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▶ To cite this version:

Guy Fayolle, Kilian Raschel. Random walks in the quarter plane with zero drift: an explicit criterion for the finiteness of the associated group. Markov Processes And Related Fields, 2011, 17 (4), pp.619-636. inria-00572276

HAL Id: inria-00572276

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Submitted on 1 Mar 2011

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N° 7555

Mars 2011

Thème NUM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--7555--FR+ENG

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE
INRIA
centre de recherche PARIS - ROCQUENCOURT

Random walks in the quarter-plane with zero drift: an explicit criterion for the finiteness of the associated group

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Thème NUM — Systèmes numériques
Équipe-Projet Imara

Rapport de recherche n° 7555 — Mars 2011 — 15 pages

Abstract: In many recent studies on random walks with small jumps in the quarter plane, it has been noticed that the so-called *group of the walk* governs the behavior of a number of quantities, in particular through its *order*. In this paper, when the *drift* of the random walk is equal to 0, we provide an effective criterion giving the order of this group. More generally, we also show that in all cases where the *genus* of the algebraic curve defined by the *kernel* is 0, the group is infinite, except precisely for the zero drift case, where finiteness is quite possible.

Key-words: Automorphism, generating function, genus, piecewise homogeneous random walk, quarter-plane, Weierstrass elliptic function.

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Marches aléatoires dans le quart de plan avec dérive nulle: un critère explicite de finitude pour le groupe associé

Résumé : Dans plusieurs études récentes sur les marches aléatoires dans le quart de plan avec des sauts vers les huit plus proches voisins, il apparat que le comportement de certaines quantits d'intérêt est directement lié au *groupe de la marche*, notamment à la finitude de son ordre. Pour les marches dérive nulle, nous donnons une formule pour l'ordre de ce groupe, en fonction explicite des probabilits de saut. De façon générale, lorsque le *genre* de la courbe algébrique définie par le *noyau* est 0, le groupe est toujours infini, sauf précisément lorsque le saut moyen est nul, auquel cas la finitude est parfaitement possible.

Mots-clés : Automorphisme, fonction génératrice, genre, marche aléatoire homogène par morceaux, quart de plan, fonction elliptique de Weierstrass.

1 Introduction and main results

For several decades, lattice random walks in the Euclidean quarter plane with unit jumps (or steps) have been at the crossroads of several mathematical areas: probability, complex analysis and, more recently, combinatorics. In this context, one of the basic problems often amounts to solve functional equations of two complex variables x, y, which have the following typical form

$$K(x,y)Q(x,y) = k(x,y)Q(x,0) + \widetilde{k}(x,y)Q(0,y) + k_0(x,y)Q(0,0) + \kappa(x,y), \tag{1.1}$$

where

- $K(x,y)$ [usually called the *kernel*], $k(x,y)$, $\widetilde{k}(x,y)$, $k_0(x,y)$, and $\kappa(x,y)$ are known functions;
- $Q(x,y)$ is sought to be analytic in the region $\{|x|, |y| \leqslant 1\}$ and continuous up to the boundary.

We briefly describle two situations where equations of type (1.1) appear.

Example 1.1. This example deals with the random walks with jumps of unit length inside the quarter plane, but arbitrary big on the axes. The question is then to calculate the related invariant measure $\{\pi_{i,j}\}_{i,j\geqslant 0}$ as well as to provide conditions for its existence. Letting

$$Q(x,y) = \textstyle\sum_{i,j\geqslant 0} \pi_{i,j} x^i y^j,$$

the classical Kolmogorov's equations yield (see, e.g., [4, 5]) equation (1.1) with $\kappa(x,y) = 0$, whereas $k(x,y)$, $\widetilde{k}(x,y)$ and $k_0(x,y)$ correspond to the generating functions of the jump probabilities on the horizontal axis, on the vertical axis and at $(0,0)$, respectively. Letting $\{p_{i,j}\}_{-1\leqslant i,j\leqslant 1}$ denote the jumps in $\mathbb{Z}_+^2$, the *kernel* is then given by

$$K(x,y) = xy[\textstyle\sum_{-1\leqslant i,j\leqslant 1} p_{i,j} x^i y^j - 1]. \tag{1.2}$$

Example 1.2. The enumeration of planar lattice walks—a classical topic in combinatorics—is the topic of this second example. For a given set $\mathcal{S}$ of allowed jumps, it is a matter of counting the number of paths of a given length, starting from a fixed origin and ending at an arbitrary point, and possibly restricted to certain regions of the plane. A first basic and natural question arises: how many such paths exist? If the paths are confined to $\mathbb{Z}_+^2 = \{0, 1, \ldots\}^2$ and if $\mathcal{S}$ is included in the set of the eight nearest neighbors, let $q(i,j;n)$ denote the number of paths of length n, which start from $(0,0)$ and end at (i,j). Then

$$Q(x,y) = \sum_{i,j,n\geqslant 0} q(i,j;n) x^i y^j z^n$$

does satisfy equation (1.1), see [2], where the kernel is now equal to

$$K(x,y) = xyz[\textstyle\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z],$$

$k(x,y) = K(x,0)$, $\widetilde{k}(x,y) = K(0,y)$, $k_0(x,y) = -K(0,0)$ and $\kappa(x,y) = -xy$. Here $Q(x,y)$ does exist at least in the domain $\{|x|, |y| \leqslant 1; |z| \leqslant 1/|\mathcal{S}|\}$, and the third variable z essentially plays the role of a parameter. This situation would also occur in the context of example 1.1, provided we would be interested in the transient behavior of the process. ■

Throughout this paper, we shall essentially remain in the framework of example 1.1. An exhaustive original method of solution of equation (1.1) has been given in the book [4], allowing to get explicit expressions for $Q(x,y)$. It mainly resorts to a reduction to boundary value problems of Riemann-Hilbert type as in [3], together with the analysis of a certain *group* of Galois automorphisms introduced in [9] and acting on the algebraic curve (and its related Riemann surface)

$$\mathscr{K} = \{(x,y) \in \mathbb{C}^2 : K(x,y) = 0\}. \tag{1.3}$$

Let $\mathbb{C}(x,y)$ be the field of rational functions in x,y over $\mathbb{C}$. Since we shall assume that K, given by (1.2), is irreducible (see [4, Lemma 2.3.2] for an interpretation of this hypothesis in terms of the parameters $\{p_{i,j}\}_{-1\leqslant i,j\leqslant 1}$), the quotient field $\mathbb{C}(x,y)$ denoted by $\mathbb{C}_Q(x,y)$ is also a field.

Definition 1.3. The *group of the walk* is the Galois group $W = \langle \xi, \eta \rangle$ of automorphisms of $\mathbb{C}_Q(x,y)$ generated by ξ and η given by

$$\xi(x,y) = \left(x, \frac{1}{y} \frac{\sum_{-1\leqslant i\leqslant 1} p_{i,-1} x^i}{\sum_{-1\leqslant i\leqslant 1} p_{i,+1} x^i} \right), \qquad \eta(x,y) = \left(\frac{1}{x} \frac{\sum_{-1\leqslant j\leqslant 1} p_{-1,j} y^j}{\sum_{-1\leqslant j\leqslant 1} p_{+1,j} y^j}, y \right).$$

Let

$$\delta = \eta \circ \xi. \tag{1.4}$$

Then W has a normal cyclic subgroup $W_0 = \{\delta^\ell, \ell \in \mathbb{Z}\}$, which is possibly infinite, and such that W/W_0 is a group of order 2. Thus, when W is finite, say of order $2n$, δ^n is the identity.

In many studies, stemming either from probability theory [4, 7, 11] and, more recently, from combinatorics [1, 2, 6, 10], the analysis of type (1.1) equations is often crucial, and the importance of the group W has repeatedly been noticed. Broadly speaking, it appears that the question of the finiteness of W can be very useful in the three following respects.

- To find an explicit form of the generating function $Q(x,y)$, see [1, 2, 4].
- To prove the holonomy—or even the algebraicity—of $Q(x,y)$, see [1, 2, 4, 6].
- To derive asymptotics of the Taylor coefficients of $Q(x,y)$, see [7, 11].

1.1 Brief overview

In spite of the facts recalled above, very few criteria exist in the literature to decide about the finiteness of the group W. Up the knowledge of the authors, the main known results can be found in [4, Chapters 3 and 4]. For instance, it is proved that the group W has order 4 if and only if

$$\Delta = \begin{vmatrix} p_{1,1} & p_{1,0} & p_{1,-1} \\ p_{0,1} & p_{0,0} - 1 & p_{0,-1} \\ p_{-1,1} & p_{-1,0} & p_{-1,-1} \end{vmatrix} = 0. \tag{1.5}$$

For groups of order 6, a criterion, still in terms of certain determinants, is also given in [4, Section 4.1.1].

It has been shown in [4, Lemma 2.3.10] that for all non-singular random walks (see Definition A.1) the Riemann surface associated with (1.3) has always *genus* 1, except in five cases, listed in Lemma A.2 of the appendix, where it has *genus* 0.

In the case of genus 1, a general criterion has been derived in [4, Section 4.1.2] for W to be of order $2n$, $4 \leqslant n \leqslant \infty$. This characterization says that the group W is finite if, and

only if, the ratio of two elliptic integrals ω_3/ω_2 (see (2.4) for more details) is *rational*, in which case, setting $\mathbb{Z}_+^* = \{1, 2, \ldots\}$, the order of the group is equal to

$$2 \inf\{\ell \in \mathbb{Z}_+^* : \ell\omega_3/\omega_2 \in \mathbb{Z}\}. \tag{1.6}$$

On other hand, this nice theoretical formula is not easy to rewrite concretely in terms of the parameters $\{p_{i,j}\}_{-1\leqslant i,j\leqslant 1}$. For instance, an expression (if any!) by means of determinants, as in (1.5), is absolutely unclear.

As for the genus 0 case, although the solution of equation (1.1) was constructed in [4, Chapter 6], no criterion was yet provided to decide about the finiteness of the group: this is precisely the subject of this paper, where we give an explicit criterion for W to be finite. The method relies mainly on a *continuity* argument from the genus 1 case, in a sense to be made precise in Sections 2 and 3.

1.2 A criterion for the finiteness of W in the genus 0 case

Now we state our main result, according to the classification of Lemma A.2 listed in the appendix. As the reader will realize, most of the analysis will be devoted to walks having a zero drift, i.e., $\overrightarrow{\mathbf{M}} = 0$, since it appears to be the most difficult (and maybe interesting!) situation. Introduce the correlation coefficient

$$R = \frac{\sum_{-1\leqslant i,j\leqslant 1} ijp_{i,j}}{[\sum_{-1\leqslant i,j\leqslant 1} i^2 p_{i,j}]^{1/2} \cdot [\sum_{-1\leqslant i,j\leqslant 1} j^2 p_{i,j}]^{1/2}},$$

and define the angle

$$\theta = \arccos(-R). \tag{1.7}$$

Theorem 1.4.

(I) *When $\overrightarrow{\mathbf{M}} = 0$, the group W is finite if, and only if, θ/π is rational, in which case the order of W is equal to*

$$2 \inf\{\ell \in \mathbb{Z}_+^* : \ell\theta/\pi \in \mathbb{Z}\}. \tag{1.8}$$

(II) *When $\overrightarrow{\mathbf{M}} \neq 0$, the order of W is always infinite in the four cases* (A.1), (A.2), (A.3) *and* (A.4). ■

In Section 2.4, we shall propose another theoretical form of the angle (1.7), merely involving characteristic values related to the uniformization of curves of genus 0.

2 Proof of Part (I) of Theorem 1.4

The proof proceeds in stages, the key idea being to consider the genus 0 case as a *continuous limit* of the genus 1 case. In particular, we shall consider ad hoc *limit conformal gluing* and *limit uniformizing* functions, allowing to connect the ratio of certain *limit periods* with the finiteness of the *limit group*.

2.1 Basic properties of the kernel

To render the paper as self-contained as possible, we recall hereafter some important results proved in [4]. They are needed for our purpose and they will be stated without further comment.

First, let us rewrite the kernel (1.2) in the two equivalent forms

$$K(x,y) = a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y),$$

where

$$a(x) = p_{1,1}x^2 + p_{0,1}x + p_{-1,1},\ b(x) = p_{1,0}x^2 + (p_{0,0}-1)x + p_{-1,0},\ c(x) = p_{1,-1}x^2 + p_{0,-1}x + p_{-1,-1},$$
$$\widetilde{a}(y) = p_{1,1}y^2 + p_{1,0}y + p_{1,-1},\ \widetilde{b}(y) = p_{0,1}y^2 + (p_{0,0}-1)y + p_{0,-1},\ \widetilde{c}(y) = p_{-1,1}y^2 + p_{-1,0}y + p_{-1,-1}.$$

Then we set

$$D(x) = b^2(x) - 4a(x)c(x), \qquad \widetilde{D}(y) = \widetilde{b}^2(y) - 4\widetilde{a}(y)\widetilde{c}(y).$$

The polynomials D and $\widetilde{D}$ are of degree 4, with respective dominant coefficients

$$C = p_{1,0}^2 - 4p_{1,1}p_{1,-1}, \qquad \widetilde{C} = p_{0,1}^2 - 4p_{1,1}p_{-1,1}. \tag{2.1}$$

Let $X(y)$ [resp. $Y(x)$] be the algebraic function defined by (1.2), so that $K(X(y),y) = 0$ [resp. $K(x,Y(x)) = 0$]. This function has two *branches*, say X_0 and X_1 [resp. Y_0 and Y_1], and we fix the notation by taking $|X_0| \leqslant |X_1|$ [resp. $|Y_0| \leqslant |Y_1|$].
Denote by $\{y_\ell\}_{1\leqslant \ell \leqslant 4}$ the four roots of $\widetilde{D}(y)$—they are branch points of the Riemann surface (1.3). They are enumerated in such a way that $|y_1| \leqslant |y_2| \leqslant |y_3| \leqslant |y_4|$. Moreover $y_1 \leqslant y_2$, $[y_1y_2] \subset [-1,+1]$ and $0 \leqslant y_2 \leqslant y_3$. The branches X_0 and X_1 are meromorphic on the complex plane cut along $[y_1y_2] \cup [y_3y_4]$. Similar results hold for $D(x)$, exchanging y and x.
On the cut $[y_1y_2] \cup [y_3y_4]$, the roots X_0 and X_1 are complex conjugate. Then we consider the two components of the quartic curve

$$\begin{cases} \mathcal{M}_1 &= X_0([\overrightarrow{y_1y_2}]) &= \overline{X}_1([\overleftarrow{y_1y_2}]), \\ \mathcal{M}_2 &= X_0([\overrightarrow{y_3y_4}]) &= \overline{X}_1([\overleftarrow{y_3y_4}]), \end{cases}$$

where $\overrightarrow{y_1y_2}$ stands for the contour $[y_1y_2]$, traversed from y_1 to y_2 along the upper edge of the slit $[y_1y_2]$, and then back to y_1 along the lower edge of the slit. Similarly, $\overleftarrow{y_1y_2}$ is defined by exchanging "upper" and "lower". In particular, the simply connected domain $\mathcal{D}_1$ bounded by $\mathcal{M}_1$ will play here a fundamental role, for reasons explained in Section 2.1.2.

2.1.1 Uniformization in the genus 1 case

When the associated Riemann surface is of genus 1, the algebraic curve $K(x,y) = 0$ admits a uniformization in terms of the Weierstrass $\wp$ function with periods ω_1, ω_2 (see equation (2.4)) and its derivatives. Indeed, setting

$$\begin{aligned} D(x) &= b^2(x) - 4a(x)c(x) &= d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0, \\ z &= 2a(x)y + b(x), \end{aligned}$$

the following formulae hold [noting that for notational convenience d_4 corresponds to C defined in equation (2.1)].

- If $d_4 \neq 0$ (four finite branch points $\{x_\ell\}_{1\leqslant \ell \leqslant 4}$), then $D'(x_4) > 0$ and

$$\begin{cases} x(\omega) = x_4 + \dfrac{D'(x_4)}{\wp(\omega) - D''(x_4)/6}, \\ z(\omega) = \dfrac{D'(x_4)\wp'(\omega)}{2[\wp(\omega) - D''(x_4)/6]^2}. \end{cases} \tag{2.2}$$

- If $d_4 = 0$ (three finite branch points $\{x_\ell\}_{1\leqslant\ell\leqslant 3}$ and $x_4 = \infty$), then

$$\begin{cases} x(\omega) = \dfrac{\wp(\omega) - d_2/3}{d_3}, \\ z(\omega) = -\dfrac{\wp'(\omega)}{2d_3}. \end{cases} \tag{2.3}$$

Let us now introduce the three key quantities

$$\omega_1 = 2i\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{-D(x)}}, \qquad \omega_2 = 2\int_{x_2}^{x_3} \frac{\mathrm{d}x}{\sqrt{D(x)}}, \qquad \omega_3 = 2\int_{X(y_1)}^{x_1} \frac{\mathrm{d}x}{\sqrt{D(x)}}. \tag{2.4}$$

The period ω_1 is clearly purely imaginary, and $0 < \omega_3 < \omega_2$, see [4, Lemma 3.3.3]. In this respect, for $\ell \in \{2,3\}$, it will be convenient to denote by $\wp_{1,\ell}$ the Weierstrass elliptic function with periods ω_1, ω_ℓ and series expansion

$$\wp_{1,\ell}(\omega) = \frac{1}{\omega^2} + \sum_{(p_1,p_\ell)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left[\frac{1}{(\omega - p_1\omega_1 - p_\ell\omega_\ell)^2} - \frac{1}{(p_1\omega_1 + p_\ell\omega_\ell)^2}\right]. \tag{2.5}$$

On the universal covering, we have $\delta(\omega + \omega_3) = \delta(\omega)$, see (1.4) and [4, equation (3.1.10)], and the necessary and sufficient condition for the group to be finite has been quoted in (1.6).

2.1.2 Conformal gluing in the genus 1 case

Definition 2.1. Let $\mathcal{D} \subset \mathbb{C} \cup \{\infty\}$ be an open and simply connected set, symmetrical with respect to the real axis, and different from $\emptyset$, $\mathbb{C}$ and $\mathbb{C} \cup \{\infty\}$. A function w is said to be a *conformal gluing function* (CGF) for the set $\mathcal{D}$ if

- w is meromorphic in $\mathcal{D}$;
- w establishes a conformal mapping of $\mathcal{D}$ onto the complex plane cut along a segment;
- For all t in the boundary of $\mathcal{D}$, $w(t) = w(\overline{t})$.

For instance, $w(t) = t + 1/t$ is a CGF for the unit disc, and any non-degenerate linear transformation of $w(t)$, namely

$$\frac{ew(t) + f}{gw(t) + h}, \qquad (e,g) \neq (0,0), \qquad eh - fg \neq 0$$

is also a CGF for the unit disc. Incidentally, this implies one can choose arbitrarily the pole of w within the unit disc—in particular, taking $e = 0$, $f = 1$, $g = 1$ and $h = -2$, we get the CGF $t/(t-1)^2$ with a pole at 1. Conversely, two CGFs for a same domain are fractional linear transformations of each other, see [8].

2.1.3 Some topological facts in the genus 0 case

As $\overrightarrow{\mathbf{M}} = 0$, the branch points x_2 and x_3 coincide and we have

$$x_2 = x_3 = 1, \quad x_1 \in [-1,1), \quad x_4 \in (1,\infty) \cup \{\infty\} \cup (-\infty,-1],$$

Similar results hold for the $\{y_\ell\}_{1\leqslant\ell\leqslant 4}$.
Here the curve $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect at the point $x = 1$, which is a corner point. In the figure 2.1, borrowed from [4, Chapter 6], the dotted curve is the unit circle, and one has drawn the contour $\mathcal{M}_1 \cup \mathcal{M}_2$, which has a self-intersection and is the image of the cut $[\overrightarrow{y_1y_4}]$ by the mapping $y \mapsto X(y)$, remembering that $X_0([\overrightarrow{y_1y_4}]) = \overline{X}_1([\overleftarrow{y_1y_4}])$.
In the sequel, we shall prove that the angle of the tangent line at the corner point is indeed deeply related to the order of the group.

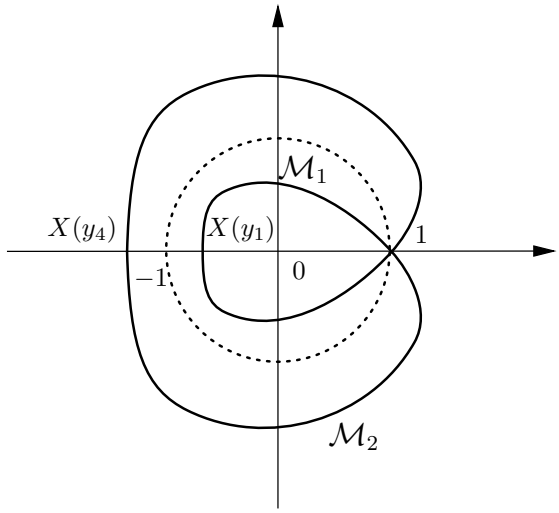

Fig. 2.1: The contour $\mathcal{M}_1 \cup \mathcal{M}_2$ for $R < 0$

2.2 Limit conformal gluing when passing from genus 1 to genus 0

Our approach resides in viewing genus 0 as a topological deformation of genus 1, when the parameters are modified in such a way that $\overrightarrow{\mathbf{M}} \to 0$.

According to the uniformization (2.2) or (2.3), and together with the notations of Section 2.1.1, let

$$f(t) = \begin{cases} \dfrac{D''(x_4)}{6} + \dfrac{D'(x_4)}{t - x_4} & \text{if} \quad x_4 \neq \infty, \\ \dfrac{1}{6}(D''(0) + D'''(0)t) & \text{if} \quad x_4 = \infty. \end{cases}$$

To avoid technicalities of minor importance, we shall assume in the sequel $C \neq 0$, with C given by (2.1), so that $x_4 \neq \infty$. Then, in the genus 1 case, it was shown in [4] that a CGF for the domain $\mathcal{D}_1$ bounded by $\mathcal{M}_1$ is obtained via the function

$$w(t) = \wp_{1,3}(\wp_{1,2}^{-1}(f(t)) - [\omega_1 + \omega_2]/2).$$

Since any fractional linear transformation of a CGF is again a CGF, the well-known addition theorem for $\wp_{1,3}$ involving translation by the half-period $\omega_1/2$ entails that

$$\wp_{1,3}(\wp_{1,2}^{-1}(f(t)) - \omega_2/2) \tag{2.6}$$

is also a CGF for $\mathcal{D}_1$.

Now letting $\overrightarrow{\mathbf{M}} \to 0$, so that $x_2, x_3 \to 1$, and by using the continuity with respect to the parameters $\{p_{i,j}\}_{-1 \leqslant i,j \leqslant 1}$, a direct calculation in the integral formulae (2.4) gives that $\omega_1 \to i\infty$ and that ω_2, ω_3 converge to certain non-degenerate quantities as below:

$$\begin{cases} \omega_1 \to i\infty, \\ \omega_2 \to \alpha_2 = \dfrac{\pi}{[C(x_4 - 1)(1 - x_1)]^{1/2}}, \\ \omega_3 \to \alpha_3 = \displaystyle\int_{X_0(y_1)}^{x_1} \frac{\mathrm{d}x}{(1 - x)[C(x - x_1)(x - x_4)]^{1/2}}. \end{cases} \tag{2.7}$$

Lemma 2.2. *Letting* $\overrightarrow{\mathbf{M}} \to 0$*, we have*

$$\frac{\theta}{\pi} = \lim_{\overrightarrow{\mathbf{M}}\to 0} \frac{\omega_2}{\omega_3} = \frac{\alpha_2}{\alpha_3}.$$

Proof. At the corner point of the curve $\mathcal{M}_1$, we shall calculate the angle of the tangent line with the horizontal axis in two different ways.

First, when $\overrightarrow{\mathbf{M}} \to 0$, the derivative $X_0'(1)$ is a root of a second degree polynomial, since

$$\widetilde{a}(1){X_0'}^2(1) + RX_0'(1) + a(1) = 0,$$

see [4, equation (6.5.3)]. Since

$$2a(1) = \textstyle\sum_{-1\leqslant i,j\leqslant 1} j^2 p_{i,j}, \quad 2\widetilde{a}(1) = \sum_{-1\leqslant i,j\leqslant 1} i^2 p_{i,j},$$

it is quite easy to check that

$$\arg(X_0'(1)) = \pm\theta, \tag{2.8}$$

where θ is given by equation (1.7).

The second way of computing θ relies on the construction of a convenient CGF, according to the definition given in Section 2.1.2. Here we shall choose w with $w(1) = \infty$. As we shall see, this implies that, in the neighborhood of $t = 1$, $w(t)$ has the form

$$w(t) = [\alpha + o(1)]/[1-t]^{\chi}, \tag{2.9}$$

for some constant $\alpha \neq 0$ and $\chi > 0$. In addition, the exponent χ must satisfy the relation

$$\chi = \pi/\theta. \tag{2.10}$$

Indeed, equation (2.8) together with the symmetry of the CGF yield $\exp(i\theta\chi) = \exp(-i\theta\chi)$, whence $\theta\chi/\pi$ is a positive integer. On the other hand, if $\theta\chi/\pi \geqslant 2$, then w would not be one-to-one. Identity (2.10) follows. Thus we are left with the proof of expansion (2.9).

In order to obtain a CGF in the zero drift case, we could use [4, Lemma 6.5.5], but it will be more convenient to use here the CGF obtained from the last paragraph, and then to let the drift go to zero.

When $\overrightarrow{\mathbf{M}} \to 0$, we know from (2.4) that $\omega_1 \to i\infty$. This way, using (2.5) as well as the well-known identity

$$\sum_{p\in\mathbb{Z}} \frac{1}{(\Omega+p)^2} = \frac{\pi^2}{\sin^2(\pi\Omega)},$$

we obtain, for $\ell \in \{2,3\}$, that uniformly in ω,

$$\wp_{1,\ell}(\omega) \to \left(\frac{\pi}{\alpha_\ell}\right)^2 \left[\frac{1}{\sin^2(\pi\omega/\alpha_\ell)} - \frac{1}{3}\right].$$

In particular, setting

$$u(t) = \sin^2\left(\frac{\alpha_2}{\alpha_3}\left[\arcsin\left\{\left[\frac{1}{3} + f(t)\left(\frac{\omega_2}{\pi}\right)^2\right]^{-1/2}\right\} - \frac{\pi}{2}\right]\right),$$

and taking the limit in (2.6), it follows that an admissible CGF for $\mathcal{M}_1$ is given by

$$\left(\frac{\pi}{\alpha_3}\right)^2 \left[\frac{1}{u(t)} - \frac{1}{3}\right].$$

Since any linear transformation of a CGF is a CGF, $u(t)$ itself is a CGF for $\mathcal{M}_1$. We shall now show the existence of $\alpha \neq 0$ such that in the neighborhood of $t = 1$,

$$u(t) = [\alpha + o(1)]/[1-t]^{\alpha_2/\alpha_3}. \tag{2.11}$$

By a direct calculation, it can be seen that for $t \in [x_1, 1]$,

$$\frac{1}{3} + f(t)\left(\frac{\alpha_2}{\pi}\right)^2 \tag{2.12}$$

belongs to the segment $[0,1]$, equals 1 at x_1 and 0 at 1. In other words, in order to understand the behavior of $u(t)$ near $t = 1$, it is necessary to analyze the asymptotics, as $T \to \infty$, of the function

$$\sin^2\left(\frac{\alpha_2}{\alpha_3}\left[\arcsin\{T\} - \frac{\pi}{2}\right]\right).$$

For $T \geqslant 1$, we have

$$\arcsin\{T\} = \int_0^1 \frac{\mathrm{d}u}{(1-u^2)^{1/2}} \pm i \int_1^T \frac{\mathrm{d}u}{(u^2-1)^{1/2}} = \frac{\pi}{2} \pm i \ln\left[T + (T^2-1)^{1/2}\right].$$

Hence, with $\sin(ix) = i\sinh(x)$, we can write

$$\begin{aligned}\sin^2\left(\frac{\alpha_2}{\alpha_3}\left[\arcsin\{T\} - \frac{\pi}{2}\right]\right) &= -\sinh^2\left(\frac{\alpha_2}{\alpha_3}\ln\left[T + (T^2-1)^{1/2}\right]\right)\\ &= -\frac{1}{4}\left(\left[T + (T^2-1)^{1/2}\right]^{2\alpha_2/\alpha_3} + \left[T - (T^2-1)^{1/2}\right]^{2\alpha_2/\alpha_3} - 2\right).\end{aligned}$$

When $T \to \infty$, $T + (T^2-1)^{1/2} = 2T + O(1/T)$ and $T - (T^2-1)^{1/2} = O(1/T)$, so that

$$u(t) = -\frac{1}{4}\left(\left[2\left[\frac{1}{3} + f(t)\left(\frac{\alpha_2}{\pi}\right)^2\right]^{-1/2}\right]^{2\alpha_2/\alpha_3} + O(1)\right).$$

As remarked earlier, the function defined in (2.12) has a zero at 1, which is simple. Equation (2.11) follows immediately, and, since conformal mapping conserves angles, the proof of the lemma is concluded. ■

2.3 Limit of the uniformization when passing from genus 1 to genus 0

The last step consists in connecting the angle θ to the group W introduced in Definition 1.3. In fact, we shall deal with the ratio α_3/α_2 and prove that, when $\overrightarrow{\mathbf{M}} = 0$, W can be interpreted as the group of transformations of $\mathbb{C}/(\alpha_2\mathbb{Z})$

$$\langle \omega \mapsto -\omega + \alpha_2, \omega \mapsto -\omega + \alpha_2 + \alpha_3 \rangle,$$

which is of order $2\inf\{\ell \in \mathbb{Z}_+^* : \ell\alpha_3/\alpha_2 \in \mathbb{Z}\}$.

As $\overrightarrow{\mathbf{M}} \to 0$, the limit of the uniformization (2.2), after some algebra using the continuity with respect to the parameters $\{p_{i,j}\}_{-1\leqslant i,j\leqslant 1}$, becomes

$$\begin{cases} x(\omega) = 1 + \dfrac{(x_4-1)(1-x_1)}{1-x_1-(x_4-x_1)\sin^2(\pi\omega/\alpha_2)}, \\ z(\omega) = \dfrac{\pi}{\alpha_2}\dfrac{(x_4-x_1)(x_4-1)(x_1-1)\sin(2\pi\omega/\alpha_2)}{[1-x_1-(x_4-x_1)\sin^2(\pi\omega/\alpha_2)]^2}, \end{cases} \tag{2.13}$$

where we have used the expression of α_2 given by equation (2.7). Setting $u = \exp(2i\pi\omega/\alpha_2)$, the obvious identity

$$\sin^2(\pi\omega/\alpha_2) = -(u + 1/u - 2)/4$$

allows to rewrite in (2.13) as $x(\omega) = \widetilde{x}(u)$, where

$$\widetilde{x}(u) = \frac{(u - z_1)(u - 1/z_1)}{(u - z_0)(u - 1/z_0)}, \tag{2.14}$$

and z_0, z_1 are complex numbers given by

$$\begin{cases} z_0 = \dfrac{2 - (x_1 + x_4) \pm 2[(1 - x_1)(1 - x_4)]^{1/2}}{x_4 - x_1}, \\ z_1 = \dfrac{x_1 + x_4 - 2x_1x_4 \pm 2[x_1x_4(1 - x_1)(1 - x_4)]^{1/2}}{x_4 - x_1}. \end{cases}$$

Clearly, $\widetilde{y}(u)$ will be also a rational function of the same form as $\widetilde{x}(u)$, but its explicit computation hinges on some general properties of Riemann surfaces. Indeed the functions (2.14) and $\widetilde{y}(u)$ are *automorphic* and provide a rational uniformization of the algebraic curve $\mathscr{K}$ defined in (1.3), which is here of genus 0. Moreover, this unifomization is unique *up to a fractional linear transformation* (see, e.g., [12]).

Before deriving the formula for $\widetilde{y}(u)$, let us quote right away a first limit automorphism, namely $\omega \mapsto -\omega + \alpha_2$, since

$$x(\omega) = x(-\omega + \alpha_2).$$

Equivalently, we have

$$\widetilde{x}(u) = \widetilde{x}(1/u),$$

which corresponds precisely to the automorphism of $\mathbb{C}$

$$\xi(u) = \frac{1}{u}. \tag{2.15}$$

Similarly, exchanging the roles of x and y, a continuity argument when $\overrightarrow{\mathbf{M}} \to 0$ in the general uniformization given in (2.2) or (2.3) yields

$$y(\omega) = y(-\omega + \alpha_2 + \alpha_3).$$

In particular, $\omega \mapsto -\omega + \alpha_2 + \alpha_3$ is the second automorphism. Indeed, $z(\omega) = -z(-\omega)$, so that $y(\omega) \neq y(-\omega)$. Moreover $x(\omega) \neq x(-\omega + \alpha_3)$, since, by taking the limit of the genus 1 in (2.4), we get $\alpha_3 < \alpha_2$ with a strict inequality (see formula (2.21) in Section 2.4).

To derive $\widetilde{y}(u)$ quickly (and nicely!), we shall make two observations.

- Exchanging the roles of x and y would lead to another rational uniformization with some parameter v, such that

  $$\widehat{y}(v) = \frac{(v - z_3)(v - 1/z_3)}{(v - z_2)(v - 1/z_2)}, \tag{2.16}$$

  where z_2, z_3 are obtained from z_0, z_1, just replacing x_1, x_4 by y_1, y_4, respectively.

- From the above remark, we necessarily have

  $$\widetilde{y}(u) = \widehat{y}(\sigma(u)),$$

  where

  $$\sigma(u) = \frac{eu + f}{gu + h}, \qquad (e, g) \neq (0, 0), \qquad eh - fg \neq 0$$

  is a linear transformation which will be completely determined.

Since $\widetilde{x}(0) = \widetilde{x}(\infty) = 1$, see (2.14), we must have $\widetilde{y}(0) = \widetilde{y}(\infty) = 1$ (in the case $\overrightarrow{\mathbf{M}} = 0$, $Y_0(1) = Y_1(1) = 1$, see Section 2.1). Similarly, (2.16) yields $\widehat{y}(0) = \widehat{y}(\infty) = 1$. These simple equalities entail at once $\sigma(u) = \rho u$, where ρ is a complex number, to be calculated later. Then, by (2.16), we obtain

$$\widetilde{y}(u) = \frac{(\rho u - z_3)(\rho u - 1/z_3)}{(\rho u - z_2)(\rho u - 1/z_2)}, \tag{2.17}$$

and the second limit automorphism η becomes on $\mathbb{C}$

$$\eta(u) = \frac{1}{\rho^2 u}. \tag{2.18}$$

Hence, combining (2.15) and (2.18), the generator δ introduced in (1.4) takes on $\mathbb{C}$ the simple form

$$\delta(u) = (\eta \circ \xi)(u) = \frac{u}{\rho^2}.$$

In the genus 0 case, the group W therefore has the order (possibly infinite)

$$2 \inf\{\ell \in \mathbb{Z}_+^* : \rho^{2\ell} = 1\}.$$

At that moment, we make a brief detour to connect (2.14) with the uniformization proposed in [4, equation 6.5.11], namely

$$\check{y}(t) = \frac{y_1 + y_4}{2} + \frac{y_4 - y_1}{4}\left(t + \frac{1}{t}\right).$$

Setting

$$T(v) = \frac{v z_2 - 1}{v - z_2},$$

one easily checks the identity $\widehat{y}(v) = \check{y}(T(v))$, which by (2.17) yields in particular

$$\widehat{y}(1) = \check{y}(-1) = y_1 = \widetilde{y}(1/\rho),$$

whence, using the properties of the algebraic curve, $\widetilde{x}(1/\rho) = X(y_1)$. Finally, by (2.14), we obtain that ρ is a root of the second degree equation

$$\rho + \frac{1}{\rho} = 2\frac{x_1 + x_4 - 2x_1x_4 + (x_1 + x_4 - 2)X(y_1)}{(x_4 - x_1)(1 - X(y_1))}. \tag{2.19}$$

The roots of (2.19) are complex conjugate and of modulus one. Choosing then the root satisfying $\arg(\rho) \in [0, \pi]$ and letting

$$\Lambda = \frac{x_1 + x_4 - 2x_1x_4 + (x_1 + x_4 - 2)X(y_1)}{2[(X(y_1) - x_1)(X(y_1) - x_4)(1 - x_1)(x_4 - 1)]^{1/2}}, \tag{2.20}$$

a direct algebra yields

$$\arg(\rho) = \frac{\pi}{2} - \arctan(\Lambda).$$

To conclude the proof of Part (I) of Theorem 1.4, it suffices to apply equation (2.21), derived in Proposition 2.4 below, which makes the link between $\arctan(\Lambda)$ and the ratio α_3/α_2. All conclusions so far obtained remain true, and even easier to prove, when $C = 0$ (see (2.1) and the assumption made at the beginning of Section 2.2). ■

2.4 A second form of the criterion for the zero drift case

By computing the limit ratio α_3/α_2 in a different manner, we shall derive another expression of the angle θ defined in equation (1.7). The following proposition holds.

Proposition 2.3. *With Λ defined in* (2.20) *and its equivalent $\widetilde{\Lambda}$ obtained from Λ by exchanging x and y, we have*

$$\theta = \frac{\pi}{2} - \arctan(\Lambda),$$

so that $\widetilde{\Lambda} = \Lambda$ and the order of the group W equals

$$2\inf\{\ell \in \mathbb{Z}_+^* : \ell[1/2 - \arctan(\Lambda)/\pi] \in \mathbb{Z}\}.$$

Proof. A straightforward calculation carried out along the same lines as for the derivation of formulae (2.7) yields, for any $t < x_1$,

$$\int_t^{x_1} \frac{\mathrm{d}x}{(1-x)[C(x-x_1)(x-x_4)]^{1/2}} = \frac{1}{[C(x_4-1)(1-x_1)]^{1/2}}\left[\frac{\pi}{2} - \arctan\left(\frac{x_1+x_4-2x_1x_4+(x_1+x_4-2)t}{2[(t-x_1)(t-x_4)(x_4-1)(1-x_1)]^{1/2}}\right)\right].$$

Instantiating now $t = X(y_1) < x_1$ in the last formula, we get exactly

$$\frac{\alpha_3}{\alpha_2} = \frac{1}{2} - \frac{\arctan(\Lambda)}{\pi}. \tag{2.21}$$

■

3 Proof of Part (II) of Theorem 1.4

In this part we prove that in all situations (A.1), (A.2), (A.3) and (A.4) listed in the appendix, the group W is infinite. In fact, it is enough to prove this result for only one case, since two groups corresponding to jump probability sets obtained one another by one of the eight symmetries of the square are necessarily isomorphic, see [4, Section 2.4]. We choose to focus on the particular case (A.2).

Like for the analysis of the case $\overrightarrow{\mathbf{M}} = 0$, we consider the genus 0 case (A.2) as a continuous limit of the genus 1 case. All the results of Section 2.1 still hold, with $x_3 = x_4 = \infty$. Accordingly, letting the parameters $\{p_{i,j}\}_{-1\leqslant i,j\leqslant 1}$ be distorted so as to yield (A.2), we get from equations (2.4)

$$\begin{cases} \omega_1 \to i\alpha_1, \text{ with } \alpha_1 \in (0,\infty), \\ \omega_2 \to \infty, \\ \omega_3 \to \alpha_3 \in (0,\infty). \end{cases}$$

In particular, for the same reasons as in Section 2.3, the limit group can be interpreted as the group of transformations

$$\langle \omega \mapsto -\omega, \omega \mapsto -\omega + \alpha_3 \rangle$$

on $\mathbb{C}/(\alpha_1\mathbb{Z})$. This group is obviously infinite, and so is W.

The proof of Theorem 1.4 is terminated. ■

4 Miscellaneous remarks

For the sake of completeness, we quote hereaftert some facts related to existing works.

- Theorem 1.4 is quite simple to check, and therefore provides a really effective criterion.
- A direct calculation gives $\Delta = -a(1)\widetilde{a}(1)\sum_{-1\leqslant i,j\leqslant 1} ijp_{i,j}$, with Δ defined in (1.5). Hence, for the group of order 4, our criterion clearly agrees with that of [4].
- From Theorem 1.4, it becomes clear that the famous *Gessel's walk* (i.e. with jump probabilities satisfying $p_{1,0} = p_{1,1} = p_{-1,0} = p_{-1,-1} = 1/4$) has a group of order 8. More generally, Theorem 1.4 leads to another proof of the (non-)finiteness of the group for all combinatorial models (with an underlying genus 0) appearing in [1, 2].
- In [2, Section 3], the non-finiteness of the group for some models (including particular instances of case (A.4)) has been proved via two approaches based on valuation and fixed point arguments, but no general criterion was really obtained. It is also worth noting that there the group was not restricted to the algebraic curve, so that Theorem 1.4 is, in a sense, more precise.
- As an other straightforward consequence of Theorem 1.4, one can check the random walk considered in [11], with
$$p_{1,0} = p_{-1,0} = 1/2 - p_{-1,1} = 1/2 - p_{1,-1} = \sin^2(\pi/n)/2,$$
has a group of order $2n$, for all $n \geqslant 3$.
- The angle θ defined in (1.7) gives the angle of the cone in which, after a suitable linear transformation, the random walk with transitions $\{p_{i,j}\}_{-1\leqslant i,j\leqslant 1}$ has a covariance which equals to some multiple of the identity.

A About the genus

Introduce the *drift* (mean jump vector)
$$\overrightarrow{\mathbf{M}} = \left(\sum_{-1\leqslant i,j\leqslant 1} ip_{i,j}, \sum_{-1\leqslant i,j\leqslant 1} jp_{i,j} \right).$$

Definition A.1. A random walk is called *singular* if the kernel K defined in (1.2) is either reducible or of degree 1 in at least one of the variables x, y.

The following classification holds, see [4, Chapter 2].

Lemma A.2. *For all non-singular random walks, the Riemann surface corresponding to* (1.3) *has genus* 0 *if, and only if, one of the five following relations takes place:*
$$\begin{aligned}
\overrightarrow{\mathbf{M}} &= \vec{0}, \\
p_{0,1} &= p_{-1,0} = p_{-1,1} = 0, && \text{(A.1)}\\
p_{1,0} &= p_{1,-1} = p_{0,-1} = 0, && \text{(A.2)}\\
p_{1,0} &= p_{0,1} = p_{1,1} = 0, && \text{(A.3)}\\
p_{0,-1} &= p_{-1,0} = p_{-1,-1} = 0. && \text{(A.4)}
\end{aligned}$$

Acknowledgments. K. Raschel would like to thank I. Kurkova and J.S.H van Leeuwaarden for stimulating discussions. His work was supported by CRC 701, Spectral Structures and Topological Methods in Mathematics at the University of Bielefeld.

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http://www.inria.fr
ISSN 0249-6399