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Key Reduction of McEliece's Cryptosystem Using List Decoding

Morgan Barbier

Computer science laboratory of École Polytechnique - LIX

INRIA Saclay - Île de France

`morgan.barbier@lix.polytechnique.fr`

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Abstract

Different variants of the code-based McEliece cryptosystem were proposed to reduce the size of the public key. All these variants use very structured codes, which open the door to new attacks exploiting the underlying structure. In this paper, we show that the dyadic variant can be designed to resist all known attacks. In light of a new study on list decoding algorithms for binary Goppa codes, we explain how to increase the security level for given public key sizes. Using the state-of-the-art list decoding algorithm instead of unique decoding, we exhibit a key size gain of about 5% for the standard McEliece cryptosystem and up to 10% for the adjusted dyadic variant.

1 Introduction

The past few years have seen a renewed interest in code-based cryptosystems due to their resistance to known quantum attacks [9]. The famous McEliece asymmetric cryptosystem [19] is perhaps the most studied of them. The private key is the generator matrix of a code \mathcal{C} and the public key is obtained from this generator matrix by a permutation of its columns followed by a multiplication by an random invertible matrix. This public key is thus a generator matrix of a code \mathcal{C}' equivalent to \mathcal{C} . The encryption consists in encoding the plaintext into a codeword $c' \in \mathcal{C}'$ using the public key and randomly adding as many errors as made possible by the decoding algorithm of \mathcal{C} . The decryption step consists in decoding the ciphertext over \mathcal{C} , thanks to the private key.

The McEliece cryptosystem delivers high encryption and decryption speeds compared to other systems like RSA [20] but suffers from the large size of the associated keys which makes it unpractical. Lately, a lot of effort has been put into the design of variants based on different code families in order to reduce

the size of the keys. For example, in 2008, a solution was proposed for signature schemes using double-circulant matrices [1]. In [4], the authors proposed a key reduction for the McEliece cryptosystem using quasi-cyclic alternant codes. The same year, a method using the sub-family of classical binary Goppa codes, called quasi-dyadic codes, was introduced in [20], adapting the idea from [4]. Another key reduction technique formulated in [5] hides the structure of a subcode of generalized Reed-Solomon codes. Generally speaking, all of these key reduction techniques involve the introduction of some kind of additional structure. As a cryptographic rule of thumb, the presence of unneeded structure is often seen as a potential angle of attack. Indeed, cryptanalysts quickly proposed new structural attacks against the aforementioned variants [14, 23, 27].

Roughly speaking, we can distinguish between two types of attacks. The first type tries to recover the plaintext from the cyphertext, without the knowledge of the private key. It is clear that increasing the number of errors during the encryption step will make this kind of attacks more difficult. Bernstein, Lange and Peters contributed to assess the effectiveness of such attacks in [11] by giving asymptotic analysis of different decoding algorithms for code-based cryptography. Moreover, working within a strict complexity model, Finiasz and Sendrier exhibited lower bounds for system designers [15] by taking into account the costs of the best decoding attacks [27].

The second type of attacks consists in retrieving the private key from the available public one. Such an attack was recently introduced in [14] and boils down to computing a Groebner basis to find the structure of an alternant code. The McEliece variant with the parameters proposed in [4] is considered to be broken by this attack. While the dyadic instance from [20] is also vulnerable, this variant can be made more robust as shown in Section 3.

This paper is organised as follows. Section 2 is devoted to the decoding of binary Goppa codes, most precisely on the correction radius of different decoding algorithms. In Section 3 we show how the dyadic variant can be made more secure against [14] and present our results on keysize reduction obtained using the best known list decoding algorithm for the classical and modified, hardened variants of the McEliece cryptosystem.

2 List decoding of binary Goppa Codes

Since a major part of the cryptanalysis of code-based cryptography is intimately linked to error correction, a natural idea is to add as many errors as possible during the encryption step, provided that the recipient is still able to correct them. Decoding a random code is a hard problem; indeed it was shown that decoding general codes is NP-complete [6]. The McEliece cryptosystem originally used binary Goppa codes. Some variants are based on different types of codes

(*e.g.* [5]), but most of them have been broken (*e.g.* [23]). In the following, we briefly recall the state of the art of the decoding of binary Goppa codes, which are perhaps the most promising for McEliece cryptosystems.

The first algebraic decoding algorithm for classical Goppa codes was proposed by Patterson in 1975 [25]. This algorithm, basically a variation of the Berlekamp-Massey algorithm [7], runs in quadratic time in the code length. Patterson's method performs an unambiguous decoding, up to the error capacity t of the code. Since classical Goppa codes are alternant, that is they are subfield subcodes of generalised Reed-Solomon codes [18], we are able to perform the well-known Guruswami-Sudan list decoding (GS-LD) algorithm [16]. This method makes it possible to correct up to the generic Johnson bound given by $n \left(1 - \sqrt{1 - \frac{2t}{n}}\right)$ errors, which is larger than t (see Figure 1). Consequently, this type of decoding does not ensure the uniqueness of the returned codewords anymore. The GS-LD algorithm is originally not tailored to the binary Goppa codes. Using specific properties of binary Goppa codes, Bernstein was able to extend Patterson's algorithm to perform a list decoding up to $n \left(1 - \sqrt{1 - \frac{2t+2}{n}}\right)$ [8], which is larger than the generic Johnson bound. Recently, a technical report [2] revisits previous works to exhibit a list decoding algorithm for square-free binary Goppa codes which decodes up to the *binary* Johnson bound given by $\tau_2 \triangleq \frac{n}{2} \left(1 - \sqrt{1 - \frac{4t+2}{n}}\right)$, which is larger than the two former bounds. As shown in Figure 1, the closer the normalized distance is to 0.5, the better the binary Johnson bound is compared to the others. We will show in Section 3 that using binary Goppa codes with normalized minimum distances closer to 0.5 makes it possible to correct more errors and ultimately, to reduce the size of the keys.

List decoding algorithms basically involve two steps. The first stage finds, by interpolation, a bivariate polynomial connecting the received word with the support of the code. The second step consists in finding the roots of this polynomial. The cost of the algorithm from [2] is dominated by the interpolation step. This algorithm has an overall complexity of $\mathcal{O}(n^2\epsilon^{-5})$ and corrects up to $(1 - \epsilon)\tau_2$ errors, where τ_2 is the binary Johnson bound. Decoding τ_2 errors is obviously prohibitively expensive but trade-offs between running time and number of corrected errors are easily achieved, making it possible to keep the cost of list decoding under control.

The classical McEliece or equivalently the Niederreiter cryptosystems [17,21] suffer from chosen cyphertext attacks [28]. Indeed, since a given plaintext can be encrypted to give different cyphertexts, an attacker could compare these different cyphertexts to extract the original plaintext. Different methods were proposed to make these cryptosystems more robust to chosen cyphertext attacks [13,24,26] leading to so-called CCA2-secure variants. When adding more errors than can be uniquely corrected, the decryption step will return a list of

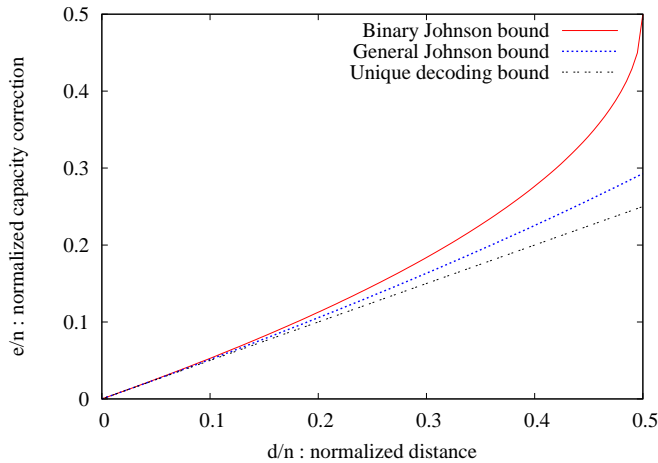


Figure 1: Comparison between the unambiguous decoding, generic and binary Johnson’s bounds.

potential plaintexts. As already remarked in [10], CCA2-secure variants make it possible to distinguish the original plaintext between all candidates returned by the list decoding algorithm used in the decryption process. Consequently, it is possible to make the task for an attacker much more difficult by adding more errors than the correction capacity. Using CCA2-secure variants and state-of-the-art list decoding algorithm, these extra-errors only add a small burden on the recipient to find the original plaintext.

3 Key reduction

Encrypting and decrypting with the McEliece cryptosystem is significantly faster than with more widespread cryptosystems based on number theory such as the ubiquitous RSA [20]. The main and perhaps only handicap holding back the McEliece cryptosystem is the substantially larger size of the public keys. We propose to address this problem not by using a well structured code as is often the case, but by adding as many errors as permitted by the best known list decoding algorithm [2]. For a given keysize, this increases the security level. Symmetrically, this makes it possible to use shorter keys while keeping a similar security level. Using a list decoding algorithm can thus lead to shorter keys at the expense of a moderately increased decryption time.

We focus on the family of square-free binary Goppa codes, which includes the traditionally used family of irreducible binary Goppa codes. In this case the error capacity t is equal to r the degree of Goppa polynomial. The algorithm decoding the largest number of errors for these codes is studied in [2]. This

list decoding algorithm works for all alternant codes, but using proposition 1 improves the correction radius and leads to even shorter keys. We numerically searched for codes parameters yielding short keys and correcting up to $\lceil \tau_2 \rceil - 1$. We illustrate the benefits of list decoding by presenting examples for both the generic and dyadic variants.

3.1 Generic variant

Tables 1, 2 and 3 show the keysize reduction obtained using the best known list decoding algorithm [2], for workfactors (WF) equal to 2^{80} , 2^{128} , 2^{156} and 2^{200} . For each workfactors, McEliece key sizes are given for Unambiguous Decoding (U.D.) and List Decoding (L.D.). The involved codes are defined by m , the degree of the extension where G and \mathcal{L} are defined, the length n , the dimension k , the degree r of the Goppa polynomial G , and τ_2 is the binary Johnson bound reached by the list decoding algorithm. The workfactors have been estimated using the complexity model and the lower bounds given in [15].

Table 1: Comparison between the public keysize of generic McEliece cryptosystem using unambiguous and list decoding for given workfactors.

Method	m	n	k	r	τ_2	WF	Keysize	gain (%)
U.D.	11	1670	1285	35		80.0064	494725	
L.D.	11	1676	1324	32	33	80.0183	466048	5.80
U.D.	12	3059	2387	56		128.001	1604064	
L.D.	12	2768	2012	63	65	128.029	1521072	5.17
U.D.	12	3681	2817	72		156.01	2433888	
L.D.	12	3531	2667	72	74	156.015	2304288	5.32
U.D.	13	5239	4056	91		200.008	4798248	
L.D.	13	5186	4055	87	89	200.015	4586205	4.42

Table 1 refers to the generic McEliece system where the size of the public keys is given by $(n - k) \times k = mkr$. As shown in figure 1, using a list decoding algorithm is all the more interesting as the normalized minimum distance $(2r + 1)/n$ gets closer to 0.5, which has apparently an adverse effect on the keysize. However, even in this unfavorable case, we were still able to exhibit a keysize reduction of about 5%.

3.2 Dyadic case

The attack proposed by Faugère, Otmani, Perret and Tillich in [14] uses Groebner basis computations to recover the private key from the only knowledge on the public one. It was specifically designed to break the compact key McEliece variants proposed in [4, 20], which use the structure of alternant codes. The variant proposed in [20] uses binary Goppa codes in dyadic form, which are also

alternant codes. The attack in [14] thus applies and can recover an equivalent private key in an alternant code form. However, this is not sufficient to break the system when using Goppa codes. Indeed, the attack does not directly retrieve the Goppa polynomial G of degree r which is crucial to decode [2, 25], but finds a generator matrix of an alternant code without a Goppa structure and with designed minimum distance $r + 1$. However, when using a Goppa code, the private key is a generator matrix of a code with designed minimum distance $2r + 1$ thanks to the following proposition, demonstrated in [2, 12]:

Proposition 1. *Let G be a square-free polynomial in \mathbb{F}_{2^m} and \mathcal{L} be a list of n elements of \mathbb{F}_{2^m} which are not roots of G . Then*

$$\Gamma(\mathcal{L}, G) = \Gamma(\mathcal{L}, G^2),$$

where $\Gamma(\mathcal{L}, G)$ is the Goppa code generated by \mathcal{L} and G .

The direct consequence is that the attacker won't be able to decode. Indeed, this attack retrieves n/r variables Y and n variables X such that $Y_i = G(X_i)^{-1}$. In order to protect against a potential interpolation of the Goppa polynomial G of degree r , we impose that $r + 1 > \frac{n}{r}$, that is $r(r + 1) > n$. Consequently, this attack does not totally break the McEliece variant based on dyadic forms. Moreover, as stated in [14], the attack becomes unpractical, for the moment, when the extension degree m is greater than 16. Working with such an extension degree slightly increases the public keysize of McEliece compared to the parameters proposed in [20], while staying drastically smaller than with the generic form, as shown in tables 1, 2 and 3.

Table 2: Comparison between the public keysize of dyadic McEliece cryptosystem with $r(r + 1) > n$ using unambiguous and list decoding for given workfactors.

Method	m	n	k	r	τ_2	WF	Keysize	gain (%)
U.D.	11	1818	707	101		80.2504	7777	
L.D.	11	1819	642	107	114	80.419	7062	9.19
U.D.	12	3654	1218	203		128.092	14616	
L.D.	12	3757	1105	221	236	128.447	13260	9.27
U.D.	13	6426	1512	378		156.356	19656	
L.D.	13	7248	1359	453	486	156.28	17667	10.11
U.D.	13	7272	2020	404		200.418	26260	
L.D.	13	6822	1895	379	403	200.241	24635	6.18

In table 2, we look at the case of the dyadic variant with countermeasure $r(r + 1) > n$. The size of the public key now becomes mk [20], removing the conflicting constraints on r . Key reduction up to 10.11% can now be achieved.

Table 3: Comparison between the public keysize of dyadic McEliece cryptosystem with $m \geq 16$ using unambiguous and list decoding for given workfactors.

Method	m	n	k	r	τ_2	WF	Keysize	gain (%)
U.D.	16	13685	805	805		80.0208	12880	
L.D.	16	12835	755	755	806	80.064	12080	6.21
U.D.	16	22950	1350	1350		128.02	21600	
L.D.	16	21471	1263	1263	1348	128.069	20208	6.44
U.D.	16	28322	1666	1666		156.075	26656	
L.D.	16	26486	1558	1558	1662	156.012	24928	6.48
U.D.	16	36720	2160	2160		200.047	34560	
L.D.	16	34357	2021	2021	2156	200.085	32336	6.43

Finally, results on the dyadic variant with countermeasure $m \geq 16$ are presented in table 3. As previously discussed, we expect better reductions than in the generic case. Indeed, our experiments showed a key reduction of more than 6%. Note that in this case, the degree r of the Goppa polynomial is the same as the dimension k of the code. This is easily explained: the large extension degree becomes such a strong constraint on the parameters that it removes all freedom when choosing the code dimension.

Table 4 displays the recommended key sizes for cryptosystems based on the Discrete Logarithm Problem over finite fields (DLP), for different security levels [3,22]. For the sake of comparison, we also include the smallest key sizes obtained with McEliece variants although in all impartiality, it should be stressed that we lack sufficient perspective to correctly assess the true security level of these fairly new variants. While the key sizes for the McEliece cryptosystem are still larger than their discrete logarithm counterparts, the gap significantly narrows when going at higher security levels. Moreover, the costs for McEliece encryption and decryption rise much more slowly with the security level than they do with DLP based or RSA systems [20].

Table 4: Keysize comparison between cryptosystem based on discrete logarithm over finite fields and McEliece cryptosystem using list decoding.

Security level	Discrete Logarithm	McEliece	ratio
80	1024	7062	6.89
128	3072	13260	4.31
156	5008	17667	3.52
200	8753	24635	2.81

4 Conclusion

In light of the recent study on the list decoding of binary Goppa codes [2], we compared the size of public keys for different variants of the McEliece cryptosystem. We showed that using list decodable codes in McEliece cryptosystems deliver compelling benefits. We explained how to secure the dyadic variant against currently known attacks while reducing the size of the keys using list decoding. For example, for a workfactor of 2^{80} , list decoding lowers the public keysize from 494,725 bits for the generic variant to 6,611 bits for the dyadic variant. It is worth mentioning that contrary to previous attempts at reducing the McEliece key sizes, using list decoding does not introduce any additional structure that could be used to attack the system.

References

- [1] Carlos Aguilar Melchor, Pierre-Louis Cayrel, and Philippe Gaborit. A new efficient threshold ring signature scheme based on coding theory. In Johannes Buchmann and Jintai Ding, editors, *Post-Quantum Cryptography*, volume 5299 of *Lecture Notes in Computer Science*, pages 1–16. Springer Berlin / Heidelberg, 2008.
- [2] Daniel Augot, Morgan Barbier, and Alain Couvreur. List-decoding of binary Goppa codes up to the binary Johnson bound. Technical report, INRIA Saclay, 2010.
- [3] Elaine Barker, William Barker, William Burr, William Polk, and Miles Smid. Recommendation for key management part 1: General (revised). *Nist Special Publication 80057*, (1/3):1–142, 2007.
- [4] Thierry Berger, Pierre-Louis Cayrel, Philippe Gaborit, and Ayoub Otmani. Reducing key length of the mceliece cryptosystem. In Bart Preneel, editor, *Progress in Cryptology AFRICACRYPT 2009*, volume 5580 of *Lecture Notes in Computer Science*, pages 77–97. Springer Berlin / Heidelberg, 2009.
- [5] Thierry P. Berger and Pierre Loidreau. How to mask the structure of codes for a cryptographic use. *Designs, Codes and Cryptography*, 35:63–79, 2005.
- [6] Elwin Berlekamp, Robert McEliece, and Henk Van Tilborg. On the inherent intractability of certain coding problems. *Information Theory, IEEE Transactions on*, 24(3):384 – 386, May 1978.
- [7] Elwyn Berlekamp. *Algebraic coding theory*. Aegean Park Press, 2 edition, 1984.
- [8] Daniel Bernstein. List decoding for binary Goppa codes. <http://cr.ypt.org/codes/goppalist-20081107.pdf>, 2008.

- [9] Daniel Bernstein, Johannes Buchmann, and Erik Dahmen, editors. *Post-Quantum cryptography*. Springer Berlin / Heidelberg, 2009.
- [10] Daniel Bernstein, Tanja Lange, and Christiane Peters. Attacking and defending the McEliece cryptosystem. In Johannes Buchmann and Jintai Ding, editors, *Post-Quantum Cryptography*, volume 5299 of *Lecture Notes in Computer Science*, pages 31–46. Springer Berlin / Heidelberg, 2008.
- [11] Daniel Bernstein, Tanja Lange, and Christiane Peters. Explicit bounds for generic decoding algorithms for code-based cryptography. *WCC 2009*, pages 168–180, May 2009.
- [12] Daniel Bernstein, Tanja Lange, and Christiane Peters. Wild McEliece. Cryptology ePrint Archive, Report 2010/410, 2010. accepted at SAC 2010.
- [13] Daniela Engelbert, Raphael Overbeck, and Arthur Schmidt. A summary of McEliece-type cryptosystems and their security. Cryptology ePrint Archive, Report 2006/162, 2006.
- [14] Jean-Charles Faugère, Ayoub Otmani, Ludovic Perret, and Jean-Pierre Tillich. Algebraic cryptanalysis of McEliece variants with compact keys. In Henri Gilbert, editor, *Advances in Cryptology EUROCRYPT 2010*, volume 6110 of *Lecture Notes in Computer Science*, pages 279–298. Springer Berlin / Heidelberg, 2010.
- [15] Matthieu Finiasz and Nicolas Sendrier. Security bounds for the design of code-based cryptosystems. In Mitsuru Matsui, editor, *Advances in Cryptology ASIACRYPT 2009*, volume 5912 of *Lecture Notes in Computer Science*, pages 88–105. Springer Berlin / Heidelberg, 2009.
- [16] Venkatesan Guruswami and Madhu Sudan. Improved decoding of Reed-Solomon and algebraic-geometry codes. *Information Theory, IEEE transactions on*, 45(6):1757–1767, 1999.
- [17] Yuan Xing Li, R.H. Deng, and Xin Mei Wang. On the equivalence of McEliece’s and Niederreiter’s public-key cryptosystems. *Information Theory, IEEE Transactions on*, 40(1):271–273, January 1994.
- [18] Florence Jessie. MacWilliams and Neil James Alexander Sloane. *The theory of error-correcting codes. II*. North-Holland Publishing Co., Amsterdam, 1977. North-Holland Mathematical Library, Vol. 16.
- [19] Robert McEliece. A public-key cryptosystem based on algebraic coding theory. *Deep Space Network Progress Report*, 44:114–116, 1978.
- [20] Rafael Misoczki and Paulo Barreto. Compact McEliece keys from Goppa codes. Cryptology ePrint Archive, Report 2009/187, 2009.
- [21] Harald Niederreiter. Knapsack-type cryptosystems and algebraic coding theory. *Problems of Control and Information Theory*, pages 15(2):159–166, 1986.

- [22] Hilarie Orman and Paul Hoffman. *Determining Strengths For Public Keys Used For Exchanging Symmetric Keys*. Purple Streak Development and VPN Consortium, april 2004.
- [23] Ayoub Otmani, Jean-Pierre Tillich, and Lonard Dallot. Cryptanalysis of two McEliece cryptosystems based on quasi-cyclic codes. *Mathematics in Computer Science*, 3:129–140, 2010.
- [24] Raphael Overbeck and Nicolas Sendrier. Code-based cryptography. In Daniel Bernstein, Johannes Buchmann, and Erik Dahmen, editors, *Post-Quantum Cryptography*, pages 95–145. Springer Berlin / Heidelberg, 2009.
- [25] Nicholas Patterson. The algebraic decoding of Goppa codes. *Information Theory, IEEE Transactions on*, 21(2):203 – 207, March 1975.
- [26] David Pointcheval. Chosen-ciphertext security for any one-way cryptosystem. In Hideki Imai and Yuliang Zheng, editors, *Public Key Cryptography*, volume 1751 of *Lecture Notes in Computer Science*, pages 129–146. Springer Berlin / Heidelberg, 2000.
- [27] Jacques Stern. A method for finding codewords of small weight. In Grard Cohen and Jacques Wolfmann, editors, *Coding Theory and Applications*, volume 388 of *Lecture Notes in Computer Science*, pages 106–113. Springer Berlin / Heidelberg, 1989.
- [28] Eric Verheul, Jeroen Doumen, and Henk Van Tilborg. Sloppy Alice attacks! adaptive chosen ciphertext attacks on the McEliece public-key cryptosystem. In *Information, coding and mathematics: proceedings of workshop honoring prof. Bob McEliece on his 60th birthday / ed. by Mario Blaum*, pages 99–119, Boston, 2002. Kluwer Academic Publishers.