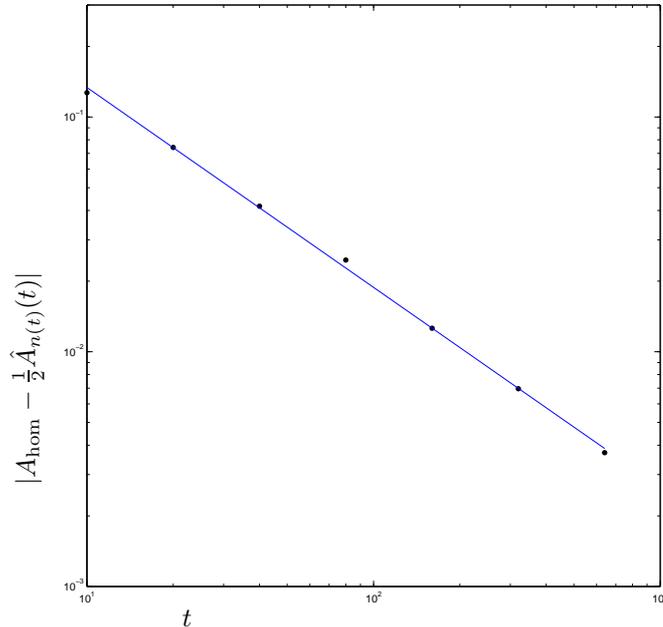
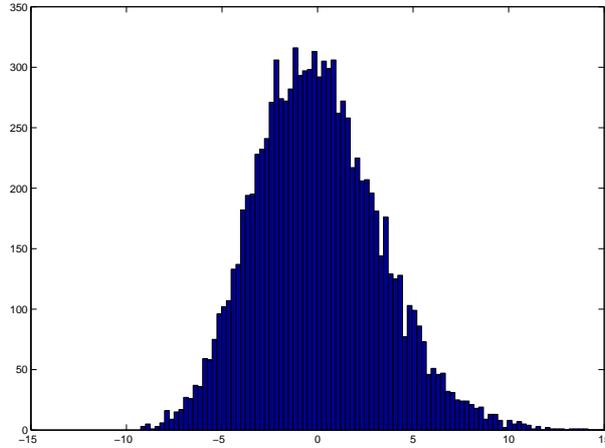
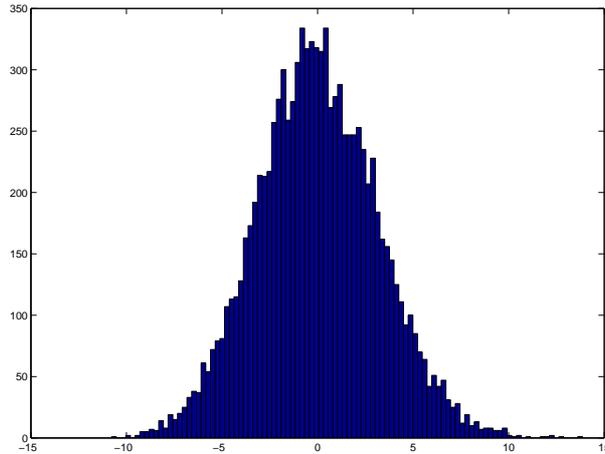


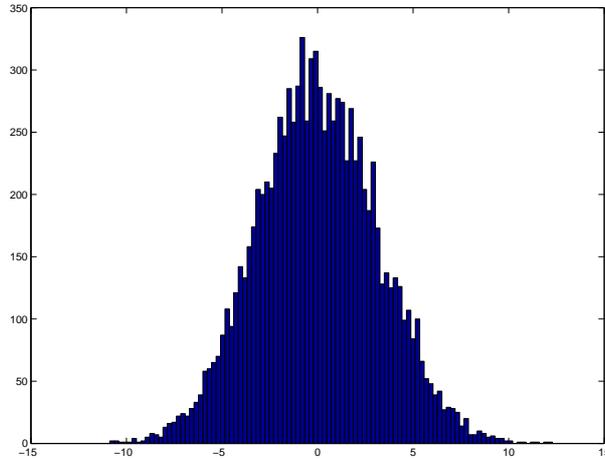
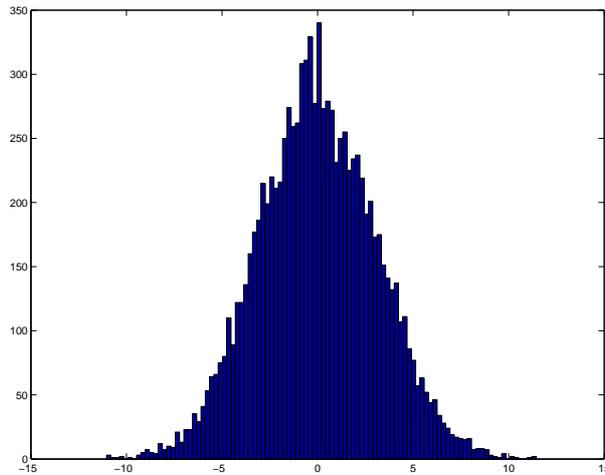
FIGURE 1. Systematic error in function of the final time t for $n(t) = K(t)t^2$ realizations

To conclude this article, let us quickly compare the Monte-Carlo approach under consideration here to other approaches to approximate homogenized coefficients. Another possibility to approximate effective coefficients is to directly solve the so-called corrector equation. In this approach, a first step towards the derivation of error estimates is a quantification of the qualitative results proved by Künnemann [Kü83] (and inspired by Papanicolaou and Varadhan's treatment of the continuous case [PV79]) and Kozlov [Ko87]. In the stochastic case, such an equation is posed on the whole \mathbb{Z}^d , and we need to localize it on a bounded domain, say the hypercube Q_R of side $R > 0$. As shown in a series of papers by Otto and the first author [GO10a, GO10b], and the first author [Gl10], there are three contributions to the \mathbb{L}^2 -error in probability between the true homogenized coefficients and its approximation. The dominant error in small dimensions takes the form of a variance: it measures the fact that the approximation of the homogenized coefficients by the average of the energy density of the corrector on a box Q_R fluctuates. This error decays at the rate of the central limit theorem R^{-d} in any dimension (with a logarithmic correction for $d = 2$). The second error is a systematic error: it is due to the fact that we have modified the corrector equation by adding a zero-order term of strength $T^{-1} > 0$ (as is standard in the analysis of the well-posedness of the corrector equation). The scaling of this error depends on the dimension and saturates at dimension 4. It is of higher order than the random error up to dimension 8. The last error is due to the use of boundary conditions on the bounded domain Q_R . Provided there is a buffer region, this error is exponentially small in the distance to the buffer zone measured in units of \sqrt{T} .

This approach has two main drawbacks. First the numerical method only converges at the central limit theorem scaling in terms of R up to dimension 8, which

FIGURE 2. Histogram of the rescaled fluctuations for $t = 10$ FIGURE 3. Histogram of the rescaled fluctuations for $t = 20$

is somehow disappointing from a conceptual point of view (although this is already fine in practice). Second, although the size of the buffer zone is roughly independent of the dimension, its cost with respect to the central limit theorem scaling dramatically increases with the dimension (recall that in dimension d , the CLT scaling is R^{-d} , so that in high dimension, we may consider smaller R for a given precision, whereas the use of boundary conditions requires $R \gg \sqrt{T}$ in any dimension). Based on ideas of the second author in [Mo11], we have taken advantage of the spectral representation of the homogenized coefficients (originally introduced by Papanicolaou and Varadhan to prove their qualitative homogenization result) in order to devise and analyze new approximation formulas for the homogenized coefficients in [GM10]. In particular, this has allowed us to get rid of the restriction

FIGURE 4. Histogram of the rescaled fluctuations for $t = 40$ FIGURE 5. Histogram of the rescaled fluctuations for $t = 80$

on dimension, and exhibit refinements of the numerical method of [Gl10] which converge at the central limit theorem scaling in any dimension (thus avoiding the first mentioned drawback). Unfortunately, the second drawback is inherent to the type of method used: if the corrector equation has to be solved on a bounded domain Q_R , boundary conditions need to be imposed on the boundary ∂Q_R . Since their values are actually also part of the problem, a buffer zone seems mandatory — with the notable exception of the periodization method, whose analysis is yet still unclear to us, especially when spatial correlations are introduced in the coefficients.

In this paper we have analyzed a method which does not suffer from the drawbacks mentioned above: the random walk in random environment approach. In

particular, following [Pap83] we have obtained an approximation of the homogenized coefficients by the numerical simulation of a random walk up to some large time. Compared to the deterministic approach based on the approximate corrector equation, the advantage of the present approach is that its convergence rate and computational costs are dimension-independent. In addition, the environment only needs to be generated along the trajectory of the random walker, so that much less information has to be stored during the calculation. This may be quite an important feature of the Monte Carlo method in view of the discussion of [Gl10, Section 4.3].

A more thorough comparison of these numerical approaches in two and three dimensions, for correlated and uncorrelated examples, will be the object of a forthcoming work [EGMN].

ACKNOWLEDGEMENTS

The authors acknowledge the support of INRIA through the “Action de recherche collaborative” DISCO. This work was also supported by Ministry of Higher Education and Research, Nord-Pas de Calais Regional Council and FEDER through the “Contrat de Projets Etat Region (CPER) 2007-2013”.

REFERENCES

- [AKS82] V.V. Anshelevich, K.M. Khanin, and Ya.G. Sinai. Symmetric random walks in random environments. *Comm. Math. Phys.*, 85(3), 449–470, 1982.
- [BD10] M.T. Barlow, and J.-D. Deuschel. Invariance principle for the random conductance model with unbounded conductances. *Ann. Probab.*, 38(1), 234–276, 2010.
- [BB07] N. Berger, and M. Biskup. Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields*, 137(1-2), 83–120, 2007.
- [BP07] M. Biskup, and T.M. Prescott. Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.*, 12(49), 1323–1348, 2007.
- [DFGW89] A. De Masi, P.A. Ferrari, S. Goldstein, and W.D. Wick. An invariance principle for reversible Markov processes. Applications to random motions in random environments. *J. Statist. Phys.*, 55(3-4), 787–855, 1989.
- [EGMN] A.-C. Eglouff, A. Gloria, J.-C. Mourrat, T. N. Nguyen. In preparation.
- [Gl10] A. Gloria. Numerical approximation of effective coefficients in stochastic homogenization of discrete elliptic equations. *M2AN*, 46, 1–38(2012).
- [GO10a] A. Gloria and F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.*, 39(3), 779–856, 2011.
- [GO10b] A. Gloria and F. Otto. An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab.*, to appear.
- [GO11] A. Gloria and F. Otto. Quantitative estimates in stochastic homogenization of linear elliptic equations. In preparation.
- [GM10] A. Gloria and J.-C. Mourrat. Spectral measure and approximation of homogenized coefficients. *Probab. Theory Related Fields*, to appear.
- [HS93] W. Hebisch and L. Saloff-Coste. Gaussian estimates for Markov chains and random walks on groups. *Ann. Probab.*, 21(2), 673–709, 1993.
- [KV86] C. Kipnis and S.R.S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Commun. Math. Phys.*, 104, 1–19, 1986.
- [Ko85] S.M. Kozlov. The averaging method and walks in inhomogeneous environments. *Uspekhi Mat. Nauk*; 40(2), 61–120, 1985. English transl.: *Russian Math. Surveys*, 40(2), 73–145, 1985.
- [Ko87] S.M. Kozlov. Averaging of difference schemes. *Math. USSR Sbornik*, 57(2):351–369, 1987.
- [Kü83] R. Künnemann. The diffusion limit for reversible jump processes on \mathbb{Z}^d with ergodic random bond conductivities. *Commun. Math. Phys.*, 90:27–68, 1983.
- [Ma08] P. Mathieu. Quenched invariance principles for random walks with random conductances. *J. Statist. Phys.*, 130(5), 1025–1046, 2008.
- [MP07] P. Mathieu and A. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. A*, 463(2085), 2287–2307, 2007.
- [Mo11] J.-C. Mourrat. Variance decay for functionals of the environment viewed by the particle. *Ann. Inst. H. Poincaré Probab. Statist.*, 47(11), 294–327, 2011.

- [PV79] G.C. Papanicolaou and S.R.S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam, 1981.
- [Pap83] G. Papanicolaou. Diffusions and random walks in random media. In *The mathematics and physics of disordered media (Minneapolis, Minn., 1983)*, volume 1035 of *Lecture Notes in Math.*, pages 391–399, Springer, Berlin, 1983.
- [SS04] V. Sidoravicius and A.-S. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Related Fields*, 129(2), 219–244, 2004.
- [Wo] W. Woess. *Random walks on infinite graphs and groups*. Cambridge tracts in Mathematics **138**, Cambridge university press (2000).

(Antoine Gloria) PROJET SIMPAF, INRIA LILLE-NORD EUROPE, FRANCE
E-mail address: `antoine.gloria@inria.fr`

(Jean-Christophe Mourrat) EPFL, INSTITUT DE MATHÉMATIQUES, STATION 8, 1015 LAUSANNE, SWITZERLAND
E-mail address: `jean-christophe.mourrat@epfl.ch`