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An excursion into Nonsmooth Dynamics: from Mechanics, to Electronics, through Control

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From Mechanics...

to Control,...

To Electronics.

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An excursion into Nonsmooth Dynamics: from Mechanics, to Electronics, through Control

Vincent Acary

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Fifty Years of Finite Freedom Mechanics.
On the occasion of Michel Jean's 70th birthday
Marseille, 25–27 October 2010

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From Mechanics of divided materials to multi-body and robotic systems,
- History and Motivations
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To control (Sliding mode control Theory)

To electronics (Nonsmooth modeling of switched Electrical circuits)

Mechanical systems with contact, impact and friction

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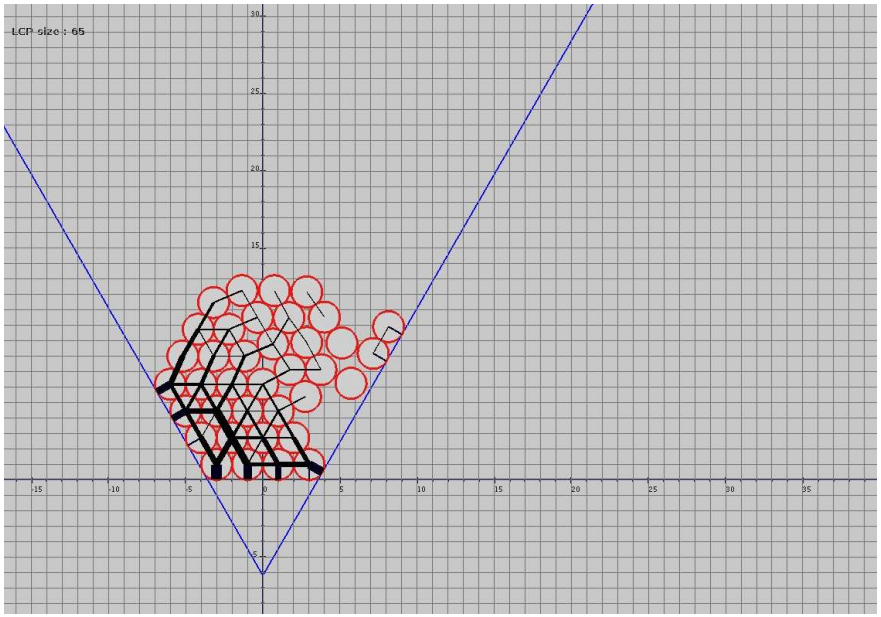
Mechanical systems with contact, impact and friction

Stack of beads with perturbation



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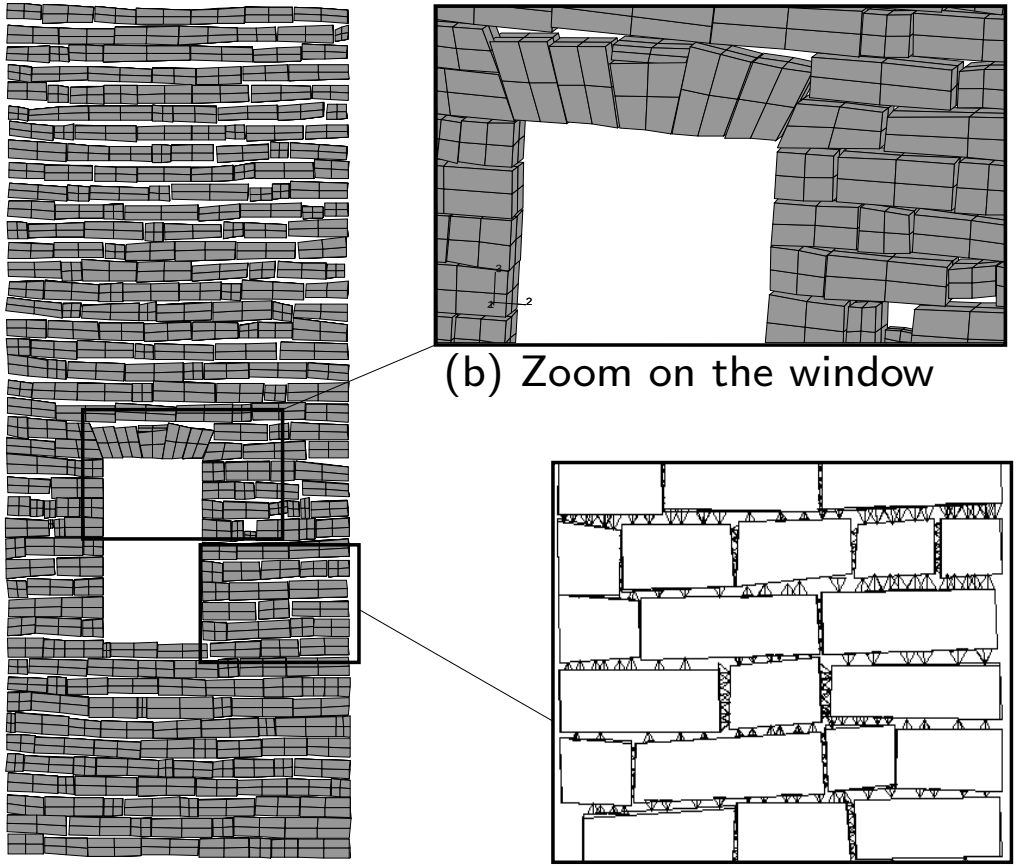

(a) FEM H8 meshing

(b) Zoom on the window

(c) Contact detection

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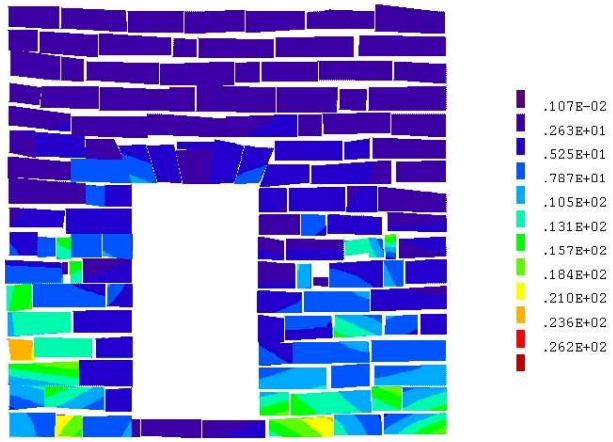

Figure: VON MISES stresses

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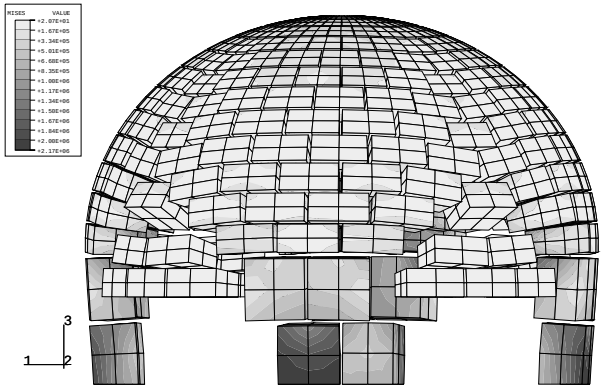
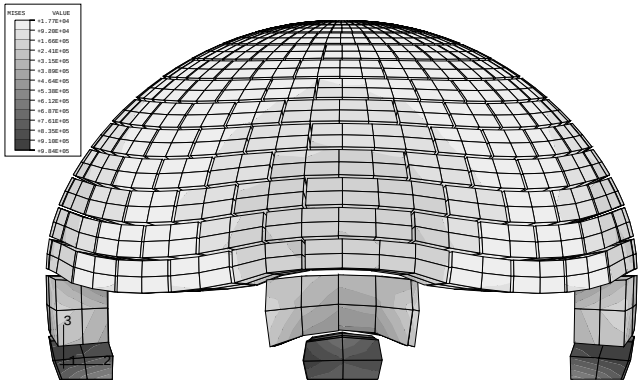
Divided Materials and Masonry





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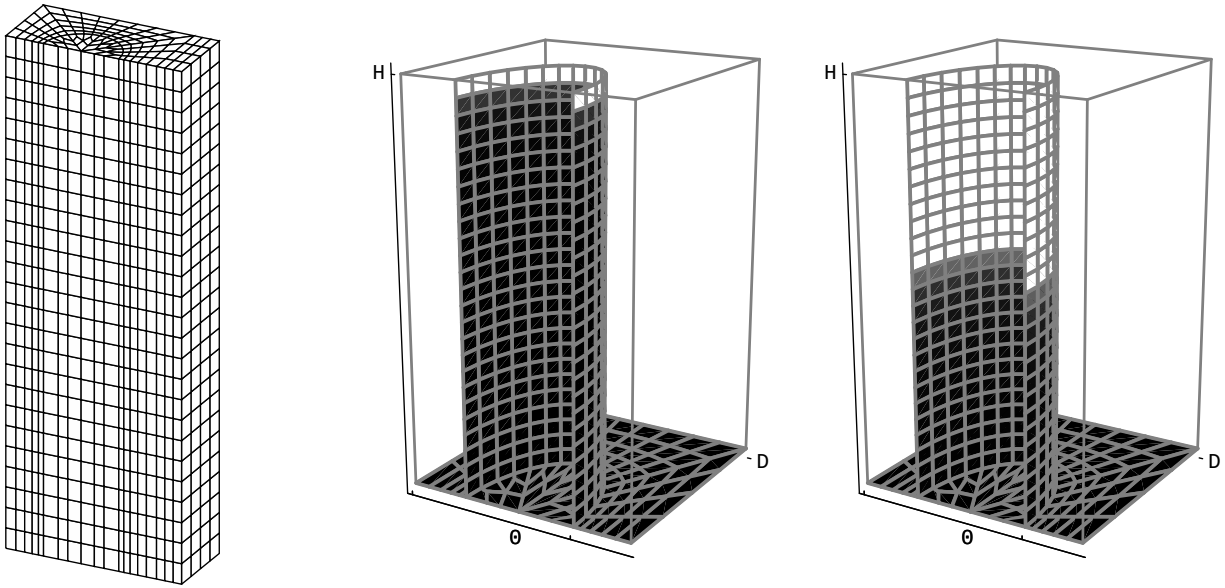
FEM models with contact, friction cohesion, etc...



Joint work with Y. Monerie, IRSN.

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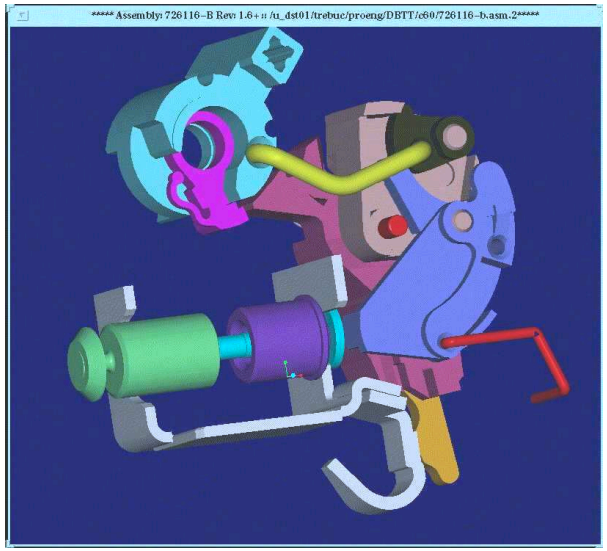
Simulation of Circuit breakers (INRIA/Schneider Electric)



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Bipedal Robot INRIA BIPOP

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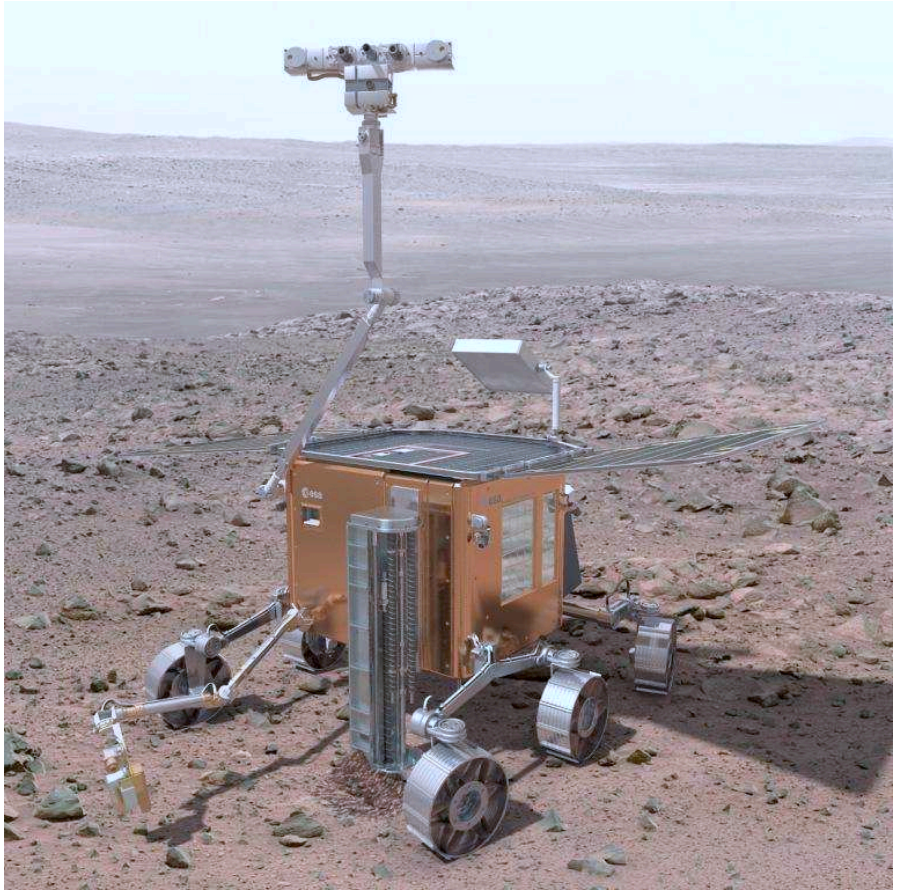
Simulation of the ExoMars Rover (INRIA/Trasys Space/ESA)



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Mechanical systems with contact, impact and friction

They are all nonsmooth mechanical systems but they differ in

- the presence of perfect nonlinear joints,
- the presence of finite rotations,
- the presence of Control (sensors & actuators)
- the desired properties in design and development which influence the numerical simulation and prototyping

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Unilateral constraints as an inclusion

Definition (Perfect unilateral constraints on the smooth dynamics)

$$\begin{cases} M(q)\dfrac{dv}{dt} + F(t,q,v) = r \\ \\ -r \in N_{\mathcal{C}(t)}(q(t)) \end{cases} \tag{1}$$

where

- $r = \nabla_q g(q,t)\,\lambda$ is the generalized reactions due to the constraints.
- Finite set of ν unilateral constraints on the generalized coordinates :

$$g(q,t) = [g_\alpha(q,t) \geqslant 0, \quad \alpha \in \{1 \dots \nu\}]^T . \tag{2}$$

- Admissible set $\mathcal{C}(t)$

$$\mathcal{C}(t) = \{q \in \mathcal{M}(t), g_\alpha(q,t) \geqslant 0, \alpha \in \{1 \dots \nu\}\} . \tag{3}$$

- Normal Cone to $\mathcal{C}(t)$

$$N_{\mathcal{C}(t)}(q(t)) = \left\{ \begin{array}{r} y \in \mathbb{R}^n \mid y = -\displaystyle\sum_\alpha \lambda_\alpha \nabla g_\alpha(q,t), \\ \lambda_\alpha \geqslant 0,\ \lambda_\alpha g_\alpha(q,t) = 0 \end{array} \right\} \tag{4}$$

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Non Smooth Lagrangian Dynamics

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that

$$v^+ = \dot{q}^+ \tag{5}$$

- q is related to this velocity by

$$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\, dt \tag{6}$$

- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that

$$dv(]a, b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{7}$$

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Non Smooth Lagrangian Dynamics

Definition (Non Smooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{8}$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- The non smooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References

[Schatzman, 1973, 1978, Moreau, 1983, 1988]

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Non Smooth Lagrangian Dynamics

Decomposition of measure

$$\left\{ \begin{array}{rlrr} dv = & \gamma\, dt + & (v^+ - v^-)\, d\nu + & dv_s \\ di = & f\, dt + & p\, d\nu + & di_s \end{array} \right. \tag{9}$$

where

- $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- f is the Lebesgue measurable force,
- $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v, i.e. where $(v^+ - v^-) \neq 0$,i.e. $d\nu = \sum_i \delta_{t_i}$
- p is the purely atomic impact percussions such that $pd\nu = \sum_i p_i \delta_{t_i}$
- dv_S and di_S are singular measures with the respect to $dt + d\eta$.

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Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the non smooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{10}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{11}$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t,q,v)dt = fdt \tag{12}$$

or

$$M(q)\gamma^+ + F(t,q,v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{13}$$

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The Moreau's sweeping process of second order

Definition (Moreau [1983, 1988])

A key stone of this formulation is the inclusion in terms of velocity. Indeed, the inclusion (1) is "replaced" by the following inclusion

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ \\ v^+ = \dot{q}^+ \\ \\ -di \in N_{T_C(q)}(v^+) \end{cases} \tag{14}$$

Comments

This formulation provides a common framework for the non smooth dynamics containing inelastic impacts without decomposition.
➜ Foundation for the time–stepping approaches.

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Comments

- *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- *The inclusion in terms of velocity* v^+ rather than of the coordinates q.

Interpretation

- Inclusion of measure, $-di \in K$
  - Case $di = r'dt = fdt$.
$$-f \in K \tag{15}$$
  - Case $di = p_i \delta_i$.
$$-p_i \in K \tag{16}$$
- Inclusion in terms of the velocity. Viability Lemma
  If $q(t_0) \in C(t_0)$, then
$$v^+ \in T_C(q), t \geqslant t_0 \Rightarrow q(t) \in C(t), t \geqslant t_0$$

  ➜ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

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The Newton-Moreau impact rule

$$-di \in N_{T_C(q(t))}(v^+(t) + ev^-(t)) \tag{17}$$

where e is a coefficient of restitution.

Velocity level formulation. Index reduction

$$\begin{array}{ccc} -\lambda \in N_{\mathbb{R}^+}(y) & \rightsquigarrow & -\lambda \in N_{T_{\mathbb{R}^+}}(\dot{y}) \\ & \Updownarrow & \\ 0 \leqslant y \perp \lambda \geqslant 0 & \rightsquigarrow & \text{if } y \leqslant 0 \text{ then } 0 \leqslant \dot{y} \perp \lambda \geqslant 0 \end{array} \tag{18}$$

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State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

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Objectives & means

Objectives

Design nonsmooth event capturing methods with

- same properties as standard methods (robustness, accumulation, . . .)
- Higher resolution (ratio error/computational cost)
- Higher order of accuracy

Means

1. Adaptive time–step size and order strategies for standard methods
2. Mixed integrators with rough pre-detection of events
3. Splitting strategies
4. Ad hoc detection of discontinuity and order of discontinuity methods.

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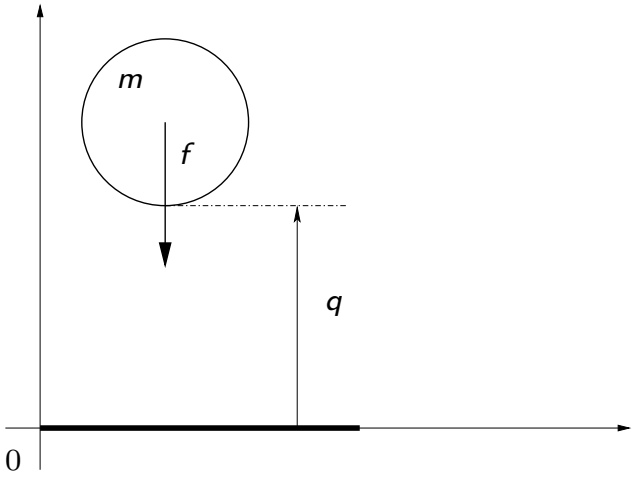
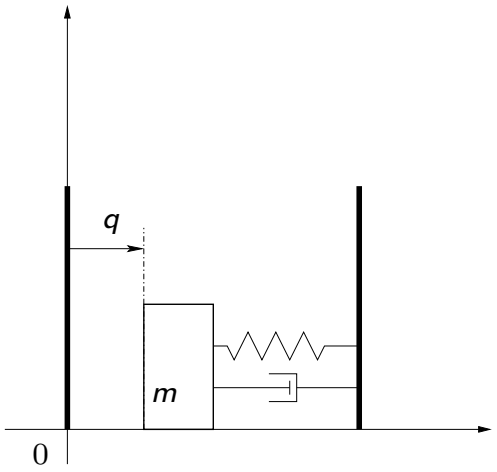
NonSmooth Multibody Systems (NSMBS)

Academic examples



(a) Bouncing ball example



(b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

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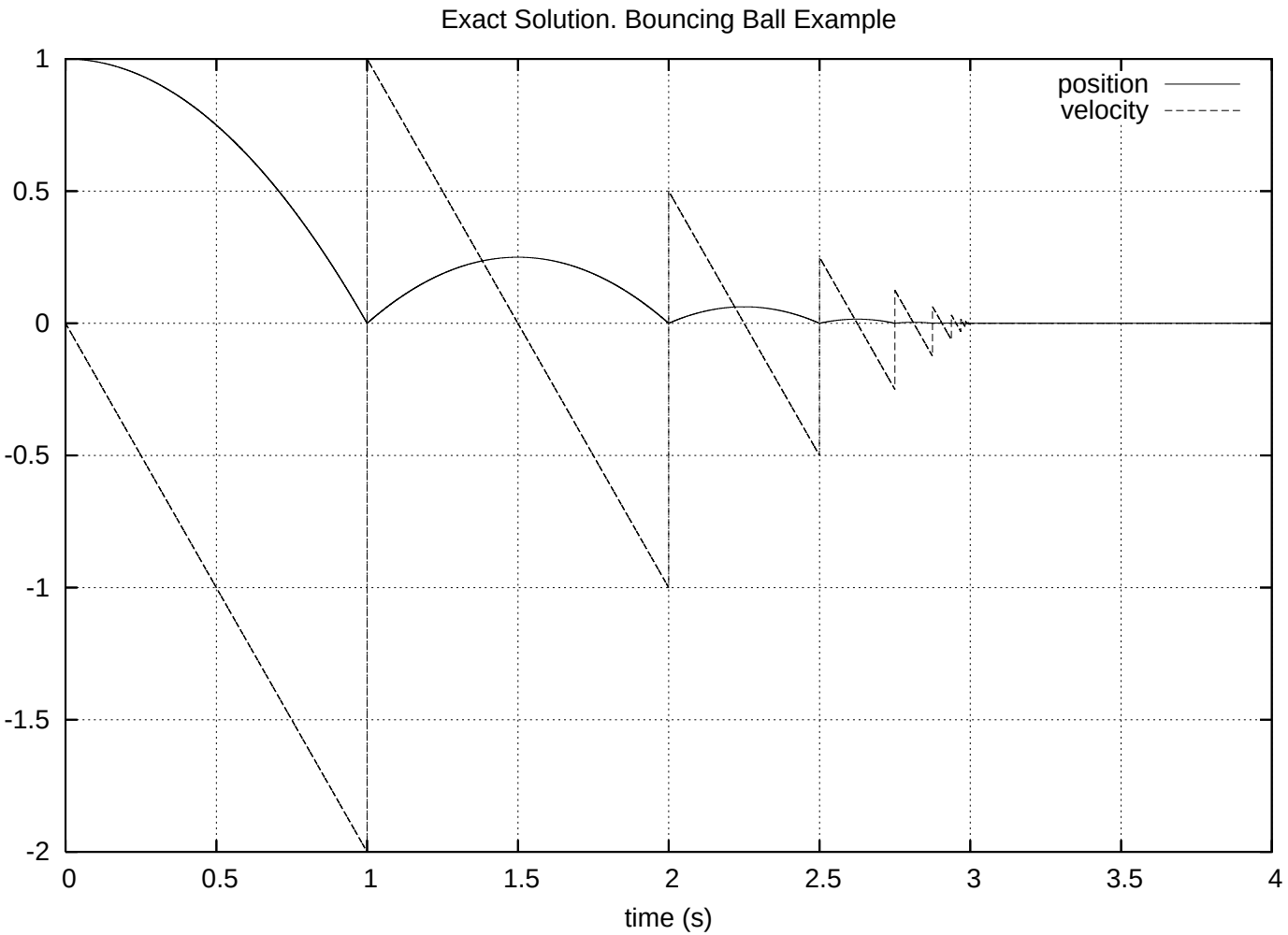

Figure: Analytical solutions. Bouncing ball example]

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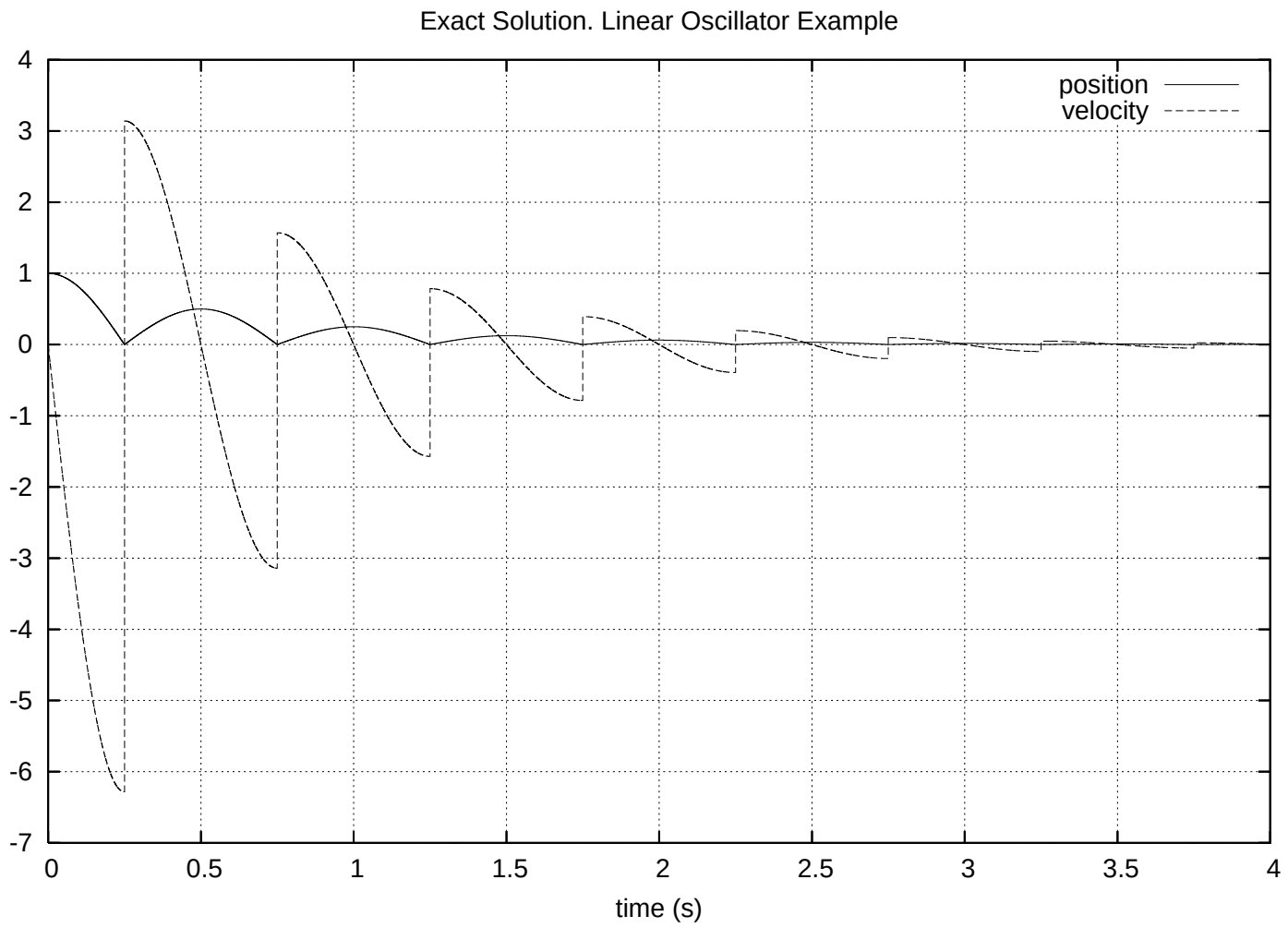

Figure: Analytical solutions. Linear Oscillator

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Moreau–Jean's Time stepping scheme [Moreau, 1988] and [Jean, 1999]

Principle of NSCD

$$
\begin{cases}
M(q_{k+\theta})(v_{k+1} - v_k) - h\tilde{F}_{k+\theta} = G(q_{k+\theta})P_{k+1}, & \text{(19a)} \\
q_{k+1} = q_k + hv_{k+\theta}, & \text{(19b)} \\
U_{k+1} = G^T(q_{k+\theta})\, v_{k+1} & \text{(19c)} \\
-P_{k+1} \in N_{T_{\mathbb{R}^m_+}(\tilde{y}_{k+\gamma})}(U_{k+1} + eU_k), & \text{(19d)} \\
\tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma \in [0,1]. & \text{(19e)}
\end{cases}
$$

with $\theta \in [0,1], \gamma \geqslant 0$ and $x_{k+\alpha} = (1-\alpha)x_{k+1} + \alpha x_k$ and $\tilde{y}_{k+\gamma}$ is a prediction of the constraints.

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Schatzman's Time stepping scheme [Paoli and Schatzman, 2002]

Principle

$$\begin{cases} M(q_k+1)(q_{k+1}-2q_k+q_{k-1})-h^2F(t_{k+\theta},q_{k+\theta},v_{k+\theta})=p_{k+1}, & (20a)\\ v_{k+1}=\dfrac{q_{k+1}-q_{k-1}}{2h}, & (20b)\\ -p_{k+1}\in N_K\left(\dfrac{q_{k+1}+eq_{k-1}}{1+e}\right), & (20c)\end{cases}$$

where N_K defined the normal cone to K.
For $K=\{q\in\mathbb{R}^n, y=g(q)\geqslant 0\}$

$$0\leqslant g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right)\perp\nabla g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right)P_{k+1}\geqslant 0 \qquad (21)$$

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Comparison

Shared mathematical properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Mechanical properties

- Position vs. velocity constraints
- Respect of the impact in one step (Moreau–Jean) vs. Two-steps(Schatzman)
- Linearized constraints rather than nonlinear.

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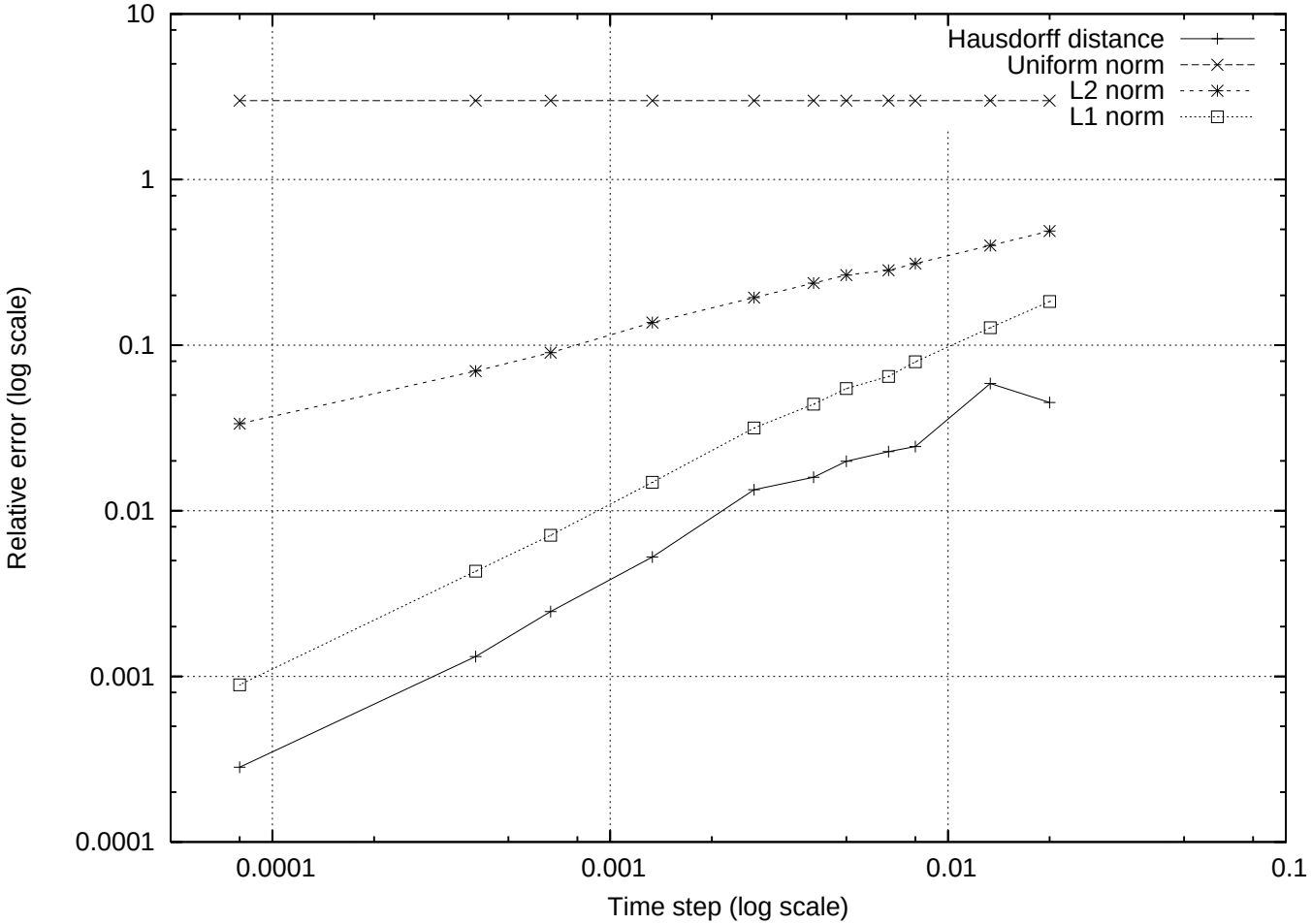
Empirical order of convergence. Moreau–Jean's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Moreau–Jean's time-stepping scheme.

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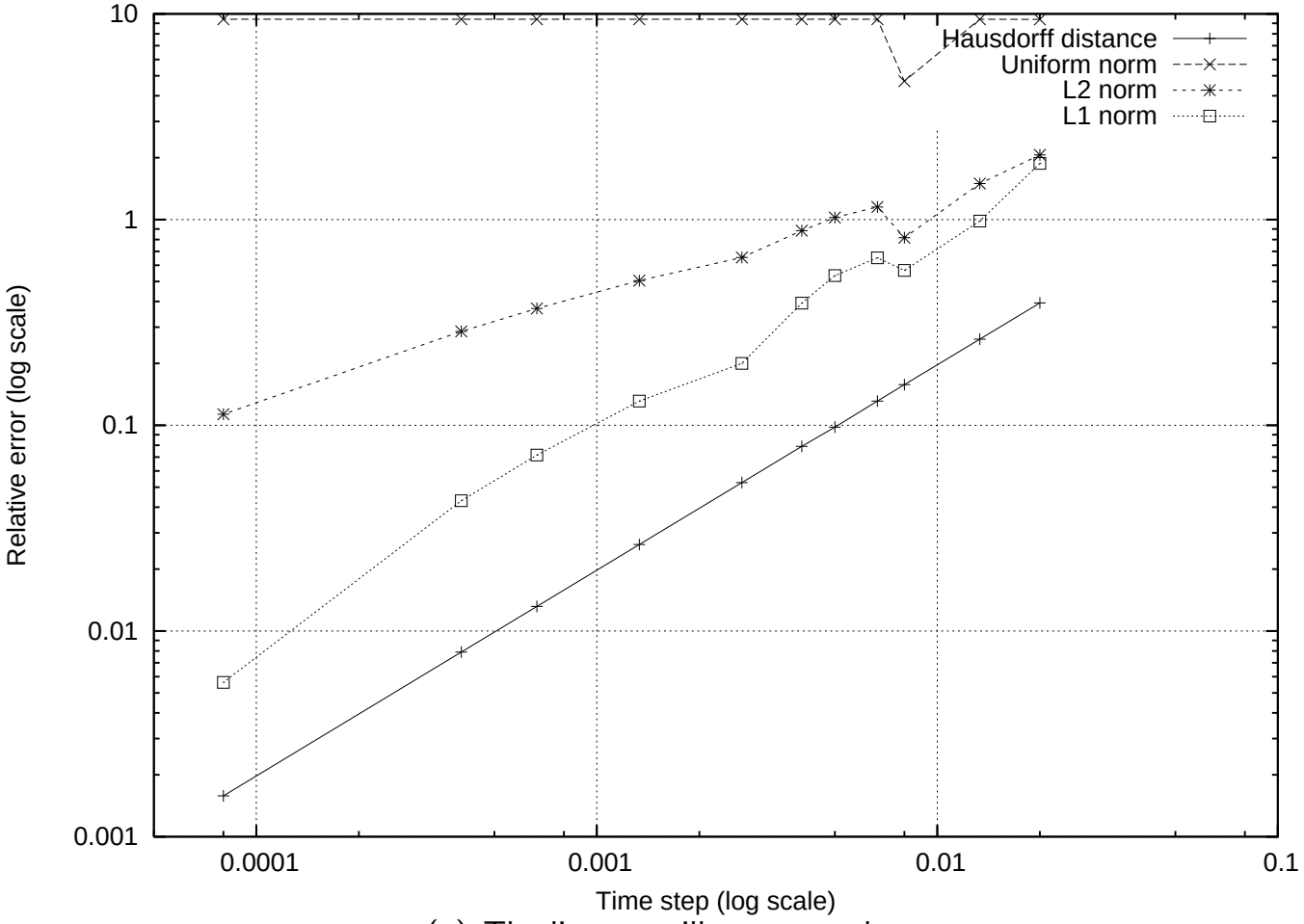

(a) The linear oscillator example

Figure: Empirical order of convergence of the Moreau–Jean's time-stepping scheme.

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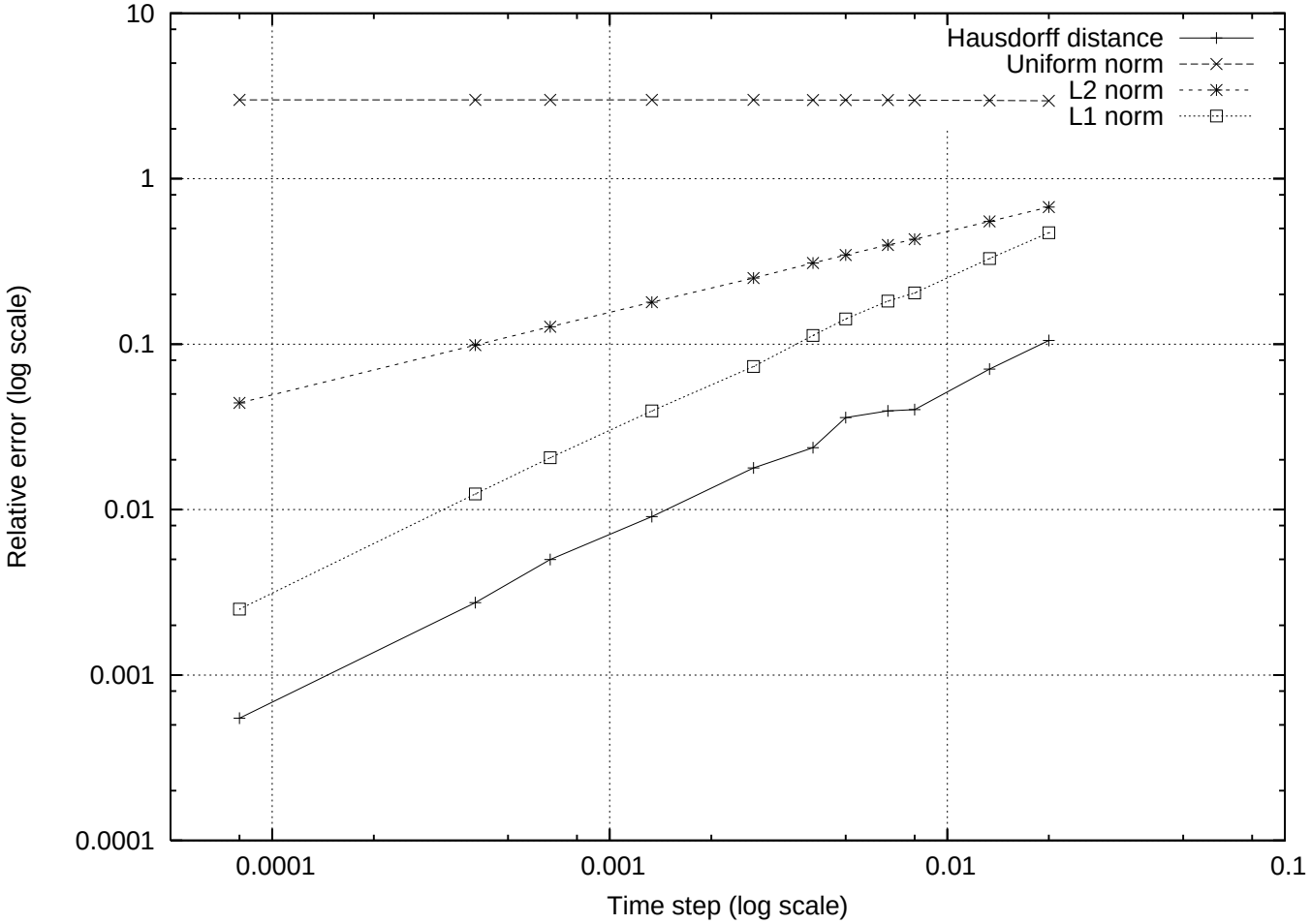
Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

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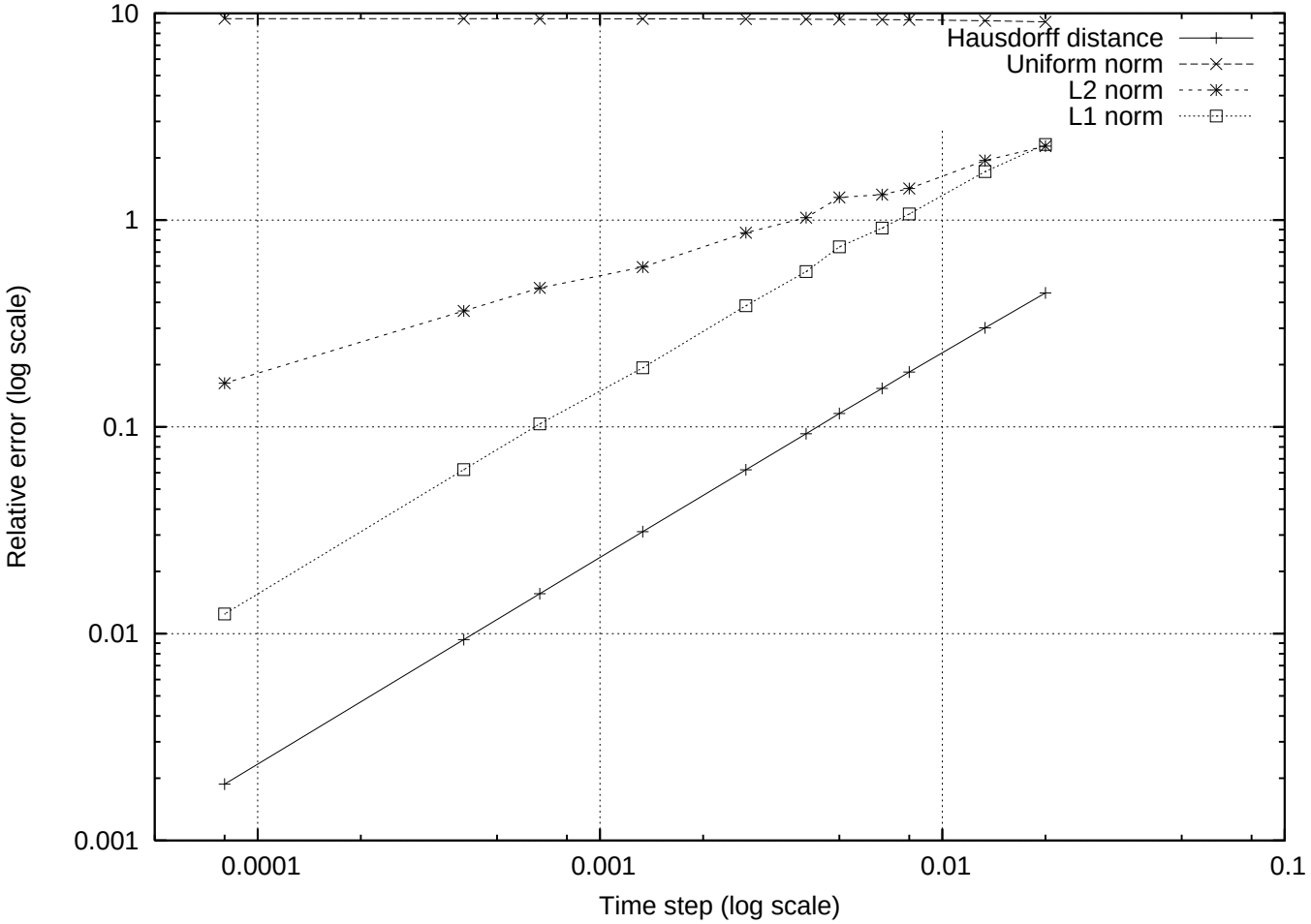

(a) The linear oscillator example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

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Local error estimates for the Moreau-Jean's time–stepping

Assumption 1 : Existence and uniqueness

A unique global solution over $[0, T]$ for Moreau's sweeping process is assumed such that $q(\cdot)$ is absolutely continuous and admits a right velocity $v^+(\cdot)$ at every instant t of $[0, T]$ and such that the function $v^+ \in LBV([0, T], \mathbb{R}^n)$.

➜ Assumption 1 is ensured in the framework introduced by Ballard [Ballard, 2000] who proves the existence and uniqueness of a solution in a general framework mainly based on the analyticity of data.

Assumption 2 : Smoothness of data

The following smoothness on the data will be assumed: a) the inertia operator $M(q)$ is assumed to be of class $\mathcal{C}^p$ and definite positive, b) the force mapping $F(t, q, v)$ is assumed to be of class $\mathcal{C}^p$, c) the constraint functions $g(q)$ are assumed to be of class $\mathcal{C}^{p+1}$ and d) the Jacobian matrix $G(q) = \nabla_q^T g(q)$ is assumed to have full-row rank.

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Local error estimates for the Moreau-Jean's time–stepping

Lemma

Let $I = [t_k, t_{k+1}]$. Let us assume that the function $f \in BV(I, \mathbb{R}^n)$. Then we have the following inequality for the θ–method, $\theta \in [0,1]$,

$$\left\| \int_{t_k}^{t_{k+1}} f(s)\, ds - h(\theta f(t_{k+1}) + (1-\theta) f(t_k)) \right\| \leqslant C(\theta)(t_{k+1} - t_k)\, var(f, I), \tag{22}$$

where $var(f, I) \in \mathbb{R}$ is the variation of f on I and $C(\theta) = \theta$ if $\theta \geqslant 1/2$ and $C(\theta) = 1 - \theta$ otherwise. Furthermore, the value of $C(\theta)$ yields a sharp bound in (22).

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Local error estimates for the Moreau-Jean's time–stepping

Proposition

Under Assumptions 1 and 2, the local order of consistency of the Moreau-Jean time–stepping scheme for the generalized coordinates is

$$e_q = q_{k+1} - q(t+h) = \mathcal{O}(h)$$

and at least for the velocities

$$e_v = v^+(t_k + h) - v_{k+1} = \mathcal{O}(1)$$

.

Comments

The bounds are reached if an impact is located within the time–step and the activation of the constraint is not correct.

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Higher Order Time–stepping schemes

Background

Work of Mannshardt [1978] on time–integration schemes of any order for ODE/DAEs with discontinuities (with tranversality assumption)

Principle

- Let us assume only one event per time–step at instants t_*.
- Choose any ODE/DAE solvers of order p
- Perform a rough location of the event inside the time step of length h Find an interval $[t_a, t_b]$ such that

$$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \tag{23}$$

  Dichotomy, Newton, Local Interpolants, Dense output,...
- Perform an integration on $[t_k, t_a]$ with the ODE solver of order p
- Perform an integration on $[t_a, t_b]$ with Moreau's time–stepping scheme
- Perform an integration on $[t_b, t_{k+1}]$ with the ODE solver of order p

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Integration of the smooth dynamics

Mainly for the sake of simplicity, the numerical integration over a smooth period is made with a Runge–Kutta (RK) method on the following index-1 DAE,

$$\begin{cases} M(q(t))\dot{v}(t) = F(t, q(t), v(t)) + G(q)\lambda(t), \\ \dot{q}(t) = v(t), \\ \gamma(t) = G(q(t))\dot{v}(t) = 0. \end{cases} \tag{24}$$

In practice, the time–integration is performed for the following system

$$\begin{cases} M(q(t))\dot{v}(t) = F(t, q(t), v(t)) + G(q)\lambda(t), \\ \dot{q}(t) = v(t), \\ 0 \leqslant \gamma(t) = G(q(t))\dot{v}(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{25}$$

on the time–interval I where the index set $\mathcal{I}(t)$ of active constraints is assumed to be constant on I and $\lambda(t) > 0$ for all $t \in I$.

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Integration of the smooth dynamics

Using the standard notation for the RK methods (see Hairer et al. [1993] for details), the complementarity problem that we have to solve at each time–step reads

$$\begin{cases} t_{ki} = t_k + c_i h, \\ v_{k+1} = v_k + h \sum_{i=1}^{s} b_i V'_{ki}, \\ q_{k+1} = q_k + h \sum_{i=1}^{s} b_i V_{ki}, \\ V'_{ki} = M^{-1}(Q_{ki}) \left[F(t_{ki}, Q_{ki}, V_{ki}) + G(Q_{ki}) \lambda_{ki} \right], \\ V_{ki} = v_k + h \sum_{j=1}^{s} a_{ij} V'_{nj}, \\ Q_{ki} = q_k + h \sum_{j=1}^{s} a_{ij} V_{nj}, \\ 0 \leqslant \gamma_{ki} = G(Q_{ki}) V'_{ki} \perp \lambda_{ki} \geqslant 0. \end{cases} \tag{26}$$

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Assumption 3

Let I a smooth period time–interval. We assume that

1. the local order of the RK method (26) is p that is

$$e_q = e_v = \mathcal{O}(h^{p+1}) \tag{27}$$

2. starting from inconsistent initial value $\tilde{q}_k$ such that $\tilde{q}_k - q_k = \mathcal{O}(h^{p+1})$, the error made by the RK method (26) is

$$\tilde{q}_{k+1} - q_{k+1} = \mathcal{O}(h^{p+1}) \tag{28}$$

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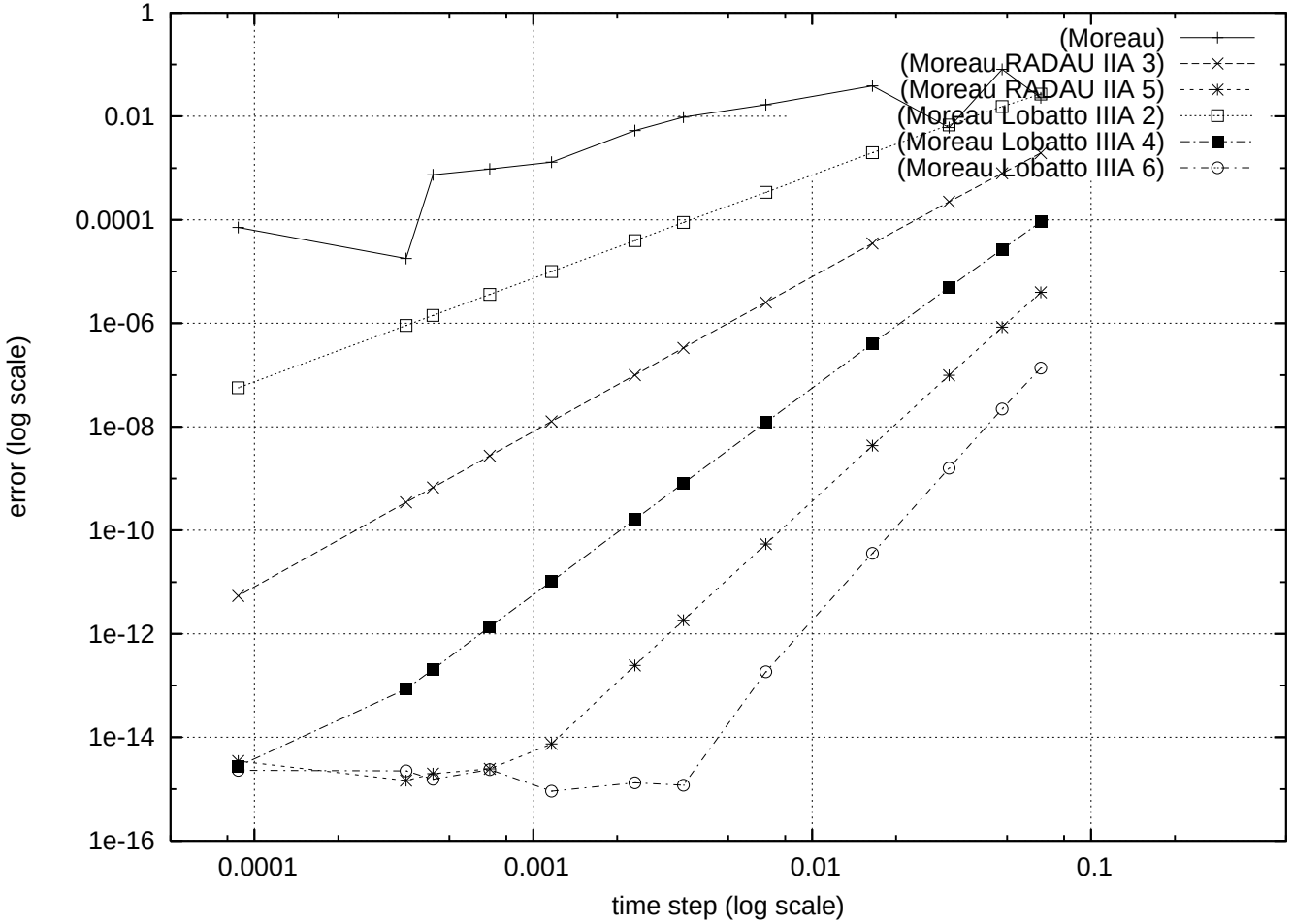
Theorem

Let us assume that Assumptions 1, 2 and 3 hold. The local error of consistency of the scheme is of order p in the generalized coordinates that is

$$e_q = \mathcal{O}(h^{p+1}). \tag{29}$$

Results on the linear oscillator



(a) The linear oscillator example with implicit Runge Kutta Method

Figure: Precision Work diagram for the Moreau's time-stepping scheme coupled with Runge–Kutta method.

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Higher Order Time–stepping schemes

Finite accumulation

- Repeat the whole process on the remaining part of the interval $[t_b, t_k]$
- By induction, repeat this process up to the end of the original time step.

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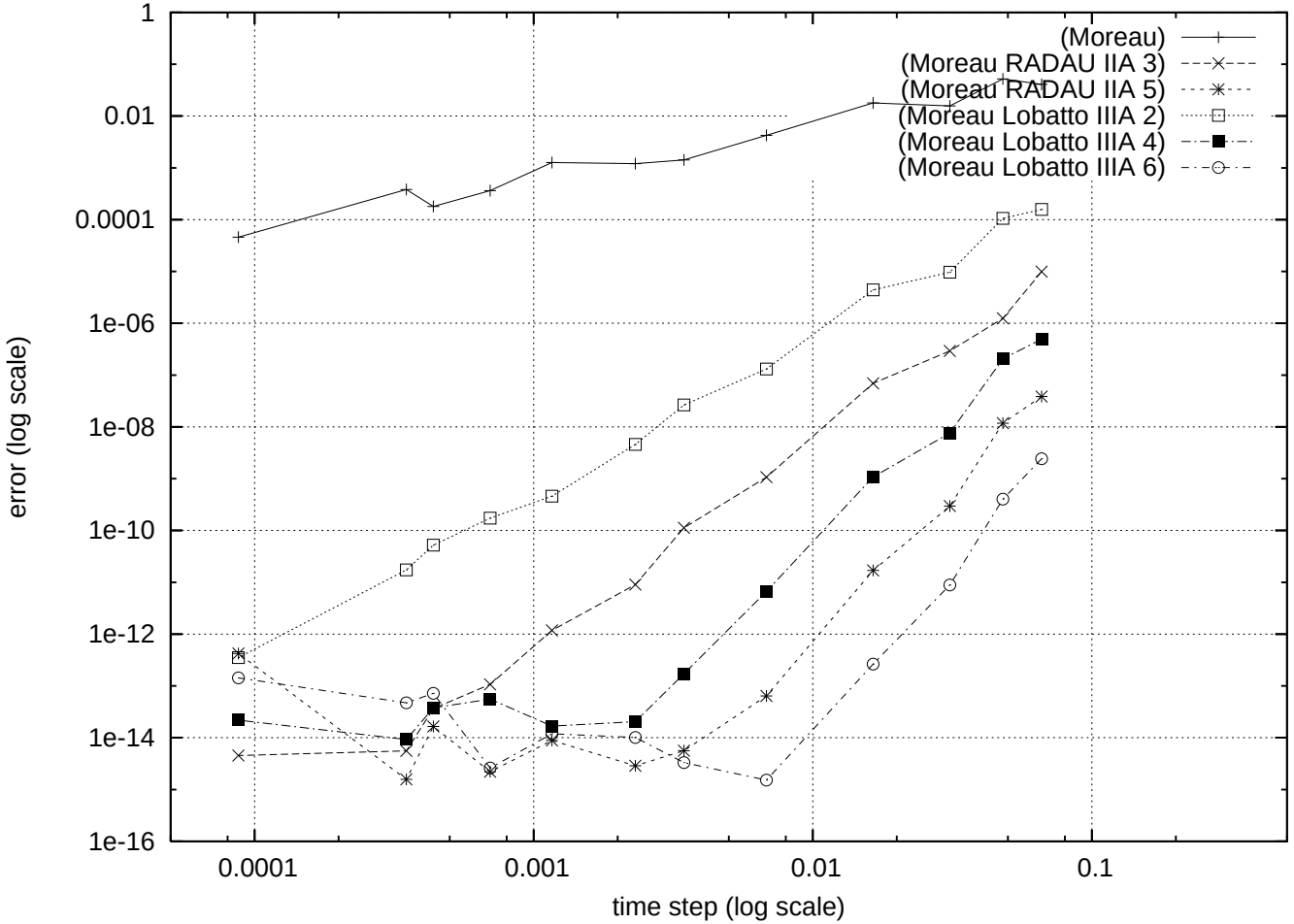
Results on the Bouncing Ball



(a) The Bouncing Ball example with implicit Runge Kutta Method

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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To control (Sliding mode control Theory)
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To electronics (Nonsmooth modeling of switched Electrical circuits)

Sliding Mode Control for dummies

Basic principles on a naive example

Problem: Stabilization of this simple dynamics

$$\left\{ \begin{array}{l} x(t_0) = x_0 \in \mathbb{R} \\ \dot{x} = f, \quad |f| \leqslant 1, \end{array} \right. \tag{30}$$

at the origin $x = 0$.

Naive solution:

$$\left\{ \begin{array}{l} x(t_0) = x_0 \in \mathbb{R} \\ \dot{x} = f + u, \quad |f| < 1, \end{array} \right. \tag{31}$$

- "Push on right" if the state is at the right of 0

$$u = -1 \text{ if } x > 0 \tag{32}$$

- "Push on right" if the state is at the left of 0

$$u = +1 \text{ if } x > 0 \tag{33}$$

- "balance the external load" in 0

$$u = -f \text{ if } x = 0 \tag{34}$$

Sliding Mode Control for dummies

Basic principles on a naive example

- Switched control based on the sign function

$$u = -\text{sign}(x) = \begin{cases} -1 \text{ for } x > 0 \\ +1 \text{ for } x < 0 \\ ? \text{ for } x = 0 \end{cases} \tag{35}$$

  Definition of u at $x = 0$?

- Discontinuous ODEs

$$\dot{x} = f - \text{sign}(x) \tag{36}$$

  Notion of solutions ?

Mathematical framework

- Multivalued maximal monotone operator

$$u = -\text{sgn}(x) = \begin{cases} -1 \text{ for } x > 0 \\ +1 \text{ for } x < 0 \\ [-1, 1] \text{ for } x = 0 \end{cases} \tag{37}$$

- Filippov's differential inclusions

In the continuous setting

- Robust control w.r.t external uncertainties
- Finite time convergence to target

➜ SMC is the most widely used non linear control in industrial practice.

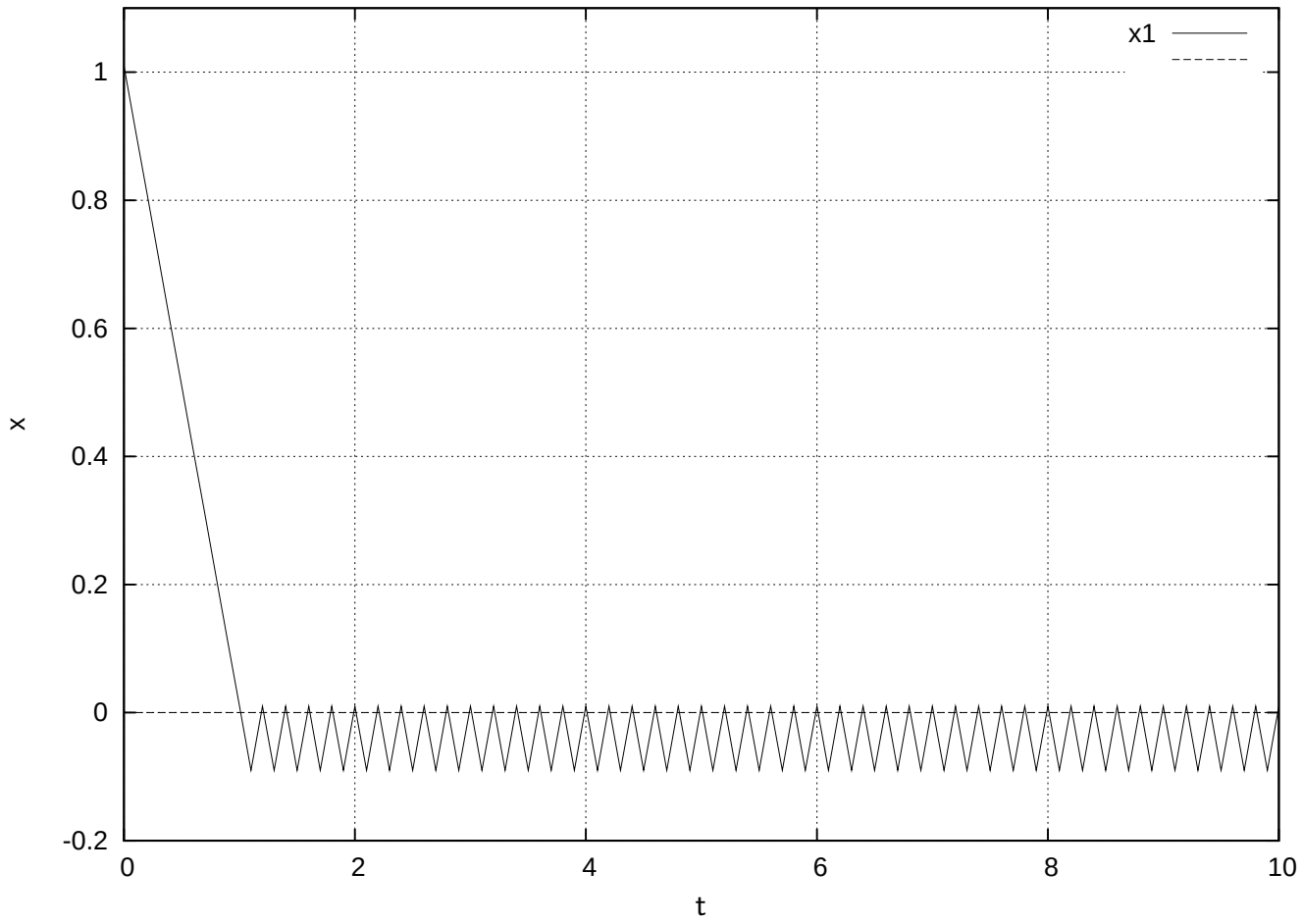
In the discrete setting

Digital implementation of SMC suffers from "chattering" due to explicit approximation

$$x_{k+1} - x_k = f - \mathrm{sgn}(x_k) \tag{38}$$

This causes

- Wear and damage in actuators
- Need for complex filtering systems which entails the good properties of continuous SMC.



(a) $h = 0.2$

Figure: A simple example for $x_0 = 1.01$ at $t_0 = 0$.

Implicit Implementation of SMC

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Our background

- Nonsmooth modelling of Friction
- Well–posedness analysis of Monotone Differential Inclusions
- Implicit numerical time integration for DI.

Objectives

- Study the implicit Euler discretization of a class of differential inclusions with sliding surfaces ($\subset$ Filippov's systems)
- Show that this numerical method permits a smooth stabilization on the sliding surface, in a finite number of steps
- Show how this may be used in real-time implementations of sliding mode control

To start with we consider the simplest case:

$$\dot{x}(t) \in -\mathrm{sgn}(x(t)) = \begin{cases} 1 & \text{if} \quad x(t) < 0 \\ -1 & \text{if} \quad x(t) > 0 \\ [\text{-}1,1] & \text{if} \quad x(t) = 0 \end{cases} \quad , \quad x(0) = x_0 \tag{39}$$

with $x(t) \in \mathbb{R}$. This system possesses a unique Lipschitz continuous solution for any x_0. The backward Euler discretization of (39) reads as:

$$\begin{cases} x_{k+1} - x_k = -hs_{k+1} \\ s_{k+1} \in \mathrm{sgn}(x_{k+1}) \end{cases} \tag{40}$$

As is known the *explicit* Euler discretization of such discontinuous systems yields spurious oscillations around the switching surface [Galias et al, IEEE TAC and CAS 2006, 2007, 2008].

$\rightsquigarrow$ this means that the derivative of the switching function while sliding occurs, is very badly estimated.

Both the explicit and the implicit methods converge (the approximated solution $x^N(\cdot)$ tends to the Filippov's solution as $h \to 0$).
However or the backward Euler method the following holds:

Lemma

For all $h > 0$ and $x_0 \in \mathbb{R}$, there exists k_0 such that $x_{k_0+n} = 0$ and $\dfrac{x_{k_0+n+1} - x_{k_0+n}}{h} = 0$ for all $n \geqslant 1$.

On this simple case this has the following graphical interpretation, as the intersection of two graphs:

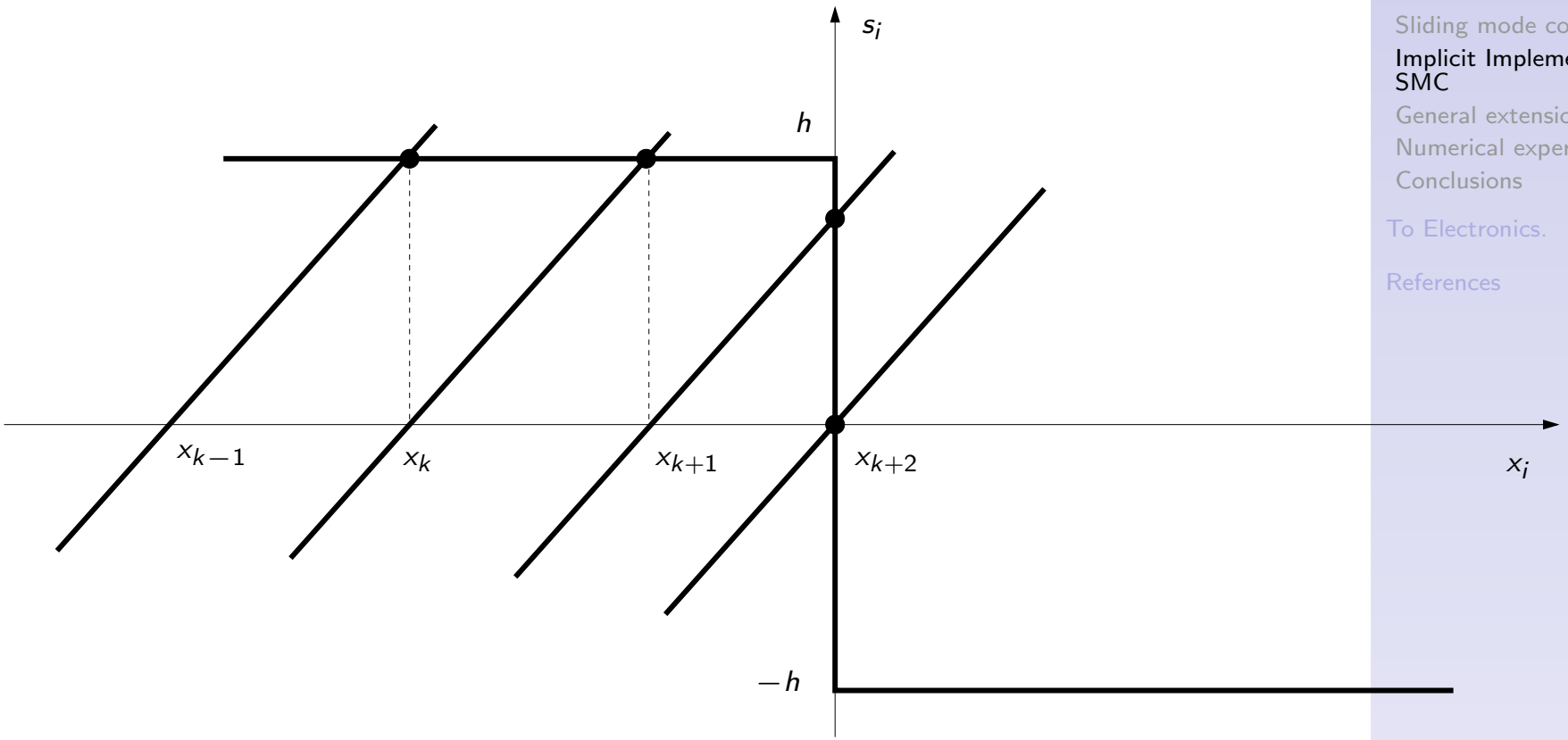
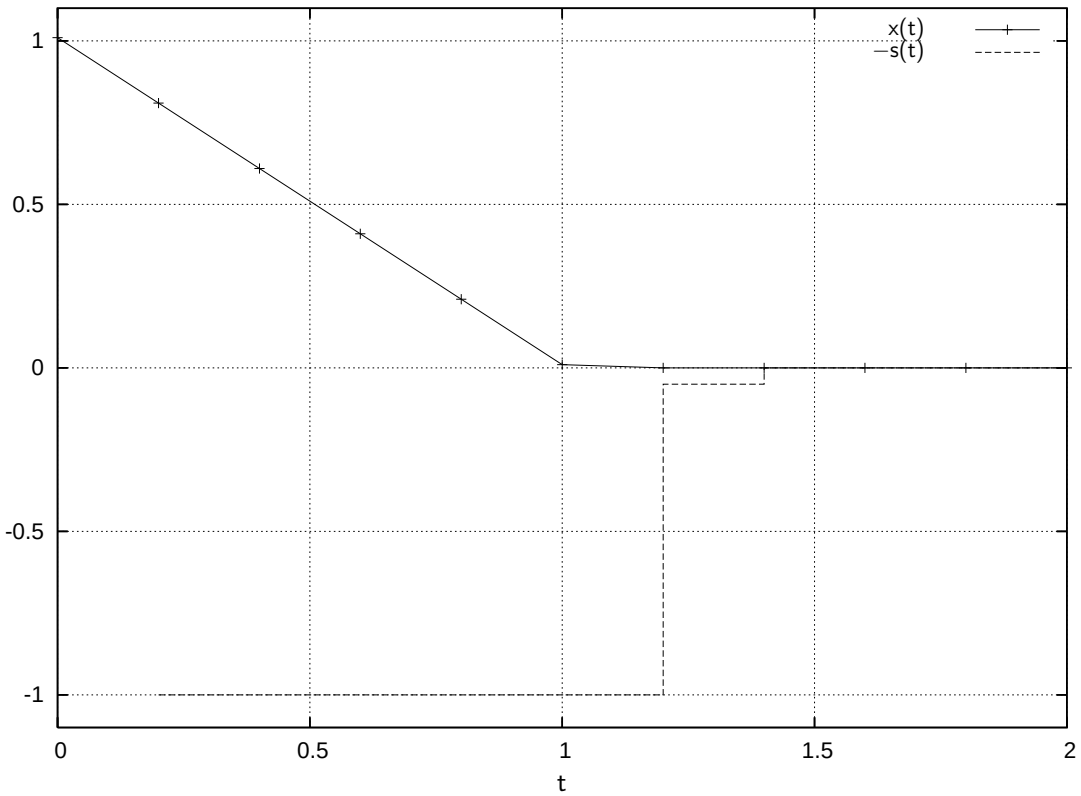


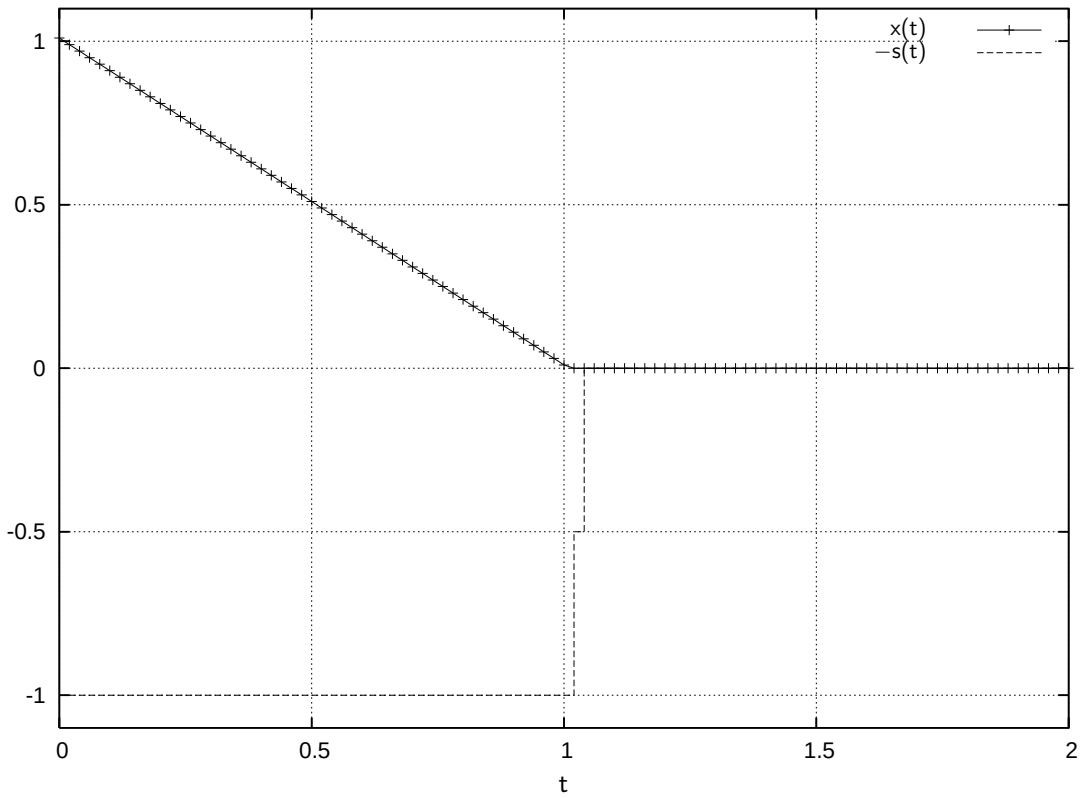
Figure: Iterations of the backward Euler method.

An interesting property is that the smooth stabilization and the finite-time convergence on the switching surface, hold (more or less) independently of the step $h > 0$:



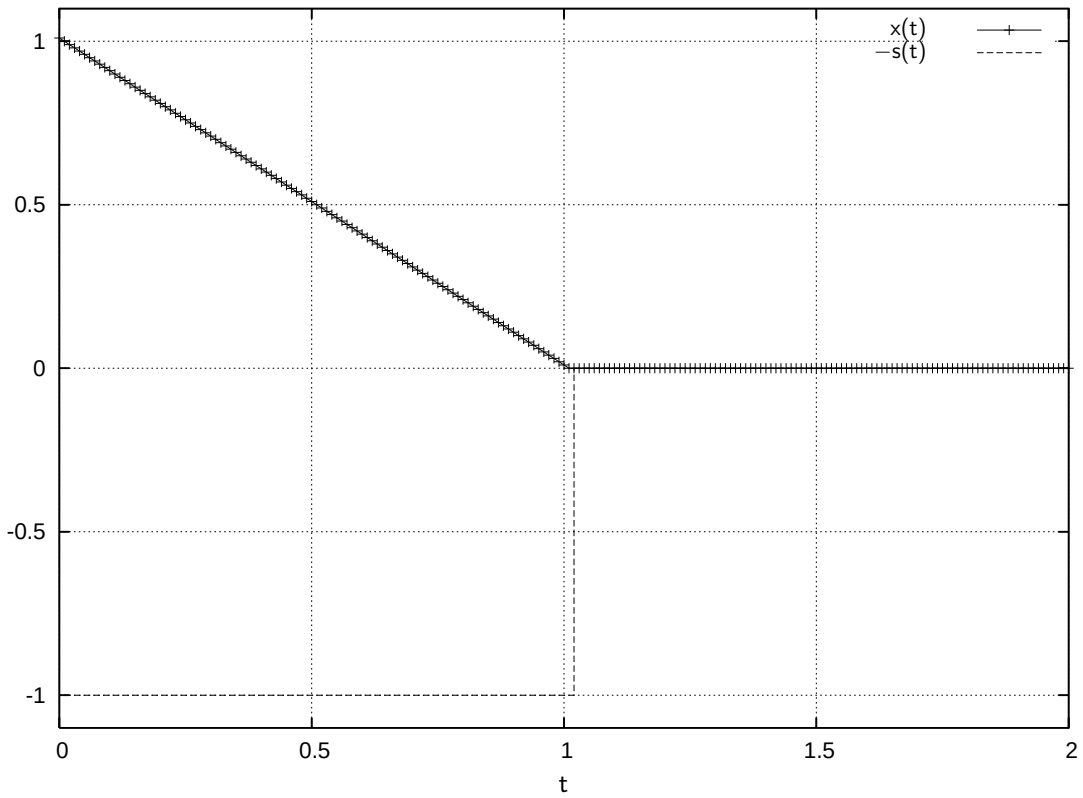
(a) $h = 0.2$

Figure: A simple example for $x_0 = 1.01$ at $t_0 = 0$.



(a) $h = 0.02$

Figure: A simple example for $x_0 = 1.01$ at $t_0 = 0$.



(a) $h = 0.01$

Figure: A simple example for $x_0 = 1.01$ at $t_0 = 0$.

General Extensions

We shall focus on inclusions of the form:

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) - B\ \mathrm{Sgn}(Cx(t) + D), \ \text{ a.e. on } (0, T) \\ x(0) = x_0 \end{cases} \tag{41}$$

with

$B \in \mathbb{R}^{n \times m}$

$\mathrm{Sgn}(Cx(t) + D) \triangleq (\mathrm{sgn}(C_1 x + D_1), ..., \mathrm{sgn}(C_m x + D_m))^T \in \mathbb{R}^m$, where $sgn(\cdot)$ is multivalued at 0.

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Well-posedness of the differential inclusions (41)

Proposition

Consider the differential inclusion in (41). Suppose that

- *There exists $L \geqslant 0$ such that for all $t \in [0, T]$, for all $x_1, x_2 \in \mathbb{R}^n$, one has $||f(t, x_1) - f(t, x_2)|| \leqslant L||x_1 - x_2||$.*
- *There exists a function $\Phi(\,\cdot\,)$ such that for all $R \geqslant 0$:*

$$\Phi(R) = \sup \left\{ \| \frac{\partial f}{\partial t}(\,\cdot\,, v) \|_{\mathcal{L}^2((0,T);\mathbb{R}^n)} \mid \| v \|_{\mathcal{L}^2((0,T);\mathbb{R}^n)} \leqslant R \right\} < +\infty.$$

If there exists an $n \times n$ matrix $P = P^T > 0$ such that

$$PB_{\bullet i} = C_{i\bullet}^T \tag{42}$$

for all $1 \leqslant i \leqslant m$, then for any initial data the differential inclusion (41) has a unique solution $x : (0, T) \to \mathbb{R}^n$ that is Lipschitz continuous.

Sketch of the proof

- Change of state variables $z = Rx$ where $R = R^T > 0$ and $R^2 = P$.
- Use a result in [Bastien-Schatzman ESAIM M2AN 2002] to conclude.

- The existence of a positive definite P such that $PB = C^T$ is satisfied in many instances of sliding-mode control: observer-based sliding-mode control, Lyapunov-based discontinuous robust control.
- This is an "input-output" constraint on the system, constraining the relative degree of the triple (A, B, C).
- It is satisfied when (A, B, C) is positive real (dissipative).

Time-discretization of (41)

The differential inclusion in (41) is therefore discretized as follows:

$$\begin{cases} \dfrac{x_{k+1} - x_k}{h} \in f(t_k, x_k) - B\mathrm{Sgn}(Cx_{k+1} + D), \text{ a.e. on } (0, T) \\ x(0) = x_0 \end{cases} \tag{43}$$

From [Bastien-Schatzman ESAIM M2AN 2002] we have that:

Proposition

Under Proposition 2 conditions, there exists η such that for all $h > 0$ one has

$$\textit{For all } t \in [0, T], \ ||x(t) - x^N(t)|| \leqslant \eta \sqrt{h} \tag{44}$$

Moreover
$\lim_{h \to 0^+} \max_{t \in [0,T]} ||x(t) - x^N(t)||^2 + \int_0^t ||x(s) - x^N(s)||^2 ds = 0.$

However we have more: the discrete state reaches the sliding surface (when it exists) in a finite number of steps, and stabilizes on it in a smooth way.

Let $y(t) \triangleq Cx(t) + D$.

Lemma

Let us assume that a sliding mode occurs for the index $\alpha \subset \{1 \dots m\}$, that is $y_\alpha(t) = 0, t > t_*$. Let C and B be such that (42) holds and $C_{\alpha\bullet} B_{\bullet\alpha} > 0$. Then there exists $h_c > 0$ such that $\forall h < h_c$, there exists $k_0 \in \boldsymbol{N}$ such that $y_{k_0+n} = Cx_{k_0+n+1} + D = 0$ for all integers $n \geqslant 1$.

Such algorithms are similar to proximal algorithms which possess finite-time stabilization properties [Baji and Cabot, Set-Valued Analysis 2006].

Remarks

- Contrarily to other methods that reduce (not suppress...) chattering, the discrete-time sliding surface is equal to the continuous-time sliding surface.
- At each step one has to solve a generalized equation with unknown x_{k+1} that takes the form of a mixed linear complementarity system (MLCP).
- Specific MLCP solvers are needed to implement the method.

Numerical experiments

Let us consider the following two examples:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -c_1 \end{bmatrix} x - \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \mathrm{sgn}(\begin{bmatrix} c_1 & 1 \end{bmatrix} x). \tag{45}$$

(codimension one sliding surface)

$$B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad D = 0, \quad f(x(t), t) = 0 \tag{46}$$

(codimension two sliding surface)

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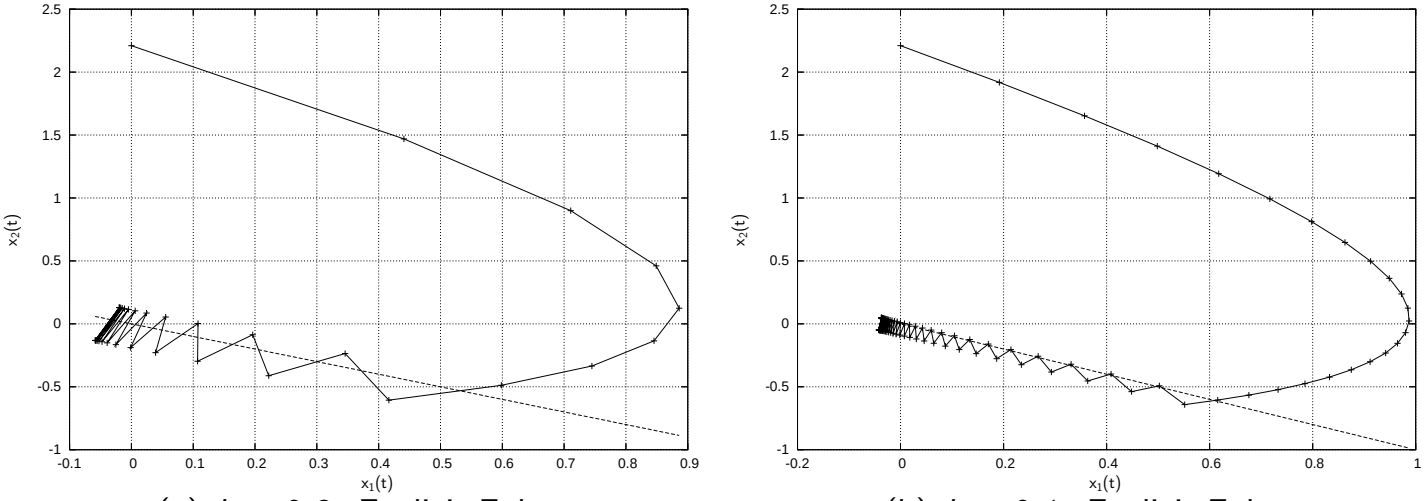
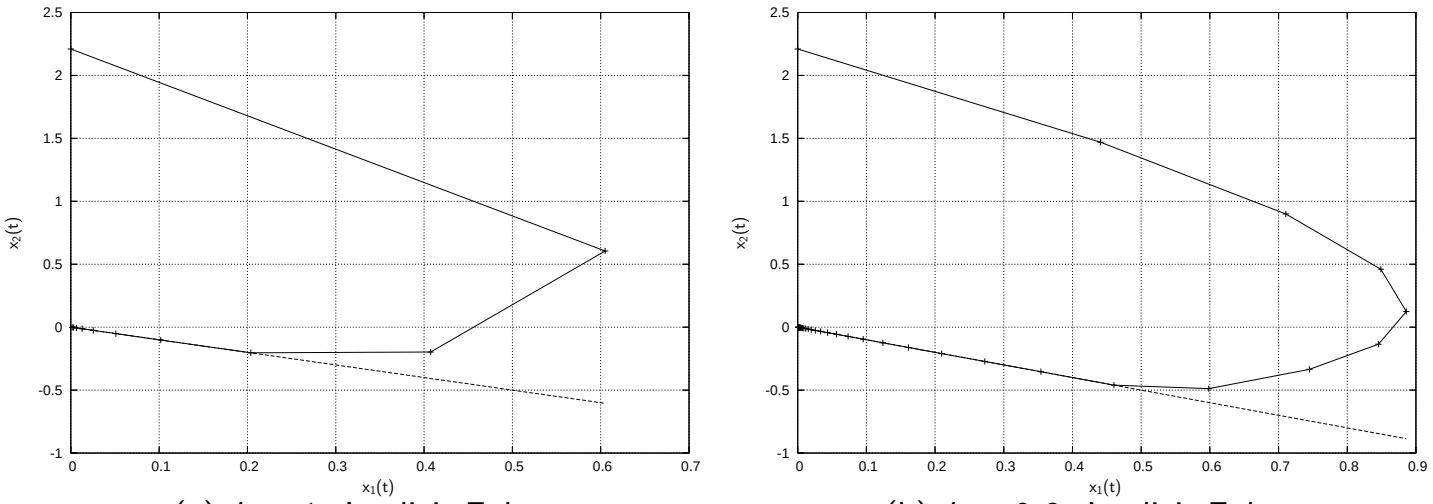
Numerical experiments



(a) $h = 0.3$. Explicit Euler (b) $h = 0.1$. Explicit Euler

Figure: Equivalent control based SMC, $c_1 = 1, \alpha = 1$ and $x_0 = [0, 2.21]^T$. State $x_1(t)$ versus $x_2(t)$.

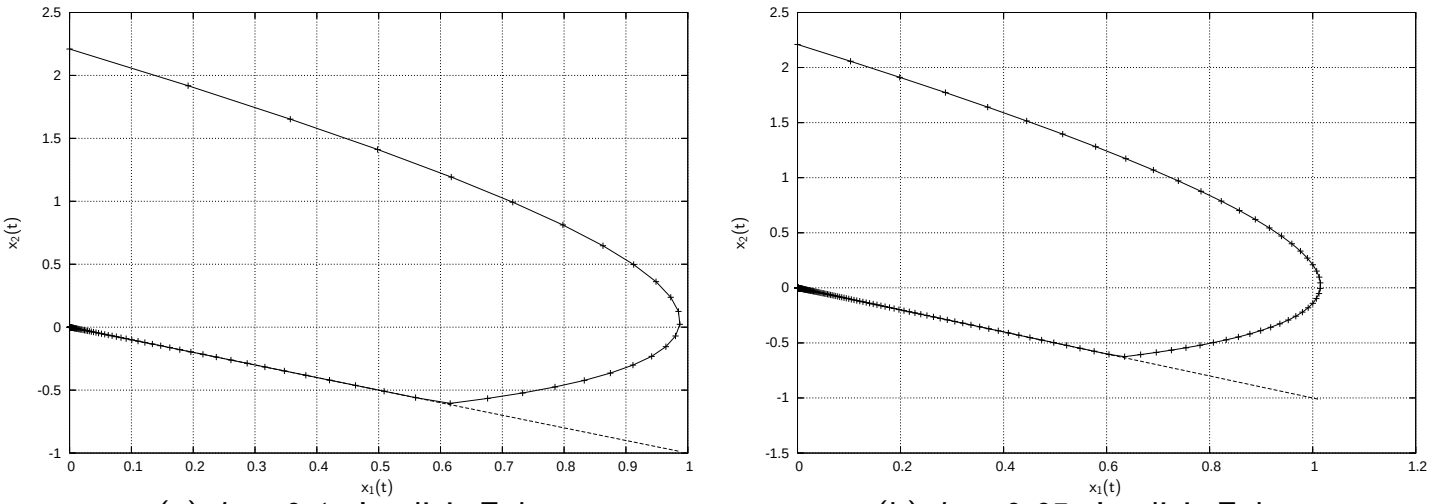
Numerical experiments



(a) $h = 1$. Implicit Euler (b) $h = 0.3$. Implicit Euler

Figure: Equivalent control based SMC, $c_1 = 1, \alpha = 1$ and $x_0 = [0, 2.21]^T$. State $x_1(t)$ versus $x_2(t)$.

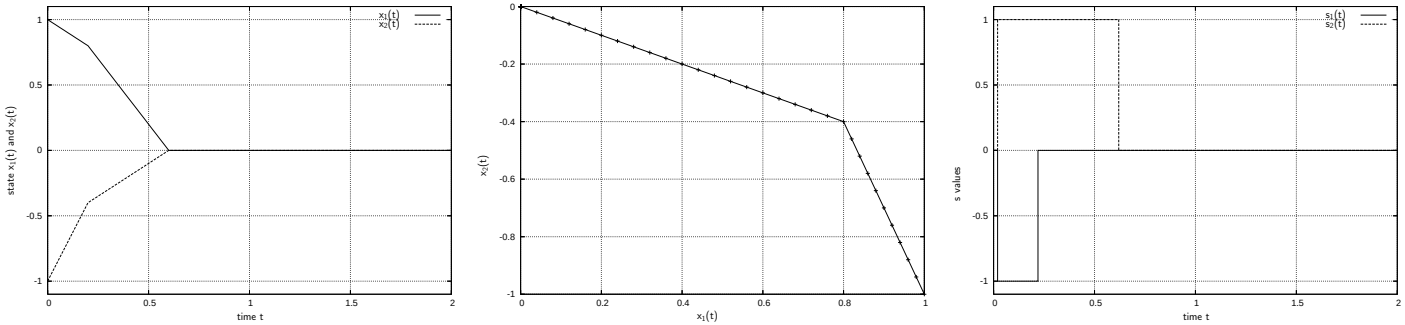
Numerical experiments



(a) $h = 0.1$. Implicit Euler (b) $h = 0.05$. Implicit Euler

Figure: Equivalent control based SMC, $c_1 = 1, \alpha = 1$ and $x_0 = [0, 2.21]^T$. State $x_1(t)$ versus $x_2(t)$.

Numerical experiments



(a) state $x_1(t)$ and $x_2(t)$ versus time

(b) phase portrait $x_2(t)$ versus $x_1(t)$

(c) sgn function $s_1(t)$ and $s_2(t)$

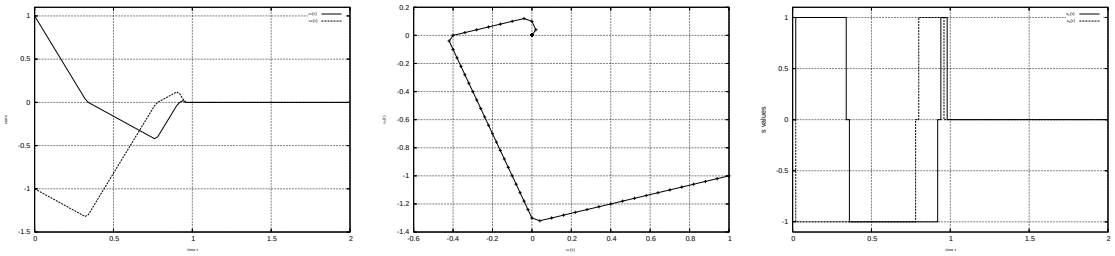
Figure: Multiple Sliding surface. $h = 0.02$, $x(0) = [1.0, -1.0]^T$

The system reaches firstly the sliding surface $2x_2 + x_1 = 0$ *without any chattering, The system then slides on the surface up to reaching the second sliding surface* $2x_1 - x_2 = 0$ *and comes to rest at the origin.*

The Filippov's example with switches accumulation

$$B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 0, \quad f(x(t), t) = 0. \quad (47)$$

The trajectories may slide on the codimension 2 surface given by $Cx = 0$. The origin is attained after an infinite number of switches in finite time.



(a) state $x_1(t)$ and $x_2(t)$ versus time (b) phase portrait $x_2(t)$ versus $x_1(t)$ (c) sgn function $s_1(t)$ and $s_2(t)$

Figure: Multiple Sliding surface. Filippov Example. $h = 0.002$, $x(0) = [1.0, -1.0]^T$

The results show that the system reaches the origin without any chattering.

Conclusions

The implicit Euler method allows one to nicely simulate the main features of sliding-mode systems:

- Finite-time stabilization on the switching surface (of codimension $\geqslant 1$)
- Smooth stabilization on the switching surface

It extends to the discrete-time implementation with ZOH discretization: looks like a promising solution for discrete-time sliding modes.

From Mechanics...

to Control,...

Sliding mode control

Implicit Implementation of SMC

General extensions

Numerical experiments.

Conclusions

To Electronics.

References

Contents

From Mechanics of divided materials to multi-body and robotic systems,

To control (Sliding mode control Theory)

To electronics (Nonsmooth modeling of switched Electrical circuits)

The RLC circuit with a diode

Example

A LC oscillator supplying a load resistor through a half-wave rectifier (see figure 14).

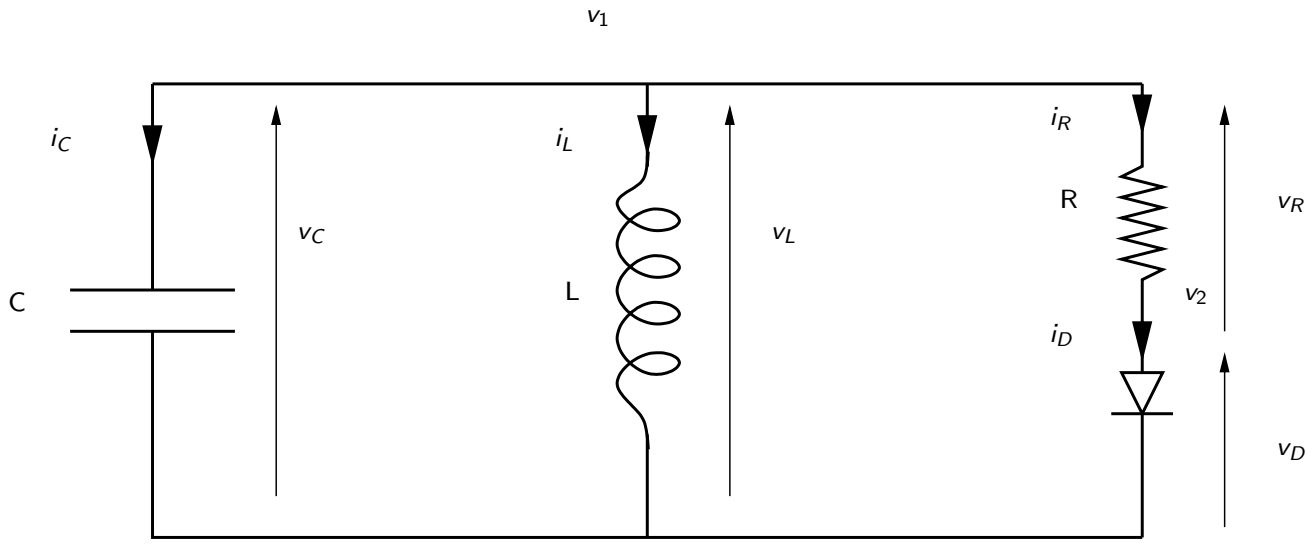
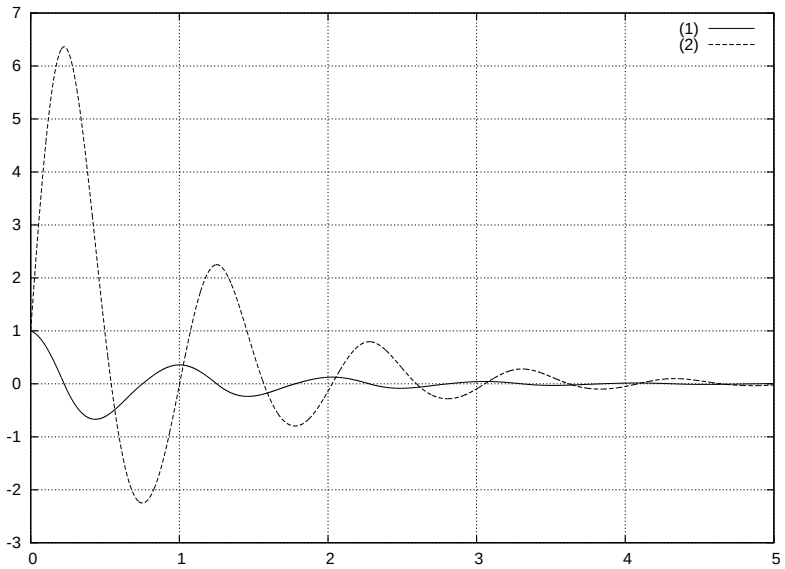


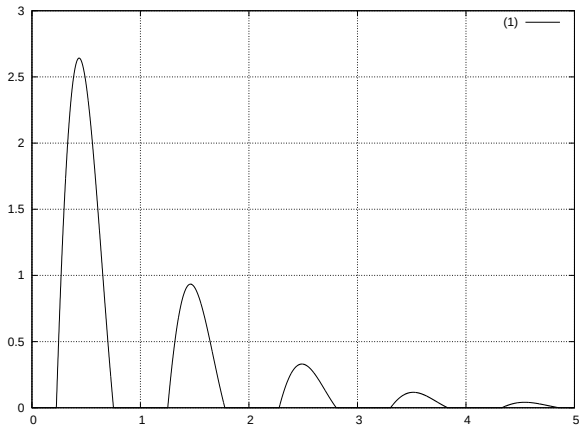
Figure: Electrical oscillator with half-wave rectifier

The RLC circuit with a diode

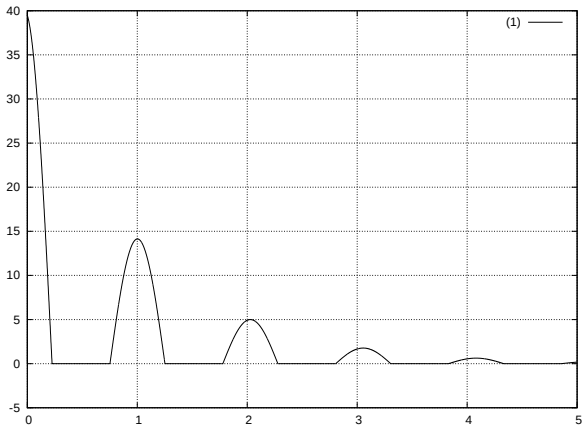
Example



(a) state versus time v_L and i_L



(b) Diode current i_D



(c) Diode voltage v_D

The RLC circuit with a diode

Example

- Kirchhoff laws :

$$\begin{aligned} v_L &= v_C \\ v_R + v_D &= v_C \\ i_C + i_L + i_R &= 0 \\ i_R &= i_D \end{aligned}$$

- Branch constitutive equations for linear devices are :

$$\begin{aligned} i_C &= C\dot{v}_C \\ v_L &= L\dot{i}_L \\ v_R &= Ri_R \end{aligned}$$

- "branch constitutive equation" of the diode

$$0 \in \mathcal{F}(i_D, V_D)$$

The RLC circuit with a diode

Example

The following dynamical system is obtained :

$$\begin{pmatrix} \dot{v}_L \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

$$v_D = v_L - R i_D$$

$$0 \in \mathcal{F}(v_D, i_D)$$

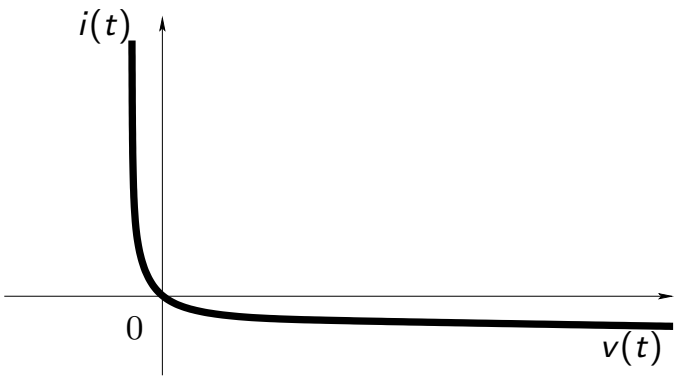
with the state variable $x \stackrel{\Delta}{=} \begin{pmatrix} v_L \\ i_L \end{pmatrix}$ and $\lambda \stackrel{\Delta}{=} i_D$, $y \stackrel{\Delta}{=} v_D$, we get

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \in \mathcal{F}(y, \lambda) \end{cases} \tag{48}$$

Diode behavior

A modeling choice

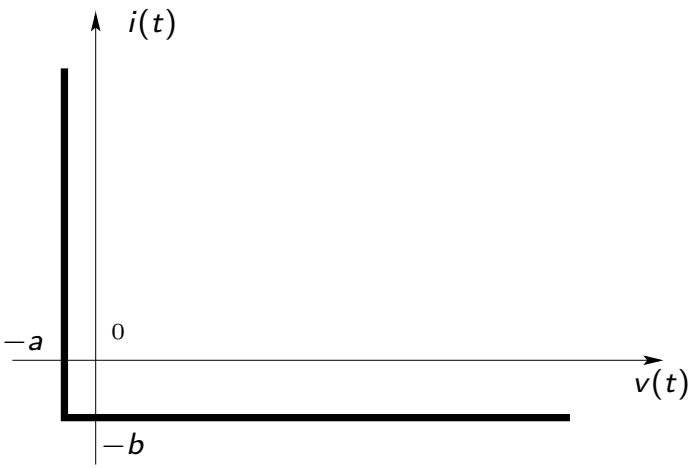
smooth modeling



(a)

$$i(t) = i_s \exp(-\frac{v(t)}{\alpha} - 1)$$

nonsmooth modeling



(b)

$$0 \leqslant i(t) + b \perp v(t) + a \geqslant 0$$

Figure: Two models of diodes.

Diode behavior

Why a nonsmooth modeling ?

- To avoid stiff nonlinear models by using ideal constraints.
- To model the ideal behavior of switched components without articifial regularization

The diode–bridge rectifier

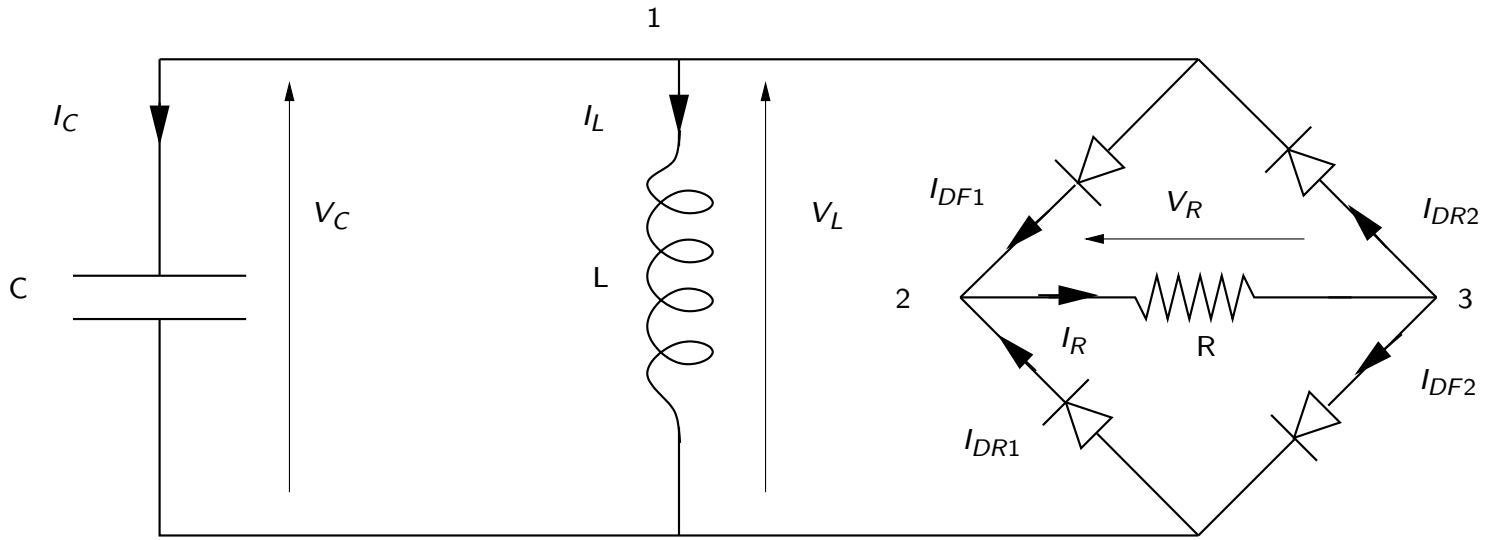


Figure: The Diode-bridge rectifier

The diode–bridge rectifier

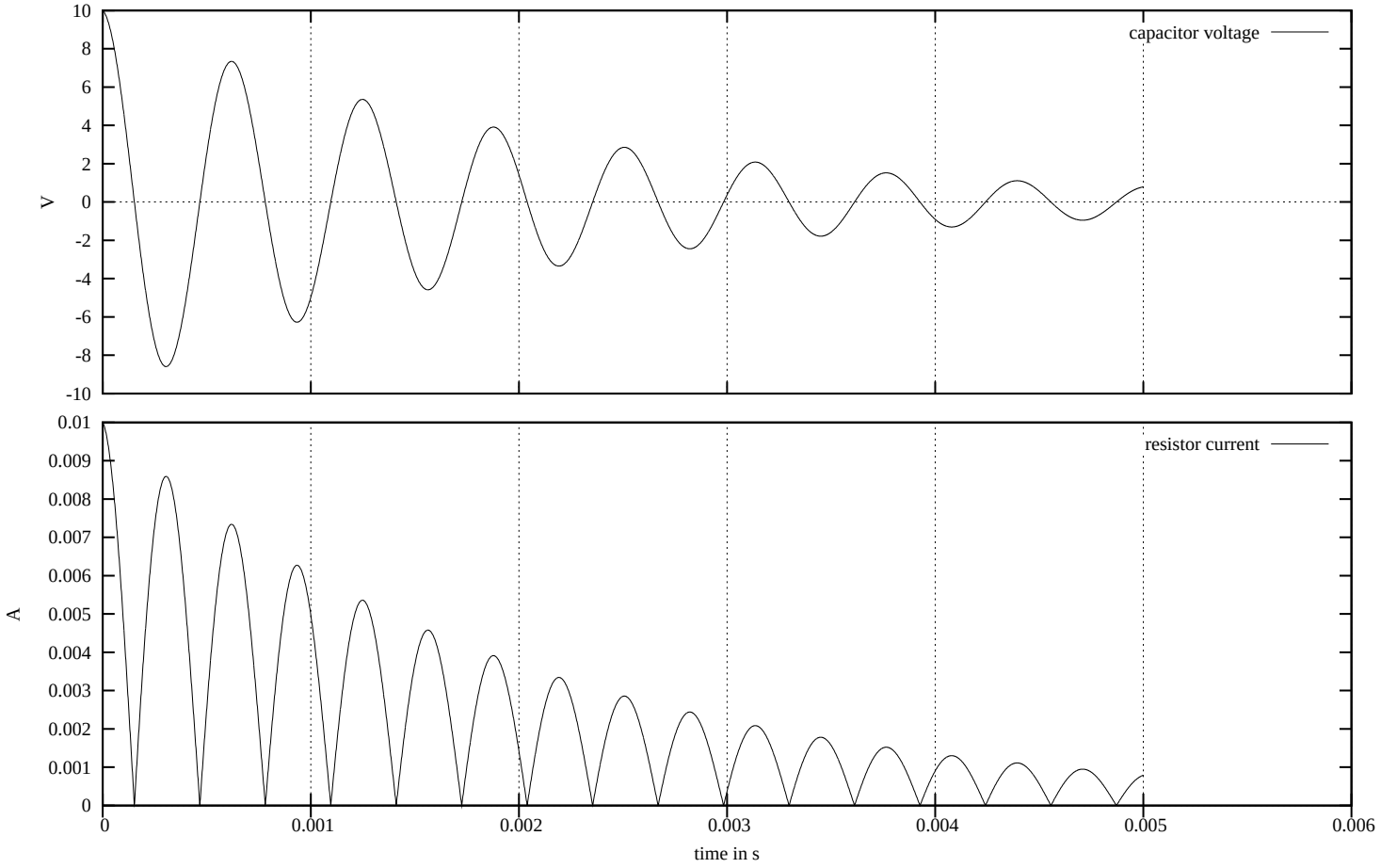


Figure: The Diode-bridge rectifier. Standard results

The diode–bridge rectifier

Differential systems

The dynamical equations are formulated as

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{49}$$

choosing :

$$x = \begin{bmatrix} V_L \\ I_L \end{bmatrix}, \quad \text{and } y = \begin{bmatrix} I_{DR1} \\ I_{DF2} \\ V_2 - V_1 \\ V_1 - V_3 \end{bmatrix}, \quad \lambda = \begin{bmatrix} V_2 \\ -V_3 \\ I_{DF1} \\ I_{DR2} \end{bmatrix}, \tag{50}$$

and with

$$A = \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1/C & 1/C & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

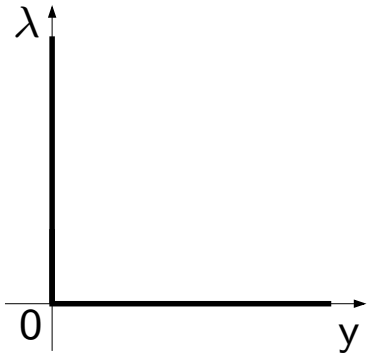
$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1/R & 1/R & -1 & 0 \\ 1/R & 1/R & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{51}$$

A typical example of nonsmooth systems

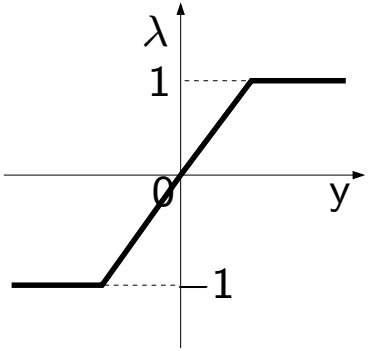
Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{52}$$

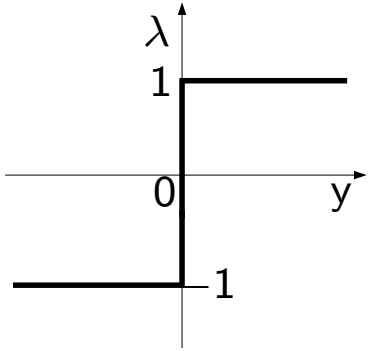
with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$
$C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, for m constraints.



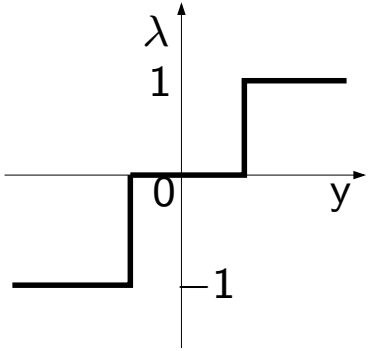
Piecewise linear systems



Saturation



Relay



Relay with dead zone

A slightly more general class of nonsmooth systems

Differential inclusion into normal cones

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ -y \in N_K(\lambda) \end{cases} \tag{53}$$

where K is a convex set and $N_K(\lambda)$ stands for the normal cone to K taken at λ

Usual examples for K

- $K = \mathbb{R}^m$, then we obtain linear time invariant DAE

$$-y \in N_{\mathbb{R}^m}(\lambda) \Longleftrightarrow y = 0, \quad \lambda \in \mathbb{R}^m \tag{54}$$

- $K = \mathbb{R}^m_+$, then we obtain Linear Complementarity Systems (LCS)

$$-y \in N_{\mathbb{R}^m_+}(\lambda) \Longleftrightarrow 0 \leqslant y \perp \lambda \geqslant 0 \tag{55}$$

- $K = [-1, 1]^m$, then we obtain linear relay systems (related to Filippov's DI and sliding mode control).

$$-y \in N_{[-1,1]^m}(\lambda) \Longleftrightarrow \lambda \in \mathrm{sgn}(y) \tag{56}$$

Objectives

Our background

- Nonsmooth modeling of unilateral constraints and friction
- Nonsmooth analysis of dynamics with jumps.

Our Objectives

- Understand what can be the nature of the solutions (uniqueness, smoothness).
- How perform the numerical time–integration ?
- Open issues for the time–integration of large dynamical systems arising in electrical network applications.

Some results

Nature of solutions for $K \in \mathbb{R}^m_+$

The nature of solutions depends on

- the relative degree (index) between y and λ
- the possible consistency of the solution

The main types of solutions are

- $\mathcal{C}^1$ solutions when λ is a lipschitz function of x (relative degree 0)
- absolutely continuous solutions (relative degree 1)
- solutions of Bounded Variations (relative degree 2)

Some results

Numerical time–integration methods

The time integration methods depends on the solution

- $\mathcal{C}^1$ solutions : Standard DAE integrators of low order
- absolutely continuous solutions : Implicit first order scheme
- solutions of Bounded Variations : Moreau's catching up algorithm

Some results

Industrial circuits and automatic circuit equations formulation

- Adaptation of the standard Modified Nodal Analysis (MNA) to the nonsmooth elements to obtain

Problem (DGE)

$$
\begin{array}{ll}
M(X,t)\dot{X} = D(X,t) + U(t) + R & \text{] Differential Algebraic Equations} \\
\left.\begin{array}{l} y = G(X,\lambda,t) \\ R = H(X,\lambda,t) \end{array}\right] & \text{Input/output relations on nonsmooth components} \\
0 \in F(y,\lambda,t) + T(y,\lambda,t) & \text{] Generalized equation} \\
X = [V, I_{\mathsf{L}}, I_{\mathsf{V}}, I_{\mathsf{NS}}]^T & \text{] Variable definition}
\end{array}
\tag{57}
$$

➜ Difficulties to discuss the nature of solution and then to adapt the time numerical method

➜ In electrical circuits, the main difficulty is induced by the topology of the circuit rather than the inherent non–linearity of the components.

Applications to industrial electrical networks

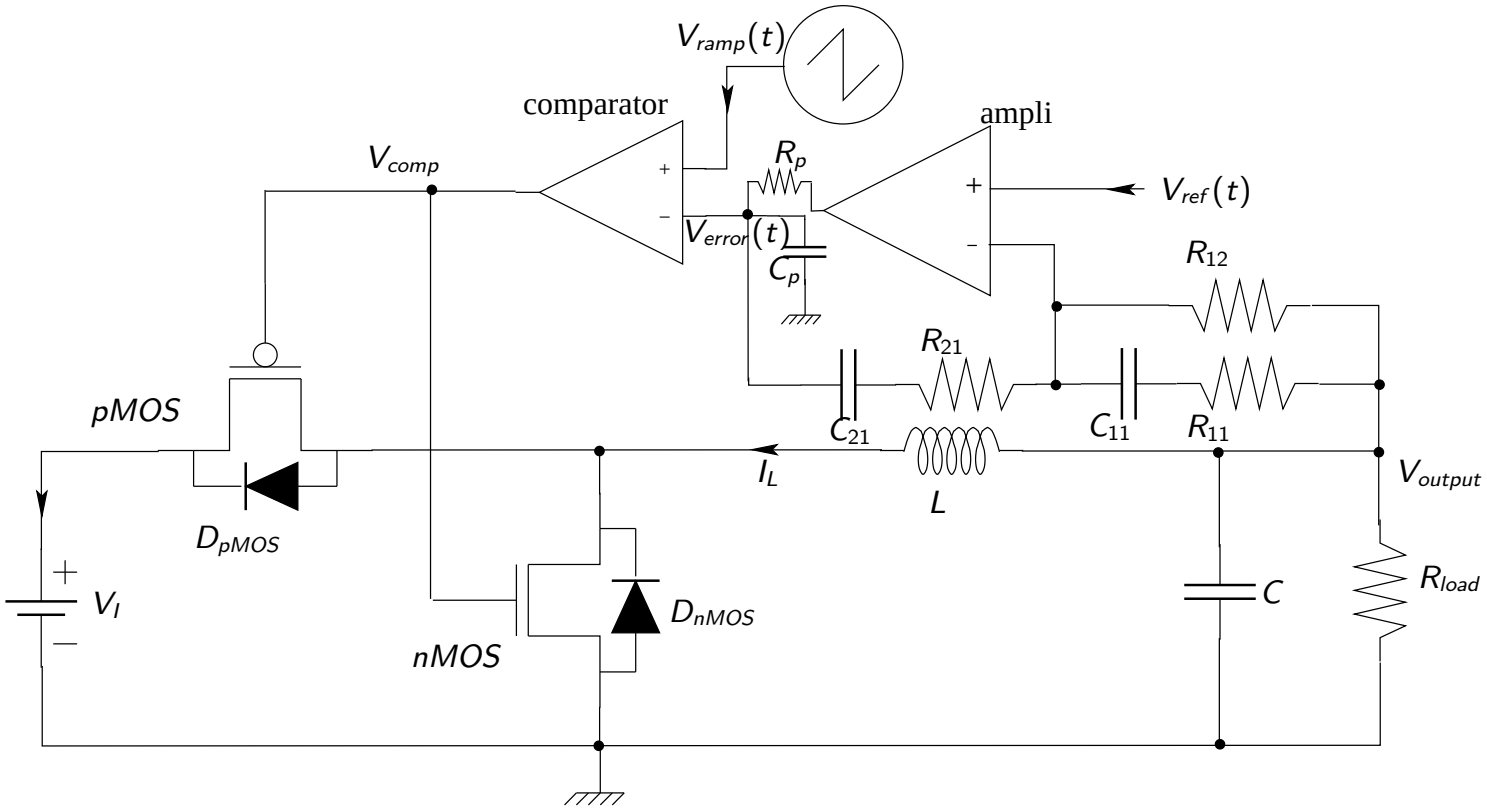
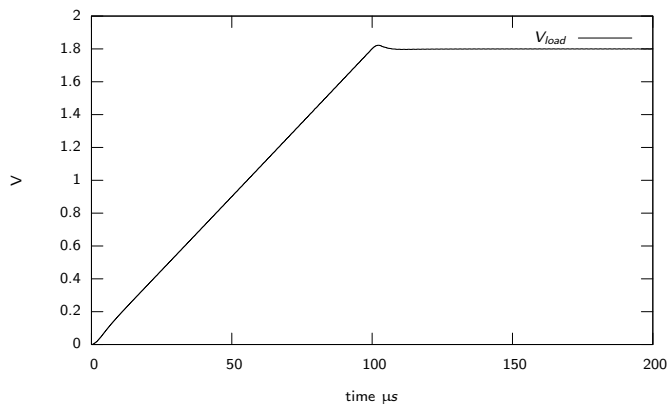
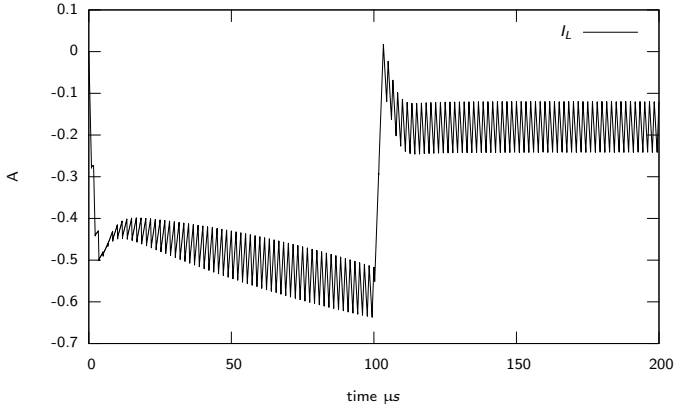


Figure: Buck converter.

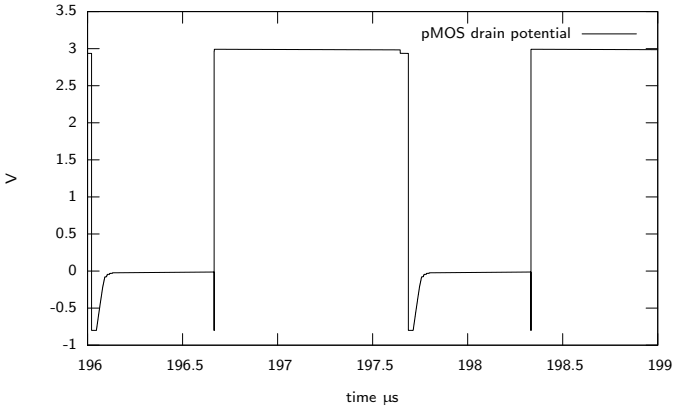
Applications to industrial electrical networks



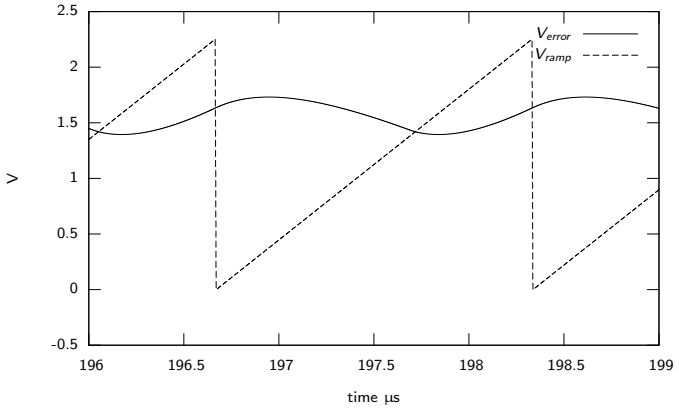
(a) V_{load}



(b) I_L



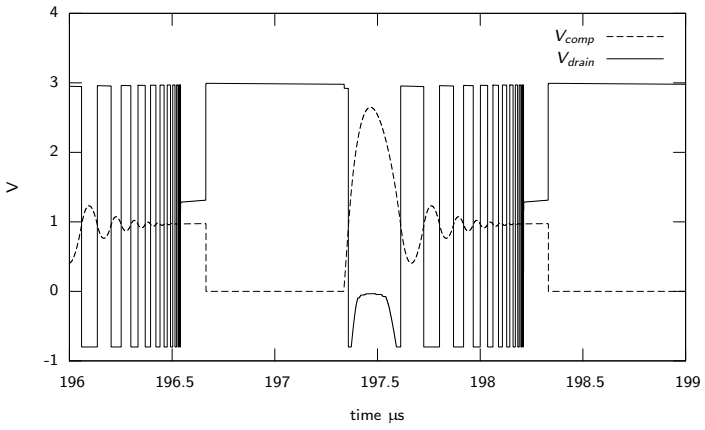
(c) pMOS drain potential



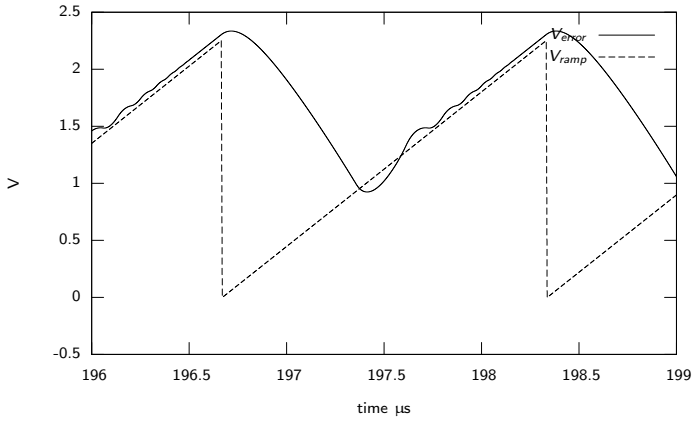
(d) V_{ramp} and V_{error}

Figure: SICONOS buck converter simulation using standard parameters.

Applications to industrial electrical networks



(a) V_{comp} and V_{drain}



(b) V_{ramp} and V_{error}

Figure: SICONOS buck converter simulation using sliding mode parameters.

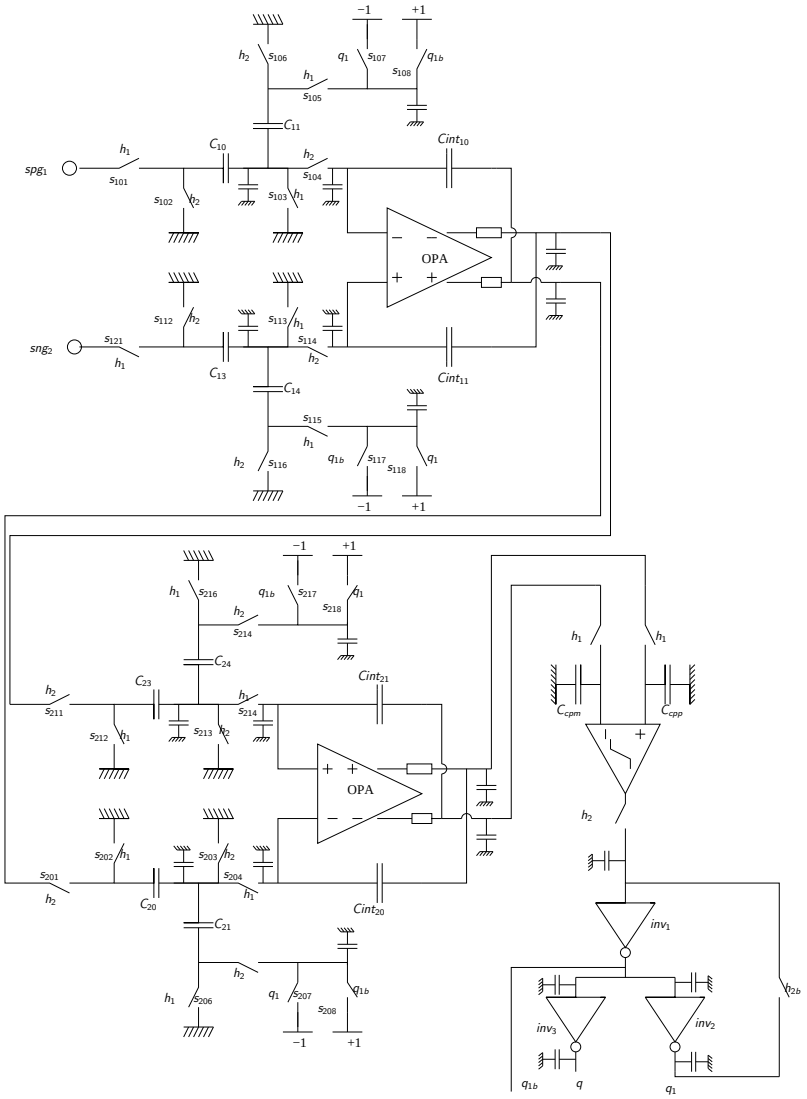


Figure: Delta-Sigma converter.

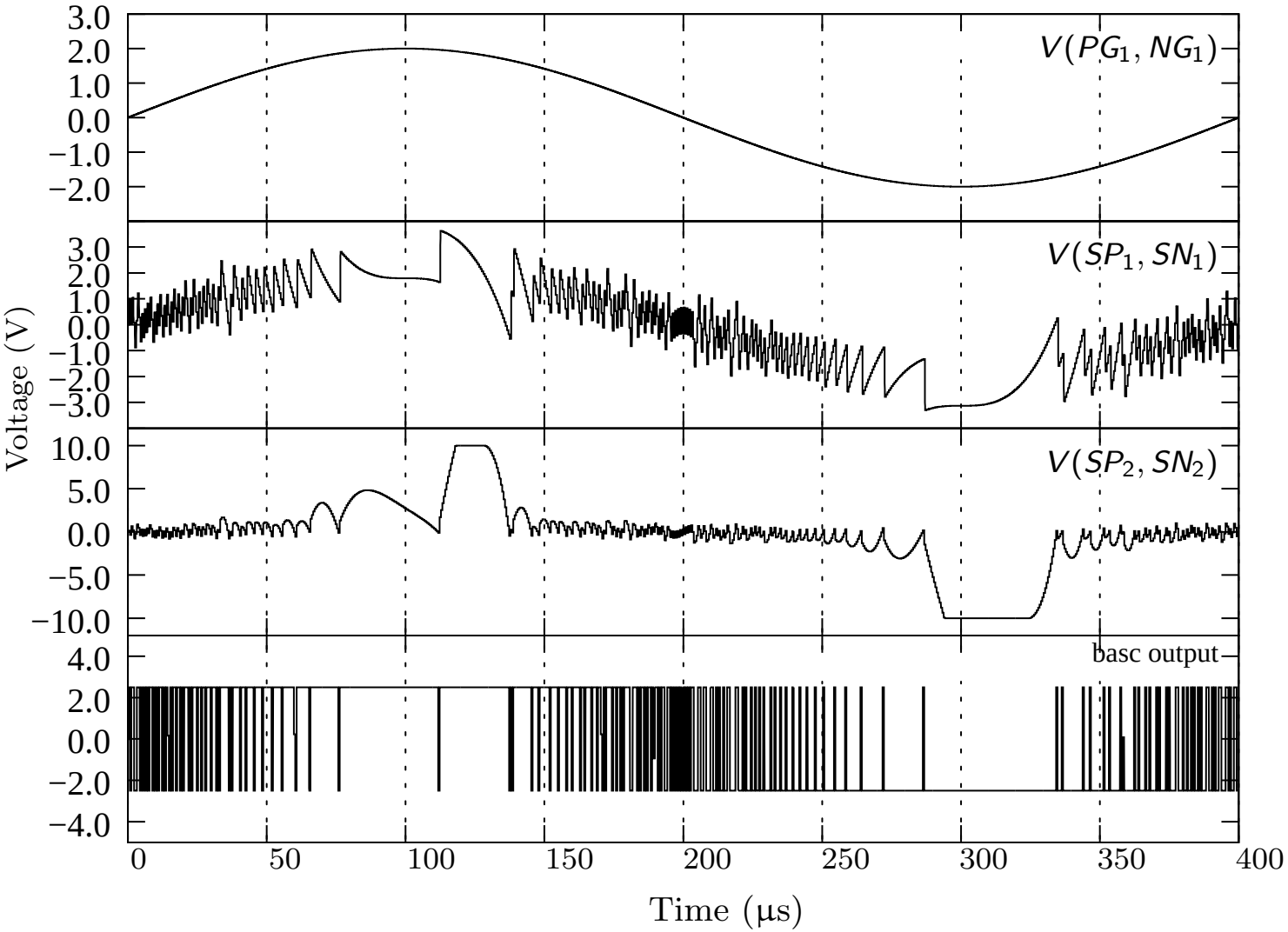


Figure: SICONOS simulation.

Open issues

For more general formulations and more complex systems, are we able to infer the nature of the solutions? That is to say,

- Define and predict an equivalent notion to index and relative degree for instance, for a matrix D semi-definite positive.
- Given passive components, are we able to forecast the nature of the solutions from some topological considerations ? (as for the DAE case.)
- Adapt the time–stepping schemes in an hierarchical way in taking into account the "index" of each variable.

and towards

- Dynamics of gene regulatory networks (cell physiology)
- ...

Thank you for your attention.
Happy Birthday Michel and thank you again

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