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An excursion into Nonsmooth Dynamics: from Mechanics, to Electronics, through Control

Vincent Acary

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An excursion into Nonsmooth Dynamics: from Mechanics, to Electronics, through Control

Vincent Acary

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Fifty Years of Finite Freedom Mechanics.
On the occasion of Michel Jean's 70th birthday
Marseille, 25–27 October 2010

From Mechanics of divided materials to multi-body and robotic systems,

History and Motivations

The smooth multibody dynamics

The Non smooth Lagrangian Dynamics

The Moreau's sweeping process

State-of-the-art

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To control (Sliding mode control Theory)

To electronics (Nonsmooth modeling of switched Electrical circuits)

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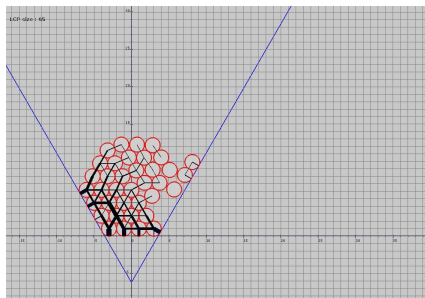
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Stack of beads with perturbation



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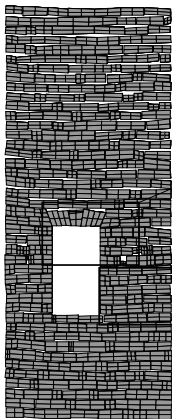
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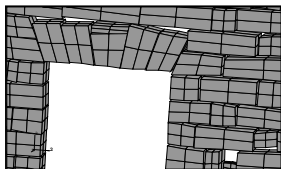
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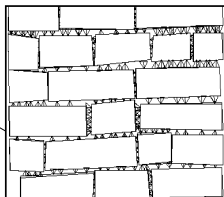
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(a) FEM H8 meshing



(b) Zoom on the window



(c) Contact detection

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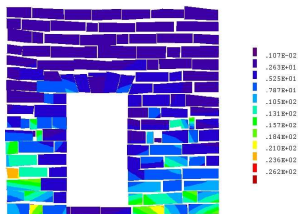
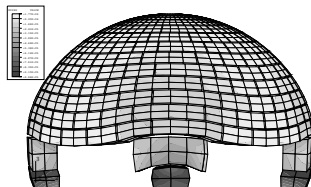
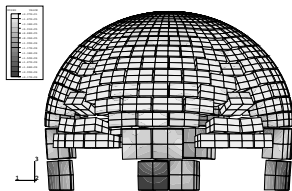


Figure: VON MISES stresses

Divided Materials and Masonry



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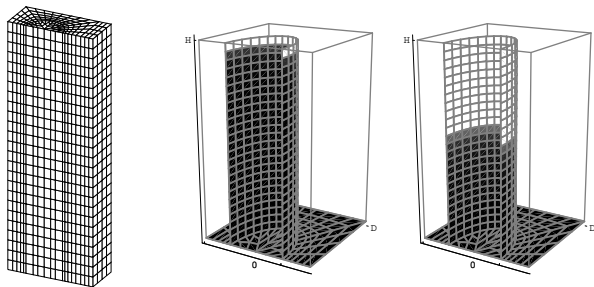
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FEM models with contact, friction cohesion, etc...



Joint work with Y. Monerie, IRSN.

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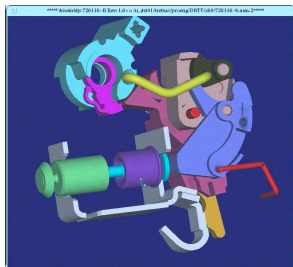
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Bipedal Robot INRIA BIPOP



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Towards controlled robotic systems on granular materials

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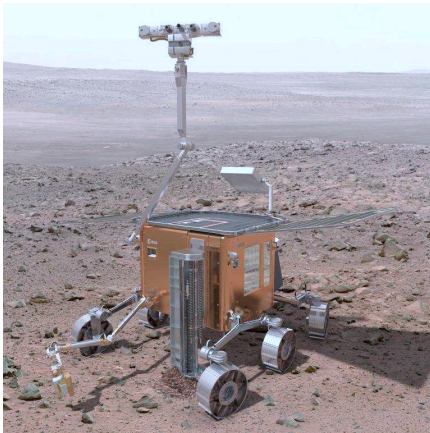
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Simulation of the ExoMars Rover (INRIA/Trasys Space/ESA)



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They are all nonsmooth mechanical systems but they differ in

- ▶ the presence of perfect nonlinear joints,
- ▶ the presence of finite rotations,
- ▶ the presence of Control (sensors & actuators)
- ▶ the desired properties in design and development which influence the numerical simulation and prototyping

Unilateral constraints as an inclusion

Definition (Perfect unilateral constraints on the smooth dynamics)

$$\begin{cases} M(q) \frac{dv}{dt} + F(t, q, v) = r \\ -r \in N_{\mathcal{C}(t)}(q(t)) \end{cases} \quad (1)$$

where

- ▶ $r = \nabla_q g(q, t) \lambda$ is the generalized reactions due to the constraints.
- ▶ Finite set of ν unilateral constraints on the generalized coordinates :

$$g(q, t) = [g_\alpha(q, t) \geq 0, \quad \alpha \in \{1 \dots \nu\}]^T. \quad (2)$$

- ▶ Admissible set $\mathcal{C}(t)$

$$\mathcal{C}(t) = \{q \in \mathcal{M}(t), g_\alpha(q, t) \geq 0, \alpha \in \{1 \dots \nu\}\}. \quad (3)$$

- ▶ Normal Cone to $\mathcal{C}(t)$

$$N_{\mathcal{C}(t)}(q(t)) = \left\{ y \in \mathbb{R}^n \mid y = - \sum_{\alpha} \lambda_{\alpha} \nabla g_{\alpha}(q, t), \right. \\ \left. \lambda_{\alpha} \geq 0, \lambda_{\alpha} g_{\alpha}(q, t) = 0 \right\} \quad (4)$$

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Fundamental assumptions.

- ▶ The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
→ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that

$$v^+ = \dot{q}^+ \quad (5)$$

- ▶ q is related to this velocity by

$$q(t) = q(t_0) + \int_{t_0}^t v^+(t) dt \quad (6)$$

- ▶ The acceleration, (\ddot{q} in the usual sense) is hence a differential measure dv associated with v such that

$$dv(]a, b]) = \int_{]a, b]} dv = v^+(b) - v^+(a) \quad (7)$$

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Definition (Non Smooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ v^+ = \dot{q}^+ \end{cases} \quad (8)$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- ▶ The non smooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- ▶ The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- ▶ This formulation is sound from a mathematical Analysis point of view.

References

[Schatzman, 1973, 1978, Moreau, 1983, 1988]

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Decomposition of measure

$$\begin{cases} dv = \gamma dt + (v^+ - v^-) d\nu + dv_S \\ di = f dt + p d\nu + di_S \end{cases} \quad (9)$$

where

- ▶ $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- ▶ f is the Lebesgue measurable force,
- ▶ $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- ▶ $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v , i.e. where $(v^+ - v^-) \neq 0$, i.e. $d\nu = \sum_i \delta_{t_i}$,
- ▶ p is the purely atomic impact percussions such that $p d\nu = \sum_i p_i \delta_{t_i}$,
- ▶ dv_S and di_S are singular measures with the respect to $dt + d\eta$.

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Substituting the decomposition of measures into the non smooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = p d\nu, \quad (10)$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \quad (11)$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t, q, v)dt = f dt \quad (12)$$

or

$$M(q)\gamma^+ + F(t, q, v^+) = f^+ [dt - a.e.] \quad (13)$$

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The Moreau's sweeping process of second order

Definition (Moreau [1983, 1988])

A key stone of this formulation is the inclusion in terms of velocity. Indeed, the inclusion (1) is “replaced” by the following inclusion

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ v^+ = \dot{q}^+ \\ -di \in N_{T_C(q)}(v^+) \end{cases} \quad (14)$$

Comments

This formulation provides a common framework for the non smooth dynamics containing inelastic impacts without decomposition.

→ Foundation for the time-stepping approaches.

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Comments

- ▶ *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- ▶ *The inclusion in terms of velocity* v^+ rather than of the coordinates q .

Interpretation

- ▶ Inclusion of measure, $-di \in K$

- ▶ Case $di = r' dt = f dt$.

$$-f \in K \quad (15)$$

- ▶ Case $di = p_i \delta_i$.

$$-p_i \in K \quad (16)$$

- ▶ Inclusion in terms of the velocity. Viability Lemma

If $q(t_0) \in C(t_0)$, then

$$v^+ \in T_C(q), t \geq t_0 \Rightarrow q(t) \in C(t), t \geq t_0$$

→ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

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The Newton-Moreau impact rule

$$-di \in N_{T_C(q(t))}(v^+(t) + ev^-(t)) \quad (17)$$

where e is a coefficient of restitution.

Velocity level formulation. Index reduction

$$\begin{aligned} -\lambda \in N_{\mathbb{R}^+}(y) &\rightsquigarrow & -\lambda \in N_{T_{\mathbb{R}^+}}(\dot{y}) \\ &\Updownarrow & \\ 0 \leq y \perp \lambda \geq 0 &\rightsquigarrow & \text{if } y \leq 0 \text{ then } 0 \leq \dot{y} \perp \lambda \geq 0 \end{aligned} \quad (18)$$

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Numerical time-integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time-stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event-driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

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Objectives

Design nonsmooth event capturing methods with

- ▶ same properties as standard methods (robustness, accumulation, ...)
- ▶ Higher resolution (ratio error/computational cost)
- ▶ Higher order of accuracy

Means

1. Adaptive time-step size and order strategies for standard methods
2. Mixed integrators with rough pre-detection of events
3. Splitting strategies
4. Ad hoc detection of discontinuity and order of discontinuity methods.

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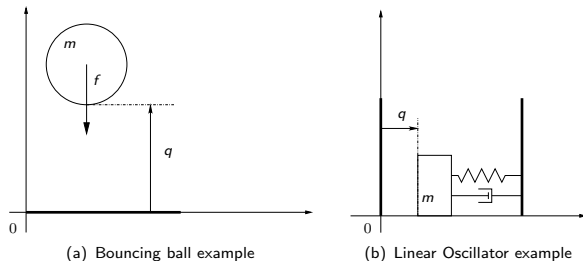


Figure: Academic test examples with analytical solutions

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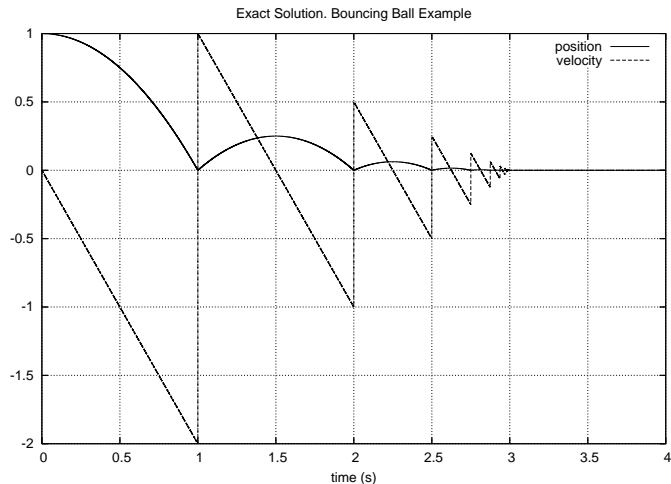


Figure: Analytical solutions. Bouncing ball example]

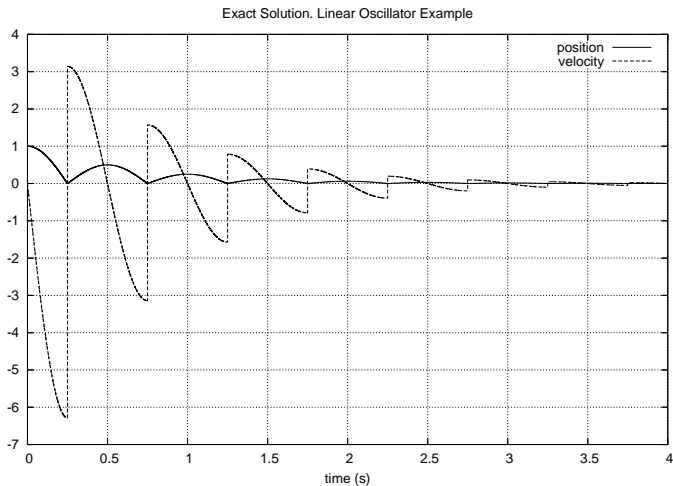


Figure: Analytical solutions. Linear Oscillator

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Moreau–Jean's Time stepping scheme [Moreau, 1988] and [Jean, 1999]

Principle of NSCD

$$\left\{ \begin{array}{l} M(\mathbf{q}_{k+\theta})(\mathbf{v}_{k+1} - \mathbf{v}_k) - h\tilde{\mathbf{F}}_{k+\theta} = G(\mathbf{q}_{k+\theta})\mathbf{P}_{k+1}, \end{array} \right. \quad (19a)$$

$$\mathbf{q}_{k+1} = \mathbf{q}_k + h\mathbf{v}_{k+\theta}, \quad (19b)$$

$$\mathbf{U}_{k+1} = G^T(\mathbf{q}_{k+\theta})\mathbf{v}_{k+1} \quad (19c)$$

$$-\mathbf{P}_{k+1} \in N_{T_{\mathbb{R}_+^m}(\tilde{\mathbf{y}}_{k+\gamma})}(\mathbf{U}_{k+1} + \mathbf{e}\mathbf{U}_k), \quad (19d)$$

$$\tilde{\mathbf{y}}_{k+\gamma} = \mathbf{y}_k + h\gamma\mathbf{U}_k, \quad \gamma \in [0, 1]. \quad (19e)$$

with $\theta \in [0, 1], \gamma \geq 0$ and $\mathbf{x}_{k+\alpha} = (1 - \alpha)\mathbf{x}_{k+1} + \alpha\mathbf{x}_k$ and $\tilde{\mathbf{y}}_{k+\gamma}$ is a prediction of the constraints.

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Principle

$$\left\{ \begin{array}{l} M(q_k + 1)(q_{k+1} - 2q_k + q_{k-1}) - h^2 F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta}) = p_{k+1} \quad (20a) \\ v_{k+1} = \frac{q_{k+1} - q_{k-1}}{2h}, \quad (20b) \\ -p_{k+1} \in N_K \left(\frac{q_{k+1} + eq_{k-1}}{1+e} \right), \quad (20c) \end{array} \right.$$

where N_K defined the normal cone to K .

For $K = \{q \in \mathbb{R}^n, y = g(q) \geq 0\}$

$$0 \leq g \left(\frac{q_{k+1} + eq_{k-1}}{1+e} \right) \perp \nabla g \left(\frac{q_{k+1} + eq_{k-1}}{1+e} \right) p_{k+1} \geq 0 \quad (21)$$

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Shared mathematical properties

- ▶ Convergence results for one constraints
- ▶ Convergence results for multiple constraints problems with acute kinetic angles
- ▶ No theoretical proof of order

Mechanical properties

- ▶ Position vs. velocity constraints
- ▶ Respect of the impact in one step (Moreau–Jean) vs. Two-steps(Schatzman)
- ▶ Linearized constraints rather than nonlinear.

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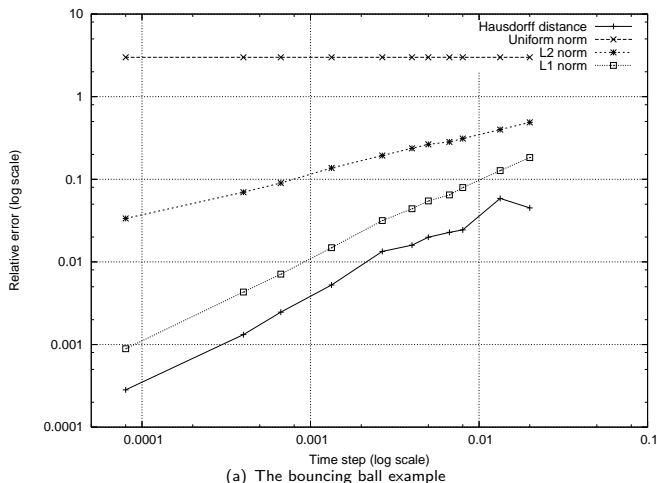


Figure: Empirical order of convergence of the Moreau–Jean's time-stepping scheme.

Empirical order of convergence. Moreau–Jean's time-stepping scheme

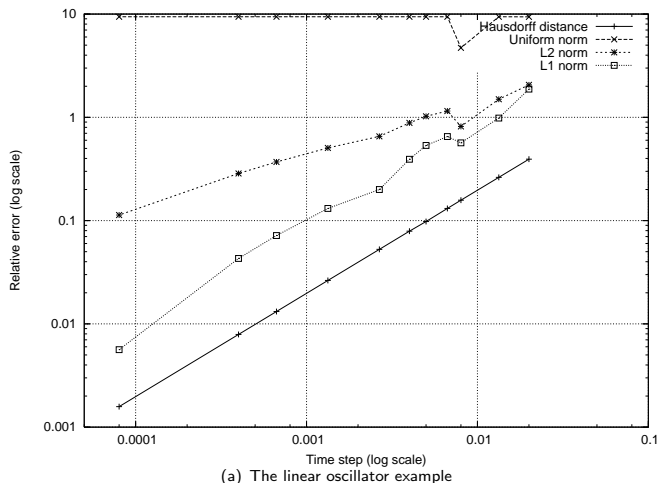


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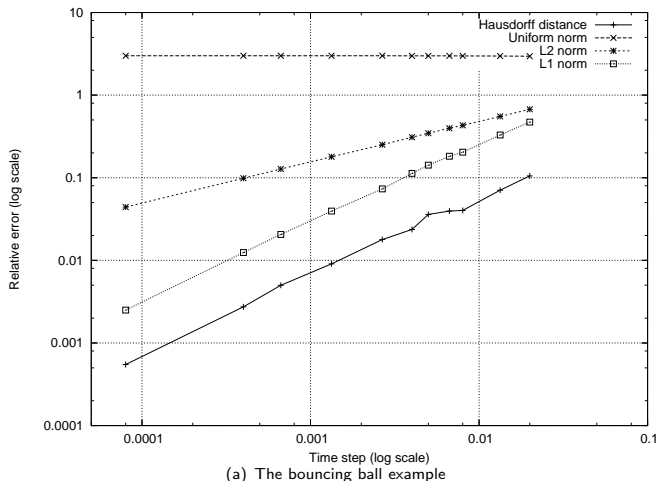
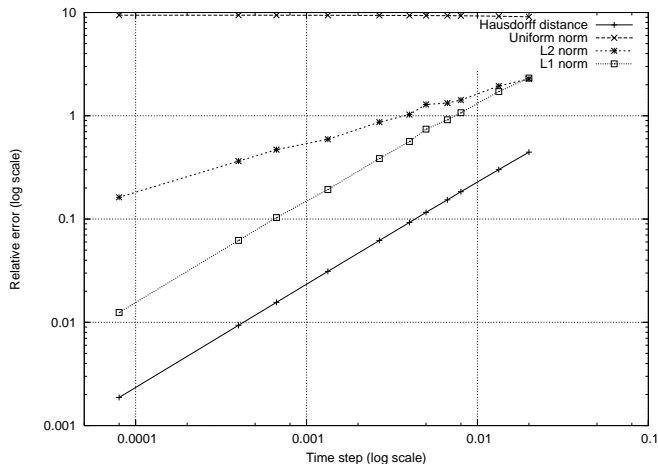


Figure: Empirical order of convergence of the Schatzman–Paoli's time-stepping scheme.

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(a) The linear oscillator example

Figure: Empirical order of convergence of the Schatzman–Paoli's time-stepping scheme.

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Assumption 1 : Existence and uniqueness

A unique global solution over $[0, T]$ for Moreau's sweeping process is assumed such that $q(\cdot)$ is absolutely continuous and admits a right velocity $v^+(\cdot)$ at every instant t of $[0, T]$ and such that the function $v^+ \in LBV([0, T], \mathbb{R}^n)$.

→ Assumption 1 is ensured in the framework introduced by Ballard [Ballard, 2000] who proves the existence and uniqueness of a solution in a general framework mainly based on the analyticity of data.

Assumption 2 : Smoothness of data

The following smoothness on the data will be assumed: a) the inertia operator $M(q)$ is assumed to be of class C^p and definite positive, b) the force mapping $F(t, q, v)$ is assumed to be of class C^p , c) the constraint functions $g(q)$ are assumed to be of class C^{p+1} and d) the Jacobian matrix $G(q) = \nabla_q^T g(q)$ is assumed to have full-row rank.

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Lemma

Let $I = [t_k, t_{k+1}]$. Let us assume that the function $f \in BV(I, \mathbb{R}^n)$. Then we have the following inequality for the θ -method, $\theta \in [0, 1]$,

$$\left\| \int_{t_k}^{t_{k+1}} f(s) ds - h(\theta f(t_{k+1}) + (1 - \theta)f(t_k)) \right\| \leq C(\theta)(t_{k+1} - t_k) \text{var}(f, I), \quad (22)$$

where $\text{var}(f, I) \in \mathbb{R}$ is the variation of f on I and $C(\theta) = \theta$ if $\theta \geq 1/2$ and $C(\theta) = 1 - \theta$ otherwise. Furthermore, the value of $C(\theta)$ yields a sharp bound in (22).

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Proposition

Under Assumptions 1 and 2, the local order of consistency of the Moreau-Jean time-stepping scheme for the generalized coordinates is

$$e_q = q_{k+1} - q(t+h) = \mathcal{O}(h)$$

and at least for the velocities

$$e_v = v^+(t_k + h) - v_{k+1} = \mathcal{O}(1)$$

Comments

The bounds are reached if an impact is located within the time-step and the activation of the constraint is not correct.

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Background

Work of Mannshardt [1978] on time–integration schemes of any order for ODE/DAEs with discontinuities (with transversality assumption)

Principle

- ▶ Let us assume only one event per time–step at instants t_* .
- ▶ Choose any ODE/DAE solvers of order p
- ▶ Perform a rough location of the event inside the time step of length h
Find an interval $[t_a, t_b]$ such that

$$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \quad (23)$$

Dichotomy, Newton, Local Interpolants, Dense output,...

- ▶ Perform an integration on $[t_k, t_a]$ with the ODE solver of order p
- ▶ Perform an integration on $[t_a, t_b]$ with Moreau's time–stepping scheme
- ▶ Perform an integration on $[t_b, t_{k+1}]$ with the ODE solver of order p

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Mainly for the sake of simplicity, the numerical integration over a smooth period is made with a Runge–Kutta (RK) method on the following index-1 DAE,

$$\begin{cases} M(q(t))\dot{v}(t) = F(t, q(t), v(t)) + G(q)\lambda(t), \\ \dot{q}(t) = v(t), \\ \gamma(t) = G(q(t))\dot{v}(t) = 0. \end{cases} \quad (24)$$

In practice, the time–integration is performed for the following system

$$\begin{cases} M(q(t))\dot{v}(t) = F(t, q(t), v(t)) + G(q)\lambda(t), \\ \dot{q}(t) = v(t), \\ 0 \leq \gamma(t) = G(q(t))\dot{v}(t) \perp \lambda(t) \geq 0 \end{cases} \quad (25)$$

on the time–interval I where the index set $\mathcal{I}(t)$ of active constraints is assumed to be constant on I and $\lambda(t) > 0$ for all $t \in I$.

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Using the standard notation for the RK methods (see Hairer et al. [1993] for details), the complementarity problem that we have to solve at each time-step reads

$$\left\{ \begin{array}{l} t_{ki} = t_k + c_i h, \\ v_{k+1} = v_k + h \sum_{i=1}^s b_i V'_{ki}, \\ q_{k+1} = q_k + h \sum_{i=1}^s b_i V_{ki}, \\ V'_{ki} = M^{-1}(Q_{ki}) [F(t_{ki}, Q_{ki}, V_{ki}) + G(Q_{ki}) \lambda_{ki}], \\ V_{ki} = v_k + h \sum_{j=1}^s a_{ij} V'_{nj}, \\ Q_{ki} = q_k + h \sum_{j=1}^s a_{ij} V_{nj}, \\ 0 \leq \gamma_{ki} = G(Q_{ki}) V'_{ki} \perp \lambda_{ki} \geq 0. \end{array} \right. \quad (26)$$

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Assumption 3

Let I a smooth period time–interval. We assume that

1. the local order of the RK method (26) is p that is

$$e_q = e_v = \mathcal{O}(h^{p+1}) \quad (27)$$

2. starting from inconsistent initial value \tilde{q}_k such that $\tilde{q}_k - q_k = \mathcal{O}(h^{p+1})$, the error made by the RK method (26) is

$$\tilde{q}_{k+1} - q_{k+1} = \mathcal{O}(h^{p+1}) \quad (28)$$

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Theorem

Let us assume that Assumptions 1, 2 and 3 hold. The local error of consistency of the scheme is of order p in the generalized coordinates that is

$$e_q = \mathcal{O}(h^{p+1}). \quad (29)$$

Results on the linear oscillator

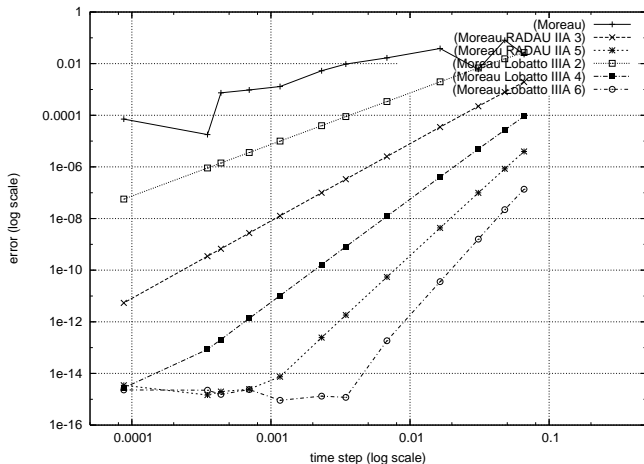


Figure: Precision Work diagram for the Moreau's time-stepping scheme coupled with Runge-Kutta method.

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Finite accumulation

- ▶ Repeat the whole process on the remaining part of the interval $[t_b, t_k]$
- ▶ By induction, repeat this process up to the end of the original time step.

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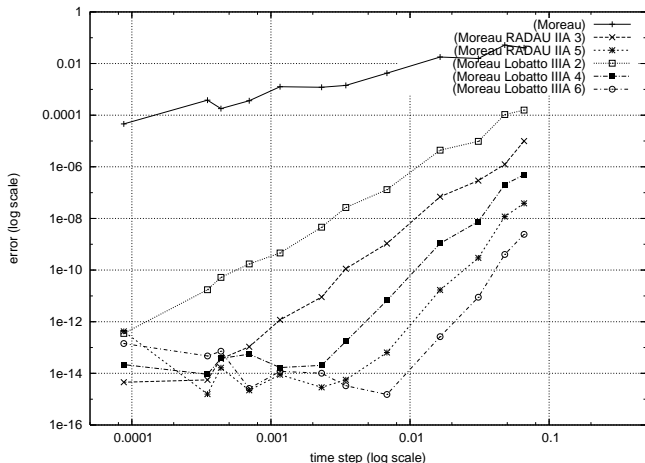
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Results on the Bouncing Ball



(a) The Bouncing Ball example with implicit Runge Kutta Method

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Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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Basic principles on a naive example

Problem: Stabilization of this simple dynamics

$$\begin{cases} x(t_0) = x_0 \in \mathbb{R} \\ \dot{x} = f, \quad |f| \leq 1, \end{cases} \quad (30)$$

at the origin $x = 0$.

Naive solution:

$$\begin{cases} x(t_0) = x_0 \in \mathbb{R} \\ \dot{x} = f + u, \quad |f| < 1, \end{cases} \quad (31)$$

- ▶ “Push on right” if the state is at the right of 0

$$u = -1 \text{ if } x > 0 \quad (32)$$

- ▶ “Push on left” if the state is at the left of 0

$$u = +1 \text{ if } x < 0 \quad (33)$$

- ▶ “balance the external load” in 0

$$u = -f \text{ if } x = 0 \quad (34)$$

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Basic principles on a naive example

- ▶ Switched control based on the sign function

$$u = -\text{sign}(x) = \begin{cases} -1 & \text{for } x > 0 \\ +1 & \text{for } x < 0 \\ ? & \text{for } x = 0 \end{cases} \quad (35)$$

Definition of u at $x = 0$?

- ▶ Discontinuous ODEs

$$\dot{x} = f - \text{sign}(x) \quad (36)$$

Notion of solutions ?

Mathematical framework

- ▶ Multivalued maximal monotone operator

$$u = -\text{sgn}(x) = \begin{cases} -1 & \text{for } x > 0 \\ +1 & \text{for } x < 0 \\ [-1, 1] & \text{for } x = 0 \end{cases} \quad (37)$$

- ▶ Filippov's differential inclusions

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In the continuous setting

- ▶ Robust control w.r.t external uncertainties
- ▶ Finite time convergence to target

→ SMC is the most widely used non linear control in industrial practice.

In the discrete setting

Digital implementation of SMC suffers from “chattering” due to explicit approximation

$$x_{k+1} - x_k = f - \text{sgn}(x_k) \quad (38)$$

This causes

- ▶ Wear and damage in actuators
- ▶ Need for complex filtering systems which entails the good properties of continuous SMC.

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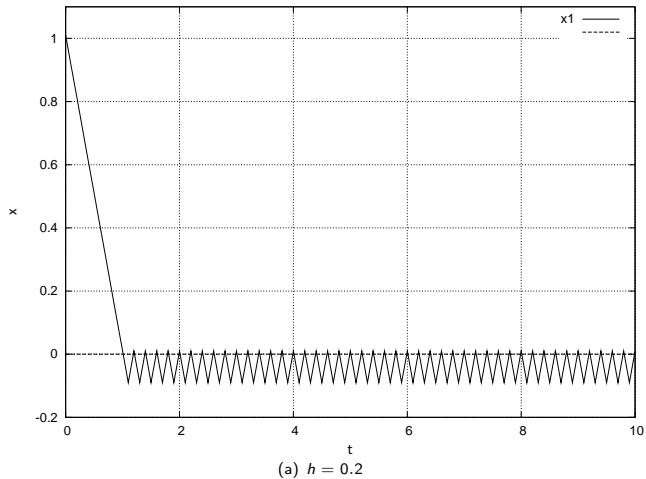


Figure: A simple example for $x_0 = 1.01$ at $t_0 = 0$.

Our background

- ▶ Nonsmooth modelling of Friction
- ▶ Well-posedness analysis of Monotone Differential Inclusions
- ▶ Implicit numerical time integration for DI.

Objectives

- ▶ Study the implicit Euler discretization of a class of differential inclusions with sliding surfaces (\subset Filippov's systems)
- ▶ Show that this numerical method permits a smooth stabilization on the sliding surface, in a finite number of steps
- ▶ Show how this may be used in real-time implementations of sliding mode control

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To start with we consider the simplest case:

$$\dot{x}(t) \in -\text{sgn}(x(t)) = \begin{cases} 1 & \text{if } x(t) < 0 \\ -1 & \text{if } x(t) > 0 \\ [-1,1] & \text{if } x(t) = 0 \end{cases}, \quad x(0) = x_0 \quad (39)$$

with $x(t) \in \mathbb{R}$. This system possesses a unique Lipschitz continuous solution for any x_0 . The backward Euler discretization of (39) reads as:

$$\begin{cases} x_{k+1} - x_k = -hs_{k+1} \\ s_{k+1} \in \text{sgn}(x_{k+1}) \end{cases} \quad (40)$$

As is known the *explicit* Euler discretization of such discontinuous systems yields spurious oscillations around the switching surface [Galias et al, IEEE TAC and CAS 2006, 2007, 2008].

~> this means that the derivative of the switching function while sliding occurs, is very badly estimated.

Both the explicit and the implicit methods converge (the approximated solution $x^N(\cdot)$ tends to the Filippov's solution as $h \rightarrow 0$). However or the backward Euler method the following holds:

Lemma

For all $h > 0$ and $x_0 \in \mathbb{R}$, there exists k_0 such that $x_{k_0+n} = 0$ and

$$\frac{x_{k_0+n+1} - x_{k_0+n}}{h} = 0 \text{ for all } n \geq 1.$$

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On this simple case this has the following graphical interpretation, as the intersection of two graphs:

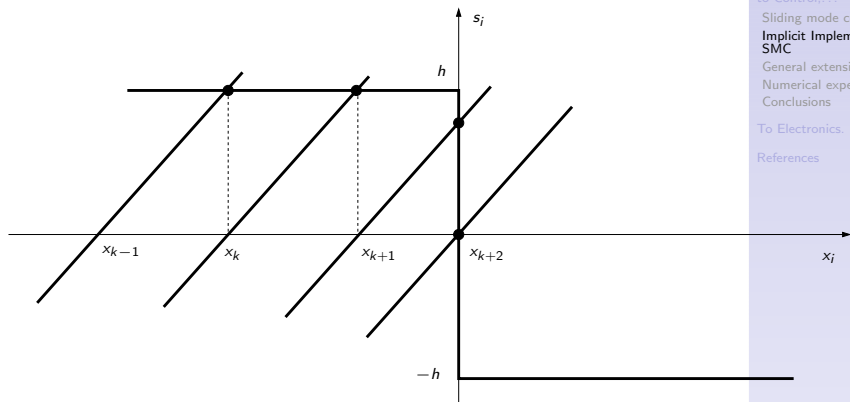


Figure: Iterations of the backward Euler method.

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An interesting property is that the smooth stabilization and the finite-time convergence on the switching surface, hold (more or less) independently of the step $h > 0$:

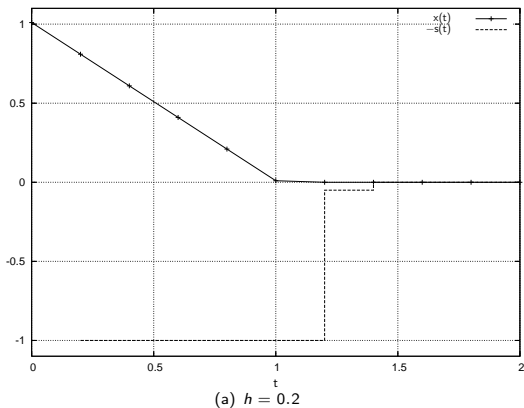


Figure: A simple example for $x_0 = 1.01$ at $t_0 = 0$.

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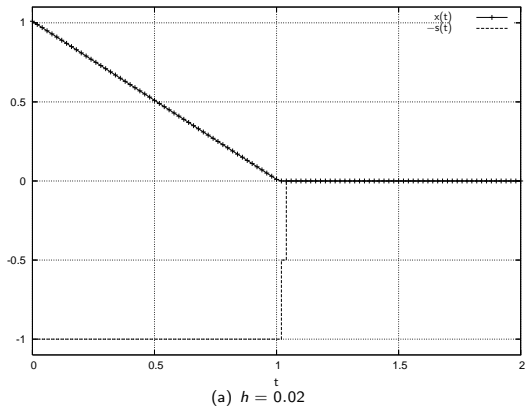


Figure: A simple example for $x_0 = 1.01$ at $t_0 = 0$.

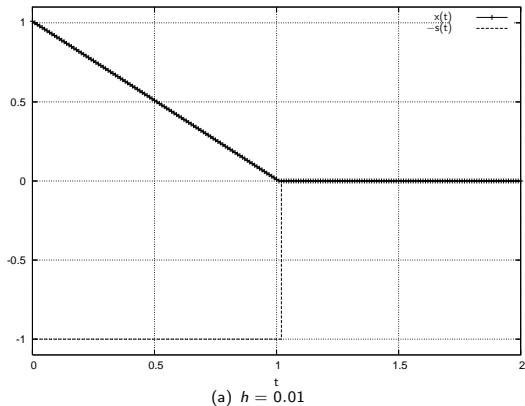


Figure: A simple example for $x_0 = 1.01$ at $t_0 = 0$.

We shall focus on inclusions of the form:

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) - B \operatorname{Sgn}(Cx(t) + D), & \text{a.e. on } (0, T) \\ x(0) = x_0 \end{cases} \quad (41)$$

with

$$B \in \mathbb{R}^{n \times m}$$

$\operatorname{Sgn}(Cx(t) + D) \triangleq (\operatorname{sgn}(C_1x + D_1), \dots, \operatorname{sgn}(C_mx + D_m))^T \in \mathbb{R}^m$, where $\operatorname{sgn}(\cdot)$ is multivalued at 0.

Proposition

Consider the differential inclusion in (41). Suppose that

- ▶ There exists $L \geq 0$ such that for all $t \in [0, T]$, for all $x_1, x_2 \in \mathbb{R}^n$, one has $\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|$.
- ▶ There exists a function $\Phi(\cdot)$ such that for all $R \geq 0$:

$$\Phi(R) = \sup \left\{ \left\| \frac{\partial f}{\partial t}(\cdot, v) \right\|_{\mathcal{L}^2((0, T); \mathbb{R}^n)} \mid \|v\|_{\mathcal{L}^2((0, T); \mathbb{R}^n)} \leq R \right\} < +\infty.$$

If there exists an $n \times n$ matrix $P = P^T > 0$ such that

$$PB_{\bullet i} = C_i^T \quad (42)$$

for all $1 \leq i \leq m$, then for any initial data the differential inclusion (41) has a unique solution $x : (0, T) \rightarrow \mathbb{R}^n$ that is Lipschitz continuous.

Sketch of the proof

- ▶ Change of state variables $z = Rx$ where $R = R^T > 0$ and $R^2 = P$.
- ▶ Use a result in [Bastien-Schatzman ESAIM M2AN 2002] to conclude.

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- ▶ The existence of a positive definite P such that $PB = C^T$ is satisfied in many instances of sliding-mode control: observer-based sliding-mode control, Lyapunov-based discontinuous robust control.
- ▶ This is an “input-output” constraint on the system, constraining the relative degree of the triple (A, B, C) .
- ▶ It is satisfied when (A, B, C) is positive real (dissipative).

Time-discretization of (41)

The differential inclusion in (41) is therefore discretized as follows:

$$\begin{cases} \frac{x_{k+1} - x_k}{h} \in f(t_k, x_k) - BSgn(Cx_{k+1} + D), \text{ a.e. on } (0, T) \\ x(0) = x_0 \end{cases} \quad (43)$$

From [Bastien-Schatzman ESAIM M2AN 2002] we have that:

Proposition

Under Proposition 2 conditions, there exists η such that for all $h > 0$ one has

$$\text{For all } t \in [0, T], \|x(t) - x^N(t)\| \leq \eta \sqrt{h} \quad (44)$$

Moreover

$$\lim_{h \rightarrow 0^+} \max_{t \in [0, T]} \|x(t) - x^N(t)\|^2 + \int_0^t \|x(s) - x^N(s)\|^2 ds = 0.$$

However we have more: the discrete state reaches the sliding surface (when it exists) in a finite number of steps, and stabilizes on it in a smooth way.

Let $y(t) \triangleq Cx(t) + D$.

Lemma

Let us assume that a sliding mode occurs for the index $\alpha \subset \{1 \dots m\}$, that is $y_\alpha(t) = 0, t > t_$. Let C and B be such that (42) holds and $C_\alpha \bullet B_\bullet \alpha > 0$. Then there exists $h_c > 0$ such that $\forall h < h_c$, there exists $k_0 \in \mathbf{N}$ such that $y_{k_0+n} = Cx_{k_0+n+1} + D = 0$ for all integers $n \geq 1$.*

Such algorithms are similar to proximal algorithms which possess finite-time stabilization properties [Baji and Cabot, Set-Valued Analysis 2006].

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Remarks

- ▶ Contrarily to other methods that reduce (not suppress...) chattering, the discrete-time sliding surface is equal to the continuous-time sliding surface.
- ▶ At each step one has to solve a generalized equation with unknown x_{k+1} that takes the form of a mixed linear complementarity system (MLCP).
- ▶ Specific MLCP solvers are needed to implement the method.

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Let us consider the following two examples:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -c_1 \end{bmatrix} x - \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \operatorname{sgn}([c_1 \quad 1] x). \quad (45)$$

(codimension one sliding surface)

$$B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad D = 0, \quad f(x(t), t) = 0 \quad (46)$$

(codimension two sliding surface)

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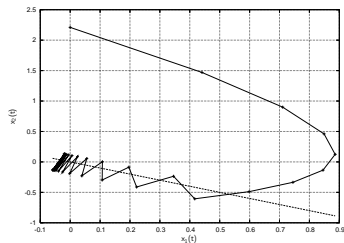
General extensions

Numerical experiments.

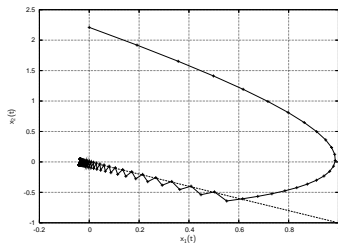
Conclusions

To Electronics.

References

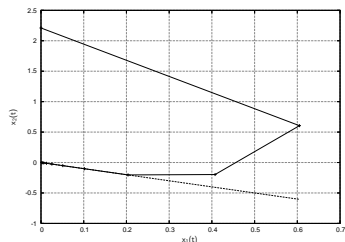


(a) $h = 0.3$. Explicit Euler

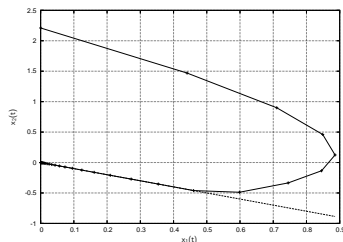


(b) $h = 0.1$. Explicit Euler

Figure: Equivalent control based SMC, $c_1 = 1$, $\alpha = 1$ and $x_0 = [0, 2.21]^T$. State $x_1(t)$ versus $x_2(t)$.

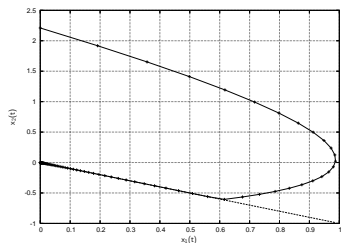


(a) $h = 1$. Implicit Euler

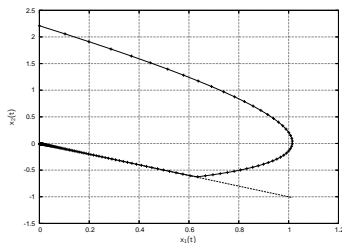


(b) $h = 0.3$. Implicit Euler

Figure: Equivalent control based SMC, $c_1 = 1$, $\alpha = 1$ and $x_0 = [0, 2.21]^T$. State $x_1(t)$ versus $x_2(t)$.



(a) $h = 0.1$. Implicit Euler



(b) $h = 0.05$. Implicit Euler

Figure: Equivalent control based SMC, $c_1 = 1$, $\alpha = 1$ and $x_0 = [0, 2.21]^T$. State $x_1(t)$ versus $x_2(t)$.

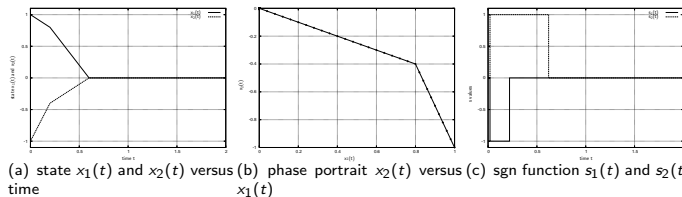


Figure: Multiple Sliding surface. $h = 0.02$, $x(0) = [1.0, -1.0]^T$

*The system reaches firstly the sliding surface $2x_2 + x_1 = 0$ without any chattering,
The system then slides on the surface up to reaching the second sliding surface
 $2x_1 - x_2 = 0$ and comes to rest at the origin.*

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The Filippov's example with switches accumulation

$$B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 0, \quad f(x(t), t) = 0. \quad (47)$$

The trajectories may slide on the codimension 2 surface given by $Cx = 0$.
The origin is attained after an infinite number of switches in finite time.

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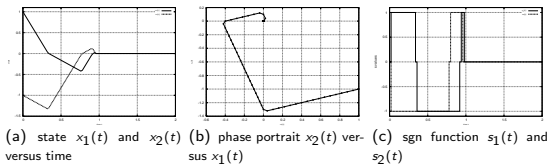


Figure: Multiple Sliding surface. Filippov Example. $h = 0.002$, $x(0) = [1.0, -1.0]^T$

The results show that the system reaches the origin without any chattering.

The implicit Euler method allows one to nicely simulate the main features of sliding-mode systems:

- ▶ Finite-time stabilization on the switching surface (of codimension ≥ 1)
- ▶ Smooth stabilization on the switching surface

It extends to the discrete-time implementation with ZOH discretization: looks like a promising solution for discrete-time sliding modes.

Contents

An excursion into
Nonsmooth Dynamics

Vincent Acary

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to Control,...

To Electronics.

References

From Mechanics of divided materials to multi-body and robotic systems,

To control (Sliding mode control Theory)

To electronics (Nonsmooth modeling of switched Electrical circuits)

The RLC circuit with a diode

Example

A LC oscillator supplying a load resistor through a half-wave rectifier (see figure 14).

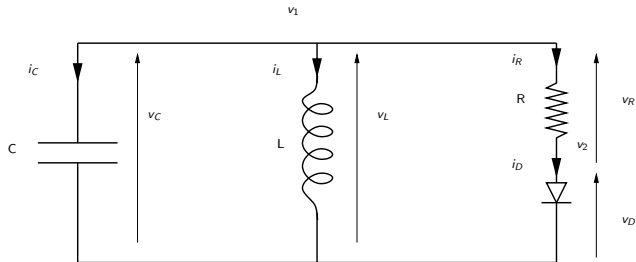


Figure: Electrical oscillator with half-wave rectifier

From Mechanics...

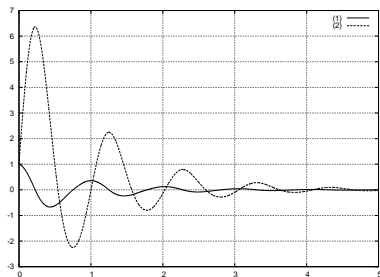
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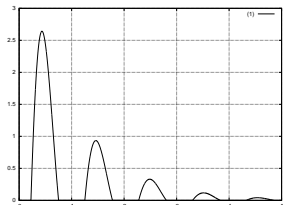
References

The RLC circuit with a diode

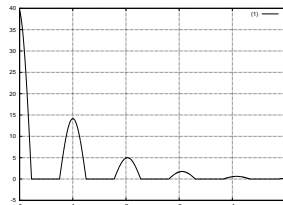
Example



(a) state versus time v_L and i_L



(b) Diode current i_D



(c) Diode voltage v_D

Example

- ▶ Kirchhoff laws :

$$\begin{aligned}v_L &= v_C \\v_R + v_D &= v_C \\i_C + i_L + i_R &= 0 \\i_R &= i_D\end{aligned}$$

- ▶ Branch constitutive equations for linear devices are :

$$\begin{aligned}i_C &= C\dot{v}_C \\v_L &= L\dot{i}_L \\v_R &= Ri_R\end{aligned}$$

- ▶ "branch constitutive equation" of the diode

$$0 \in \mathcal{F}(i_D, v_D)$$

The RLC circuit with a diode

Example

The following dynamical system is obtained :

$$\begin{pmatrix} \dot{v}_L \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

$$v_D = v_L - Ri_D$$

$$0 \in \mathcal{F}(v_D, i_D)$$

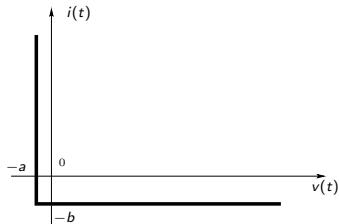
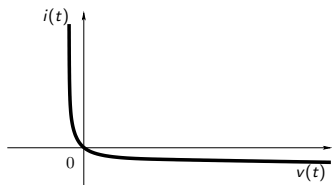
with the state variable $x \triangleq \begin{pmatrix} v_L \\ i_L \end{pmatrix}$ and $\lambda \triangleq i_D$, $y \triangleq v_D$, we get

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \in \mathcal{F}(y, \lambda) \end{cases} \quad (48)$$

A modeling choice

smooth modeling

nonsmooth modeling



(a)

$$i(t) = i_s \exp\left(-\frac{v(t)}{\alpha} - 1\right)$$

(b)

$$0 \leq i(t) + b \perp v(t) + a \geq 0$$

Figure: Two models of diodes.

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References

Why a nonsmooth modeling ?

- ▶ To avoid stiff nonlinear models by using ideal constraints.
- ▶ To model the ideal behavior of switched components without artificial regularization

The diode-bridge rectifier

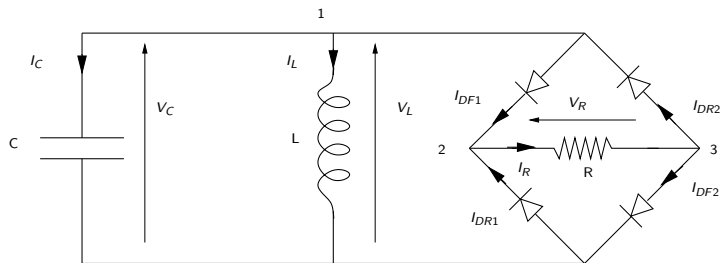


Figure: The Diode-bridge rectifier

The diode-bridge rectifier

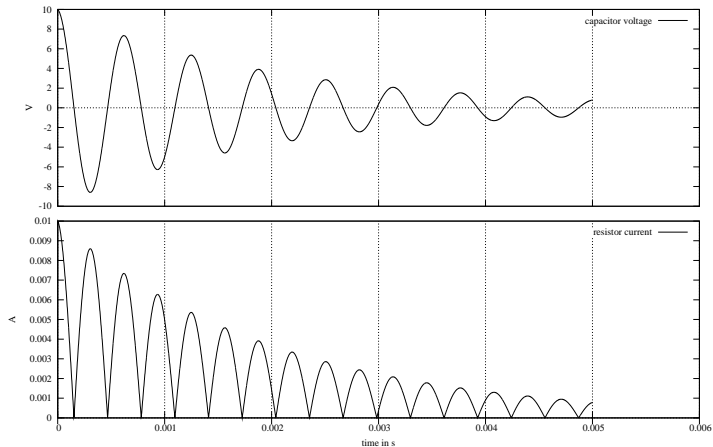


Figure: The Diode-bridge rectifier. Standard results

The diode-bridge rectifier

Differential systems

The dynamical equations are formulated as

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \quad (49)$$

choosing :

$$x = \begin{bmatrix} V_L \\ I_L \end{bmatrix}, \quad \text{and } y = \begin{bmatrix} I_{DR1} \\ I_{DF2} \\ V_2 - V_1 \\ V_1 - V_3 \end{bmatrix}, \quad \lambda = \begin{bmatrix} V_2 \\ -V_3 \\ I_{DF1} \\ I_{DR2} \end{bmatrix}, \quad (50)$$

and with

$$A = \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1/C & 1/C & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

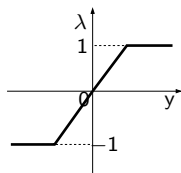
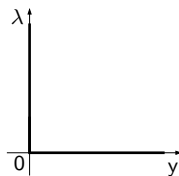
$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1/R & 1/R & -1 & 0 \\ 1/R & 1/R & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (51)$$

A typical example of nonsmooth systems

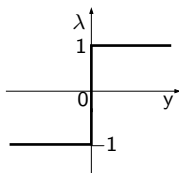
Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \quad (52)$$

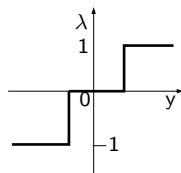
with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$
 $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$, for m constraints.



Saturation



Relay



Relay with dead zone

A slightly more general class of nonsmooth systems

Differential inclusion into normal cones

$$\begin{cases} \dot{x} = Ax + B\lambda, & x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y = Cx + D\lambda \\ -y \in N_K(\lambda) \end{cases} \quad (53)$$

where K is a convex set and $N_K(\lambda)$ stands for the normal cone to K taken at λ

Usual examples for K

- ▶ $K = \mathbb{R}^m$, then we obtain linear time invariant DAE

$$-y \in N_{\mathbb{R}^m}(\lambda) \iff y = 0, \quad \lambda \in \mathbb{R}^m \quad (54)$$

- ▶ $K = \mathbb{R}_+^m$, then we obtain Linear Complementarity Systems (LCS)

$$-y \in N_{\mathbb{R}_+^m}(\lambda) \iff 0 \leq y \perp \lambda \geq 0 \quad (55)$$

- ▶ $K = [-1, 1]^m$, then we obtain linear relay systems (related to Filippov's DI and sliding mode control).

$$-y \in N_{[-1,1]^m}(\lambda) \iff \lambda \in \text{sgn}(y) \quad (56)$$

From Mechanics...

to Control,...

To Electronics.

References

Our background

- ▶ Nonsmooth modeling of unilateral constraints and friction
- ▶ Nonsmooth analysis of dynamics with jumps.

Our Objectives

- ▶ Understand what can be the nature of the solutions (uniqueness, smoothness).
- ▶ How perform the numerical time-integration ?
- ▶ Open issues for the time-integration of large dynamical systems arising in electrical network applications.

Nature of solutions for $K \in \mathbb{R}_+^m$

The nature of solutions depends on

- ▶ the relative degree (index) between y and λ
- ▶ the possible consistency of the solution

The main types of solutions are

- ▶ \mathcal{C}^1 solutions when λ is a lipschitz function of x (relative degree 0)
- ▶ absolutely continuous solutions (relative degree 1)
- ▶ solutions of Bounded Variations (relative degree 2)

Numerical time-integration methods

The time integration methods depends on the solution

- ▶ C^1 solutions : Standard DAE integrators of low order
- ▶ absolutely continuous solutions : Implicit first order scheme
- ▶ solutions of Bounded Variations : Moreau's catching up algorithm

Industrial circuits and automatic circuit equations formulation

- ▶ Adaptation of the standard Modified Nodal Analysis (MNA) to the nonsmooth elements to obtain

Problem (DGE)

$$M(X, t)\dot{X} = D(X, t) + U(t) + R \quad] \text{ Differential Algebraic Equations}$$

$$\begin{aligned} y &= G(X, \lambda, t) \\ R &= H(X, \lambda, t) \end{aligned} \quad] \text{ Input/output relations} \\ \text{on nonsmooth components}$$

$$0 \in F(y, \lambda, t) + T(y, \lambda, t) \quad] \text{ Generalized equation}$$

$$X = [V, I_L, I_V, I_{NS}]^T \quad] \text{ Variable definition}$$

(57)

- Difficulties to discuss the nature of solution and then to adapt the time numerical method
- In electrical circuits, the main difficulty is induced by the topology of the circuit rather than the inherent non-linearity of the components.

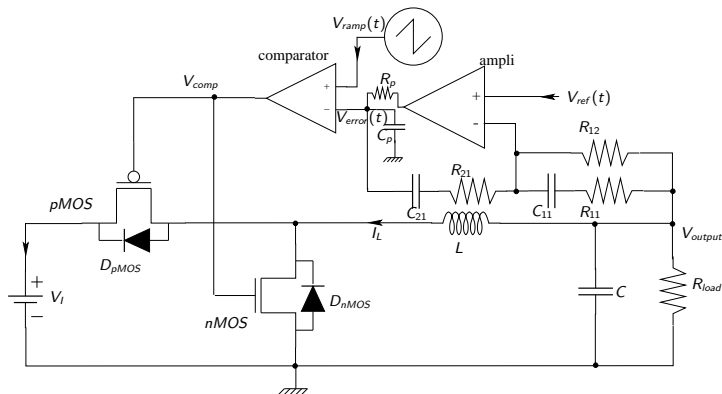


Figure: Buck converter.

Applications to industrial electrical networks

An excursion into
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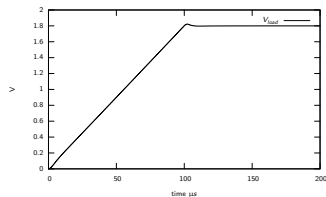
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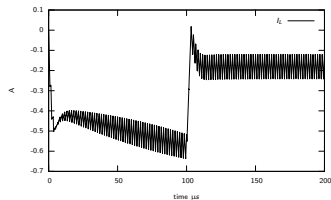
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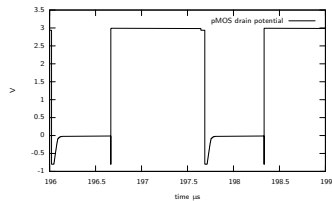
References



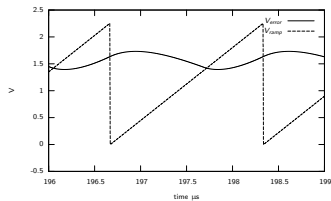
(a) V_{load}



(b) I_L



(c) pMOS drain potential



(d) V_{ramp} and V_{error}

Figure: SICONOS buck converter simulation using standard parameters.

Applications to industrial electrical networks

An excursion into
Nonsmooth Dynamics

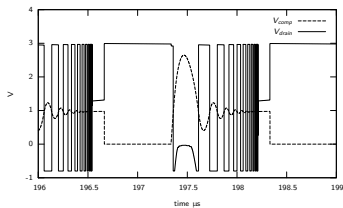
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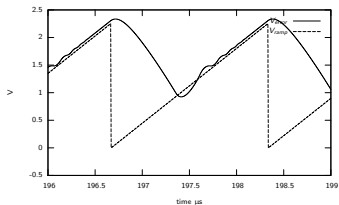
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(a) V_{comp} and V_{drain}



(b) V_{ramp} and V_{error}

Figure: SICONOS buck converter simulation using sliding mode parameters.

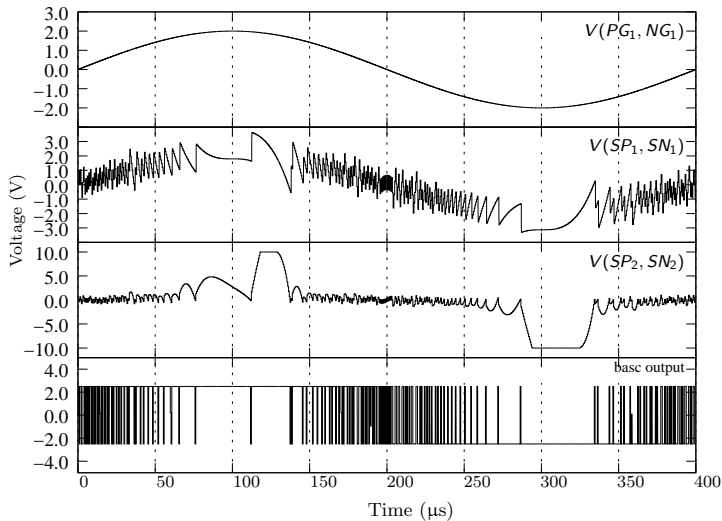


Figure: SICONOS simulation.

For more general formulations and more complex systems, are we able to infer the nature of the solutions? That is to say,

- ▶ Define and predict an equivalent notion to index and relative degree for instance, for a matrix D semi-definite positive.
- ▶ Given passive components, are we able to forecast the nature of the solutions from some topological considerations ? (as for the DAE case.)
- ▶ Adapt the time-stepping schemes in an hierarchical way in taking into account the "index" of each variable.

and towards

From Mechanics...

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References

- ▶ Dynamics of gene regulatory networks (cell physiology)
- ▶ ...

From Mechanics...

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References

Thank you for your attention.
Happy Birthday Michel and thank you again

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