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Julio Araujo, V. Campos, Frédéric Giroire, Leonardo Sampaio, R. Soares. On the hull number of some graph classes. [Research Report] RR-7567, 2011, pp.12. inria-00576581v1

**HAL Id: inria-00576581**

**<https://inria.hal.science/inria-00576581v1>**

Submitted on 14 Mar 2011 (v1), last revised 14 Sep 2011 (v2)

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***On the hull number of some graph classes.***

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**N° 7567**

Mars 2011

Domaine 2

 ***rapport  
de recherche***



# On the hull number of some graph classes. \*

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Thème : Algorithmique, programmation, logiciels et architectures  
Équipe-Projet Mascotte

Rapport de recherche n° 7567 — Mars 2011 — 9 pages

**Abstract:** Given a graph  $G = (V, E)$ , the *closed interval* of a pair of vertices  $u, v \in V$ , denoted by  $I[u, v]$ , is the set of vertices that belongs to some shortest  $(u, v)$ -path. For a given  $S \subseteq V$ , let  $I[S] = \bigcup_{u, v \in S} I[u, v]$ . We say that  $S \subseteq V$  is a *convex set* if  $I[S] = S$ .

The *convex hull*  $I_h[S]$  of a subset  $S \subseteq V$  is the smallest convex set that contains  $S$ . We say that  $S$  is a *hull set* if  $I_h[S] = V$ . The cardinality of a minimum hull set of  $G$  is the *hull number* of  $G$ , denoted by  $hn(G)$ .

We show that deciding if  $hn(G) \leq k$  is a NP-complete problem, even if  $G$  is bipartite and we prove that  $hn(G)$  can be computed in polynomial time for cactus and  $P_4$ -sparse graphs.

**Key-words:** graph convexity, hull number, bipartite graph, cactus graph,  $P_4$ -sparse graph.

\* Research supported by the INRIA Equipe Associée EWIN.

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<sup>§</sup> Partially supported by CAPES/Brazil and ANR Blanc AGAPE.

## Le nombre enveloppe de quelques classes de graphes

**Résumé :** Étant donné un graphe  $G = (V, E)$ , l'intervalle fermé d'une paire de sommets  $u, v \in V$ , noté  $I[u, v]$ , est l'ensemble des sommets qui appartiennent à tous les plus courts chemins entre  $u$  et  $v$ . Pour un ensemble  $S \subseteq V$ , on note  $I[S]$  l'ensemble  $\bigcup_{u,v \in S} I[u, v]$ . De plus,  $S \subseteq V$  est dit *convexe* si  $I[S] = S$ .

L'*enveloppe convexe*  $I_h[S]$  d'un sous-ensemble  $S \subseteq V$  de  $G$  est défini comme le plus petit ensemble convexe qui contient  $S$ .  $S \subseteq V$  est une *enveloppe* si  $I_h[S] = V$ . Le *nombre enveloppe* de  $G$ , noté  $hn(G)$ , est la cardinalité minimum d'une enveloppe de graphe  $G$ .

Nous montrons que décider si  $hn(G) \leq k$  est un problème NP-complet, même si  $G$  est biparti et nous prouvons que  $hn(G)$  peut être calculé en temps polynomial pour les cactus et pour les graphes  $P_4$ -épars.

**Mots-clés :** convexité des graphes, nombre enveloppe, graphes bipartis, graphes  $P_4$ -épars, graphes cactus

## 1 Introduction

In this paper, we adopt the graph terminology defined in [2]. All the graphs studied in this work are considered to be simple and undirected.

Given a graph  $G = (V, E)$ , the *closed interval* of a pair of vertices  $u, v \in V$ , denoted by  $I[u, v]$ , is the set of vertices that belong to some shortest  $(u, v)$ -path. If  $S \subseteq V$ , let  $I[S] = \bigcup_{u, v \in S} I[u, v]$ . We say that  $S \subseteq V$  is a *convex set* if  $I[S] = S$ . Given a subset  $S \subseteq V$ , the *convex hull*  $I_h[S]$  of  $S$  is the smallest convex set that contains  $S$ . We say that  $S$  is a *hull set* if  $I_h[S] = V$ . The cardinality of a minimum hull set of  $G$  is the *hull number* of  $G$ , denoted by  $hn(G)$ . Determining the hull number of an arbitrary graph is a NP-hard problem [7].

These definitions bring concepts from Euclidian space to Graph Theory. There are several works in the literature about these notions [3–9, 11].

In Section 3, we answer an open question of Dourado et al. [7] by showing that it is NP-hard to compute the hull number of a given bipartite graph. We then present polynomial time algorithms to compute the hull number of cacti, in Section 4, and for  $P_4$ -sparse graphs in Section 5. This last result extends the previous known result for cographs [7]. Finally, we present some concluding remarks.

## 2 Preliminaries

In order to present our main results, it is necessary to first point some observations.

**Lemma 1** ([7]). *Every simplicial vertex of a graph  $G$  belongs to any hull set of  $G$ .*

**Lemma 2** ([7]). *If a graph  $G$  is not complete, then every universal vertex  $u$  of  $G$  does not belong to any minimum hull set of  $G$ .*

**Lemma 3** ([7]). *Let  $G$  be a graph and  $H$  be an isometric subgraph of  $G$ . Then for every hull set  $S$  of  $H$  it holds that  $V(H) \subseteq I_h(S)_G$ .*

**Lemma 4** ([7]). *Let  $G$  be a graph and  $S$  a proper and non-empty subset of  $V(G)$ . If  $V(G) \setminus S$  is convex, then every hull set of  $G$  contains at least one vertex of  $S$ .*

**Lemma 5.** *Let  $G = (V, E)$  be an arbitrary graph such that  $V$  can be partitioned into three sets  $K$ , a clique,  $R$ , a non-empty set which is not a clique, and  $S = V \setminus (K \cup R)$ . Moreover, for every vertex  $v \in R$ ,  $(N(v) \cap (V \setminus R)) = K$ . Then, any minimum hull set of  $G$  contains a minimum hull set of  $H = G[K \cup R]$ .*

*Proof.* First, observe that either in  $G$  or in  $H$  if there is shortest  $(u, v)$ -path containing an internal vertex in  $R$ , then both  $u$  and  $v$  are, necessarily, in  $R$  because every vertex of  $R$  is neighbor of every vertex in  $K$  and of no vertex in  $S$ .

By the above remark, all vertices of  $R$  can only be generated by two vertices in  $R$  and every vertex of  $K$  is also generated by two non adjacent vertices of  $R$ . Consequently, to generate all the vertices of  $R$  in  $G$ , a subset  $S_R \subseteq R$  must be chosen which is also a hull set of  $H$ . Therefore, any minimum hull set of  $G$  contains a minimum hull set of  $H$ .  $\square$

## 3 Bipartite graphs

In this section, we study the complexity of deciding if  $hn(G) \leq k$ , for a bipartite graph  $G$  and an integer  $k$  given as input. However, we first show some approximability results.

Let  $I(G)$  be the *incidence graph* of  $G$ , obtained from  $G$  by adding one vertex  $s_{uv}$ , for each edge  $uv \in E(G)$ , and replacing the edge  $uv$  by the edges  $us_{uv}, s_{uv}v$ .

**Proposition 6.**  $hn(I(G)) \leq hn(G) \leq 2hn(I(G))$ .

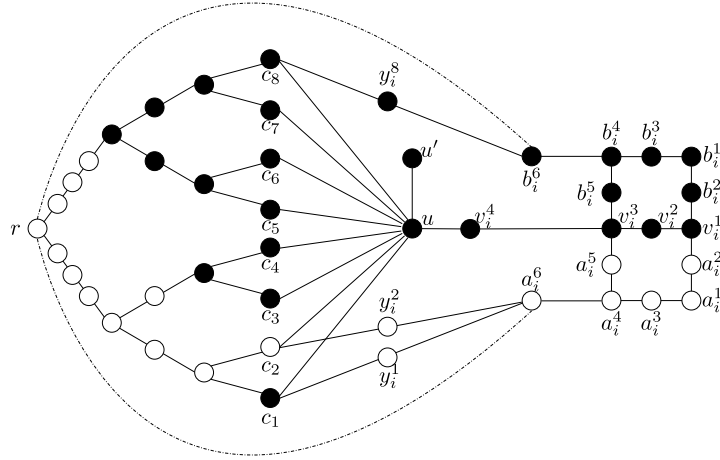


Figure 1: Subgraph of the bipartite instance  $G(\mathcal{F})$  containing the gadget of a variable  $x_i$  that appears positively in clauses  $C_1$  and  $C_2$  and negatively in  $C_8$ . White vertices belong to the set  $U$  in case  $C_2$  is not satisfied.

*Proof.* Let  $I(G)$  be the incidence graph of  $G$ . Observe that any hull set of  $G$  is a hull set of  $I(G)$ , since any shortest path in  $G$  is a shortest path in  $I(G)$  (the edges were subdivided). Consequently,  $hn(I(G)) \leq hn(G)$ . However, given a hull set  $S_h$  of  $I(G)$ , one may find a hull set of  $G$  by simply replacing each vertex of  $S_h$  that represents an edge of  $G$  by its neighbors (vertices of  $G$ ). Thus,  $hn(G) \leq 2hn(I(G))$ .  $\square$

**Corollary 7.** *If there exists a  $k$ -approximation algorithm  $B$  to bipartite graphs, then  $B$  is a  $2k$ -approximation algorithm for any graph.*

Due to the next result, one may not expect to find an exact polynomial time algorithm to calculate a hull number of a bipartite graph, unless  $P = NP$ .

**Theorem 8.** *Given a bipartite graph  $G$  and an integer  $k$ , deciding if  $hn(G) \leq k$  is a NP-complete problem.*

*Proof.* This proof for bipartite graphs is just a slight modification of the proof for general graphs [7]. Further, it is no longer necessary to show that this problem is in NP, since it was already done for any graph.

We reduce the SATisfiability Problem to Hull Number in bipartite graphs. In this problem the input is a set of clauses  $\mathcal{F} = \{C_1, C_2, \dots, C_m\}$  in the conjunctive normal form over a set of boolean variables  $X = \{x_1, x_2, \dots, x_n\}$ . The goal is to decide if there is an assignment of the variables such that each clause is satisfied.

We assume that  $m$  is a power of 2, since one can trivially add dummy variables and clauses without changing the satisfiability of  $\mathcal{F}$ . Moreover, we also assume that each variable  $x_i$  and its negation appear at least once in  $\mathcal{F}$ , otherwise the clauses that  $x_i$  appears could always be satisfied.

Given an instance  $\mathcal{F}$  to the SAT problem, we construct a bipartite graph  $G(\mathcal{F})$  as follows. First, we build a full binary tree  $T$  rooted in  $r$  with  $m$  leaves, and we denote by  $L = \{c_1, c_2, \dots, c_m\}$  the set of leaves of  $T$ . We then construct a graph  $H$  by adding a vertex  $u$  that is adjacent to every vertex in  $L$  and replace every edge joining a vertex  $v \in V(T)$  to its parent in  $T$  by a path with  $2^{h(v)}$  edges, where  $h(v)$  is the distance of  $v$  to any of its decendent leaves.

The following claims are proved in [7].

Claim I. Let  $v, w \in V(T) \setminus \{r\}$ . The closed interval of  $v, w$  in  $H$  contains the parents of  $v$  in  $T$  if and only if  $v$  and  $w$  are siblings in  $T$ .

Claim II. The set  $L$  is a *minimal* hull set of  $H$ .

Finally, we build a graph  $G(\mathcal{F})$  by adding to  $H$  the vertex  $u'$  whose only neighbor is  $u$  and by adding, for each variable  $x_i$ , a variable subgraph in  $G(\mathcal{F})$  with vertex set  $\bigcup_{j=1}^6 \{a_i^j, b_i^j\} \cup \{\bigcup_{j=1}^4 v_i^j\} \cup \{\bigcup_{\{j \mid x_i \in C_j\}} y_i^j\}$  and edges as depicted in Figure 1.

Furthermore, we complete the construction of  $G(\mathcal{F})$  by adding, for every variable  $x_i$ , one edge joining  $u$  to  $v_i^4$  and creating two paths with  $2^{h(r)} - 3$  edges, one joining  $a_i^6$  to  $r$  and the other joining  $b_i^6$  to  $r$ . Denote by  $D$  the set of all internal vertices of these paths.

Claim III.  $G(\mathcal{F})$  is a bipartite graph.

Let us present a proper 2-coloring  $c$  of  $G(\mathcal{F})$ . Let  $c(r) = 1$ , and for each vertex  $w$  in  $V(H)$ , define  $c(w)$  as 1, if  $w$  is in an even distance from  $r$ , and 2 otherwise. Clearly,  $c$  is a partial *proper* coloring of  $G(\mathcal{F})$  and moreover we have  $c(u) = 1$  and  $c(c_j) = 2$ , for any  $j \in \{1, \dots, m\}$ . Let  $c(u') = 2$ . For each  $i \in \{1, \dots, n\}$ , let  $c(y_i^j) = 1$ , for every  $j$  such that  $x_i \in C_j$ , and  $c(b_i^6) = c(a_i^6) = c(v_i^4) = 2$ . Again, this partial coloring of  $G(\mathcal{F})$  is proper. One can easily verify that this coloring can be extended to each variable subgraph. Moreover, since  $c(r) = 1$  and  $c(a_i^6) = 2$  ( $c(b_i^6) = 2$ ), for every  $i \in \{1, \dots, n\}$ , and since the path that we add in  $G(\mathcal{F})$  between  $r$  and  $a_i^6$  ( $b_i^6$ ) is of odd size, one can completely extend  $c$  in order to get a proper 2-coloring of  $G(\mathcal{F})$ .

Set  $k = n + 1$ . We show that  $\mathcal{F}$  is satisfiable if and only if the  $hn(G(\mathcal{F})) \leq n + 1$ .

First, consider that  $\mathcal{F}$  is satisfiable. Given an assignment  $A$  that turns  $\mathcal{F}$  true, define a set  $S$  as follows. For  $1 \leq i \leq n$ , if  $x_i$  is true in  $A$  add  $a_i^1$  to  $S$ , otherwise add  $b_i^1$  to  $S$ . Finally, add  $u'$  to  $S$ . Note that  $|S| = n + 1$ . We show that  $S$  is a hull set of  $G(\mathcal{F})$ . First note that  $a_i^6, c_j \in I[a_i^1, u']$ , for every clause  $C_j$  containing the positive literal of  $x_i$ . Similarly, observe that  $b_i^6, c_j \in I[b_i^1, u']$ , for every clause  $C_j$  containing the negative literal of  $x_i$ . Since  $A$  satisfies  $\mathcal{F}$ , it follows  $L \subseteq I_h[S]$ . Therefore,  $H$  being an isometric subgraph of  $G(\mathcal{F})$ , Lemma 3 and Claim II imply that  $V(H) \subseteq I_h[S]$ . Furthermore, the shortest paths between  $r$  and  $u$  have length  $2^{h(r)}$ , which implies that all vertices  $a_{i,6}$  and  $b_{i,6}$  are included in  $I_h[S]$ . It remains to observe that  $I_h[a_i^6, b_i^6, w]$ , where  $w \in [a_i^1, b_i^1]$ , contains the variable subgraph of  $x_i$ . Therefore we have that  $S$  is a hull set of  $G(\mathcal{F})$ .

To prove the sufficiency, we need the following claim:

Claim IV. The set  $V(G(\mathcal{F})) \setminus \{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$  is convex, for any  $i \in \{1, \dots, n\}$ .

Denote  $W = \{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$ , for some  $i$ , and  $W' = \{a_i^3, b_i^3, v_i^2\}$ . It is easy to verify that for any pair  $x, y \in W'$ ,  $I[x, y]$  contains no vertex of  $W$ .

We prove the sufficiency by contradiction. Suppose that  $G(\mathcal{F})$  contains a hull set  $S$  with at most  $n + 1$  vertices and that  $\mathcal{F}$  is not satisfiable.

Clearly  $u' \in S$ , because  $u'$  is a simplicial vertex of  $G(\mathcal{F})$ . Furthermore, Claim IV and Lemma 4 imply that  $S$  must contain at least one vertex  $w_i$  of the set  $\{a_i^1, a_i^2, v_i^1, b_i^1, b_i^2\}$ , for every  $i \in \{1, \dots, n\}$ . Hence,  $|S| = n + 1$ . Since  $v_i^1 \in I[u', a_i^1]$ ,  $v_i^1 \in I[u', b_i^1]$ ,  $a_i^2 \in I[u', a_i^1]$  and  $b_i^2 \in I[u', b_i^1]$ , we can assume, without loss of generality, that either  $w_i = a_i^1$  or  $w_i = b_i^1$ , for every  $i \in \{1, \dots, n\}$ . Therefore  $S$  defines the following truth assignment  $A$  to  $\mathcal{F}$ . If  $w_i = a_i^1$  set  $x_i$  true, else set  $x_i$  as false.

Using the hypothesis that  $\mathcal{F}$  is not satisfiable, we complete the proof by showing that there is a set  $U$  such that  $V(G(\mathcal{F})) \setminus U$  is a convex set and  $U \cap S = \emptyset$ .

As  $\mathcal{F}$  is not satisfiable, there exists at least one clause  $C_j$  not satisfied by  $A$ . Let  $U'$  be the set of vertices belonging to the path from  $c_j$  to  $r$  in  $G(\mathcal{F})$ ; and  $U''$  formed by the vertices of  $G(\mathcal{F})$  which are inner vertices of a path between a vertex from  $U' \cap T$  and its child in  $T$  that is not in  $U'$ . Define  $U \subset V(G(\mathcal{F}))$  as the union of the following sets (we suggest the reader to see Figure 1):

- the set  $D$ ;
- $U' \cup U''$ ;
- the union of the sets  $\{a_i^p, 1 \leq p \leq 6\}$  (resp.  $\{b_i^p, 1 \leq p \leq 6\}$ ) for each  $i$  such that  $x_i$  occurs positively (resp. negatively) in  $C_j$ ;



- the union of the sets  $\{y_i^k \mid x_i \text{ appears positively in } C_k\}$  (resp.  $\{y_i^k \mid x_i \text{ appears negatively in } C_k\}$ ) for each  $i$  such that  $x_i$  occurs positively (resp. negatively) in  $C_j$ .

Observe that  $U \cap S = \emptyset$ . It remains to prove that  $V(G(\mathcal{F})) \setminus U$  is a convex set. In order to do this, we show that  $I[w, w'] \cap U = \emptyset$  for every pair  $w, w' \in V(G(\mathcal{F})) \setminus U$ . Consider the following partition  $A_1 \cup A_2 \cup A_3$  of  $V(G(\mathcal{F})) \setminus U$ :

- $A_1 = V(H) \setminus (U \cup \{u\})$ ;
- $A_2$  contains the vertices of the variable subgraphs which are not in  $U$ ;
- $A_3 = \{u, u'\}$

Case 1:  $w, w' \in A_1$ . We can assume that  $w, w' \in V(T)$ . Suppose by contradiction that there exists  $x \in I[w, w'] \cap U$ . It is easy to see that  $x \notin D$ , and, by using Claim I,  $x \notin U \cap V(H)$ . The only possibility is that  $x = a_i^6$  or  $x = b_i^6$ , for some  $i$ . But in this case, there would be a strictly shorter path between  $w$  and  $w'$  passing through  $u$ .

Case 2:  $w \in A_1$  and  $w' \in A_2$ . If  $w'$  does not belong to the gadget of a variable  $x_i$  that appears in  $C_j$ , then it is trivial to check that the shortest  $(w, w')$ -paths contain only vertices of  $V(G(\mathcal{F})) \setminus U$ . Otherwise, without loss of generality, we may assume that  $w = c_k$ , for some  $k \in \{1, \dots, m\} \setminus \{j\}$  and that  $w' \in \{v_i^1, v_i^3, v_i^4\} \cup \{y_i^l \mid \text{for every } l \text{ s.t. } y_i^l \notin U\}$ . In any case, it is easy to check that  $I[w, w']$  does not contain any vertex of  $U$ .

Moreover, all the other cases can be trivially verified. □

## 4 Cacti graphs

It is easy to observe that the hull number a tree is size of the set of its leaves. In this section, we generalize this simple result to the class of cacti. A *cactus* is a graph in which every pair of cycles have at most one common vertex. In order, to compute the hull number of a cactus, we present which vertices must be chosen.

**Lemma 9.** *Let  $C$  be an induced cycle in  $G$ . If  $C$  has at most one vertex  $v$  such that  $d(v) > 2$ , then any minimum hull set of  $G$  contains either one or two vertices of  $V(C)$ .*

*Proof.* The proof follows from the simple observation that  $(V(G) \setminus V(C)) \cup \{v\}$  is a convex set. Thus, if  $C$  is odd, it is necessary to choose two vertices of  $C$ , otherwise, one suffices. □

**Lemma 10.** *Let  $C$  be an induced cycle in  $G$ . Let  $v_1, \dots, v_k$  be the vertices of  $C$  having degree greater than two, for  $k \geq 2$ . If there is a set of vertices  $S$  of  $C$  such that they are not in any minimum  $(v_i, v_j)$ -path,  $1 \leq i, j \leq k$ , in  $C$ , then any minimum hull set of  $G$  contains either one or two vertices of  $S$ .*

*Proof.* Again, observe that  $V(G) \setminus S$  is a convex set and that  $S$  induces a path  $P_k$  on  $k$  vertices. If  $k$  is odd, then one vertex of  $S$  suffices to generate the others, otherwise, it is necessary to select two of them. □

**Theorem 11.** *Given a cactus  $G$ ,  $hn(G)$  can be found in linear time.*

*Proof.* By a simple breadth-first search all the cycles of a cactus can be detected. For each one, it is necessary to detect if it has at most one vertex of degree greater than two. In case it has at least two vertices of degree greater than two, it is necessary to also verify if there is a set  $S$  satisfying Lemma 10. Finally, every vertex of degree one must be chosen. These tests be done in linear time and the result follows by Lemmas 1, 9 and 10. □

## 5 $P_4$ -sparse graphs

A graph is said to be a *cograph* if it is  $P_4$ -free. Computing the hull number of cographs can be done in polynomial time [7]. In this section, we extend this result to the class of  $P_4$ -sparse graphs which properly contains the cographs.

A graph is  $P_4$ -sparse if, for every  $S \subseteq V$  such that  $|S| = 5$ ,  $G[S]$  contains at most one induced  $P_4$ . Clearly, a cograph is a  $P_4$ -sparse graph. We call  $G$  a *strict  $P_4$ -sparse graph* if  $G$  is  $P_4$ -sparse and contains at least one  $P_4$  as induced subgraph.

The algorithm we present to compute the hull number of any  $P_4$ -sparse graph  $G$  uses its modular decomposition tree  $T(G)$ . The idea is to perform a post-order traversal in  $T(G)$  calculating  $hn(G)$ . The definition and classical results about the modular decomposition of graphs can be found in [12].

It is well-known that the neighborhood nodes of the modular decomposition of  $P_4$ -sparse graphs are spiders [10].

A *spider* is a graph whose vertex set can be partitioned into disjoint sets  $S$ ,  $K$ , and  $R$  such that:  $|S| = |K| \geq 2$ ,  $S$  is a stable set,  $K$  is a clique; every vertex in  $R$  is adjacent to all the vertices in  $K$  and to no vertex in  $S$ ; there exists a bijection  $f : S \rightarrow K$  such that either  $N_G(s) \cap K = K - \{f(s)\}$ , for all vertices  $s \in S$ , in this case the spider is called *thick spider*; or  $N_G(s) \cap K = \{f(s)\}$ , for all vertices  $s \in S$ , and it is called *thin spider*.

Note that  $R$  may be empty, it is the only possible non-trivial strong submodule of a spider, and that if  $|S| = |K| = 2$ , the spider is, by definition, thin and thick. However, in order to simplify the notation in the proofs, we consider that a spider is called *thick* if, and only if,  $|S| = |K| \geq 3$ . Moreover, we remark that it is possible to find a modular decomposition tree in such a way that the series and parallel nodes of the tree have only two children each.

Finally, we present how to compute the hull number of each inner node of the modular decomposition tree of a  $P_4$ -sparse graph.

**Lemma 12** ([7]). *If  $G = G_1 \cup G_2$ , for any two graphs  $G_1$  and  $G_2$ , then  $hn(G) = hn(G_1) + hn(G_2)$ .*

**Lemma 13.** *Let  $G_1$  and  $G_2$  be two  $P_4$ -sparse graphs and  $G = G_1 \oplus G_2$ . Thus,  $hn(G)$  can be computed in polynomial-time.*

*Proof.* If  $G_1$  and  $G_2$  are both cographs, then  $hn(G)$  can be found in polynomial time according to Dourado et al. [7]. Let  $G_1$  be a graph which is not complete and  $G_2$  be a strict  $P_4$ -sparse. We claim that two non adjacent vertices of  $G_1$  form a minimum hull set of  $G$  because they generate all the vertices of  $G_2$  and  $G_2$  also have two non adjacent vertices that generate all vertices of  $G_1$ .

It remains to consider the case in which  $G_1$  is a complete graph and  $G_2$  is a strict  $P_4$ -sparse graph. First, observe that no vertex of  $G_1$  belongs to any minimum hull set of  $G$ . Let  $r$  be the root of the modular decomposition tree of  $G_2$ . Let us consider all the possible cases for  $r$ .

**Case 1:**  $r$  is a series node. We claim that in this case  $hn(G) = hn(G_2)$ .

As a matter of fact, since  $G_2$  is composed by the join of graphs, the distance between any two vertices is at most 2. Consequently, any shortest  $(u, v)$ -path between vertices in  $G_2$  is also a shortest  $(u, v)$ -path in  $G$ . Moreover, a minimum hull set from  $G_2$  generates all vertices of  $V(G_2)$  in  $G$ . Finally, since  $G_2$  has two vertices  $u$  and  $v$  that are non-adjacent they generate every vertex of  $G_1$ .

**Case 2:**  $r$  is a parallel node. Let  $G_2 = \bigcup_{i=1}^k G_2^i$ . In this case, we claim that  $hn(G) = \sum_{i=1}^k f(i)$  such that

$$f(i) = \begin{cases} hn(G_1 \oplus G_2^i), & \text{if } G_2^i \text{ is not complete;} \\ |V(G_i)|, & \text{otherwise.} \end{cases}$$

We prove the claim by showing that any minimum hull set of  $G$  in this case is composed by all the vertices of  $G_2^i$ , for every  $i$  such that  $G_2^i$  is a complete graph, union all the minimum hull sets of  $G_1 \oplus G_2^j$ , for every  $j$  such that  $G_2^j$  is not complete.

Recall that, by Lemma 2, no vertex of  $G_1$  belongs to a minimum hull set of  $G$ . By Lemma 1, every vertex of  $G_2^i$  such that  $G_2^i$  is complete must belong to every minimum hull set of  $G$ .

It remains to show that it is necessary and sufficient to also include a minimum hull set of every  $hn(G_1 \oplus G_2^j)$  such that  $G_2^j$  is not complete, in every minimum hull set of  $G$ .

The necessity can be easily proved by just applying Lemma 5 to every  $G_2^i$  that is not a complete graph.

The sufficiency follows again from the fact that  $G_1$  is generated by two non adjacent vertices of  $G_2$  and for each  $G_2^i$  either  $V(G_2^i)$  is chosen or  $V(G_2^i)$  is generated by a minimum hull set of  $G_1 \oplus G_2^i$ .

**Case 3:**  $r$  is a neighborhood node. Thus,  $G_2 = (S \cup K \cup R, E_2)$  is a spider. We claim that  $hn(G) = hn(G_2) + 1$  if  $G_2$  is a thin spider and  $R = \emptyset$ , otherwise  $hn(G) = hn(G_2)$ .

First, no matter if  $G_2$  is a thin or a thick spider, by Lemma 1, every vertex of  $S$  must belong to any minimum hull set of  $G_2$  and also of  $G$  and, again, no vertex of  $G_1$  belong to any minimum hull set of  $G$ . Moreover, all vertices in  $S$  are simplicial in  $G_2$  and by the same argument they must belong to any minimum hull set of  $G_2$ .

Observe that, if  $G_2$  is thin and  $R = \emptyset$ ,  $S$  is a hull set of  $G_2$ , but it is not a hull set of  $G$  since the vertices of  $K$  are not generated by them in  $G$ . Consequently, it is necessary and sufficient to add one, and just one, vertex of  $K$  to have a minimum hull set of  $G$ .

In the case that  $G_2$  is thick and  $R = \emptyset$ , the vertices of  $S$  are sufficient to generate  $G_2$  and also  $G$ , because every vertex of  $K$  is adjacent to at least two vertices of  $S$  that are necessarily non adjacent.

On the other hand, if  $R \neq \emptyset$ , then by Lemma 1 any minimum hull set of  $G_2$  contains  $S$  and, by Lemma 5 a minimum hull set of  $K \cup R$  which is composed by vertices of  $R$ .

By the same Lemmas, a minimum hull set of  $G_2$  is a minimum hull set of  $G$  since, by Lemma 2, no vertex of  $G_1$  belongs to any minimum hull set of  $G$  and  $G_1$  is generated by non adjacent vertices of  $G_2$ .  $\square$

**Lemma 14.** Let  $G = (S \cup K \cup R, E)$  be a spider. Then,

$$hn(G) = \begin{cases} |S| + hn(G[K] \oplus G[R]), & \text{if } R \neq \emptyset \text{ and } R \text{ is not a clique;} \\ |S| + |R|, & \text{otherwise.} \end{cases}$$

*Proof.* Since all the vertices in  $S$  are simplicial vertices, we apply Lemma 1 to conclude that they are all contained in any hull set of  $G$ .

By the structure of a spider, every vertex of  $K$  belongs to a shortest path between two vertices in  $S$  and are therefore generated by them in any minimum hull set of  $G$ . Consequently, if  $R = \emptyset$ ,  $S$  is a minimum hull set of  $G$ . If  $R$  is a clique,  $S \cup R$  is the set of simplicial vertices of  $G$  and also a minimum hull set of  $G$ .

Finally, if  $R \neq \emptyset$  and  $R$  is not a clique, then Lemma 5 completes the proof.  $\square$

**Theorem 15.** Given a  $P_4$ -sparse graph  $G$ , the hull number of  $G$  can be found in polynomial time.

*Proof.* The modular decomposition tree of any graph  $G$  can be found in linear time [13].

The complexity of the post-order traversal is  $O(n)$ . For each inner node, the algorithm may spend  $O(n^2)$  operations in case 2 of Lemma 13 or when  $R \neq \emptyset$  in Lemma 14. Consequently, the complexity of the algorithm is  $O(n^3)$ .  $\square$

## 6 Conclusions

In this paper, we simplified the reduction of Dourado et al. [7] to answer two questions they asked about the complexity of computing the hull number of triangle-free graphs and, in particular, of bipartite graphs. However, the complexity of this problem for interval graphs and chordal graphs remains open.

Finally, we used the modular decomposition tree of  $P_4$ -sparse graphs to compute the hull number of these graphs. The  $P_4$ -sparse graphs are contained in the class of  $(q, q-4)$ -graphs for which every set of  $q$  vertices induce at most  $q-4$  disjoint  $P_4$ 's [1]. It would be nice to extend our results to this class of graphs.

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ISSN 0249-6399