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The Emptiness Problem for Tree Automata with Global Constraints

Luis Barguñó, Carles Creus, Guillem Godoy
Universitat Politècnica de Catalunya
Barcelona, Spain

Florent Jacquemard
INRIA Saclay
LSV-CNRS/ENS Cachan

Camille Vacher
France Telecom R&D
LSV-CNRS/ENS Cachan

Abstract—We define tree automata with global constraints (TAGC), generalizing the class of tree automata with global equality and disequality constraints [1] (TAGED). TAGC can test for equality and disequality between subtrees whose positions are defined by the states reached during a computation. In particular, TAGC can check that all the subtrees reaching a given state are distinct. This constraint is related to monadic key constraints for XML documents, meaning that every two distinct positions of a given type have different values.

We prove decidability of the emptiness problem for TAGC. This solves, in particular, the open question of decidability of emptiness for TAGED. We further extend our result by allowing global arithmetic constraints for counting the number of occurrences of some state or the number of different subtrees reaching some state during a computation. We also allow local equality and disequality tests between sibling positions and the extension to unranked ordered trees. As a consequence of our results for TAGC, we prove the decidability of a fragment of the monadic second order logic on trees extended with predicates for equality and disequality between subtrees, and cardinality.

I. INTRODUCTION

Tree automata techniques are widely used in several domains like automated deduction (see e.g. [2]), static analysis of programs [3] or protocols [4], [5], and XML processing [6]. A severe limitation of standard tree automata (TA) is however that they are not able to test for equality (isomorphism) or disequality between subtrees in an input tree. For instance, the language of trees described by a non-linear pattern of the form \( f(x, x) \) is not regular (i.e. there exists no TA recognizing this language). Similar problems are also frequent in the context of XML documents processing. XML documents are commonly represented as labeled trees, and they can be constrained by XML schemas, which define both typing restrictions and integrity constraints. All the typing formalisms currently used for XML, are based on finite tree automata. The key constraints for databases are common integrity constraints expressing that every two distinct positions of a given type have different values.

This is typically the kind of constraints that can not be characterized by TA.

One first approach to overcome this limitation of TA consists in adding the possibility to make equality or disequality tests at each step of the computation of the automaton. The tests are performed locally, between subtrees at a bounded distance from the current computation position in the input tree. The emptiness problem, whether the language recognized by a given automaton is empty, is undecidable with such tests [7]. A decidable subclass is obtained by restricting the tests to sibling subtrees [8] (see [2] for a survey).

Another approach was proposed more recently in [9], [1] with the definition of tree automata with global equality and disequality tests (TAGED). The TAGED do not perform the tests during the computation steps but globally on the tree, at the end of the computation, at positions which are defined by the states reached during the computation. For instance, they can express that all the subtrees that reached a given state \( q \) are equal, or that every two subtrees that reached respectively the states \( q \) and \( q' \) are different. The emptiness has been shown decidable for several subclasses of TAGED [9], [1], but the decidability of emptiness for the whole class remained a challenging open question.

In this paper, we answer this question positively, even for a class of tree recognizers more general than TAGED. We define (in Section II) a class of tree automata with global constraints (TAGC) which, roughly, corresponds to TAGED extended with the possibility to express disequalities between subtrees that reached the same state (specifying key constraints, which are not expressible with TAGEDs), and with arbitrary Boolean combinations (including negation) of constraints. We show in Section III that emptiness is decidable for TAGC. The decision algorithm uses an involved pumping argument: every sufficiently large tree recognized by the given TAGC can be reduced by an operation of parallel pumping into a smaller tree which is still recognized. The existence of the bound is based on a particular well quasi-ordering.

In Section IV-A, we study the extension of TAGC with global counting constraints on the number \( |q| \) of occurrences of a given state \( q \) in a computation, or the number \( \|q\| \) of distinct subtrees reaching a given state \( q \) in a computation. We show that emptiness is decidable for this extension when
counting constraints are only allowed to compare states to
counts, like in $|q| \leq 5$ or $||q|| + 2||q'|| \geq 9$ (actually
in this case, the counting constraints do not improve
the expressiveness of TAGC). With counting constraints being
able to compare state cardinalities (like in $|q| = |q'|$),
emptiness becomes undecidable. We show that the emptiness
decision algorithm can also be applied to the combination
of TAGC with local tests between sibling subtrees a la [8]
(Section IV-B), and to unranked ordered labeled trees (Section
IV-C). This demonstrates the robustness of the method.

As an application of our results, in Section V we present a
(strict) extension of the monadic second order logic on trees
whose existential fragment corresponds exactly to TAGC. In
particular, we conclude its decidability. The full version of
this paper including all proofs can be found in [10].

**Related Work.** The languages of TAGC and tree au-
tomata with local equality and disequality constraints are incomparable (see e.g. [11]). We show in Section IV-B that
the local tests between sibling subtrees of [8] can be added
to TAGC while preserving the decidability emptiness. The
tree automata of [8] have been generalized from ranked
trees to unranked ordered trees [12], [13]. The decidable
generalization of TAGC to unranked ordered trees proposed
in Section IV-C and the automata of [12], [13] are incompara-
ble. A combination of both formalisms could be the object
of a further study.

Another way to handle subtree equalities is to use au-
tomata computing on DAG representation of trees [14], [15].
This model is incomparable to TAGC whose constraints are
conjunctions of equalities [11]. The decidable extension of
TA with one tree shaped memory [16] can simulate TAGC
with equality constraints only, providing that at most one
state per run can be used to test equalities [9].

As explained in Section II-B, the TAGC strictly generalize
the TAGEDs of [9], [1]. The latter have been introduced as a
tool to decide a fragment of the spatial logic TQL [9].
Decidable subclasses of TAGEDs were also shown in cor-
respondence with fragments of monadic second order logic
on the tree extended with predicates for subtree (dis)equality
tests. In Section V, we generalize this correspondence to
TAGC and a more natural extension of MSO.

There have been several approaches to extend TA with
arithmetic constraints on cardinalities $|q|$ described above:
the constraints can be added to transitions in order to count
between siblings [17], [18] (in this case we could call
them local by analogy with equality tests) or they can be
global [19]. We compare in Section IV-A the latter approach
(closer to our settings) with our extension of TAGC, wrt
emptiness decision. To our knowledge, this is the first time
that arithmetic constraints on cardinalities of the form $||q||$
are studied.

II. PRELIMINARIES

A. Terms, Positions, Tree Automata

We use the standard notations for terms and positions, see [20]. A signature $\Sigma$ is a finite set of function sym-

dols with arity. We sometimes denote $\Sigma$ explicitly as
$\{ f_1 : a_1, \ldots, f_n : a_n \}$ where $f_1, \ldots, f_n$ are the function

symbols, and $a_1, \ldots, a_n$ are the corresponding arities, or
as $\{ f_1, \ldots, f_n \}$ when the arities are omitted. We denote the
subset of function symbols of $\Sigma$ of arity $m$ as $\Sigma_m$. The set
of (ranked) terms over the signature $\Sigma$ is defined recursively
as $T(\Sigma) := \{ f : 0 \in \Sigma \} \cup \{ f(t_1, \ldots, t_m) : f : m \in
\Sigma, t_1, \ldots, t_m \in T(\Sigma) \}$.

Positions in terms are denoted by sequences of natural
numbers. With $\Lambda$ we denote the empty sequence (root
position), and $p,p'$ denotes the concatenation of positions
$p$ and $p'$. The set of positions of a term $t$ is defined
recursively as $Pos(f(t_1, \ldots, t_m)) = \{ \Lambda \} \cup \{ i.p | i \in
\{1, \ldots, m\} \land p \in Pos(t_i) \}$. A term $t$ in $T(\Sigma)$ can be seen
as a function from its set of positions $Pos(t)$ into $\Sigma$. For
this reason, the symbol labeling the position $p$ in $t$ shall
be denoted by $t(p)$. By $p < p'$ and $p \leq p'$ we denote that
$p$ is a proper prefix of $p'$, and that $p$ is a prefix of
$p'$, respectively. In this cases, $p'$ is necessarily of the form
$p.p''$, and we define $p' - p = p''$. Two positions $p_1, p_2$ incomparable with respect to the prefix ordering are called
parallel, and it is denoted by $p_1 \parallel p_2$. The subterm $t$ at
position $p$, denoted $t|_p$, is defined recursively as $t|_\Lambda = t$ and
$f(t_1, \ldots, t_m)|_{i,p} = t|_p$. The replacement in $t$ of the subterm
at position $p$ by $s$, denoted $t|_s$, is defined recursively
as $t|_s|_\Lambda = s$ and $f(t_1, \ldots, t_i-1, t_i, t_i+1, \ldots, t_m)|_{i,p} =
(f(t_1, \ldots, t_i-1, t_i|_s, t_i+1, \ldots, t_m)$. The fact $t = t|_s$ may
also be used to emphasis that $t|_p$ is $s$. The height of a term $t$,
denoted $h(t)$, is the maximal length of a position of $Pos(t)$.
In particular, the length of $\Lambda$ is 0.

A tree automaton (TA, see e.g. [2]) is a tuple $A = \langle Q, \Sigma, F, \Delta \rangle$ where $Q$ is a finite set of states, $\Sigma$ is a
signature, $F \subset Q$ is the subset of final states and $\Delta$

is a set of transitions rules of the form $f(q_1, \ldots, q_m) \rightarrow q$
where $f : m \in \Sigma, q_1, \ldots, q_m, q \in Q$. Sometimes, we shall
refer to $A$ as a subcircuit of its components, like in $Q_A$
to indicate that $Q$ is the state set of $A$.

A run of $A$ is a pair $r = (t, M)$ where $t$ is a term in
$T(\Sigma)$ and $M : Pos(t) \rightarrow Q_A$ is a matching satisfying, for
all $p \in Pos(t)$, that the rule $t(p)(M(p.1), \ldots, M(p.m)) \rightarrow
M(p)$ is in $\Delta_A$, where $m$ is the arity of the symbol $t(p)$
in $\Sigma$. By abuse of notation we write $r(p)$ for $M(p)$, and
say that $r$ is a run of $A$ on $t$. Moreover, by $\tau(e)$ we refer to $t$, and by $\tau(e) \Delta \tau(r)$ we refer to $(A)$. The run $r$

is called successful (or accepting) if $r(\Lambda)$ is in $F_A$.
The language $L(A)$ of $A$ is the set of terms $t$ for which there
exists a successful run of $A$. A language $L$ is called regular
if there exists a TA $A$ satisfying $L = L(A)$. For facility
of explanations, we shall use term-like notations for runs
defined as follows in the natural way. For a run \( r = \langle t, M \rangle \)
by \( \text{Pos}(r) \) we denote \( \text{Pos}(t) \), and by \( h(r) \) we denote \( h(t) \).
Similarly, by \( r|_p \) we denote the run \( \langle t|_p, M|_p \rangle \), where \( M|_p \)
is defined as \( M|_p(p') = M(p,p') \) for each \( p' \) in \( \text{Pos}(t|_p) \),
and say that \( r|_p \) is a subrun of \( r \). Moreover, for a run \( r' = \langle t', M' \rangle \),
by \( r|r' \) we denote the run \( \langle t|t'|_p, M[M|_p] \rangle \), where
\( M[M|_p] \) is defined as \( M[M|_p] = M[p,p'] = M(p') \) for each \( p' \)
in \( \text{Pos}(t') \), and as \( M[M|_p] = M(p') \) for each \( p' \) with \( p \not\approx p' \).

A well quasi-ordering [21] \( \leq \) on a set \( S \) is a reflexive and
transitive relation such that any infinite sequence of elements
\( e_1, e_2, \ldots \) of \( S \) contains an increasing pair \( e_i \leq e_j \) with
\( i < j \).

B. Tree Automata with Global Constraints

In this subsection, we define a class of tree automata with
global constraints which generalizes the class of
TAGEDs [1].

Definition II.1 A tree automaton with global constraints
(TAGC) over a signature \( \Sigma \) is a tuple
\( A = \langle Q, \Sigma, F, C, \Delta \rangle \) such that
\( (Q, \Sigma, F, \Delta) \) is a TA, denoted \( \text{ta}(A) \), and \( C \)
is a Boolean combination of atomic constraints of the form
\( q \approx q' \) or \( q \not\approx q' \), where \( q, q' \in Q \). A TAGC \( A \) is called
positive if \( C_A \) is a conjunction of conjunctive atomic constraints.
A TAGC \( A \) is called positive conjunctive if
\( C_A \) is a conjunction of atomic constraints. The subclasses
of positive and positive conjunctive TAGC are denoted by
PTAGC and PCTAGC, respectively.

A run \( r \) of the TAGC \( A \) is a run of \( \text{ta}(A) \) such that \( r \)
satisfies \( C_A \), denoted \( r \models C_A \), where the satisfiability of
constraints is defined as follows, where \( t \) is \( \text{term}(r) \). For
atomic constraints, \( r \models q \approx q' \) holds (respectively \( r \models q \not\approx q' \)
if and only if for all different positions \( p, p' \in \text{Pos}(t) \) such that \( r(p) = q \) and \( r(p') = q' \), \( t|_p = t|_{p'} \) holds
(respectively \( t|_p \not\approx t|_{p'} \) holds). This notion of satisfiability is
extended to Boolean combinations as usual. For TAGs, we say that
\( r \) is a run of \( A \) on \( t \).

A run of \( A \) on \( t \in \mathcal{T}(\Sigma) \) is successful if \( r(\Lambda) \in F_A \).
The language \( \mathcal{L}(A) \) of \( A \) is the set of terms \( t \) for which there
exists a successful run of \( A \).

It is important to note that the semantics of \( \neg(q \approx q') \) and
\( q \not\approx q' \) differ, as well as the semantics of \( \neg(q \not\approx q') \) and
\( q \approx q' \). This is because we have a “for all” quantifier in
both definitions.

We use below the notation \( \text{TAGC}[\tau] \), where \( \tau \) is either \( \approx \)
or \( 
\not\approx \), to characterise the subclass of TAGC with only atomic
constraints of type \( \tau \) (and same for PTAGC and PCTAGC).

The class of regular languages is strictly included in
the class of TAGC languages due to the constraints.

Moreover, the TAGEDs of [1] are also a particular case of
TAGC, since they can be redefined in our setting as
restricted PCTAGC. In particular \( q \) and \( q' \) are required to be distinct in
\( q \not\approx q' \) for TAGEDs. Reflexive disequivalence constraints such as
\( q \not\approx q \) correspond to monadic key constraints for XML
documents, meaning that every two distinct positions of type
\( q \) have different values. A state \( q \) of a TAGC can be used for
instance to characterize unique identifiers as in the following
example, which presents a TAGC whose language cannot be
recognized by a TAGED.

Example II.2 The TAGC of our running example accepts
(in state \( q_M \)) lists of dishes called menus, where every dish is
associated with one identifier (state \( q_{id} \)) and the time needed
to cook it (state \( q_t \)). We have other states accepting digits
\( (q_d) \), numbers \( (q_N) \) and lists of dishes \( (q_L) \).

\( \text{The TAGC } A = \langle Q, \Sigma, F, C, \Delta \rangle \) is defined as follows:
\( \Sigma = \{0, \ldots, 9 : 0, N, L_0 : 2, L, M : 3\} \),
\( Q = \{q_d, q_N, q_{id}, q_t, q_L, q_M\} \),
\( F = \{q_M\} \), and \( \Delta = \{i \rightarrow q_d | q_N | q_{id} | q_t | 0 \leq i \leq 9 \} \cup \{N(q_d, q_N) \rightarrow q_N | q_{id} | q_t | L_0(q_d, q_t) \rightarrow q_L, L(q_d, q_{id}, q_t) \rightarrow q_L, M(q_d, q_{id}, q_t) \rightarrow q_M\} \).

The constraint \( C \) ensures that all the identifiers of the dishes
in a menu are pairwise distinct (i.e. that \( q_{id} \) is a key) and
that the time to cook is the same for all dish:
\( C = q_{id} \not\approx q_{id} \land q_t \approx q_t \).

A term in \( L(A) \) together with an associated successful run
are depicted in Figure 1.

Similarly to TAGED (see [1]) TAGC can be proved to be
closed under union and intersection, but not under
complementation. The membership problem (given a term
\( t \) and a TAGC \( A \), do we have \( t \in L(A) \)?) is NP-complete
while universality (given a TAGC \( A \) over \( \Sigma \), do we have
\( L(A) = \mathcal{T}(\Sigma) \)?) is undecidable for PCTAGC\([\approx]\) [11]. The
following consequence is a new result for TAGEDs.

Proposition II.3 It is undecidable whether the language of
a given PTAGC\([\approx]\) is regular.

Proof: (Sketch) A reduction from universality to regularity
can be easily described as follows by using a new
function symbol \( f \) with arity 2, and any non-regular language \( L \) which is recognizable by a PTAGC\( ^e \).

Let \( A \) be an input of universality for PTAGC\( ^e \). It is not difficult to compute a new PTAGC\( ^e \) \( \mathcal{A} \) recognizing the language \( \{ f(t_1, t_2) \mid t_1 \in T(\Sigma) \land t_2 \in L \} \cup \{ f(t_1, t_2) \mid t_1 \in \mathcal{L}(\mathcal{A}) \land t_2 \in T(\Sigma) \} \). Then we can show that \( \mathcal{L}(\mathcal{A}) = T(\Sigma) \) if and only if \( \mathcal{L}(\mathcal{A}') \) is regular.

The \textit{emptiness} is the problem to decide, given a TAGC \( \mathcal{A} \), whether \( \mathcal{L}(\mathcal{A}) = \emptyset \)? The proof that it is decidable for TAGC is rather involved and is presented in Section III.

### III. Emptiness Decision Algorithm

In this section we prove the decidability of the \textit{emptiness} problem for TAGC. We start by stating that it suffices to prove this result for PCTAGC.

**Lemma III.1** Given a TAGC \( \mathcal{A} \), one can effectively construct a PCTAGC recognizing \( \mathcal{L}(\mathcal{A}) \).

The proof of this lemma, given in [10], is technical but also straightforward. It is based on the fact that negative literals \( \neg(q_1 \neq q_2) \) or \( \neg(q_1 \approx q_2) \) can be encoded with the addition of new states and positive literals.

The decidability of emptiness for PCTAGC is proved in three steps. In Subsection III-A, we present a new notion of pumping which allows to transform a run into a smaller run under certain conditions. In Subsection III-B, we define a well quasi-ordering \( \leq \) on a certain set \( S \). In Subsection III-C, we connect the two previous subsections by describing how to compute, for each run \( r \) with height \( h = h(r) \), a certain sequence \( e_1, \ldots, e_1 \) of elements of \( S \) satisfying the following fact: there exists a pumping on \( r \) if and only if \( e_i \leq e_j \) for some \( h \leq i > j \leq 0 \). Finally, all of these constructions are used as follows. Suppose the existence of an accepting run \( r \). If \( r \) is \textit{“too high”}, the fact that \( \leq \) is a well quasi-ordering and the property of the sequence imply the existence of such \( i, j \). Thus, it follows the existence of a pumping providing a smaller accepting run \( r' \). We conclude the existence of a computational bound for the height of an accepting run, and hence, decidability of emptiness.

#### A. Global Pumpings

Pumping is a traditional concept in automata theory, and in particular, they are very useful to reason about tree automata. The basic idea is to convert a given run \( r \) into another run by replacing a subrun at a certain position \( p \) in \( r \) by a run \( r' \), thus obtaining a run \( r[r']_p \). Pumpings are useful for deciding emptiness: if a “big” run can always be reduced by a pumping, then decision of emptiness is obtained by a search of an accepting “small” run.

For plain tree automata, a necessary and sufficient condition to ensure that \( r[r']_p \) is a run is that the resulting states of \( r[p] \) and \( r' \) coincide, since the correct application of a rule at a certain position depends only on the resulting states of the subruns of the direct children. In this case, an accepting run with height bounded by the number of states exists, whenever the accepted language is not empty.

When the tree automaton has global equality and disequality constraints, the constraints may be falsified when replacing a subrun by a new run. For PCTAGC, we will define a notion of pumping ensuring that the constraints are satisfied. This notion of pumping requires to perform several replacements in parallel. We first define the sets of positions involved in such a kind of pumping.

**Definition III.2** Let \( A \) be a PCTAGC. Let \( r \) be a run of \( A \). Let \( i \) be an integer between 0 and \( h(r) \). We define \( H_i \) as \( \{ p \in Pos(r) \mid h(r[p]) = i \} \) and \( H_i \) as \( \{ p, j \in Pos(r) \mid h(r[p]) < i \wedge h(r[p]) > j \} \).

**Example III.3** According to Definition III.2, for our running example (Example II.2), we have the \( H_i \) and \( H_i \) presented in Figure 2.

![Figure 2](image2.png)

The following lemma is rather straightforward from the previous definition.

**Lemma III.4** Let \( A \) be a PCTAGC. Let \( r \) be a run of \( A \). Let \( i, j \) be integers satisfying \( 0 \leq j < i \leq h(r) \). A pumping-injection \( I : (H_i \cup H_i) \to (H_j \cup H_j) \) is an injection function such that the following conditions hold:

(\( C_1 \)) \( I(H_i) \subseteq H_i \) and \( I(H_i) \subseteq H_i \).

(\( C_2 \)) For each \( \bar{p} \) in \( H_i \cup H_i \), \( r(\bar{p}) = r(I(\bar{p})) \).

(\( C_3 \)) For each \( \bar{p}_1, \bar{p}_2 \) in \( H_i \cup H_i \), \( \text{term}(r[\bar{p}_1]) = \text{term}(r[\bar{p}_2]) \iff \text{term}(r[\bar{p}_1]) = \text{term}(r[\bar{p}_2]) \).

Let \( \{ \bar{p}_1, \ldots, \bar{p}_n \} \) be \( H_i \cup H_i \) more explicitly written. The run \( r[\bar{p}_1] \ldots r[\bar{p}_n] \) is called a global pumping on \( r \) with indexes \( i, j \), and injection \( I \).

By Condition \( C_2 \), \( r[\bar{p}_1] \ldots r[\bar{p}_n] \) is clearly a run of \( ta(\mathcal{A}) \), but it is still necessary to prove that it
Figure 3. Pump-injection of Example III.6.

is a run of $A$. By abuse of notation, when we write $r^{|r|_{I(p_1)}}_{p_1} \ldots |r|_{I(p_n)}_{p_n}$, we sometimes consider that $I$ and $\{p_1, \ldots, p_n\}$ are still explicit, and say that it is a global pumping with some indexes $0 \leq j < i \leq h(r)$.

**Example III.6** Following our running example, we define a pump-injection $I : (H_4 \cup H_3) \rightarrow (H_3 \cup H_3)$ as follows: $I(1) = 3.1$, $I(2) = 2$, $I(3) = 3.3$. We note that $I$ is a correct pump-injection: $I(H_4) \subseteq H_3$ and $I(H_3) \subseteq H_3$ hold, thus $(C_3)$ holds. For $(C_2)$, we have $r(1) = r(I(1)) = q_{id}$, $r(2) = r(I(2)) = q_{\ell}$, and $r(3) = r(I(3)) = q_L$. Regarding $(C_3)$, for each different $p_1, p_2$ in $H_4 \cup H_3$, $\text{term}(r|_{p_1}) \neq \text{term}(r|_{p_2})$ and $\text{term}(r|_{I(p_1)}) \neq \text{term}(r|_{I(p_2)})$ hold.

After applying the pump-injection $I$, we obtain the term and run $r'$ of Figure 3.

Our goal is to prove that any global pumping $r^{|r|_{I(p_1)}}_{p_1} \ldots |r|_{I(p_n)}_{p_n}$ is a run, and in particular, that all global equality and disequality constraints are satisfied. To this end we first state the following intermediate statement, which determines the height of the terms pending at some positions after the pumping action. It can be easily proved by induction on the height of the involved term.

**Lemma III.7** Let $A$ be a PCTAGC. Let $r$ be a run of $A$. Let $r'$ be the global pumping $r^{|r|_{I(p_1)}}_{p_1} \ldots |r|_{I(p_n)}_{p_n}$ on $r$ with indexes $0 \leq j < i \leq h(r)$ and injection $I$. Let $k \geq 0$ be a natural number and let $p$ be a position of $r$ such that $h(r|_p) = i + k$.

Then, $p$ is also a position of $r'$ and $h(r'|_p) = j + k$.

**Corollary III.8** Let $A$ be a PCTAGC. Let $r$ be a run of $A$. Let $r'$ be a global pumping of $r$. Then, $h(r') < h(r)$.

The following lemma states that equality and disequality relations are preserved, not only for terms pending at the positions of the domain of $I$, but also for terms pending at prefixes of positions of such domain. Again, it is rather easy to prove by induction on the height of the involved term.

**Lemma III.9** Let $A$ be a PCTAGC. Let $r$ be a run of $A$. Let $r'$ be the global pumping $r^{|r|_{I(p_1)}}_{p_1} \ldots |r|_{I(p_n)}_{p_n}$ with indexes $0 \leq j < i \leq h(r)$ and injection $I$. Let $p_1, p_2$ be positions of $r$ satisfying $h(r|_{p_1}), h(r|_{p_2}) \geq i$.

Then, $p_1, p_2$ are also positions of $r'$ and $\text{term}(r|_{p_1}) = \text{term}(r'|_{p_1}) \Rightarrow \text{term}(r|_{p_2}) = \text{term}(r'|_{p_2})$ holds.

As a consequence of the previous lemmas, we prove that the result of a global pumping is a run.

**Lemma III.10** Let $A$ be a PCTAGC. Let $r$ be a run of $A$. Let $r'$ be the global pumping $r^{|r|_{I(p_1)}}_{p_1} \ldots |r|_{I(p_n)}_{p_n}$ with indexes $0 \leq j < i \leq h(r)$ and injection $I$.

Then, $r'$ is a run of $A$.

**Proof:** By Condition $(C_2)$ of the definition of pumping-injection, in order to see that $r'$ is a run, it suffices to see that all global constraints are satisfied. Thus, let us consider two different positions $p_1, p_2$ of $Pos(r')$ involved in an atom of the constraint of $A$, i.e. either $r'(p_1) \approx r'(p_2)$ or $r'(p_1) \neq r'(p_2)$ occurs in the constraint of $A$. According to Lemma III.4, we can distinguish the following cases:

- Suppose that a position in $H_1 \cup H_1$, say $\bar{p}_1$, is a prefix of both $p_1, p_2$. Then, $r'|_{p_1} = r|\bar{p}_1,-p_1\rangle$ and $r'|_{p_2} = r|\bar{p}_2,-p_2\rangle$ hold. Hence, $r'|_{p_1}$ and $r'|_{p_2}$ are also subruns of $r$ occurring at different positions. Thus, since $r$ is a run, they satisfy the atom involving $r'(p_1)$ and $r'(p_2)$.

- Suppose that two different positions in $H_1 \cup H_1$, say $\bar{p}_1$ and $\bar{p}_2$, are prefixes of $p_1$ and $p_2$, respectively. Then, $r'|_{p_1} = r|\bar{p}_1,-p_1\rangle$ and $r'|_{p_2} = r|\bar{p}_2,-p_2\rangle$ hold. By the injectivity of $I$, $I(\bar{p}_1) \neq I(\bar{p}_2)$ holds. Moreover, by Lemma III.4, $I(\bar{p}_1) \parallel I(\bar{p}_2)$ holds. Hence, as before, $r'|_{p_1}$ and $r'|_{p_2}$ are subruns of $r$ occurring at different (in fact, parallel) positions. Thus, they satisfy the atom involving $r'(p_1)$ and $r'(p_2)$.

- Suppose that one of $p_1, p_2$, say $p_1$, is a proper prefix of a position in $H_1 \cup H_1$, and that $p_2$ satisfies that some position in $H_1 \cup H_1$ is a prefix of $p_2$. It follows that $h(r'|_{p_1})$ is smaller than or equal to $j$, and $r'|_{p_2}$ is also a subrun of $r$. Moreover, $p_1$ is also a position of $r$, $r'(p_1) = r(p_1)$ holds, and $h(r|_{p_1}) = i + k$ holds for some $k > 0$. Hence, $\text{term}(r|_{p_1}) \neq \text{term}(r'|_{p_1})$ holds. Since $r$ is a run and $r'|_{p_2}$ is a subrun of $r$, the atom involving $r'(p_1)$ and $r'(p_2)$ is necessarily of the form $r(p_1) \neq r(p_2)$. Thus, the atom involving $r'(p_1)$ and $r'(p_2)$ is necessarily of the form $r'(p_1) \neq r'(p_2)$. By Lemma III.7, $h(r|_{p_1}) = j + k$. Therefore, $\text{term}(r'|_{p_1}) \neq \text{term}(r'|_{p_2})$ holds, and hence, such an atom is satisfied for such positions in $r'$.

- Suppose that both $p_1, p_2$ are proper prefixes of positions in $H_1 \cup H_1$. Then, $p_1, p_2$ are positions of $r$ satisfying $h(r|_{p_1}), h(r|_{p_2}) \geq i$. Moreover, $r(p_1) = r'(p_1)$ and $r(p_2) = r'(p_2)$ hold. Since $r$ is a run, the atom involving $r(p_1)$ and $r(p_2)$ is satisfied in the run $r$ for positions $p_1$ and $p_2$. By Lemma III.9, $\text{term}(r|_{p_1}) = \text{term}(r|_{p_2}) \Rightarrow \text{term}(r'|_{p_1}) = \text{term}(r'|_{p_2})$ holds. Thus, the atom involving $r'(p_1)$ and $r'(p_2)$ is satisfied in the run $r'$ for positions $p_1$ and $p_2$. ■
B. A well quasi-ordering

In this subsection we define a well quasi-ordering. It assures the existence of a computational bound for certain sequences of elements of the corresponding well quasi-ordered set. It will be connected with global pumpings in the next subsection.

**Definition III.11** Let \( \leq \) denote the usual quasi-ordering on natural numbers. Let \( n \) be a natural number.

We define the extension of \( \leq \) to \( n \)-tuples of natural numbers as \( (x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \) if \( x_i \leq y_i \) for each \( i \) in \( \{1, \ldots, n\} \). We define \( \sum(x_1, \ldots, x_n) := x_1 + \cdots + x_n \).

We define the extension of \( \leq \) to multisets of \( n \)-tuples of natural numbers as \( \{e_1, \ldots, e_n\} \leq \{e'_1, \ldots, e'_n\} \) if there is an injection \( I : \{1, \ldots, n\} \rightarrow \{1, \ldots, \beta\} \) satisfying \( e_i \leq e'_{I(i)} \) for each \( i \) in \( \{1, \ldots, \alpha\} \). We define \( \sum\{e_1, \ldots, e_n\} := \sum(e_1) + \cdots + \sum(e_n) \).

We define the extension of \( \leq \) to pairs of multisets of \( n \)-tuples of natural numbers as \( \langle P_1, P_1 \rangle \leq \langle P_2, P_2 \rangle \) if \( P_1 \leq P_2 \) and \( P_1 \leq P_2 \).

As a direct consequence of Higman’s Lemma [21] we have the following:

**Lemma III.12** Given \( n, \leq \) is a well quasi-ordering for pairs of multisets of \( n \)-tuples of natural numbers.

In any infinite sequence \( e_1, e_2, \ldots \) of elements from a well quasi-ordered set there always exist two indexes \( i < j \) satisfying \( e_i \leq e_j \). In general, this fact does not imply the existence of a bound for the length of sequences without such indexes. For example, the relation \( \leq \) between natural numbers is a well quasi-ordering, but there may exist arbitrarily long sequences \( x_1, \ldots, x_k \) of natural numbers such that \( \langle x_i \rangle \) \( x_i \succ x_j \) for all \( 1 \leq i < j \leq k \). In order to bound the length of sequences satisfying \( \langle \rangle \), it is sufficient to force that the first element and each next element of the sequence are chosen among a finite number of possibilities. Indeed in this case, by König’s lemma, the prefix trees describing all such (finite) sequences is finite. As a particular case of this fact we have the following result (proved in [10]).

**Lemma III.13** There exists a computable function \( B : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) such that, given two natural numbers \( a, n \), \( B(a,n) \) is a bound for the length \( \ell \) of the maximum-length sequence \( \langle T_1, T_1 \rangle, \ldots, \langle T_\ell, T_\ell \rangle \) of pairs of multisets of \( n \)-tuples of natural numbers such that the following conditions hold:

1. The tuple \( (0,\ldots,0) \) does not occur in any \( T_i, T_i \) for \( i \) in \( \{1, \ldots, \ell\} \).
2. \( \sum(T_1) = 1 \) and \( \sum(T_1) = 0 \).
3. For each \( i \) in \( \{1, \ldots, \ell-1\} \), \( \sum(T_{i+1})+\sum(T_1) \leq \alpha \cdot \sum(T_i) + \sum(T_i) \).
4. There are no \( i, j \) satisfying \( 1 \leq i < j \leq \ell \) and \( \langle T_i, T_i \rangle \not\leq \langle T_j, T_j \rangle \).

In order to bound the height of a term accepted by a given PCTAGC \( A \) (and of minimum height), Lemma III.13 will be used by making \( a \) to be the maximum arity of the signature of \( A \), and making \( n \) to be the number of states of \( A \).

C. Mapping a run to a sequence of the well quasi-ordered set

We will associate, to each number \( i \) in \( \{0, \ldots, b(r)\} \), a pair of multisets of tuples of natural numbers, which can be compared with other pairs according to the definition of \( \leq \) in the previous subsection. To this end, we first associate tuples to terms and multisets of tuples to sets of positions.

**Definition III.14** Let \( A \) be a PCTAGC. Let \( q_1, \ldots, q_n \) be the states of \( A \). Let \( r \) be a run of \( A \). Let \( P \) be a set of positions of \( r \). Let \( t \) be a term. We define \( r_{t,P} \) as the following tuple of natural numbers: \( \langle \{ p \in P \mid \text{term}(r_{|p}) = t \wedge r(p) = q_i \} \rangle \). In order to bound the length of sequences satisfying \( \langle \rangle \), it is sufficient to force that the first element and each next element of the sequence are chosen among a finite number of possibilities. Indeed in this case, by König’s lemma, the prefix trees describing all such (finite) sequences is finite. As a particular case of this fact we have the following result (proved in [10]).

**Lemma III.17** Let \( A \) be a PCTAGC. Let \( r \) be a run of \( A \). Let \( \{ t_1, \ldots, t_k \} \) be the set of terms \( \{ t \mid \exists p \in P : \text{term}(r_{|p}) = t \} \). We define \( r_{P} \) as the multiset \( r_{t_1,P}, \ldots, r_{t_k,P} \).

**Example III.16** Following our running example, for the representation of the tuples of natural numbers we order the states as \( \langle q_{q}, q_{N}, q_{d}, q_{l}, q_{L}, q_{M} \rangle \). The multisets \( r_{H_i} \) are presented in Figure 4.

The following lemma connects the existence of a pump-injection with the quasi-ordering relation.

**Lemma III.17** Let \( A \) be a PCTAGC. Let \( r \) be a run of \( A \). Let \( i, j \) be integers satisfying \( 0 \leq j < i \leq h(r) \).

Then, there exists a pump-injection \( I : (H_i \cup \tilde{H}_i) \rightarrow (H_j \cup \tilde{H}_j) \) if and only if \( r_{H_i} \leq r_{H_j} \).

**Proof:** We just prove the right-to-left direction: the other direction is technical but not conceptually difficult, and it is not necessary for the rest of the paper. Hence
assume that \( \langle r_{H_i}, r_{\tilde{H}_i} \rangle \leq \langle r_{H_j}, r_{\tilde{H}_j} \rangle \) holds. We have to construct a pump-injection \( I : (H_i \cup \tilde{H}_i) \rightarrow (H_j \cup \tilde{H}_j) \). We just define \( I : H_i \rightarrow H_j \) and prove Conditions (C2) and (C3) for \( \tilde{p}, \tilde{p}_1, \tilde{p}_2 \) in \( H_i \). This is because \( I : H_i \rightarrow H_j \) can be defined analogously, and Conditions (C2) and (C3) for the corresponding positions can be checked analogously. Moreover, for positions \( \tilde{p}_1 \in H_i \) and \( \tilde{p}_2 \in H_i \), Condition (C4) holds whenever Condition (C1) holds since in this case \( \text{term}(r|_{\tilde{p}_1}) \neq \text{term}(r|_{\tilde{p}_2}) \) and \( \text{term}(r|_{\tilde{p}_1}) \neq \text{term}(r|_{\tilde{p}_2}) \) hold. Hence, this simple case is enough to prove the whole statement.

We write \( \{ \text{term}(r|_p) \mid p \in H_j \} \) and \( \{ \text{term}(r|_p) \mid p \in H_j \} \) more explicitly as \( \{ t_{i,1,\ldots, t_{i, \alpha}} \} \) and \( \{ t_{j,1, \ldots, t_{j, \beta}} \} \), respectively. Since \( \langle r_{H_i}, r_{\tilde{H}_i} \rangle \leq \langle r_{H_j}, r_{\tilde{H}_j} \rangle \) holds, \( r_{H_i} \leq r_{H_j} \) also holds. Thus, there exists an injective function \( I' : \{ 1, \ldots, \alpha \} \rightarrow \{ 1, \ldots, \beta \} \) satisfying the following statement for each \( \delta \in \{ 1, \ldots, \alpha \} \) and each state \( q \in A \):

\[
\left| \{ p \in H_j \mid \text{term}(r|_p) = t_{i,\delta} \wedge r(p) = q \} \right| \leq \left| \{ p \in H_j \mid \text{term}(r|_p) = t_{j,\delta} \wedge r(p) = q \} \right|.
\]

To prove \( I : H_i \rightarrow H_j \), we define \( I \) for each of such sets \( \{ p \in H_j \mid \text{term}(r|_p) = t_{i,\delta} \wedge r(p) = q \} \) as an injective function \( I : \{ p \in H_j \mid \text{term}(r|_p) = t_{i,\delta} \wedge r(p) = q \} \rightarrow \{ p \in H_j \mid \text{term}(r|_p) = t_{j,\delta} \wedge r(p) = q \} \), which is possible by the above inequality (ï). The global \( I \) is then injective thanks to the injectivity of \( I' \). Conditions (C2) and (C3) trivially follow from this definition.

Example III.18 Following our running example, we first prove \( \langle r_{H_i}, r_{\tilde{H}_i} \rangle \leq \langle r_{H_j}, r_{\tilde{H}_j} \rangle \). To this end just note that \( \{ 0, 0, 0, 1, 0 \} \leq \{ 0, 0, 0, 1, 0 \} \leq \{ 0, 0, 0, 1, 0 \} \leq \{ 0, 0, 0, 1, 0 \} \leq \{ 0, 0, 0, 1, 0 \} \leq \{ 0, 0, 0, 2, 0 \} \) hold. We can define \( I : (H_i \cup \tilde{H}_i) \rightarrow (H_j \cup \tilde{H}_j) \) from this relation according to Lemma III.17. Doing the adequate guess we obtain the following definition: \( I(1) = 3.1, I(2) = 2, I(3) = 3.3 \) which is the pump-injection considered for our running example.

The following lemma follows directly from the definition of the sets \( H_i \) and \( \tilde{H}_i \), and allows to connect such definitions with Lemma III.13.

Lemma III.19 Let \( A \) be a PCTAGC. Let \( a \) be the maximum arity of the symbols in the signature of \( A \). Let \( r \) be a run of \( A \). Then, the following conditions hold:

1. \( |H_{a(r)}| = 1 \) and \( |H_{\tilde{a}(r)}| = 0 \).
2. For each \( i \) in \( \{ 1, \ldots, h(r) \} \), \( |H_{i-1}| + |H_{i-1}| \leq a \cdot |H_i| + |H_{\tilde{i}}| \).
3. For each \( i \) in \( \{ 0, \ldots, h(r) \} \), \( |H_i| = \sum_{r|H_i} \).

Lemma III.20 Let \( B : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) be the computable function of Lemma III.13. Let \( A \) be a PCTAGC. Let \( a \) be the maximum arity of the symbols in the signature of \( A \). Let \( n \) be the number of states of \( A \). Let \( r \) be a run of \( A \) satisfying \( h(r) \geq B(a, n) \). Then, there is a global pumping on \( r \).

Proof: Consider the sequence \( \langle r_{H_{\tilde{a}(r)}}, r_{H_{\tilde{a}(r)}} \rangle \). Note that the \( n \)-tuple \( \{ 0, \ldots, 0 \} \) does not appear in the multisets of the pairs of this sequence. By Lemma III.19, \( |H_{\tilde{a}(r)}| = 1 \) and \( |H_{\tilde{a}(r)}| = 0 \), and for each \( i \) in \( \{ 1, \ldots, h(r) \} \), \( |H_{i-1}| + |H_{i-1}| \) holds. Moreover, for each \( i \) in \( \{ 0, \ldots, h(r) \} \), \( |H_i| = \sum_{r|H_i} \) and \( |H_i| = \sum_{r|H_i} \). Thus, \( \sum_{r|H_i} \) is 1, \( \sum_{r|H_{\tilde{a}(r)}} \) is 0, and for each \( i \) in \( \{ 1, \ldots, h(r) \} \), \( \sum_{r|H_{\tilde{a}(r)}} + \sum_{r|H_{\tilde{a}(r)}} \leq a \cdot \sum_{r|H_{\tilde{a}(r)}} + \sum_{r|H_{\tilde{a}(r)}} \). Hence, since \( h(r) \geq B(a, n) \) holds, by Lemma III.13 there exist \( i, j \) satisfying \( h(r) \geq i > j \geq 0 \) and \( \langle r_{H_i}, r_{\tilde{H}_i} \rangle \leq \langle r_{H_j}, r_{\tilde{H}_j} \rangle \). By Lemma III.17, there exists a pump-injection \( I : (H_i \cup \tilde{H}_i) \rightarrow (H_j \cup \tilde{H}_j) \). Therefore, there exists a global pumping on \( r \).

Theorem III.21 Emptiness is decidable for PCTAGC.

Proof: Let \( a \) be the maximum arity of the symbols in the signature of \( A \). Let \( n \) be the number of states of \( A \). Let \( r \) be an accepting run of \( A \) with minimum height.

Suppose that \( h(r) \geq B(a, n) \) holds. Then, by Lemma III.20, there exists a global pumping \( r' \) on \( r \). By Corollary III.8, \( h(r') < h(r) \) holds. Moreover, by the definition of global pumping, \( r'(\Lambda) = \Lambda \) holds. Finally, by Lemma III.10, \( r' \) is a run of \( A \). Thus, \( r' \) contradicts the minimality of \( r \). We conclude that \( h(r) < B(a, n) \) holds.

The decidability of emptiness of \( A \) follows, since the existence of successful runs implies that one of them can be found among a computable and finite set of possibilities.

Using Lemma III.1 and Theorem III.21, we can conclude the decidability of emptiness for TAGC.

Corollary III.22 Emptiness is decidable for TAGC.

IV. Extensions

In this section, we extend the emptiness result by considering the addition of new constraints to TAGC. We shall extend the notation TAGC[\( \tau_1, \ldots, \tau_k \)] introduced in Section to these new type of constraints, in addition to \( \approx \) and \( \neq \).

A. Arithmetic Constraints

We study first the addition of counting constraints to TAGC. Let \( Q \) be a set of states. A linear inequality over \( Q \) is an expression of the form \( \sum_{q \in Q} a_q|q| \geq a \) or \( \sum_{q \in Q} a_q||q|| \geq a \) where every \( a_q \) and \( a \) belong to \( \mathbb{Z} \).

Let \( r \) be a run on a term \( t \) of a TA or TAGC \( A \) over \( \Sigma \) and with state set \( Q \), and let \( q \in Q \). The interpretations of \( |q| \) and \( ||q|| \) wrt \( r \) (and \( t \)) are defined respectively by the following cardinals \( |q| = |r^{-1}(q)| \) and \( ||q|| = \{ s \in T(\Sigma) \mid \exists p \in Pos(t), r(p) = q, s = t(p) \} \). This permits to define the satisfiability of linear equalities wrt \( t \) and \( r \) and the notion of successful runs for extensions of TAGC with atomic constraints which can have the form of the above linear inequalities.
Let us denote by $|.|_\mathbb{Z}$ and $|||.||$ the types of the above linear inequalities, seen as atomic constraints of TAGC. The class TAGC$[|.|_\mathbb{Z}]$ has been studied under different names (e.g. Parikh automata in [19], linear constraint tree automata in [22]) and it has a decidable emptiness test.

Combining constraints of type $\approx$ and counting constraints of type $|.|_\mathbb{Z}$ however leads to undecidability.

**Theorem IV.1** Emptiness is undecidable for PTAGC$[|.|, |.|_\mathbb{Z}]$.

**Proof:** (Sketch) The proof is based on a reduction from the Hilbert’s tenth problem. The main point is the fact that we can encode (non negative) integer multiplication by combining $\approx$ and $|.|_\mathbb{Z}$. We can indeed build a PTAGC$[|.|, |.|_\mathbb{Z}]$ $A_r$, recognizing terms of the form $(t_1, t_2, t_{12})$ such that in a run of $A_r$, all the leaves of $t_1$, $t_2$ and $t_{12}$ respectively are labeled by the state $q_1$, $q_2$ and $q_{12}$. Moreover, $t_{12}$ is of the form $f(t_1, \ldots, f(t_i, \ldots))$, and in a run of $A_r$, the position of the first $t_1$ (first argument of $*$) is labelled by a state $r_1$ and the positions of all the $t_1$’s in $t_{12}$ are labeled by $r_1$. The constraint $r_1 \approx r_1' \land |r_1'| = |q_2|$ ensures that $t_{12}$ contains $|q_{12}|$ occurrences of $t_1$, hence that $|q_{12}| = |q_1| \cdot |q_2|$.

We present now a restriction on linear equalities which enables a decidable emptiness test when combined with $\approx$ and $\not\approx$ as global constraints. A natural linear inequality over $Q$ is a linear inequality as above whose coefficients $a_q$ and $a_1$ have all the same sign. The types of the natural linear inequalities are denoted by $|.|_\mathbb{N}$ and $|||.||_\mathbb{N}$. Below, we shall abbreviate these two types by $\mathbb{N}$.

We show in [10] that the class PTAGC$[|.|, \not\approx, \mathbb{N}]$ has the same expressiveness as TAGC$[|.|, \not\approx]$. The proofs works in two steps, using the intermediate class (equally expressivable) PTAGC$[|.|, \not\approx, \mathbb{N}]$. We add new states and replace: in the first step, the negative constraints by counting constraints and positive constraints (of type $\approx$ and $\not\approx$), and in the second step, the counting constraints by positive constraints.

Both Lemma III.1 and the next theorem follow immediately from this result.

**Theorem IV.2** Emptiness is decidable for TAGC$[|.|, \not\approx, \mathbb{N}]$.

B. Equality Tests Between Brothers

The constraints of TAGC are checked once for all on a whole run. There exists another kind of equality and disequality constraints for extending TA which are tested locally at every transition step. One example of TA with such local constraints defined in [8] are tree automata with constraints between brothers (TACB).

A TACB is a tuple $A = (Q, \Sigma, F, \Delta)$ where $Q$, $F$, $\Sigma$ are defined as with TA and the transitions rules of $\Delta$ have the form: $f(q_1, \ldots, q_n) \leq \rightarrow q$, where $c$ is a conjunction of atoms of the form $i = j$ or $i \not= j$ with $1 \leq i, j \leq n$. A run of the TACB $A$ on a term $t \in T(\Sigma)$ is a function $r$ from $\text{Pos}(t)$ into $Q$ such that, for all $p \in \text{Pos}(t)$, there exists a rule $t(p)(r(p.1), \ldots, r(p.m)) \leq \rightarrow r(p) \in \Delta$ satisfying $t(p) \in \Sigma_m$, and moreover, for all $i = j$ in the conjunction $c$, $t_{|p.i} = t_{|p.j}$ holds, and for all $i \not= j$ in the conjunction $c$, $t_{|p.i} \not= t_{|p.j}$ holds.

The notions of successful runs and languages can be extended straightforwardly from TA to TACB. Global constraints can also be added to TACB in the natural way. The automata of the resulting classes TACB$[|.|, \not\approx, \mathbb{N}]$ will therefore perform global and local test during their computations.

**Theorem IV.3** Emptiness is decidable for TACB$[|.|, \not\approx, \mathbb{N}]$.

C. Unranked Ordered Trees

Our tree automata models and results can be generalized from ranked to unranked ordered terms. In this setting, $\Sigma$ is called an unranked signature, meaning that there is no arity fixed for its symbols, i.e. that in a term $a(t_1, \ldots, t_n)$, the number $n$ of children is arbitrary and does not depend on $a$. Let us denote by $U(\Sigma)$ the set of unranked ordered terms over $\Sigma$. The notions of positions, subterms etc are defined for unranked terms of $U(\Sigma)$ as for ranked terms of $T(\Sigma)$.

We extend the definition of automata for unranked ordered terms, called hedge automata [23], with global constraints. A hedge automaton with global constraints (HAGC) is a tuple $A = (Q, \Sigma, F, C, \Delta)$ where $Q$, $F$ and $C$ are as in the definition of TAGC in Section II-B and the transitions of $\Delta$ have the form $a(L) \rightarrow q$ where $a \in \Sigma$, $q \in Q$ and $L$ is a regular (word) language over $Q^\ast$, assumed given by an NFA with input alphabet $Q$. The notion of run is extended to HAGC in the natural way.

The emptiness decision results of Corollary III.22 can be extended from TAGC to HAGC using a standard transformation from unranked to ranked binary terms, like the extension encoding described in [2], Chapter 8.

**Theorem IV.4** Emptiness is decidable for HAGC$[|.|, \not\approx, \mathbb{N}]$.

V. MONADIC SECOND ORDER LOGIC

A ranked term $t \in T(\Sigma)$ over $\Sigma$ can be seen as a model for logical formulae, with an interpretation domain which is the set of positions $\text{Pos}(t)$. We consider monadic second order formulae interpreted on such models, with quantifications over first order variables (interpreted as positions), denoted $x, y, \ldots$ and over unary predicates (i.e. set variables interpreted as sets of positions), denoted $X, Y, \ldots$.

The formulae are built with the following predicates:

1) equality $x = y$, and membership $X(x)$ – the position $x$ belongs to the set $X$,
The extension MSO expressiveness as TA [24] and therefore it is decidable. Above predicates (the form \(a(x, y)\)), for all \(i\) smaller than or equal to the maximal arity of symbols of \(\Sigma\), which is true on every pair of positions \((p, p.i)\), (we call +1 the type of such predicates).

Term equality \(X \approx Y\), resp. disequality \(X \not\approx Y\), which is true when for all \(x\) in \(X\) and all \(y\) in \(Y\), \(t|_x = t|_y\), resp. \(t|_x \neq t|_y\), (predicate types \(\approx\) and \(\not\approx\)).

5) Linear inequalities \(\sum a_i |X_i| \geq a\) or \(\sum a_i \|X_i\| \geq a\), where every \(a_i\) and \(a\) belong to \(\mathbb{Z}\); \(|X_i|\) is interpreted as the cardinality of \(X_i\) and \(\|X_i\|\) as the cardinality of \(\{t|_p | p \in X_i\}\), (predicate types \(\|\|\) and \(\|\|\)).

We write MSO[\(\tau_1, \ldots, \tau_k\)] for the set of monadic second order logic formulae with equality, membership, labeling predicates and other predicates of type \(\tau_1, \ldots, \tau_k\), amongst the above types +1, \(\approx\), \(\not\approx\), and \(\|\|\), \(\|\|\). We also use the notations \(|\cdot|\) and \(\|\cdot\|\) for natural linear inequalities and the abbreviations \(\mathbb{Z}\) and \(\mathbb{N}\) as in Section IV-A. Let EMSO[\(\tau\)] be the fragment of MSO[\(\tau\)] of the formula with the form \(\exists X_1 \ldots \exists X_n \phi\) where all the set variables in \(\phi\) belong to \(\{X_1, \ldots, X_n\}\).

It is well known that MSO[+1] has exactly the same expressiveness as TA [24] and therefore it is decidable. The extension MSO[+1, \(\approx\), \(\not\approx\), \(\|\|\), \(\|\|\)] is undecidable, see e.g. [9], as well as MSO[+1, \(|\|\), \(\|\|\)] [19] (the latter extension is also undecidable for unranked ordered terms when counting constraints are applied to sibling positions [17]), but the fragment EMSO[+1, \(|\|\), \(\|\|\)] is decidable [19].

In [1] the fragment EMSO with \(\approx\) and a restricted form of \(\not\approx\) is shown decidable, with a two way correspondence between these formulae and a decidable subclass of TAGEDs. This construction can be straightforwardly adapted to establish a two way correspondence between EMSO[+1, \(\approx\), \(\not\approx\), \(\|\|\), \(\|\|\)] and TAGC[\(\approx\), \(\not\approx\), \(\|\|\), \(\|\|\)]. With Theorem IV.2, it gives the following result.

**Theorem V.2** EMSO[+1, \(\approx\), \(\not\approx\), \(\|\|\), \(\|\|\)] is decidable on unranked ordered terms.

### VI. Conclusion

We have answered (positively) the open problem of decidability of the emptiness problem for the TAGEDs [1], by proposing a decision algorithm for a class TAGC of tree automata with global constraints extending TAGEDs. Our method for emptiness decision, presented in Section III appeared to be robust enough to deal with several extensions like global counting constraints, local equality tests between sibling subterms and extension to unranked terms. It could perhaps be extended to equality modulo commutativity and other equivalence relations. Another interesting subject mentioned in the introduction is the combination of the HAGC of Section IV-C with the unranked tree automata with tests between siblings [12], [13].

A challenging question would be to investigate the precise complexity of the problem, avoiding the use of Higman’s Lemma in the algorithm. For instance, in [1], it is shown, using a direct reduction into solving positive and negative set constraints [25], [26], that emptiness is decidable in NEXPTIME for PCTAGC[\(\not\approx\)] such that in each atomic constraints \(q \not\approx q’\), \(q\) and \(q’\) are distinct states. On the other hand, the best known lower bound for emptiness decision for TAGC is EXPTIME-hardness (this holds already for PCTAGC[\(\not\approx\)] as shown in [1]).

Another branch of research related to TAGC concerns automata and logics for data trees, i.e. trees labeled over an infinite (countable) alphabet (see [27] for a survey). Indeed, data trees can be represented by terms over a finite alphabet, with an encoding of the data values into terms. This can be done in several ways, and with such encodings, the data equality relation becomes the equality between subterms. Therefore, this could be worth studying in order to relate our results on TAGC to decidability results on automata or logics on data trees like those in [28], [22].

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### References

