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# Fuel-Optimal Trajectories for Continuous-Thrust Orbital Rendezvous with Path Constraints

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# Fuel-Optimal Trajectories for Continuous-Thrust Orbital Rendezvous with Path Constraints

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ENSTA ParisTech, Paris, March 2, 2011



# Outline

- 1 Problem statement
  - Dynamical equations
  - Optimal control formulation
- 2 Solving the path-constrained rendezvous problem
  - Smoothing the bang-off-bang control
  - A new approach to deal with the state constraint
- 3 Numerical results - A rendezvous in Highly Elliptical Orbit
  - Statement of the test case
  - Unconstrained rendezvous
  - Rendezvous under collision avoidance constraint
- 4 Conclusion and future prospects

# Problem statement

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# Problem statement

## Dynamical equations

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## Tschauner-Hempel equations in Hill's frame

- Keplerian motions - Small intersatellite distance
- $(a, e, \nu)$ : semi-major axis, eccentricity and true anomaly of the target satellite

- $X(\nu), Y(\nu), Z(\nu)$ : relative coordinates of the chaser

$$\begin{bmatrix} x_1(\nu) \\ x_2(\nu) \\ x_3(\nu) \end{bmatrix} = (1 + e \cos(\nu)) \begin{bmatrix} X(\nu) \\ Y(\nu) \\ Z(\nu) \end{bmatrix}$$

- $x_4(\nu), x_5(\nu), x_6(\nu)$ : derivatives of  $x_i(\nu)$ , ( $i = 1, \dots, 3$ ) w.r.t.  $\nu$
- $m(\nu)$ : mass of the chaser at true anomaly  $\nu$
- $u(\nu)$ : normalized thrust vector of the chaser at true anomaly  $\nu$

# Problem statement

## Optimal control formulation

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## State-constrained minimum-fuel rendezvous (1/2)

## The problem to solve

$$(P) \left\{ \begin{array}{l} \text{Find } \bar{u} = \underset{u}{\operatorname{argmin}} J(u) = -m(v_f) \\ \text{s.t.} \\ \dot{x}(v) = A(v)x(v) + B(v) \frac{u(v)}{m(v)} \\ \dot{m}(v) = -c(v) \|u(v)\| \\ \|u(v)\| \leq 1 \quad v \in [v_0, v_f] \\ g(v, x(v)) \leq 0 \quad v \in [v_0, v_f] \\ x(v_0) = x_0 \quad h(x(v_f)) = 0 \\ m(v_0) = m_0 \end{array} \right.$$



## State-constrained minimum-fuel rendezvous (2/2)

### Collision avoidance constraint

$$g(v, x(v)) = 1 - \frac{\sqrt{x_1(v)^2 + x_2(v)^2 + x_3(v)^2}}{d_{min}(1 + e\cos(v))} \leq 0 \quad v \in [v_0, v_f]$$

### Key parameter

$d_{min}$ : minimum safety distance between the chaser and the target

### Main issues for shooting methods

- The control is bang-off-bang  $\implies$  numerical difficulties
- State constraint  $\implies$  the number and location of constrained arcs must be defined beforehand in order to build the MPBVP

# Solving the path-constrained rendezvous problem

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# Solving the path-constrained rendezvous problem

## Smoothing the bang-off-bang control

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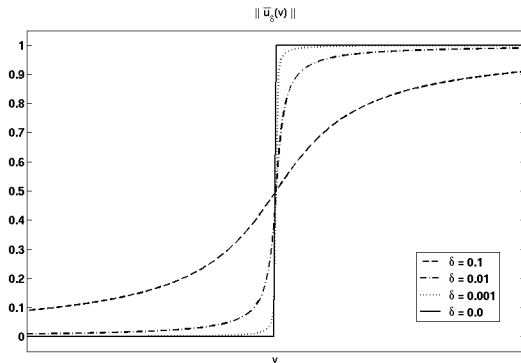
# Problem regularization (1/2)

## A logarithmic barrier approach

$$(P)_\delta \left\{ \begin{array}{l} \text{Find } \bar{u}_\delta = \underset{u}{\operatorname{argmin}} J_\delta(u) = J(u) - \delta \int_{v_0}^{v_f} F(v, u(v)) dv \\ \text{s.t.} \\ \dot{x}(v) = A(v)x(v) + B(v) \frac{u(v)}{m(v)} \\ \dot{m}(v) = -c(v) \|u(v)\| \\ \|u(v)\| \leq 1 \quad v \in [v_0, v_f] \\ g(v, x(v)) \leq 0 \quad v \in [v_0, v_f] \\ x(v_0) = x_0 \quad h(x(v_f)) = 0 \\ m(v_0) = m_0 \end{array} \right.$$

$$F(v, u(v)) = c(v) (\log(\|u(v)\|) + \log(1 - \|u(v)\|))$$

# Problem regularization (2/2)

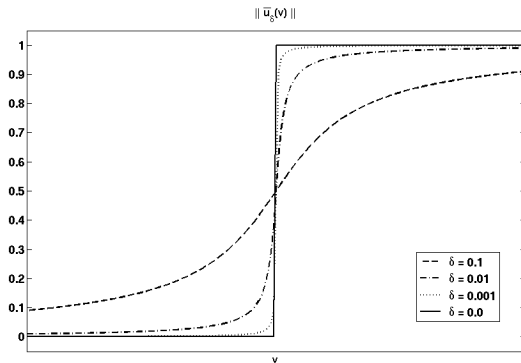


## Convergence results - No state constraint

$$J_\delta(\bar{u}_\delta) \rightarrow J(\bar{u}) \quad \text{as } \delta \rightarrow 0$$

$$\bar{u}_\delta \rightarrow \bar{u} \quad \text{for the weak-* topology on } L^\infty([v_0, v_f], \mathbb{R}^3) \quad \text{as } \delta \rightarrow 0$$

# Problem regularization (2/2)



## Reference

R. Epenoy and R. Bertrand: New Smoothing Techniques for Solving Bang-Bang Optimal Control Problems - Numerical Results and Statistical Interpretation, *Optimal Control Applications and Methods*, Vol. 23, No. 4, 2002, pp. 171-197.

# Solving the path-constrained rendezvous problem

## A new approach to deal with the state constraint

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# The smoothed exact penalty method

The penalized problem  $(P)_{\sigma,\alpha,\epsilon}^2$

$$\left\{ \begin{array}{l} \text{Find } u_{\sigma,\alpha,\epsilon}^2 = \underset{u}{\operatorname{argmin}} J_{\sigma,\alpha,\epsilon}^2(u) = J_{\delta}(u) + \int_{v_0}^{v_f} \psi_{\sigma,\alpha,\epsilon}(g(v, x(v))) dv \\ \text{s.t.} \\ \dot{x}(v) = A(v)x(v) + B(v) \frac{u(v)}{m(v)} \\ \dot{m}(v) = -c(v) \|u(v)\| \\ \|u(v)\| \leq 1 \quad v \in [v_0, v_f] \\ x(v_0) = x_0 \quad h(x(v_f)) = 0 \\ m(v_0) = m_0 \end{array} \right.$$

$$\psi_{\sigma,\alpha,\epsilon}(z) = \sigma \log \left( 1 + \exp \left[ \frac{z}{\sigma \epsilon} \left( 1 + \frac{\epsilon}{\alpha - z} \right) \right] \right) \xrightarrow{\sigma \rightarrow 0} \operatorname{Max} \left\{ 0, \frac{z}{\epsilon} \left( 1 + \frac{\epsilon}{\alpha - z} \right) \right\}$$



# The smoothed exact penalty method

The penalized problem  $(P)_{\sigma,\alpha,\epsilon}^2$

$$\left\{ \begin{array}{l} \text{Find } u_{\sigma,\alpha,\epsilon}^2 = \underset{u}{\operatorname{argmin}} J_{\sigma,\alpha,\epsilon}^2(u) = J_{\delta}(u) + \int_{v_0}^{v_f} \psi_{\sigma,\alpha,\epsilon}(g(v, x(v))) dv \\ \text{s.t.} \\ \dot{x}(v) = A(v)x(v) + B(v) \frac{u(v)}{m(v)} \\ \dot{m}(v) = -c(v) \|u(v)\| \\ \|u(v)\| \leq 1 \quad v \in [v_0, v_f] \\ x(v_0) = x_0 \quad h(x(v_f)) = 0 \\ m(v_0) = m_0 \end{array} \right.$$

G. Liuzzi and S. Lucidi: A Derivative-Free Algorithm for Inequality Constrained Nonlinear Programming via Smoothing of an  $l_{\infty}$  Penalty Function, *SIAM Journal on Optimization*, Vol. 20, No. 1, 2009, pp. 1-29.

# Algorithm

Let  $0 < q_1 < q_2 < 1$ ,  $0 < \alpha_{lim} < \alpha_0$ ,  $\epsilon_0 > 0$ ,  $\sigma_0 > \alpha_{lim}^{q_1}$ ,  $0 < \theta < 1$ ,  $0 < \tau < 1$

Let  $k = 0$ , end = false

WHILE (end = false)

Solve problem  $(P)_{\sigma_k, \alpha_k, \epsilon_k}^2 \rightarrow (x_{\sigma_k, \alpha_k, \epsilon_k}^2, m_{\sigma_k, \alpha_k, \epsilon_k}^2, u_{\sigma_k, \alpha_k, \epsilon_k}^2)$

IF  $(\alpha_k \leq \alpha_{lim})$  THEN

end = true

$(\bar{x}_\delta, \bar{m}_\delta, \bar{u}_\delta) \leftarrow (x_{\sigma_k, \alpha_k, \epsilon_k}^2, m_{\sigma_k, \alpha_k, \epsilon_k}^2, u_{\sigma_k, \alpha_k, \epsilon_k}^2)$

ELSE

IF  $\text{Min} \left\{ \epsilon_k, \int_{v_0}^{v_f} \text{Max} \left( 0, g(v, x_{\sigma_k, \alpha_k, \epsilon_k}^2(v)) \right) dv \right\} > \frac{\alpha_k^{q_2}}{\sigma_k}$  THEN

$$\epsilon_{k+1} = \tau \frac{\alpha_k^{q_2}}{\sigma_k}$$

ELSE

$$\epsilon_{k+1} = \epsilon_k$$

ENDIF

$$\alpha_{k+1} = \theta \alpha_k$$

$$\sigma_{k+1} = \text{Min} \left\{ \sigma_k, \alpha_{k+1}^{q_1} \right\}$$

$$k = k + 1$$

ENDIF

END WHILE

# Convergence results

## Lemma

- Let  $\alpha > 0$ ,  $z < \alpha$  and  $\epsilon > 0$  be given. Then,  $(\sigma \rightarrow \psi_{\sigma, \alpha, \epsilon}(z))$  is strictly increasing on  $]0, +\infty[$
- Let  $z < 0$ ,  $\epsilon > 0$  and  $\sigma > 0$  be given. Then,  $(\alpha \rightarrow \psi_{\sigma, \alpha, \epsilon}(z))$  is strictly increasing on  $]0, +\infty[$

## Convergence theorem - See Epenoy (2011) in JGCD

- $J_{\sigma_k, \alpha_k, \epsilon_k}^2 (u_{\sigma_k, \alpha_k, \epsilon_k}^2) \rightarrow J_\delta(\bar{u}_\delta)$  as  $k \rightarrow \infty$
- $u_{\sigma_k, \alpha_k, \epsilon_k}^2 \rightarrow \bar{u}_\delta$  according to the weak-\* topology on  $L^\infty([v_0, v_f], \mathbb{R}^3)$  as  $k \rightarrow \infty$
- $x_{\sigma_k, \alpha_k, \epsilon_k}^2 \rightarrow \bar{x}_\delta$  uniformly on  $[v_0, v_f]$  as  $k \rightarrow \infty$
- $m_{\sigma_k, \alpha_k, \epsilon_k}^2 \rightarrow \bar{m}_\delta$  uniformly on  $[v_0, v_f]$  as  $k \rightarrow \infty$

# Numerical results - A rendezvous in Highly Elliptical Orbit

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# Numerical results - A rendezvous in Highly Elliptical Orbit

## Statement of the test case

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# Numerical data - SIMBOL-X project

$$a = 106246.9753 \text{ km}$$

$$F_{max} = 0.1 \text{ N}$$

$$m_0 = 960.0 \text{ kg}$$

$$e = 0.798788$$

$$I_{sp} = 220 \text{ s}$$

$$v_0 = 3.317940017547 \text{ rad}$$

$$t_0 = 0.0 \text{ s}$$

$$v_f = 3.349161118514 \text{ rad}$$

$$t_f = 8000.0 \text{ s}$$

$$\begin{bmatrix} X(t_0) \\ Y(t_0) \\ Z(t_0) \end{bmatrix} = \begin{bmatrix} -100 \text{ m} \\ -100 \text{ m} \\ -100 \text{ m} \end{bmatrix}$$

$$\begin{bmatrix} X(t_f) \\ Y(t_f) \\ Z(t_f) \end{bmatrix} = \begin{bmatrix} 500 \text{ m} \\ 500 \text{ m} \\ 500 \text{ m} \end{bmatrix}$$

$$\begin{bmatrix} \frac{dX}{dt}(t_0) \\ \frac{dY}{dt}(t_0) \\ \frac{dZ}{dt}(t_0) \end{bmatrix} = \begin{bmatrix} 0.0 \text{ m/s} \\ 0.0 \text{ m/s} \\ 0.0 \text{ m/s} \end{bmatrix}$$

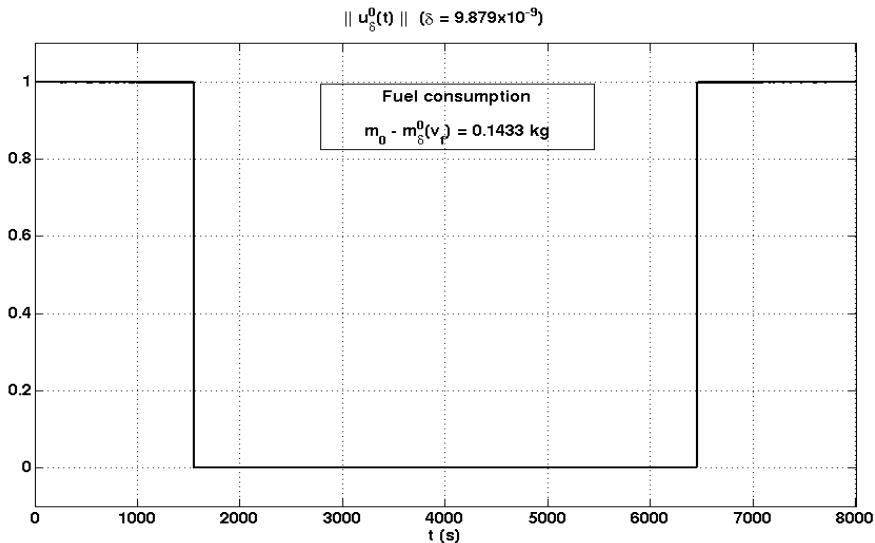
$$\begin{bmatrix} \frac{dX}{dt}(t_f) \\ \frac{dY}{dt}(t_f) \\ \frac{dZ}{dt}(t_f) \end{bmatrix} = \begin{bmatrix} 0.0 \text{ m/s} \\ 0.0 \text{ m/s} \\ 0.0 \text{ m/s} \end{bmatrix}$$

# Numerical results - A rendezvous in Highly Elliptical Orbit

## Unconstrained rendezvous

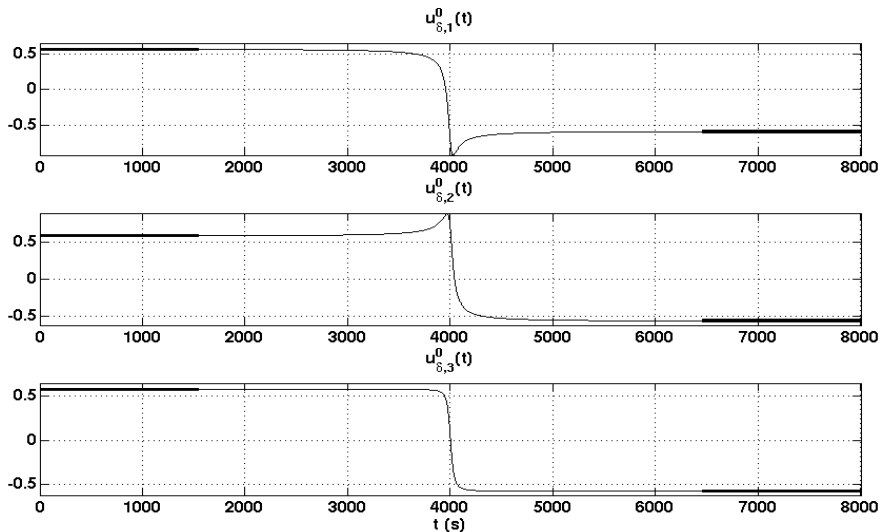
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# Norm of the normalized thrust vector

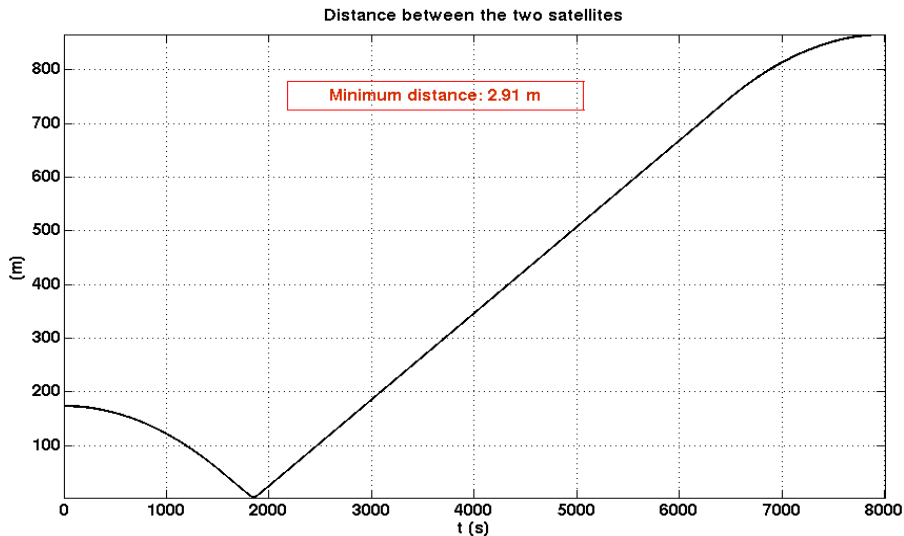




# Normalized thrust vector



# Intersatellite distance



# Numerical results - A rendezvous in Highly Elliptical Orbit

## Rendezvous under collision avoidance constraint

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# Parameters of the algorithm

Smoothing parameter: The same as for the unconstrained problem

$$\delta = 9.879 \times 10^{-9}$$

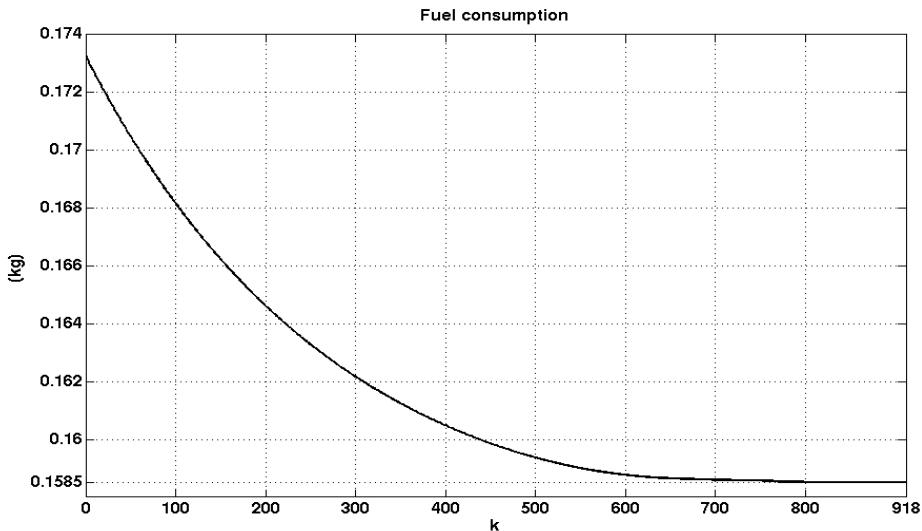
Case 1:  $d_{min} = 50.0$  m

$q_1 = 0.3$	$q_2 = 0.6$	$\alpha_{lim} = 4.73 \times 10^{-4}$	$\alpha_0 = 0.98$
$\epsilon_0 = 0.0999$	$\sigma_0 = 1.0$	$\theta = 0.99$	$\tau = 0.9$

Case 2:  $d_{min} = 140.0$  m

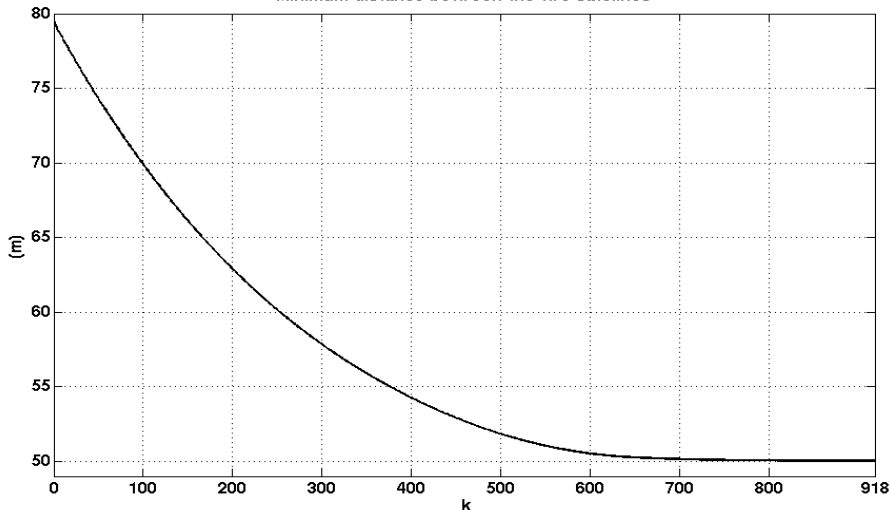
$q_1 = 0.3$	$q_2 = 0.6$	$\alpha_{lim} = 1.37 \times 10^{-4}$	$\alpha_0 = 0.98$
$\epsilon_0 = 0.0157$	$\sigma_0 = 1.0$	$\theta = 0.99$	$\tau = 0.9$

# Case 1 - Fuel consumption vs. iteration index $k$

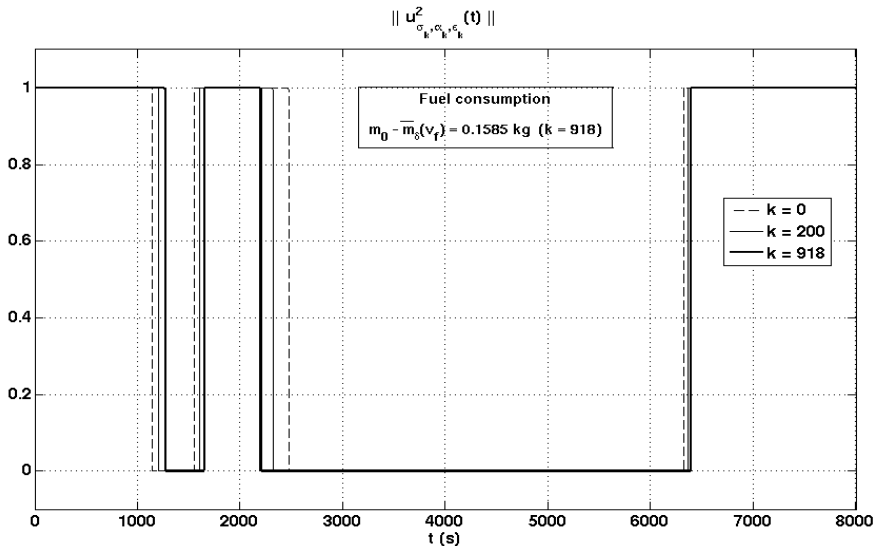


# Case 1 - Minimum distance vs. iteration index $k$

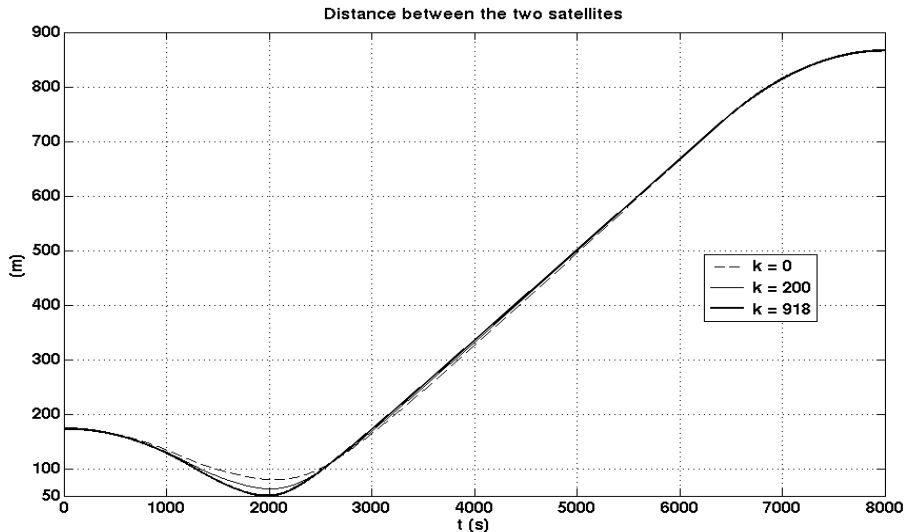
Minimum distance between the two satellites



## Case 1 - Norm of the normalized thrust vector

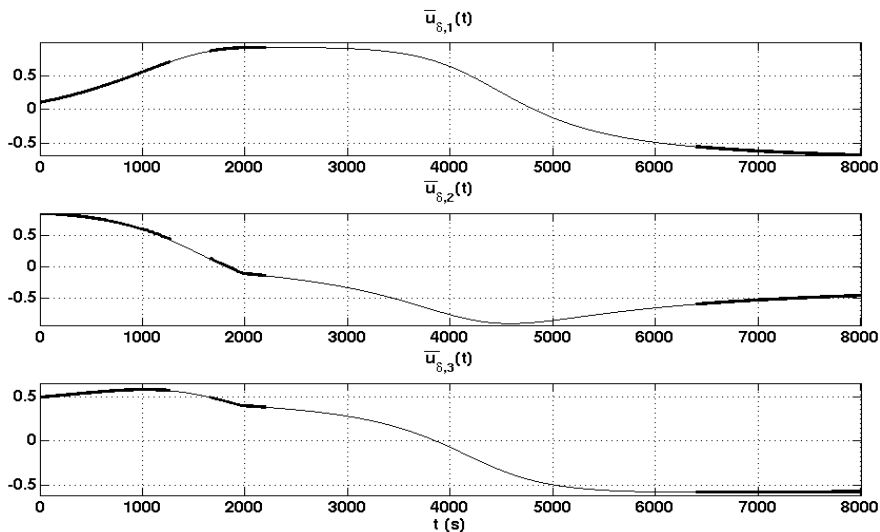


# Case 1 - Intersatellite distance

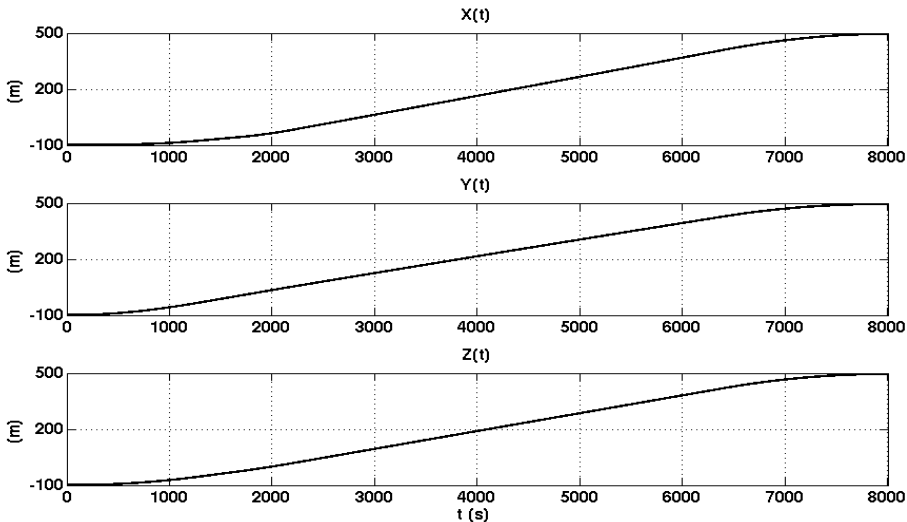




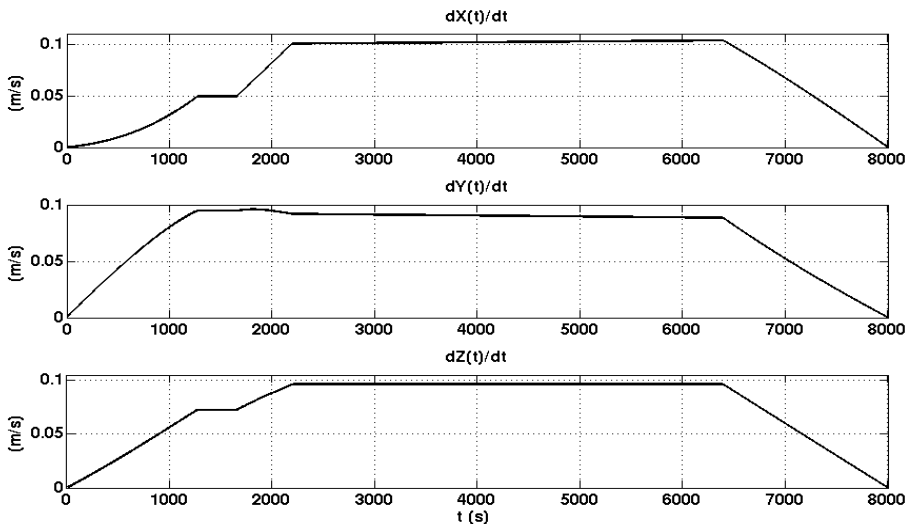
# Case 1 - Normalized thrust vector



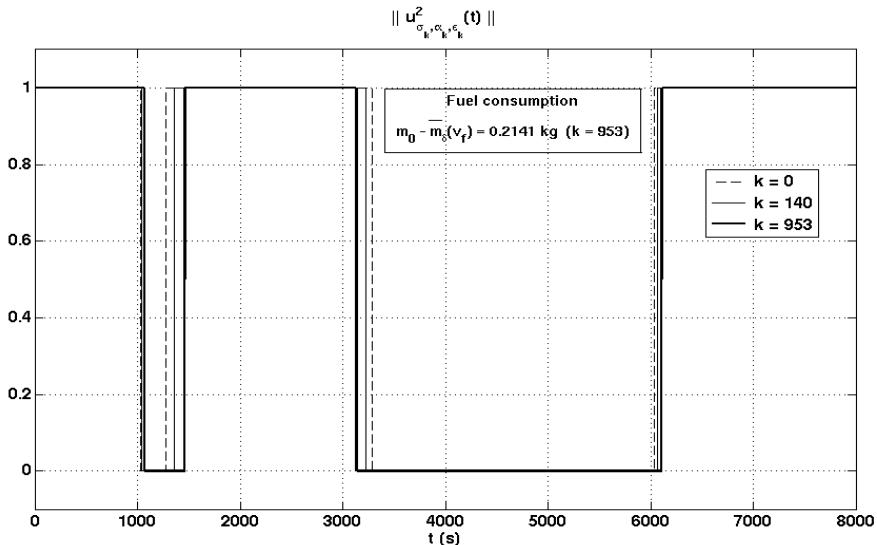
# Case 1 - Relative position vector



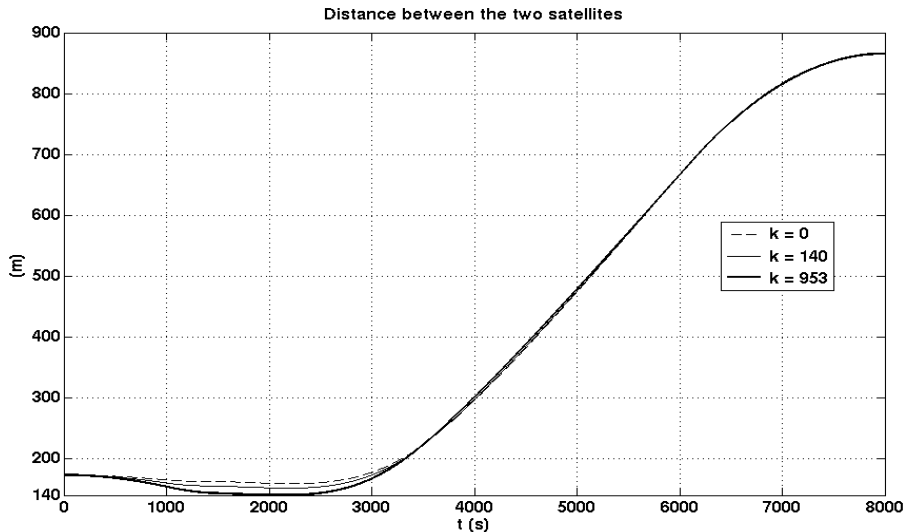
# Case 1 - Relative velocity vector



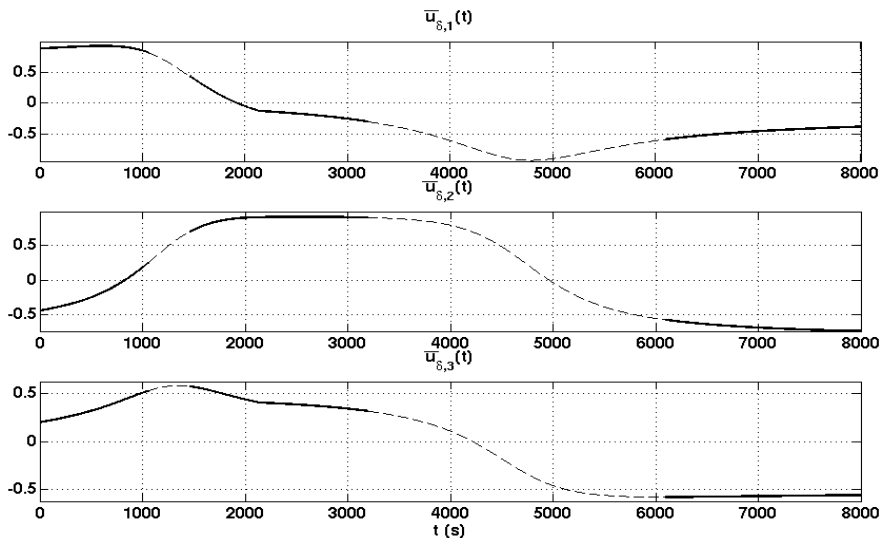
## Case 2 - Norm of the normalized thrust vector



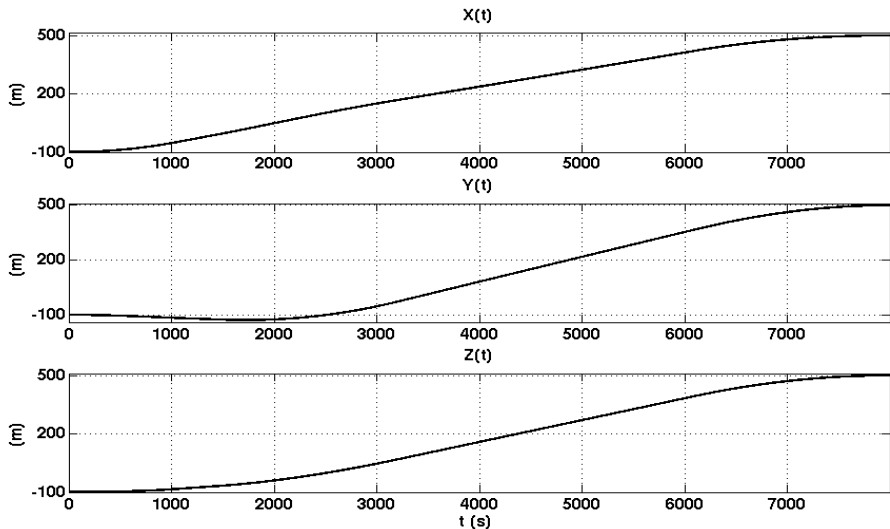
## Case 2 - Intersatellite distance



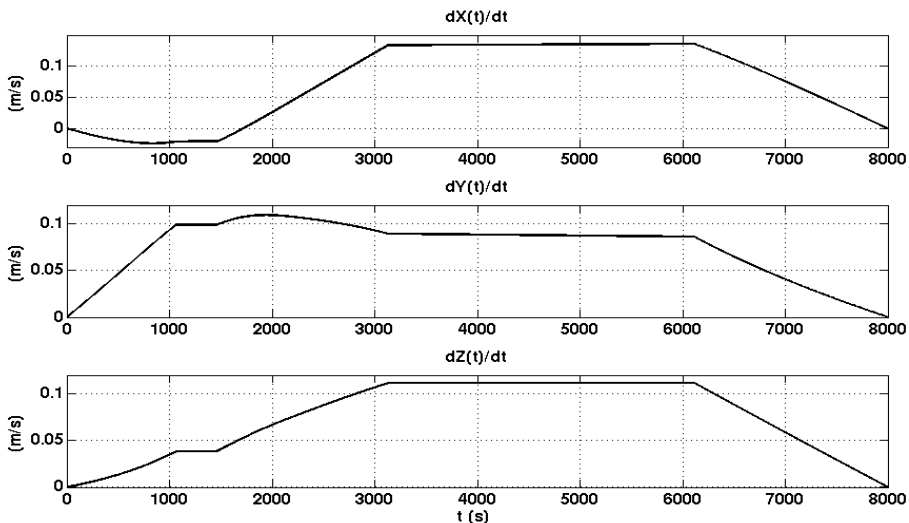
## Case 2 - Normalized thrust vector



## Case 2 - Relative position vector



## Case 2 - Relative velocity vector





# Conclusion and future prospects

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# Conclusion

## The smoothed exact penalty approach

- Applied for the first time to an optimal control problem
- Efficient to deal with the collision avoidance constraint
- Implies solving a sequence of state constraint-free problems
- More theoretically based than other penalization techniques
- Can be used to solve a large class of problems

## A just published paper

R. Epenoy: Fuel Optimization for Continuous-Thrust Orbital Rendezvous with Collision Avoidance Constraint, *Journal of Guidance, Control and Dynamics*, Vol. 34, No. 2, March-April 2011.

## Future prospects

### Minimum-fuel rendezvous in perturbed environment

Necessity to modify the dynamical equations

Low Earth Orbit applications  $\implies J_2$  term of the Earth's potential

### Application to other state-constrained problems

Reentry trajectories under thermal flux constraint

Interplanetary trajectories with minimum flyby altitude constraint

### Toward closing the loop

Model Predictive Control methodology

Neighboring extremal paths in the presence of state constraints

Hamilton-Jacobi-Bellman equation for state-constrained problems

**Thank you for your attention**