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***Temporal convergence of a locally implicit  
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Ludovic Moya — Jan Verwer

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## Temporal convergence of a locally implicit discontinuous Galerkin method for Maxwell's equations

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**Abstract:** In this note we study the temporal convergence of a locally implicit discontinuous Galerkin (DG) method for Maxwell's equations modeling electromagnetic wave propagation. Particularly, we wonder whether the method retains its second-order ordinary differential equation (ODE) convergence under stable simultaneous space-time grid refinement towards the true partial differential equation (PDE) solution. This is not a priori clear due to the component splitting which can introduce order reduction.

**Key-words:** temporal convergence, discontinuous Galerkin method, Maxwell equations, component splitting, order reduction

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# Convergence temporelle d'une méthode Galerkin discontinue localement implicite pour les équations de Maxwell

**Résumé :** Dans ce papier nous étudions la convergence temporelle d'une méthode Galerkin discontinue localement implicite pour la résolution des équations de Maxwell modélisant la propagation des ondes électromagnétiques. En particulier, nous nous demandons si pour un raffinement du maillage, simultané et stable en espace-temps, le second ordre de convergence au sens des EDO est conservé pour la solution exacte de l'EDP. Cela n'est pas à priori clair en raison de la décomposition des éléments qui peut introduire une réduction d'ordre.

**Mots-clés :** convergence temporelle, méthode Galerkin discontinue, équations de Maxwell, décomposition des éléments, réduction d'ordre

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## 1 Introduction

In recent years there has been an increasing interest in DG time-domain methods. One of the main features of DG spatial discretization is the possibility of using locally refined space grids to easily deal with complex geometries. Also the choice of the temporal integration method is a crucial step for the global efficiency of the numerical method. We distinguish two major families for the temporal integration: implicit and explicit methods. An implicit integration method to numerically solve a time-dependent PDE leads in general to unconditional stability. Then the time step can be chosen arbitrarily large. However, implicit methods require the solution of large linear systems resulting in a high computational effort. An explicit integration method results in less computational effort per time step, but leads to step size restrictions caused by the smallest grid elements. For examples of implicit and explicit integration methods for the semi-discrete Maxwell equations we refer to [2, 8] and [1, 4], respectively. Considering the strengths of both methods, the authors of [3, 6] and [7] have proposed a possible alternative: locally implicit time integration methods. More precisely, the smallest grid elements are treated implicitly and the remaining elements explicitly by a technique we call component splitting. If the ratio of fine to coarse elements is small, the most severe step size restrictions are overcome. Consequently, these methods are particularly well suited if the local refinement is strongly localized. Note that the method from [3, 6] has been especially designed for DG discretizations, while that from [7] covers the common spatial discretizations like finite-difference and various finite-element discretizations.

In this note we study the temporal convergence of the locally implicit DG method for Maxwell's equations proposed in [3, 6]. In particular we examine whether the method retains its second-order ODE convergence towards the true PDE solution under stable simultaneous space-time grid refinement. This question is legitimate because component splitting can cause order reduction.

The contents of the paper is as follows. Section 2 presents the problem and the notations. In Section 3 we introduce the component splitting and the implicit-explicit method for the time-dependent Maxwell equations spatially discretized with DG. This implicit-explicit integration method is a blend of the explicit one-step second-order Leap-Frog scheme (LF2) and the implicit one-step second-order Crank-Nicolson scheme (CN2). Next we study the temporal convergence order. The different steps of this convergence analysis are the same as in [7], where another implicit-explicit integration method for semi-discrete Maxwell equations is proposed. Note that this last method is also a blend of the same two methods LF2 and CN2. It is proved in [7] that this method retains its second-order ODE convergence towards the true PDE solution under stable simultaneous space-time grid refinement. In Section 4 we give some numerical results for 2D Maxwell problems to illustrate the previous convergence analysis. Section 5 concludes the paper with final remarks and future plans.

## 2 Problem statement

We consider the Maxwell equations

$$\begin{cases} \epsilon \partial_t E &= \text{curl } H - \sigma E - J_E, \\ \mu \partial_t H &= -\text{curl } E, \end{cases} \quad (2.1)$$

where  $E$  and  $H$  denote the electric and magnetic field, respectively.  $J_E$  is the given source current and  $\epsilon$ ,  $\mu$  and  $\sigma$  are coefficients representing dielectric permittivity, magnetic permeability and conductivity, respectively. After discretization in space by a DG method we obtain the semi-discrete Maxwell system

$$\begin{cases} M^\epsilon \partial_t E &= SH - DE + M^\epsilon f^E, \\ M^\mu \partial_t H &= -S^T E + M^\mu f^H, \end{cases} \quad (2.2)$$

where, for convenience, we use the same notation for the electric and magnetic fields  $E$  and  $H$  as in the space-continuous case. For more details on DG spatial discretization we refer to [3]. The matrices  $M^\epsilon$ ,  $M^\mu$  are the DG mass matrices which contain the values of the dielectric permittivity and magnetic permeability coefficient. The matrix  $S$  emanates from the discretization of the curl operator. The matrix  $D$  is associated with the dissipative conduction term  $-\sigma E$ . The functions  $f^E$  and  $f^H$  are associated

with source terms. More precisely  $f^E$  represents the given source current  $-J_E$ , but  $f^E$  and  $f^H$  may also contain Dirichlet boundary data.

We can give an equivalent formulation of (2.2) without mass matrix. As in [1] we introduce the Cholesky factorizations

$$M^\epsilon = L_{M^\epsilon} L_{M^\epsilon}^T \text{ and } M^\mu = L_{M^\mu} L_{M^\mu}^T, \quad (2.3)$$

where  $L_{M^\epsilon}$  and  $L_{M^\mu}$  are triangular matrices. Then with (2.2) we have

$$\begin{cases} L_{M^\epsilon} L_{M^\epsilon}^T \partial_t E &= SH - DE + L_{M^\epsilon} L_{M^\epsilon}^T f^E, \\ L_{M^\mu} L_{M^\mu}^T \partial_t H &= -S^T E + L_{M^\mu} L_{M^\mu}^T f^H. \end{cases} \quad (2.4)$$

Introducing  $\tilde{E} = L_{M^\epsilon}^T E$  and  $\tilde{H} = L_{M^\mu}^T H$ , we get

$$\begin{cases} \partial_t \tilde{E} &= L_{M^\epsilon}^{-1} S (L_{M^\mu}^T)^{-1} \tilde{H} - L_{M^\epsilon}^{-1} D (L_{M^\epsilon}^T)^{-1} \tilde{E} + L_{M^\epsilon}^T f^E, \\ \partial_t \tilde{H} &= -L_{M^\mu}^{-1} S^T (L_{M^\epsilon}^T)^{-1} \tilde{E} + L_{M^\mu}^T f^H. \end{cases} \quad (2.5)$$

Next we write

$$\begin{aligned} \tilde{S} &= L_{M^\epsilon}^{-1} S (L_{M^\mu}^{-1})^T, & \tilde{D} &= L_{M^\epsilon}^{-1} D (L_{M^\epsilon}^{-1})^T, \\ \tilde{f}^E &= L_{M^\epsilon}^T f^E, & \tilde{f}^H &= L_{M^\mu}^T f^H, \end{aligned} \quad (2.6)$$

and note that

$$\tilde{S}^T = [L_{M^\epsilon}^{-1} S (L_{M^\mu}^{-1})^T]^T = L_{M^\mu}^{-1} [L_{M^\epsilon}^{-1} S]^T = L_{M^\mu}^{-1} S^T (L_{M^\epsilon}^{-1})^T. \quad (2.7)$$

Thus we can write the semi-discrete Maxwell system equivalent to (2.2) as

$$\begin{cases} \partial_t \tilde{E} &= \tilde{S} \tilde{H} - \tilde{D} \tilde{E} + \tilde{f}^E, \\ \partial_t \tilde{H} &= -\tilde{S}^T \tilde{E} + \tilde{f}^H. \end{cases} \quad (2.8)$$

For convenience of notation and presentation we use the same notation in (2.2) and (2.8) i.e.

$$\begin{cases} \partial_t E &= SH - DE + f^E, \\ \partial_t H &= -S^T E + f^H. \end{cases} \quad (2.9)$$

We will proceed with (2.9)<sup>1</sup>. In particular results obtained for (2.9) apply to (2.2) and vice versa. Note that  $S$  emanates from an appropriate DG discretization for the Maxwell problem under consideration and further on we will prove that

$$S \sim \frac{1}{h}, \quad \text{for } h \rightarrow 0, \quad (2.10)$$

where the parameter  $h$  denotes the maximum diameter of the (non-uniform) grid elements. Throughout the remainder we will assume initial values at time  $t = 0$  and the source functions  $f^E(t), f^H(t) \in C^2[0, T]$  for an interval  $[0, T]$ , so that  $E(t), H(t) \in C^3[0, T]$ .

### 3 The implicit-explicit integration method

The implicit-explicit integration considered in this note is issued from [3, 6]. As we have previously mentioned it is a blend of the LF2 method

$$\begin{cases} \frac{H^{n+\frac{1}{2}} - H^n}{\Delta t/2} &= -S^T E^n + f^H(t_n), \\ \frac{E^{n+1} - E^n}{\Delta t} &= SH^{n+\frac{1}{2}} - \frac{1}{2}D(E^{n+1} + E^n) + \frac{1}{2}(f^E(t_{n+1}) + f^E(t_n)), \\ \frac{H^{n+1} - H^{n+\frac{1}{2}}}{\Delta t/2} &= -S^T E^{n+1} + f^H(t_{n+1}). \end{cases} \quad (3.1)$$

<sup>1</sup>Note that (2.9) is the formulation used in [7] with  $u = H$  and  $v = E$ . The meaning of  $E, H, S, D, f^E$  and  $f^H$  will always be clear from the context or will be precised.



and the CN2 method

$$\begin{cases} \frac{H^{n+\frac{1}{2}} - H^n}{\Delta t/2} &= -S^T E^n + f^H(t_n), \\ \frac{E^{n+1} - E^n}{\Delta t} &= SH^{n+1} - \frac{1}{2}D(E^{n+1} + E^n) + \frac{1}{2}(f^E(t_{n+1}) + f^E(t_n)), \\ \frac{H^{n+1} - H^{n+\frac{1}{2}}}{\Delta t/2} &= -S^T E^{n+1} + f^H(t_{n+1}). \end{cases} \quad (3.2)$$

which only differ in the middle stage in the time level for  $H$ , cf. [7]. Herein  $\Delta t = t_{n+1} - t_n$  denotes the step size and upper indices refer to time levels, as usual.

### 3.1 Component splitting

The set of grid elements is assumed to be partitioned into two subsets: one made of the smallest elements that will be treated implicitly as in CN2 and the other one of the remaining elements that will be treated explicitly as in LF2. In line with this splitting the problem unknowns are reordered as

$$E = \begin{pmatrix} E_e \\ E_i \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} H_e \\ H_i \end{pmatrix}, \quad (3.3)$$

where the indices  $i$  and  $e$  are associated to the elements of the subset treated implicitly, respectively explicitly. Likewise the semi-discrete curl operator  $S$  is split into the block form

$$S = \begin{pmatrix} S_e & -A_{ei} \\ -A_{ie} & S_i \end{pmatrix}, \quad (3.4)$$

where for the specific meaning of the block-entries of  $S$  we refer to [3]. In line with this splitting  $D$  is supposed to be split into

$$D = \begin{pmatrix} D_e & 0 \\ 0 & D_i \end{pmatrix}, \quad (3.5)$$

and we write

$$f^E = \begin{pmatrix} f_e^E \\ f_i^E \end{pmatrix}, \quad f^H = \begin{pmatrix} f_e^H \\ f_i^H \end{pmatrix}. \quad (3.6)$$

Inserting this splitting into (2.9) we obtain the system of ODEs

$$\begin{cases} \partial_t E_e &= S_e H_e - A_{ei} H_i - D_e E_e + f_e^E(t), \\ \partial_t E_i &= S_i H_i - A_{ie} H_e - D_i E_i + f_i^E(t), \\ \partial_t H_e &= -S_e^T E_e + A_{ie}^T E_i + f_e^H(t), \\ \partial_t H_i &= -S_i^T E_i + A_{ei}^T E_e + f_i^H(t). \end{cases} \quad (3.7)$$

### 3.2 The implicit-explicit time integration method

The implicit-explicit method proposed in [3, 6] is a blend of LF2 and CN2 applied to (3.7). It reads

$$\begin{cases}
 \frac{H_e^{n+1/2} - H_e^n}{\Delta t/2} = -S_e^T E_e^n + A_{ie}^T E_i^n + f_e^H(t_n), \\
 \frac{E_e^{n+1/2} - E_e^n}{\Delta t/2} = S_e H_e^{n+1/2} - A_{ei} H_i^n - D_e E_e^n + f_e^E(t_n), \\
 \frac{H_i^{n+1} - H_i^n}{\Delta t} = S_i \left( \frac{H_i^{n+1} + H_i^n}{2} \right) - A_{ie} H_e^{n+1/2} \\
 \quad - D_i \left( \frac{E_i^{n+1} + E_i^n}{2} \right) + \frac{f_i^E(t_{n+1}) + f_i^E(t_n)}{2}, \\
 \frac{H_i^{n+1} - H_i^n}{\Delta t} = -S_i^T \left( \frac{E_i^{n+1} + E_i^n}{2} \right) + A_{ei}^T E_e^{n+1/2} + \frac{f_i^H(t_{n+1}) + f_i^H(t_n)}{2}, \\
 \frac{E_e^{n+1} - E_e^{n+1/2}}{\Delta t/2} = S_e H_e^{n+1/2} - A_{ei} H_i^{n+1} - D_e E_e^{n+1} + f_e^E(t_{n+1}), \\
 \frac{H_e^{n+1} - H_e^{n+1/2}}{\Delta t/2} = -S_e^T E_e^{n+1} + A_{ie}^T E_i^{n+1} + f_e^H(t_{n+1}).
 \end{cases} \quad (3.8)$$

For the stability analysis of this method we refer to [3]. The proof is based on the conservation of a discrete electromagnetic energy. It is proved that this energy is a positive quadratic form of the numerical unknowns  $E_e^n$ ,  $E_i^n$ ,  $H_e^n$  and  $H_i^n$  under a condition on the step size. Consequently the non-dissipative nature of the method yields the stability of the method.

### 3.3 Matrix behavior for $h \rightarrow 0$

In this subsection we are interested in the behavior of the matrices in (3.8) for  $h \rightarrow 0$ . This is an essential point for convergence analysis because some of these matrices can lie at the origin of order reduction. Let us consider the general case of dimension  $n$  ( $n = 1, 2$  or  $3$ ). First we investigate the behavior of the matrices in the formulation with the mass matrices. Thereafter we will be able to deduce the behavior of the matrices in (3.8).

We reintroduce the notation with a tilde for the elements involved in the formulation without mass matrix (see (2.8)) in order to avoid confusion. The specific meaning of the block-entries of the different matrices involved in the formulation with mass matrices can be found in [3]. First we observe that the mass matrices are only composed of volumic terms, hence we have

$$M^\epsilon, M^\mu \sim h^n, \text{ for } h \rightarrow 0. \quad (3.9)$$

Thus

$$L_{M^\epsilon}, L_{M^\epsilon}^T, L_{M^\mu} \text{ and } L_{M^\mu}^T \sim h^{\frac{n}{2}}, \text{ for } h \rightarrow 0. \quad (3.10)$$

The matrices  $S_e$  and  $S_i$  are composed of volumic and surfacic terms, hence

$$S_e \text{ and } S_i \sim h^{n-1}, \text{ for } h \rightarrow 0. \quad (3.11)$$

The matrices  $A_{ei}$  and  $A_{ie}$  represent surfacic terms (interface matrices). Hence

$$A_{ei} \text{ and } A_{ie} \sim h^{n-1}, \text{ for } h \rightarrow 0 \quad (3.12)$$

and from the block form

$$S = \begin{pmatrix} S_e & -A_{ei} \\ -A_{ie} & S_i \end{pmatrix}, \quad (3.13)$$

we deduce with (3.11) and (3.12) that

$$S \sim h^{n-1}, \text{ for } h \rightarrow 0. \quad (3.14)$$

From (2.6) we get

$$\tilde{S} = L_{M^\epsilon}^{-1} S (L_{M^\mu}^{-1})^T, \quad (3.15)$$

then with the behaviors above we deduce that for  $h \rightarrow 0$

$$\begin{aligned} \tilde{S}_e, \tilde{S}_i &= \mathcal{O}(\tfrac{1}{h}), \\ \tilde{A}_{ei}, \tilde{A}_{ie} &= \mathcal{O}(\tfrac{1}{h}), \end{aligned} \quad (3.16)$$

and we have the expected behaviour (2.10) for  $\tilde{S}$ .

### 3.4 Temporal convergence

In this section we are interested in the PDE convergence of method (3.8). More precisely, we will examine whether the method retains its second-order ODE convergence under stable simultaneous space-time grid refinement  $\Delta t \sim h$ ,  $h \rightarrow 0$  towards the true PDE solution. This is not a priori clear due to the component splitting which can introduce order reduction through error constants which grow with  $h^{-1}$ , for  $h \rightarrow 0$ .

This section is organized in four subsections. In Subsection 2.4.1 we will eliminate the intermediate values of (3.8) to get an equivalent one step formula from  $t_n$  to  $t_{n+1}$  that we will use for our convergence analysis. In Subsection 2.4.2 we will introduce the perturbed method obtained by substituting the true PDE solution restricted to the assumed space grid into (3.7), and defects (space-time truncation errors) obtained by substituting this true PDE solution into the equivalent one step formula of our method. In Subsection 2.4.3 we will define the common one-step recurrence relation for the global error. In Subsection 2.4.4 we will point out the order reduction mentioned above. Finally in Subsection 2.4.5 we will see that this order reduction, affecting the local error, may (partly) cancel in the transition from the local to the global error.

#### 3.4.1 Elimination of intermediates values

First we treat  $H_e$ . From the first and last equations of (3.8) we get

$$\begin{aligned} H_e^{n+\frac{1}{2}} &= H_e^n - \frac{\Delta t}{2} S_e^T E_e^n + \frac{\Delta t}{2} A_{ie}^T E_i^n + \frac{\Delta t}{2} f_e^H(t_n), \\ H_e^{n+\frac{1}{2}} &= H_e^{n+1} + \frac{\Delta t}{2} S_e^T E_e^{n+1} - \frac{\Delta t}{2} A_{ie}^T E_i^{n+1} + \frac{\Delta t}{2} f_e^H(t_{n+1}). \end{aligned} \quad (3.17)$$

Inserting the first equation of (3.17) into the last equation of (3.8) yields

$$H_e^{n+1} = H_e^n - \frac{\Delta t}{2} S_e^T (E_e^n + E_e^{n+1}) + \frac{\Delta t}{2} A_{ie}^T (E_i^n + E_i^{n+1}) + \frac{\Delta t}{2} (f_e^H(t_n) + f_e^H(t_{n+1})). \quad (3.18)$$

Next we treat  $E_e$ . From the second and fifth equations of (3.8) we get

$$\begin{aligned} E_e^{n+\frac{1}{2}} &= E_e^n + \frac{\Delta t}{2} S_e H_e^{n+\frac{1}{2}} - \frac{\Delta t}{2} A_{ei} H_i^n - \frac{\Delta t}{2} D_e E_e^n + \frac{\Delta t}{2} f_e^E(t_n), \\ E_e^{n+\frac{1}{2}} &= E_e^{n+1} - \frac{\Delta t}{2} S_e H_e^{n+\frac{1}{2}} + \frac{\Delta t}{2} A_{ei} H_i^{n+1} + \frac{\Delta t}{2} D_e E_e^{n+1} - \frac{\Delta t}{2} f_e^E(t_{n+1}). \end{aligned} \quad (3.19)$$

Inserting the first equation of (3.19) and half of each expression of (3.17) for  $H_e^{n+\frac{1}{2}}$  into the fifth equation of (3.8) yields

$$\begin{aligned} E_e^{n+1} &= E_e^n + \frac{\Delta t}{2} S_e (H_e^n + H_e^{n+1}) - \frac{\Delta t}{2} A_{ei} (H_i^n + H_i^{n+1}) \\ &\quad - \frac{\Delta t}{2} D_e (E_e^n + E_e^{n+1}) + \frac{\Delta t}{2} (f_e^E(t_n) + f_e^E(t_{n+1})) \\ &\quad + \frac{\Delta t^2}{4} S_e (-S_e^T E_e^n + A_{ie}^T E_i^n) \\ &\quad + \frac{\Delta t^2}{4} S_e (S_e^T E_e^{n+1} - A_{ie}^T E_i^{n+1}) \\ &\quad + \frac{\Delta t^2}{4} S_e (f_e^H(t_n) + f_e^H(t_{n+1})). \end{aligned} \quad (3.20)$$

Now we consider  $H_i$ . Inserting half of each expression of (3.19) for  $E_e^{n+\frac{1}{2}}$  in the fourth equation of (3.8) gives

$$\begin{aligned} H_i^{n+1} &= H_i^n - \frac{\Delta t}{2} S_i^T (E_i^n + E_i^{n+1}) + \frac{\Delta t}{2} A_{ei}^T (E_e^n + E_e^{n+1}) + \frac{\Delta t}{2} (f_i^H(t_n) + f_i^H(t_{n+1})) \\ &\quad + \frac{\Delta t^2}{4} A_{ei}^T A_{ei} (H_i^{n+1} - H_i^n) + \frac{\Delta t^2}{4} A_{ei}^T D_e (E_e^{n+1} - E_e^n) \\ &\quad + \frac{\Delta t^2}{4} A_{ei}^T (f_e^E(t_n) + f_e^E(t_{n+1})). \end{aligned} \quad (3.21)$$

Finally we treat  $E_i$ . Inserting half of each expression of (3.17) for  $H_e^{n+\frac{1}{2}}$  in the third equation of (3.8) yields

$$\begin{aligned} E_i^{n+1} &= E_i^n + \frac{\Delta t}{2} S_i (H_i^n + H_i^{n+1}) - \frac{\Delta t}{2} A_{ie} (H_e^n + H_e^{n+1}) \\ &\quad - \frac{\Delta t}{2} D_i (E_i^n + E_i^{n+1}) + \frac{\Delta t}{2} (f_i^E(t_n) + f_i^E(t_{n+1})) \\ &\quad - \frac{\Delta t^2}{4} A_{ie} (-S_e^T E_e^n + A_{ie}^T E_i^n) \\ &\quad - \frac{\Delta t^2}{4} A_{ie} (S_e^T E_e^{n+1} - A_{ie}^T E_i^{n+1}) \\ &\quad - \frac{\Delta t^2}{4} A_{ie} (f_e^H(t_n) + f_e^H(t_{n+1})). \end{aligned} \quad (3.22)$$

The equivalent method of (3.8) with its intermediate values eliminated thus reads

$$\begin{aligned} E_e^{n+1} &= E_e^n + \frac{\Delta t}{2} S_e (H_e^n + H_e^{n+1}) - \frac{\Delta t}{2} A_{ei} (H_i^n + H_i^{n+1}) \\ &\quad - \frac{\Delta t}{2} D_e (E_e^n + E_e^{n+1}) + \frac{\Delta t}{2} (f_e^E(t_n) + f_e^E(t_{n+1})) \\ &\quad + \frac{\Delta t^2}{4} S_e [(-S_e^T E_e^n + A_{ie}^T E_i^n) - (S_e^T E_e^{n+1} - A_{ie}^T E_i^{n+1})] \\ &\quad + \frac{\Delta t^2}{4} S_e (f_e^H(t_n) + f_e^H(t_{n+1})), \\ E_i^{n+1} &= E_i^n + \frac{\Delta t}{2} S_i (H_i^n + H_i^{n+1}) - \frac{\Delta t}{2} A_{ie} (H_e^n + H_e^{n+1}) \\ &\quad - \frac{\Delta t}{2} D_i (E_i^n + E_i^{n+1}) + \frac{\Delta t}{2} (f_i^E(t_n) + f_i^E(t_{n+1})) \\ &\quad - \frac{\Delta t^2}{4} A_{ie} [(-S_e^T E_e^n + A_{ie}^T E_i^n) - (S_e^T E_e^{n+1} - A_{ie}^T E_i^{n+1})] \\ &\quad - \frac{\Delta t^2}{4} A_{ie} (f_e^H(t_n) + f_e^H(t_{n+1})), \\ H_e^{n+1} &= H_e^n - \frac{\Delta t}{2} S_e^T (E_e^n + E_e^{n+1}) + \frac{\Delta t}{2} A_{ie}^T (E_i^n + E_i^{n+1}) \\ &\quad + \frac{\Delta t}{2} (f_e^H(t_n) + f_e^H(t_{n+1})), \\ H_i^{n+1} &= H_i^n - \frac{\Delta t}{2} S_i^T (E_i^n + E_i^{n+1}) + \frac{\Delta t}{2} A_{ei}^T (E_e^n + E_e^{n+1}) \\ &\quad + \frac{\Delta t}{2} (f_i^H(t_n) + f_i^H(t_{n+1})) \\ &\quad + \frac{\Delta t^2}{4} A_{ei}^T A_{ei} (H_i^{n+1} - H_i^n) \\ &\quad + \frac{\Delta t^2}{4} A_{ei}^T [D_e (E_e^{n+1} - E_e^n) + (f_e^E(t_n) + f_e^E(t_{n+1}))]. \end{aligned} \quad (3.23)$$

### 3.4.2 The perturbed method and defects for the PDE solution

Let  $E_e^h(t)$  denote at time  $t$  the true solution of the PDE problem restricted to the assumed space grid that we have approximated with the semi-discrete system (3.7).  $E_e^h(t_n)$  thus represents the vector that is approximated by  $E_e^n$ . Assume the same notation for  $E_i$ ,  $H_e$  and  $H_i$ . Substituting  $E_e^h(t)$ ,  $E_i^h(t)$ ,  $H_e^h(t)$

and  $H_i^h(t)$  into (3.7) reveals the spatial truncation errors

$$\begin{cases} \frac{d}{dt} E_e^h(t) &= S_e H_e^h(t) - A_{ei} H_i^h(t) - D_e E_e^h(t) + f_e^E(t) + \sigma_e^E(t), \\ \frac{d}{dt} E_i^h(t) &= S_i H_i^h(t) - A_{ie} H_e^h(t) - D_i E_i^h(t) + f_i^E(t) + \sigma_i^E(t), \\ \frac{d}{dt} H_e^h(t) &= -S_e^T E_e^h(t) + A_{ie}^T E_i^h(t) + f_e^H(t) + \sigma_e^H(t), \\ \frac{d}{dt} H_i^h(t) &= -S_i^T E_i^h(t) + A_{ei}^T E_e^h(t) + f_i^H(t) + \sigma_i^H(t), \end{cases} \quad (3.24)$$

where  $\sigma_e^E(t)$ ,  $\sigma_i^E(t)$ ,  $\sigma_e^H(t)$  and  $\sigma_i^H(t)$  denote the spatial truncation errors.

Substituting  $E_e^h(t)$ ,  $E_i^h(t)$ ,  $H_e^h(t)$  and  $H_i^h(t)$  into (3.23) reveals the defects for the PDE solution (space-time truncation errors) and gives what we call the perturbed method

$$\begin{aligned} E_e^h(t_{n+1}) &= E_e^h(t_n) + \frac{\Delta t}{2} S_e (H_e^h(t_n) + H_e^h(t_{n+1})) - \frac{\Delta t}{2} A_{ei} (H_i^h(t_n) + H_i^h(t_{n+1})) \\ &\quad - \frac{\Delta t}{2} D_e (E_e^h(t_n) + E_e^h(t_{n+1})) + \frac{\Delta t}{2} (f_e^E(t_n) + f_e^E(t_{n+1})) \\ &\quad + \frac{\Delta t^2}{4} S_e [(-S_e^T E_e^h(t_n) + A_{ie}^T E_i^h(t_n)) - (S_e^T E_e^h(t_{n+1}) - A_{ie}^T E_i^h(t_{n+1}))] \\ &\quad + \frac{\Delta t^2}{4} S_e (f_e^H(t_n) + f_e^H(t_{n+1})) + \Delta t \delta_{e,n}^E, \\ E_i^h(t_{n+1}) &= E_i^h(t_n) + \frac{\Delta t}{2} S_i (H_i^h(t_n) + H_i^h(t_{n+1})) - \frac{\Delta t}{2} A_{ie} (H_e^h(t_n) + H_e^h(t_{n+1})) \\ &\quad - \frac{\Delta t}{2} D_i (E_i^h(t_n) + E_i^h(t_{n+1})) + \frac{\Delta t}{2} (f_i^E(t_n) + f_i^E(t_{n+1})) \\ &\quad - \frac{\Delta t^2}{4} A_{ie} [(-S_e^T E_e^h(t_n) + A_{ie}^T E_i^h(t_n)) - (S_e^T E_e^h(t_{n+1}) - A_{ie}^T E_i^h(t_{n+1}))] \\ &\quad - \frac{\Delta t^2}{4} A_{ie} (f_e^H(t_n) + f_e^H(t_{n+1})) + \Delta t \delta_{i,n}^E, \\ H_e^h(t_{n+1}) &= H_e^h(t_n) - \frac{\Delta t}{2} S_e^T (E_e^h(t_n) + E_e^h(t_{n+1})) + \frac{\Delta t}{2} A_{ie}^T (E_i^h(t_n) + E_i^h(t_{n+1})) \\ &\quad + \frac{\Delta t}{2} (f_e^H(t_n) + f_e^H(t_{n+1})) + \Delta t \delta_{e,n}^H, \\ H_i^h(t_{n+1}) &= H_i^h(t_n) - \frac{\Delta t}{2} S_i^T (E_i^h(t_n) + E_i^h(t_{n+1})) + \frac{\Delta t}{2} A_{ei}^T (E_e^h(t_n) + E_e^h(t_{n+1})) \\ &\quad + \frac{\Delta t}{2} (f_i^H(t_n) + f_i^H(t_{n+1})) \\ &\quad + \frac{\Delta t^2}{4} A_{ei}^T A_{ei} (H_i^h(t_{n+1}) - H_i^h(t_n)) \\ &\quad + \frac{\Delta t^2}{4} A_{ei}^T [D_e (E_e^h(t_{n+1}) - E_e^h(t_n)) + (f_e^E(t_n) + f_e^E(t_{n+1}))] + \Delta t \delta_{i,n}^H, \end{aligned} \quad (3.25)$$

where  $\delta_{e,n}^E$ ,  $\delta_{i,n}^E$ ,  $\delta_{e,n}^H$  and  $\delta_{i,n}^H$  denote the defects for the PDE solution.

Eliminating all source term  $f_e^E$ ,  $f_i^E$ ,  $f_e^H$  and  $f_i^H$  contributions through (3.24) yields

$$\begin{aligned} \delta_{e,n}^E &= \delta_{E_e^h} + \frac{\Delta t}{4} S_e \frac{d}{dt} (H_e^h(t_{n+1}) - H_e^h(t_n)) \\ &\quad + \frac{1}{2} (\sigma_e^E(t_n) + \sigma_e^E(t_{n+1})) + \frac{\Delta t}{4} S_e (\sigma_e^H(t_n) + \sigma_e^H(t_{n+1})), \\ \delta_{i,n}^E &= \delta_{E_i^h} - \frac{\Delta t}{4} A_{ie} \frac{d}{dt} (H_e^h(t_{n+1}) - H_e^h(t_n)) \\ &\quad + \frac{1}{2} (\sigma_i^E(t_n) + \sigma_i^E(t_{n+1})) - \frac{\Delta t}{4} A_{ie} (\sigma_e^H(t_n) + \sigma_e^H(t_{n+1})), \\ \delta_{e,n}^H &= \delta_{H_e^h} + \frac{1}{2} (\sigma_e^H(t_n) + \sigma_e^H(t_{n+1})) \\ \delta_{i,n}^H &= \delta_{H_i^h} - \frac{\Delta t}{4} A_{ei}^T S_e (H_e^h(t_{n+1}) - H_e^h(t_n)) + \frac{\Delta t}{4} A_{ei}^T \frac{d}{dt} (E_e^h(t_{n+1}) - E_e^h(t_n)) \\ &\quad + \frac{1}{2} (\sigma_i^H(t_n) + \sigma_i^H(t_{n+1})) + \frac{\Delta t}{4} A_{ei}^T (\sigma_e^E(t_n) + \sigma_e^E(t_{n+1})), \end{aligned} \quad (3.26)$$

where  $\delta_{E_e^h}$  denotes the implicit trapezoidal (CN) rule defect (see [7]) for variable  $E_e^h$  (similarly for  $E_i^h$ ,  $H_e^h$  and  $H_i^h$ ), i.e.

$$\delta_{E_e^h}(t) = \frac{E_e^h(t + \Delta t) - E_e^h(t)}{\Delta t} - \frac{1}{2} \frac{d}{dt} (E_e^h(t + \Delta t) + E_e^h(t)). \quad (3.27)$$

### 3.4.3 The error scheme

Let  $\varepsilon_{e,n}^E = E_e^h(t_n) - E_e^n$  denote the global error (similarly we introduce  $\varepsilon_{i,n}^E$ ,  $\varepsilon_{e,n}^H$  and  $\varepsilon_{i,n}^H$ ). Subtracting (3.23) and (3.25) we obtain the error scheme

$$\begin{aligned} \varepsilon_{e,n+1}^E &= \varepsilon_{e,n}^E + \frac{\Delta t}{2} S_e(\varepsilon_{e,n}^H + \varepsilon_{e,n+1}^H) - \frac{\Delta t}{2} A_{ei}(\varepsilon_{i,n}^H + \varepsilon_{i,n+1}^H) - \frac{\Delta t}{2} D_e(\varepsilon_{e,n}^E + \varepsilon_{e,n+1}^E) \\ &\quad + \frac{\Delta t^2}{4} S_e[(-S_e^T \varepsilon_{e,n}^E + A_{ie}^T \varepsilon_{i,n}^E) - (S_e^T \varepsilon_{e,n+1}^E - A_{ie}^T \varepsilon_{i,n+1}^E)] + \Delta t \delta_{e,n}^E, \\ \varepsilon_{i,n+1}^E &= \varepsilon_{i,n}^E + \frac{\Delta t}{2} S_i(\varepsilon_{i,n}^H + \varepsilon_{i,n+1}^H) - \frac{\Delta t}{2} A_{ie}(\varepsilon_{e,n}^H + \varepsilon_{e,n+1}^H) - \frac{\Delta t}{2} D_i(\varepsilon_{i,n}^E + \varepsilon_{i,n+1}^E) \\ &\quad - \frac{\Delta t^2}{4} A_{ie}[(-S_e^T \varepsilon_{e,n}^E + A_{ie}^T \varepsilon_{i,n}^E) - (S_e^T \varepsilon_{e,n+1}^E - A_{ie}^T \varepsilon_{i,n+1}^E)] + \Delta t \delta_{i,n}^E, \\ \varepsilon_{e,n+1}^H &= \varepsilon_{e,n}^H - \frac{\Delta t}{2} S_e^T(\varepsilon_{e,n}^E + \varepsilon_{e,n+1}^E) + \frac{\Delta t}{2} A_{ie}^T(\varepsilon_{i,n}^E + \varepsilon_{i,n+1}^E) + \Delta t \delta_{e,n}^H, \\ \varepsilon_{i,n+1}^H &= \varepsilon_{i,n}^H - \frac{\Delta t}{2} S_i^T(\varepsilon_{i,n}^E + \varepsilon_{i,n+1}^E) + \frac{\Delta t}{2} A_{ei}^T(\varepsilon_{e,n}^E + \varepsilon_{e,n+1}^E) \\ &\quad + \frac{\Delta t^2}{4} A_{ei}^T A_{ei}(\varepsilon_{i,n+1}^H - \varepsilon_{i,n}^H) + \frac{\Delta t^2}{4} A_{ei}^T D_e(\varepsilon_{e,n+1}^E - \varepsilon_{e,n}^E) + \Delta t \delta_{i,n}^H. \end{aligned} \quad (3.28)$$

Let

$$\varepsilon_n = \begin{pmatrix} \varepsilon_{e,n}^E \\ \varepsilon_{i,n}^E \\ \varepsilon_{e,n}^H \\ \varepsilon_{i,n}^H \end{pmatrix} \quad \text{and} \quad \delta_n = \begin{pmatrix} \delta_{e,n}^E \\ \delta_{i,n}^E \\ \delta_{e,n}^H \\ \delta_{i,n}^H \end{pmatrix}, \quad (3.29)$$

then from (3.28) we can write the global error in a more compact form (one-step recurrence relation)

$$\varepsilon_{n+1} = R \varepsilon_n + \Delta t \rho_n, \quad R = R_L^{-1} R_R, \quad \rho_n = R_L^{-1} \delta_n, \quad (3.30)$$

where

$$R_L = \begin{pmatrix} I + \frac{\Delta t}{2} D_e - \frac{\Delta t^2}{4} S_e S_e^T & \frac{\Delta t^2}{4} S_e A_{ie}^T & -\frac{\Delta t}{2} S_e & \frac{\Delta t}{2} A_{ei} \\ \frac{\Delta t^2}{4} A_{ie} S_e^T & I + \frac{\Delta t}{2} D_i - \frac{\Delta t^2}{4} A_{ie} A_{ie}^T & \frac{\Delta t}{2} A_{ie} & -\frac{\Delta t}{2} S_i \\ \frac{\Delta t}{2} S_e^T & -\frac{\Delta t}{2} A_{ie}^T & I & 0 \\ -\frac{\Delta t}{2} A_{ei}^T - \frac{\Delta t^2}{4} A_{ei}^T D_e & \frac{\Delta t}{2} S_i^T & 0 & I - \frac{\Delta t^2}{4} A_{ei}^T A_{ei} \end{pmatrix}, \quad (3.31)$$

$$R_R = \begin{pmatrix} I - \frac{\Delta t}{2} D_e - \frac{\Delta t^2}{4} S_e S_e^T & \frac{\Delta t^2}{4} S_e A_{ie}^T & \frac{\Delta t}{2} S_e & -\frac{\Delta t}{2} A_{ei} \\ \frac{\Delta t^2}{4} A_{ie} S_e^T & I - \frac{\Delta t}{2} D_i - \frac{\Delta t^2}{4} A_{ie} A_{ie}^T & -\frac{\Delta t}{2} A_{ie} & \frac{\Delta t}{2} S_i \\ -\frac{\Delta t}{2} S_e^T & \frac{\Delta t}{2} A_{ie}^T & I & 0 \\ \frac{\Delta t}{2} A_{ei}^T - \frac{\Delta t^2}{4} A_{ei}^T D_e & -\frac{\Delta t}{2} S_i^T & 0 & I - \frac{\Delta t^2}{4} A_{ei}^T A_{ei} \end{pmatrix}, \quad (3.32)$$

and  $\varepsilon_n$ ,  $\Delta t \rho_n$  and  $\delta_n$  are respectively the (space-time) global, local and truncation errors.

Note that the recursion (3.30) has the standard form (see e.g. [5]) for the convergence analysis of one-step integration methods. It transfers local errors to the global error, essentially by adding all local errors. Indeed, for a given time interval  $[0, T]$  we make the usual stability hypothesis

$$\|R^n\| \leq K \quad \text{for } h \rightarrow 0 \text{ and } n \geq 0, \quad n\Delta t \leq T. \quad (3.33)$$

On the other hand the elaboration of the error recursion (3.30) gives

$$\varepsilon_n = R^n \varepsilon_0 + R^{n-1} \Delta t \rho_0 + \dots + R \Delta t \rho_{n-2} + \Delta t \rho_{n-1}, \quad (3.34)$$

which leads directly (with (3.33)) to

$$\|\varepsilon_n\| \leq K \|\varepsilon_0\| + K \Delta t \sum_{j=0}^{n-1} \|\rho_j\| \quad \text{for } n \Delta t \leq T. \quad (3.35)$$

Recall that  $\rho_j = R_L^{-1} \delta_j$  and because we assume stability we may consider  $R_L$  inversely bounded for  $\Delta t \sim h$ ,  $h \rightarrow 0$ . Consequently if  $\delta_j = \mathcal{O}(\Delta t^k)$  we have  $\|\rho_j\| \leq C \Delta t^k$ . Assuming  $\varepsilon_0 = 0$ , we deduce from (3.35) that

$$\|\varepsilon_n\| \leq \tilde{K} \Delta t^k \quad \text{for } n \Delta t \leq T, \quad (3.36)$$

with constant  $\tilde{K} = KTC$ .

#### 3.4.4 Error analysis

We assume that the true PDE solutions  $E_e^h$ ,  $E_i^h$ ,  $H_e^h$  and  $H_i^h$  are sufficiently differentiable. Then may Taylor expand the trapezoidal rule defect (3.27) at the symmetry point  $t_{n+1/2}$  to get

$$\delta_{E_e^h}(t_n) = \sum_{j=2'} \frac{-j}{2^j(j+1)!} (\Delta t)^j E_e^{h(j+1)}, \quad (3.37)$$

where  $j = 2'$  means even values for  $j$  only and  $E_e^{h(j)}$  denotes the  $j$ -th derivative of  $E_e^h(t)$  at time  $t = t_{n+1/2}$ . We obtain similar expressions for  $\delta_{E_i^h}(t_n)$ ,  $\delta_{H_e^h}(t_n)$  and  $\delta_{H_i^h}(t_n)$ . Note that these defects start with  $\Delta t^2$  and the third solution derivative of the true PDE solution (in the first section we have assumed that the true PDE solution is three times differentiable). We conclude that the trapezoidal rule defects are  $\mathcal{O}(\Delta t^2)$  for  $\Delta t \sim h$  and  $h \rightarrow 0$ . Then, from (3.26), we write the truncation error as follows

$$\delta_n = b_n + \mathcal{O}(\Delta t^2), \quad (3.38)$$

where

$$b_n = \begin{pmatrix} b_{e,n}^E \\ b_{i,n}^E \\ b_{e,n}^H \\ b_{i,n}^H \end{pmatrix} = \begin{pmatrix} \frac{\Delta t}{4} S_e \frac{d}{dt} (H_e^h(t_{n+1}) - H_e^h(t_n)) \\ -\frac{\Delta t}{4} A_{ie} \frac{d}{dt} (H_e^h(t_{n+1}) - H_e^h(t_n)) \\ 0 \\ -\frac{\Delta t}{4} A_{ei}^T S_e (H_e^h(t_{n+1}) - H_e^h(t_n)) + \frac{\Delta t}{4} A_{ei}^T \frac{d}{dt} (E_e^h(t_{n+1}) - E_e^h(t_n)) \end{pmatrix}, \quad (3.39)$$

and  $\mathcal{O}(\Delta t^2)$  contains the trapezoidal rule defects. Note that we have voluntarily omitted the spatial error parts contained in the  $\sigma_e^E$ ,  $\sigma_i^E$ ,  $\sigma_e^H$ ,  $\sigma_i^H$  contributions from (3.26) because our interest lies in temporal convergence order<sup>2</sup>.

Next we Taylor expand the components of  $b_n$  at the symmetry point  $t_{n+1/2}$

$$\begin{aligned} b_{e,n}^E &= \frac{\Delta t}{4} S_e \sum_{j=1'} \frac{1}{2^{j-1} j!} (\Delta t)^j H_e^{h(j+1)}, \\ b_{i,n}^E &= -\frac{\Delta t}{4} A_{ie} \sum_{j=1'} \frac{1}{2^{j-1} j!} (\Delta t)^j H_e^{h(j+1)}, \\ b_{i,n}^H &= -\frac{\Delta t}{4} A_{ei}^T S_e \sum_{j=1'} \frac{1}{2^{j-1} j!} (\Delta t)^j H_e^{h(j)} + \frac{\Delta t}{4} A_{ei}^T \sum_{j=1'} \frac{1}{2^{j-1} j!} (\Delta t)^j E_e^{h(j+1)}, \end{aligned} \quad (3.40)$$

where  $j = 1'$  means odd values for  $j$  only. For a fixed spatial dimension we find the expected second-order ODE convergence, since  $S_e$ ,  $A_{ie}$  and  $A_{ei}^T$  are bounded for fixed dimension. Indeed, with (3.40) we

<sup>2</sup>Further carrying the spatial error contributions only complicates the formulas and will not lead to different conclusions for the temporal errors.

have  $b_n = \mathcal{O}(\Delta t^2)$  and because we assume stability we may consider  $R_L$  inversely bounded, consequently  $\rho_n = R_L^{-1} \delta_n = \mathcal{O}(\Delta t^2)$  and we conclude that we have the second-order convergence for a fixed dimension.

Now we observe, with (3.16), that under stable simultaneous space-time refinement,  $\Delta t \sim h$  and  $h \rightarrow 0$ , we might lose one unit of  $\Delta t$  in  $b_{e,n}^E$  and  $b_{i,n}^E$  (due to  $S_e$  and  $A_{ie}$ , respectively) and two units of  $\Delta t$  in  $b_{i,n}^H$  (due to the product  $A_{ei}^T S_e$  in the first terms). Then  $b_n = \mathcal{O}(1)$ , for  $\Delta t \sim h$  and  $h \rightarrow 0$  and with (3.36) we should expect a severe order reduction. However, as mentioned in [7], this result is based on common local error analysis and in the transition from local to global errors it can happen that the order reduction for local errors is (partly) canceled. Often this cancelation can be shown to exist through a transformation of the global error recurrence to one by which we may gain one unit of  $\Delta t$  in the transformed local error.

### 3.4.5 A transformed global error recursion

The transformation used in [7] emanates from [5], Lemma II.2.3. First we assume for  $\Delta t \sim h$ ,  $h \rightarrow 0$  that

$$A_{ei}^T S_e H_e^{h(1)} = \mathcal{O}\left(\frac{1}{\Delta t}\right). \quad (3.41)$$

Consequently we get  $b_n = \mathcal{O}(\Delta t)$  (because in this case  $b_{i,n}^H = \mathcal{O}(\Delta t)$ ). With the above-mentioned Lemma we can assume that if the local error  $\Delta t \rho_n$  allows a decomposition

$$\Delta t \rho_n = (I - R) \xi_n + \eta_n \quad (3.42)$$

such that  $\xi_n = \mathcal{O}(\Delta t^2)$ ,  $\eta_n = \mathcal{O}(\Delta t^3)$  for  $\Delta t \sim h$ ,  $h \rightarrow 0$ , then we have the desired second-order convergence for  $\varepsilon_n$ . So we need to verify (3.42), or equivalently,

$$\Delta t \delta_n = (R_L - R_R) \xi_n + R_L \eta_n, \quad (3.43)$$

such that  $\xi_n = \mathcal{O}(\Delta t^2)$ ,  $\eta_n = \mathcal{O}(\Delta t^3)$  for  $\Delta t \sim h$ ,  $h \rightarrow 0$ .

First we deal with the condition  $\eta_n = \mathcal{O}(\Delta t^3)$ . Recall that  $\delta_n = b_n + \mathcal{O}(\Delta t^2)$ , then (3.43) can be written as

$$\Delta t (b_n + \mathcal{O}(\Delta t^2)) = (R_L - R_R) \xi_n + R_L \eta_n. \quad (3.44)$$

Furthermore,  $R_L$  is inversely bounded, then by assigning the  $\mathcal{O}(\Delta t^2)$  terms present in (3.44) multiplied by  $\Delta t R_L^{-1}$  for  $\eta_n$  (i.e.  $\eta_n = \Delta t R_L^{-1} \mathcal{O}(\Delta t^2)$ ), we have  $\eta_n = \mathcal{O}(\Delta t^3)$ .

Consequently, we deduce from the Lemma that we have the desired second-order convergence if a vector  $\xi_n$  exists such that  $\xi_n = \mathcal{O}(\Delta t^2)$  for  $\Delta t \sim h$ ,  $h \rightarrow 0$  and

$$\Delta t b_n = (R_L - R_R) \xi_n, \quad \text{i.e.} \quad (3.45)$$

$$\begin{pmatrix} D_e & 0 & -S_e & A_{ei} \\ 0 & D_i & A_{ie} & -S_i \\ S_e^T & -A_{ie}^T & 0 & 0 \\ -A_{ei}^T & S_i^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_{e,n}^E \\ \xi_{i,n}^E \\ \xi_{e,n}^H \\ \xi_{i,n}^H \end{pmatrix} = \begin{pmatrix} b_{e,n}^E \\ b_{i,n}^E \\ b_{e,n}^H \\ b_{i,n}^H \end{pmatrix}. \quad (3.46)$$

Equivalently, we have second-order convergence if a vector  $\xi_n = [(\xi_n^E)^T, (\xi_n^H)^T]^T$  exists such that  $\xi_n = \mathcal{O}(\Delta t^2)$  for  $\Delta t \sim h$ ,  $h \rightarrow 0$  and

$$\begin{aligned} D \xi_n^E - S \xi_n^H &= b_n^E, \\ S^T \xi_n^E &= b_n^H, \end{aligned} \quad (3.47)$$

where  $\xi_n^E = [(\xi_{e,n}^E)^T, (\xi_{i,n}^E)^T]^T$ ,  $\xi_n^H = [(\xi_{e,n}^H)^T, (\xi_{i,n}^H)^T]^T$ ,  $b_n^E = [(b_{e,n}^E)^T, (b_{i,n}^E)^T]^T$ ,  $b_n^H = [(b_{e,n}^H)^T, (b_{i,n}^H)^T]^T$ .

Now we will check the existence of a such vector  $\xi_n$ . At this stage of the derivation we must be careful because the matrix  $S$  is not necessarily a square matrix (for example in 2D this is indeed not the case) and consequently  $S$  may no be invertible. More precisely, if we let  $n_{\text{dof}}$  be the number of degrees



of freedom, and if we consider the two-dimensional TM (transversal magnetic) model, then the size of the matrix  $S$  is  $n_{dof} \times 2n_{dof}$ .

Therefore we now use the notion of pseudo inverse. Let  $A \in \mathcal{R}^{m \times n}$ ,  $b \in \mathcal{R}^m$ ,  $x \in \mathcal{R}^n$  and  $A^+$  be the Moore-Penrose pseudo inverse of  $A$  which is a generalization of the inverse and exists for any  $m \times n$  matrix. If  $A$  has full rank, then

$$\begin{aligned} A^+ &= A^{-1} & (m = n) \\ A^+ &= A^T(AA^T)^{-1} & (m < n), \\ A^+ &= (A^T A)^{-1}A^T & (m > n), \end{aligned} \quad (3.48)$$

and the solution of  $Ax = b$  is  $x = A^+b$ .

Assume the size of the matrix  $S$  is  $m \times n$  with  $m \leq n$  (the case  $m > n$  can be treated similarly). From the second equation of (3.47) we derive

$$\xi_n^E = (S^T)^+ b_n^H. \quad (3.49)$$

With (2.10) we have

$$(S^T)^+ = (SS^T)^{-1}S \sim h, \quad \text{for } h \rightarrow 0, \quad (3.50)$$

and recall that  $b_{e,n}^H = 0$  and  $b_{i,n}^H = \mathcal{O}(\Delta t)$  (due to the assumption) for  $\Delta t \sim h$ ,  $h \rightarrow 0$ . We then conclude with (3.49) and (3.50) that for  $\Delta t \sim h$ ,  $h \rightarrow 0$

$$\xi_n^E = \mathcal{O}(\Delta t^2). \quad (3.51)$$

From the first equation of (3.47) we get

$$\xi_n^H = -S^+(b_n^E - D\xi_n^E) \quad (3.52)$$

and with (2.10) we get

$$S^+ = S^T(SS^T)^{-1} \sim h, \quad \text{for } h \rightarrow 0. \quad (3.53)$$

Recalling that  $b_{e,n}^E = \mathcal{O}(\Delta t)$ ,  $b_{i,n}^E = \mathcal{O}(\Delta t)$  and  $\xi_n^E = \mathcal{O}(\Delta t^2)$  for  $\Delta t \sim h$ ,  $h \rightarrow 0$ , we conclude from (3.52) and (3.53) that for  $\Delta t \sim h$ ,  $h \rightarrow 0$

$$\begin{aligned} &\text{if } D = 0 \text{ (no conduction term } -\sigma E) \text{ or } D = \mathcal{O}(h^k) \text{ with } k \geq -1, \\ &\text{then } \xi_n^H = \mathcal{O}(\Delta t^2). \end{aligned} \quad (3.54)$$

With (3.51) and (3.54) we have  $\xi_n = [(\xi_n^E)^T, (\xi_n^H)^T]^T = \mathcal{O}(\Delta t^2)$  for  $\Delta t \sim h$ ,  $h \rightarrow 0$  and consequently through the Lemma we have the second-order convergence uniformly in  $h$  under the assumption (3.41). Note that if we do not assume (3.41) (i.e.  $b_n = \mathcal{O}(1)$ ) a similar proof, based on the same Lemma, only guarantees the first-order convergence.

To sum up, we can guarantee at least the first-order convergence of method (3.8). As might be feared, component splitting can be detrimental for the temporal convergence order (order reduction). However, we have put forward a sufficient condition (3.41) on the the true solution of the PDE problem<sup>3</sup> for second-order convergence.

## 4 Numerical results

We solve the two-dimensional (2D) Transverse Magnetic (TM) model for the components  $E^z(x, y, t)$ ,  $H^x(x, y, t)$  and  $H^y(x, y, t)$ :

$$\begin{cases} \mu \frac{\partial H^x}{\partial t} &= -\frac{\partial E^z}{\partial y}, \\ \mu \frac{\partial H^y}{\partial t} &= \frac{\partial E^z}{\partial x}, \\ \epsilon \frac{\partial E^z}{\partial t} &= \frac{\partial H^y}{\partial x} - \frac{\partial H^x}{\partial y} - \sigma E^z - J_E^z. \end{cases} \quad (4.1)$$

<sup>3</sup>It would have been better if this sufficient condition would have been controllable through the source term, because in general the true solution is of course not (a-priori) known.

In the following, we set  $\epsilon = \mu = 1$  and  $\sigma = 0$ . Equations (4.1) are space discretized using a DG method formulated on quadrangular meshes. In the preliminary implementation of this DG method, the approximation of the electromagnetic field components within a quadrangle  $\tau_i$  relies on a nodal  $\mathbb{Q}_2$  interpolation method<sup>4</sup>. For the numerical experiments the integration interval in time is  $[0, 1]$  and we make use of two types of meshes : uniform and non-uniform (see Figure 1). A quadrangle  $\tau_i$  is characterized by the discretization parameters  $(\Delta x_i, \Delta y_i)$ . We will present a number of results obtained for the hybrid implicit-explicit time stepping with a DG discretization method (3.8). The step size used in the tests is given by

$$\Delta t = \frac{CFL}{\sqrt{\frac{1}{\Delta x_k^2} + \frac{1}{\Delta y_k^2}}}, \quad (4.2)$$

where the parameters  $(\Delta x_k, \Delta y_k)$  characterize the smallest quadrangle  $\tau_k$  treated explicitly. For our integration method and the chosen spatial DG scheme the  $CFL$  number is equal to 0.24.

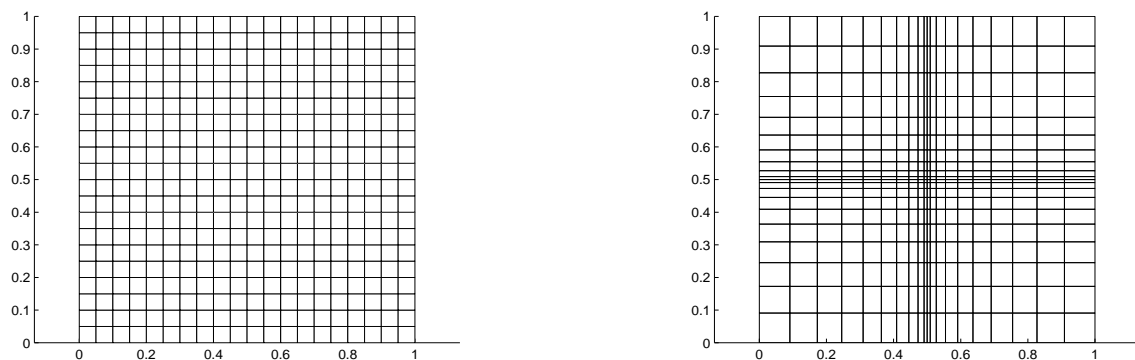


Figure 1: Uniform and non-uniform meshes.

## 4.1 Propagation of an eigenmode in a unitary PEC cavity

We recall that the exact solution of this problem is given by<sup>5</sup>:

$$\begin{cases} H^x(x, y, t) &= -\frac{k}{\sqrt{k^2 + l^2}} \sin(l\pi x) \cos(k\pi y) \sin(\sqrt{k^2 + l^2}\pi t), \\ H^y(x, y, t) &= \frac{l}{\sqrt{k^2 + l^2}} \cos(l\pi x) \sin(k\pi y) \sin(\sqrt{k^2 + l^2}\pi t), \\ E^z(x, y, t) &= \sin(l\pi x) \sin(k\pi y) \cos(\sqrt{k^2 + l^2}\pi t). \end{cases} \quad (4.3)$$

### 4.1.1 Convergence in the ODE sense

To estimate the temporal convergence in the ODE sense we fix the discretization in space and we measure the electromagnetic field for different time steps ( $W_{\Delta t}$ ) at the final time ( $T$ ). We plot the sup-norm of  $W_{\Delta t/n}(T) - W_{\Delta t/2n}(T)$  as a function of  $\Delta t/n$  (in logarithmic scale). The use of the logarithmic scale allows to visualize the rate of convergence as the slope of the curve. The results obtained are given in Table 1 and Figure 2 below. We observe that the results are in agreement with the expected behavior.

<sup>4</sup>The a-priori convergence analysis for this DGTD method based on a centered numerical flux and formulated on simplicial meshes shows that the convergence rate is  $O(h^p)$  for a  $p$ -th interpolation order. We therefore choose  $p = 2$ , in order that the spatial error will not be detrimental for the temporal convergence order in the PDE sense.

<sup>5</sup>For numerical tests we put  $k = l = 1$ .

$n$	$\Delta t/n \rightarrow \Delta t/2n$	$\ W_{\Delta t/n} - W_{\Delta t/2n}\ _\infty$
1	0.0169 $\rightarrow$ 0.0085	0.0085
2	0.0085 $\rightarrow$ 0.0042	0.0019
4	0.0042 $\rightarrow$ 0.0021	4.8669e-4
8	0.0021 $\rightarrow$ 0.0011	1.2186e-4
Conv. rate		2.0249

$n$	$\Delta t/n \rightarrow \Delta t/2n$	$\ W_{\Delta t/n} - W_{\Delta t/2n}\ _\infty$
1	0.0076 $\rightarrow$ 0.0038	0.0017
2	0.0038 $\rightarrow$ 0.0019	4.3224e-4
4	0.0019 $\rightarrow$ 0.0009	1.0728e-4
8	0.0009 $\rightarrow$ 0.0005	2.6802e-5
Conv. rate		1.9952

Table 1: Temporal convergence (in the sense) of the hybrid implicit-explicit time stepping method (3.8) based on the DGTD- $\mathbb{Q}_2$  method.

Uniform meshes (left) / non-uniform meshes (right).

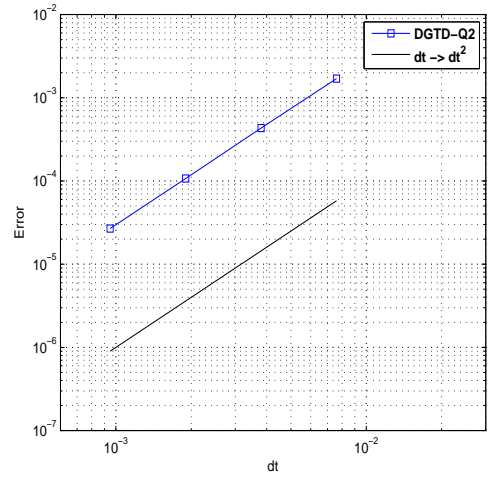
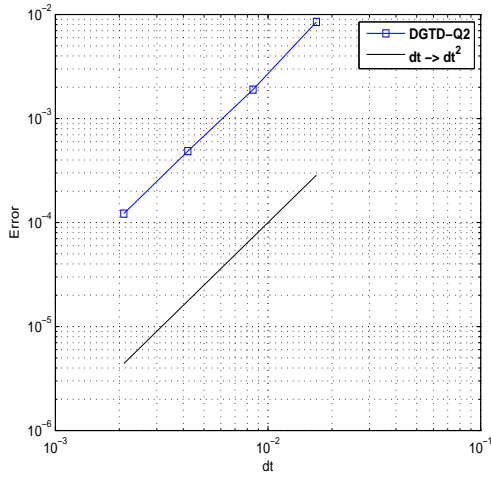


Figure 2: Temporal convergence (in the ODE sense) of the hybrid implicit-explicit time stepping method (3.8) based on the DGTD- $\mathbb{Q}_2$  method.

Uniform meshes (left) / non-uniform meshes (right).

#### 4.1.2 Convergence in the PDE sense

To estimate the temporal convergence in the PDE sense we measure the maximal  $L^2$ -norm of the error for different meshes of increased resolution. We plot this error as a function of the square root of the number of degrees of freedom (in logarithmic scale). The obtained results are given in Table 2 and Figure 3. We also conducted numerical investigations to check whether the condition (3.41) is satisfied. The following results have been obtained, for  $\Delta t \sim h$ ,  $h \rightarrow 0$

$$\begin{aligned}
 A_{et}^T S_e H_e^{h(1)} &= \mathcal{O}\left(\frac{1}{(\Delta t)^{1.3}}\right), & \text{in the uniform mesh case,} \\
 A_{et}^T S_e H_e^{h(1)} &= \mathcal{O}\left(\frac{1}{(\Delta t)^{0.7}}\right), & \text{in the non-uniform mesh case.}
 \end{aligned} \tag{4.4}$$

In the uniform mesh case the condition (3.41) is not satisfied and the obtained results are in agreement with the theoretical behavior previously presented, i.e. first-order convergence. In the non-uniform mesh case (3.41) is satisfied and the behavior is even better, which could explain the *super-convergence* phenomenon (convergence order about 2.8 instead of 2).

# DOF	Max. error $L^2$
900	0.0012
3600	5.4236e-4
8100	3.5759e-4
14400	2.7164e-4
Conv. rate	1.0762

# DOF	Max. error $L^2$
900	8.0792e-4
3600	1.1770e-4
8100	3.6686e-5
14400	1.5898e-5
Conv. rate	2.8313

Table 2: Convergence (in the PDE sense) of the hybrid implicit-explicit time stepping method based on DGTD- $\mathbb{Q}_2$  method.

Uniform meshes (left) / non-uniform meshes (right).

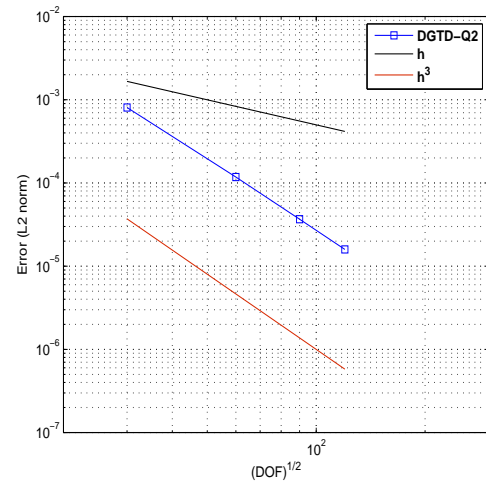
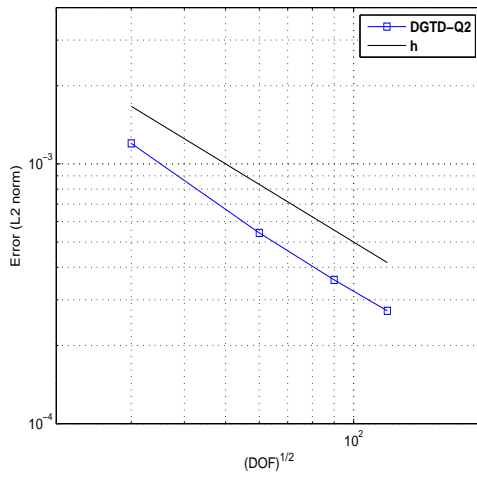


Figure 3: Convergence (in the PDE sense) of the hybrid implicit-explicit time stepping method (3.8) based on the DGTD- $\mathbb{Q}_2$  method.

Uniform meshes (left) / non-uniform meshes (right).

## 4.2 Test problem with a localized source

We solve (4.1) with a source term  $J_E^z$  defined by

$$J_E^z(x, y, t) = -e^t[x(1-x)y(1-y) + 2x(1-x) + 2y(1-y)]. \quad (4.5)$$

The exact solution of this problem is given by

$$\begin{cases} E^z(x, y, t) &= e^t x(1-x)y(1-y), \\ H^x(x, y, t) &= -e^t x(1-x)(1-2y), \\ H^y(x, y, t) &= e^t y(1-y)(1-2x). \end{cases} \quad (4.6)$$

### 4.2.1 Convergence in the ODE sense

The obtained results are given in Table 3 and Figure 4 below. We observe that the results are in agreement with the expected behavior.

$n$	$dt/n \rightarrow dt/2n$	$\ W_{dt/n} - W_{dt/2n}\ _\infty$
1	0.0169 $\rightarrow$ 0.0085	0.0049
2	0.0085 $\rightarrow$ 0.0042	0.0012
4	0.0042 $\rightarrow$ 0.0021	2.9839e-4
8	0.0021 $\rightarrow$ 0.0011	7.4222e-5
Conv. rate		2.0056

$n$	$dt/n \rightarrow dt/2n$	$\ W_{dt/n} - W_{dt/2n}\ _\infty$
1	0.0076 $\rightarrow$ 0.0038	0.0010
2	0.0038 $\rightarrow$ 0.0019	2.5045e-4
4	0.0019 $\rightarrow$ 0.0009	6.2302e-5
8	0.0009 $\rightarrow$ 0.0005	1.5577e-5
Conv. rate		2.0001

Table 3: Temporal convergence (in the ODE sense) of the hybrid implicit-explicit time stepping methods based on the DGTD- $\mathbb{Q}_2$  method.

Uniform meshes (left) / non-uniform meshes (right).

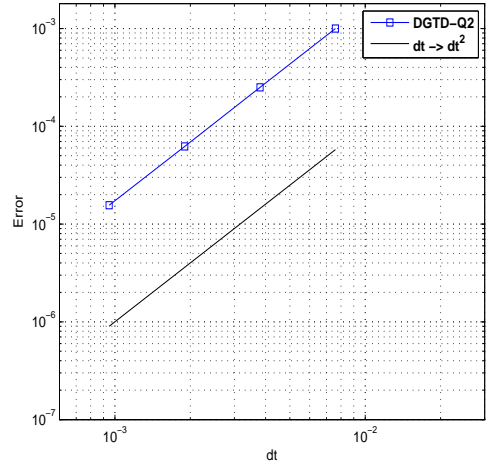
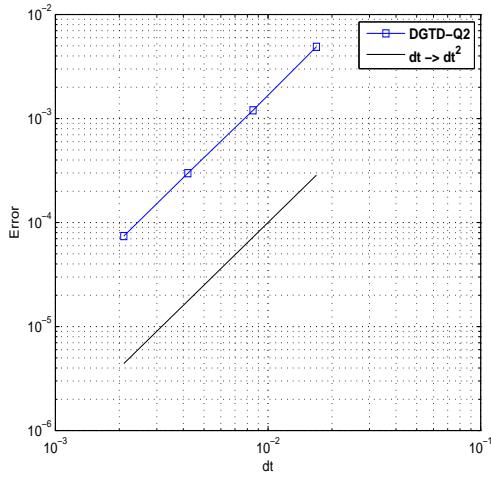


Figure 4: Temporal convergence (in the ODE sense) of the hybrid implicit-explicit time stepping method (3.8) based on the DGTD- $\mathbb{Q}_2$  method.

Uniform meshes (left) / non-uniform meshes (right).

#### 4.2.2 Convergence in the PDE sense

The obtained results are given in Table 4 and Figure 5 below. The results are similar to those obtained with the previous test problem, i.e. a first-order convergence for uniform meshes and a *super-convergence* phenomenon for non-uniform meshes. Furthermore numerical investigations conducted to check whether the condition (3.41) is satisfied give the same behaviors (4.4) than for the previous test problem. Consequently the results given in Table 4 and Figure 5 are also in agreement with theory.

# DOF	Max. error $L^2$
900	5.0424e-4
3600	2.2409e-4
8100	1.4469e-4
14400	1.0728e-4
Conv. rate	1.1191

# DOF	Max. error $L^2$
900	9.9807e-5
3600	1.4338e-5
8100	4.4703e-6
14400	2.1187e-6
Conv. rate	2.7924

Table 4: Convergence (in the PDE sense) of the hybrid implicit-explicit time stepping method based on the DGTD- $\mathbb{Q}_2$  method.

Uniform meshes (left) / non-uniform meshes (right).

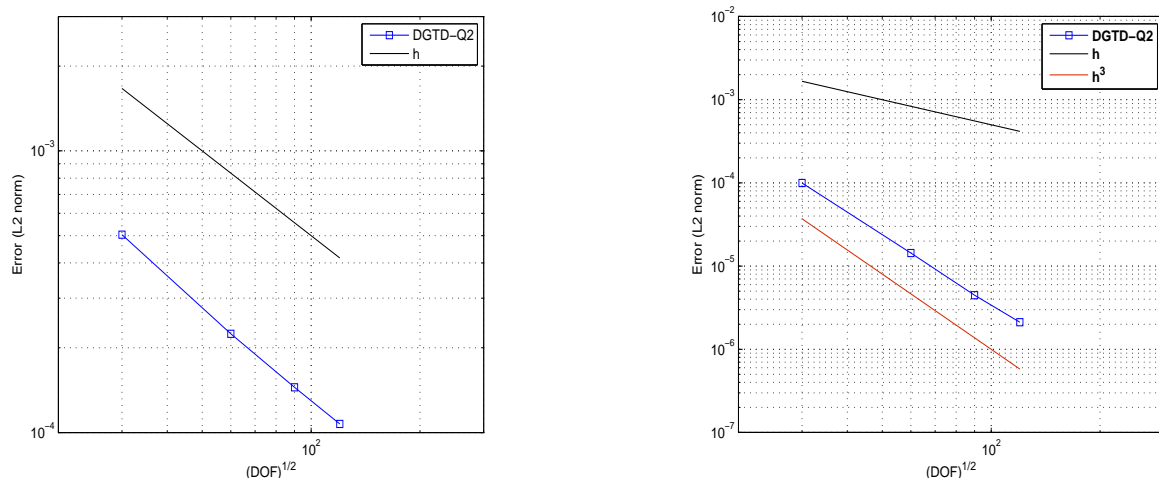


Figure 5: Convergence (in PDE sense) of the hybrid implicit-explicit time stepping method (3.8) based on DGTD- $\mathbb{Q}_2$  method.

Uniform meshes (left) / non-uniform meshes (right).

## 5 Conclusion

We have seen that for method (3.8) component splitting could be detrimental for the temporal convergence uniformly in  $h$ . We have presented the sufficient condition (3.41) on the true PDE solution to recover the second-order convergence. Thus we can only guarantee first-order convergence. In Section 4 numerical investigations in 2D have been conducted to illustrate this last theoretical result. Note that the purpose of these tests was not to illustrate the efficiency of the method, but the temporal convergence order of (3.8) (in ODE and PDE senses).

In this paper we have mentioned another component splitting method from [7] which cover the common spatial discretizations like finite-difference and various finite-element discretizations. We plan to compare this method with the implicit-explicit integration method from [3, 6] for discontinuous Galerkin discretization on unstructured meshes. The purpose of planned research is to illustrate the practical virtue of both the component splitting methods using locally refined space grids, and to highlight the most efficient one which then can be advocated for future use.

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