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On confined McKean Langevin processes satisfying the mean no-permeability boundary condition

Mireille Bossy* Jean-François Jabir†

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Abstract

We construct a confined Langevin type process aimed to satisfy a mean no-permeability condition at the boundary. This Langevin process lies in the class of conditional McKean Lagrangian stochastic models studied in [5]. The confined process considered here is a first construction of solutions to the class of Lagrangian stochastic equations with boundary condition issued by the so-called PDF methods for Computational Fluid Dynamics. We prove the well-posedness of the confined system when the state space of the Langevin process is a half-space.

Keywords McKean Langevin equation, Lagrangian stochastic model, mean no-permeability condition, specular boundary condition.

1 Introduction

We consider the nonlinear Langevin equation (1.1), describing the time evolution of the position and velocity (X, U) of a particle with the position process X confined in the upper-half plane $\mathbb{R}^{d-1} \times [0, +\infty)$,

$$\begin{cases} X_t = X_0 + \int_0^t U_s ds, \\ U_t = U_0 + \int_0^t B\left[X_s, U_s; \rho_s\right] ds + W_t + K_t, \\ K_t = -\sum_{0<s\leq t} 2\left(U_{s^-} \cdot n_{\mathcal{D}}\right) n_{\mathcal{D}} \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}, \\ \rho_t \text{ is the probability density of } (X_t, U_t), \text{ for all } t \in (0, T]. \end{cases} \tag{1.1}$$

Here $\mathcal{D} = \mathbb{R}^{d-1} \times (0, +\infty)$, $\partial\mathcal{D} = \mathbb{R}^{d-1} \times \{0\}$ and $n_{\mathcal{D}} = (0, \ldots, 0, -1)$ is the outward normal unit vector.

In this paper, we prove the well-posedness of the equation (1.1) on a finite time interval $[0, T]$. Moreover, we prove that the confined solution (X, U) satisfies a mean version of the so-called no-permeability condition, that we write formally in this introduction as

$$\mathbb{E}\left[(U_t \cdot n_{\mathcal{D}}) / X_t = x\right] = 0, \text{ for a.e. } (t, x) \in (0, T] \times \partial\mathcal{D}. \tag{1.2}$$

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More precisely, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a d-dimensional standard Brownian motion W and a $\mathcal{D} \times \mathbb{R}^d$-valued random variable (X_0, U_0) independent of W. We prove that there exists a unique solution, (X, U) valued in $\mathcal{C}([0,T]; \overline{\mathcal{D}}) \times \mathbb{D}([0,T]; \mathbb{R}^d)$, to the nonlinear McKean equation (1.1). In particular, the related sequence of hitting times

$$\tau_n = \inf\{\tau_{n-1} < t \leq T \text{ s.t. } X_t \in \partial\mathcal{D}\}, \text{ for } n \geq 1, \ \tau_0 = 0, \tag{1.3}$$

is well-defined and the sum $\sum_{0<s\leq t} \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}$ acts over a countable set of times. Then the confining term K is a càdlàg process reflecting the velocity of the outgoing particles. The drift coefficient B is the mapping from $\mathcal{D} \times \mathbb{R}^d \times L^1(\mathcal{D} \times \mathbb{R}^d)$ to $\mathbb{R}^d$ defined by

$$B[x, u; \gamma] = \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} b(v,u)\gamma(x,v)\,dv}{\displaystyle\int_{\mathbb{R}^d} \gamma(x,v)\,dv} & \text{if } \displaystyle\int_{\mathbb{R}^d} \gamma(x,v)\,dv \neq 0, \\ 0 \text{ elsewhere,} \end{cases} \tag{1.4}$$

where $b : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a given bounded and continuous interaction kernel. As noticed in Bossy, Jabir and Talay [5], formally the drift coefficient $B[x, u; \rho_t]$ in (1.1) is the function

$$(t, x, u) \mapsto \mathbb{E}\left[b(U_t, u)/X_t = x\right],$$

and the system (1.1) involves a conditional expectation in its drift term.

This present work is a first step in the analysis of the Lagrangian stochastic models involving a prescribed behaviour of the velocity when the particle hits the boundary $\partial\mathcal{D}$. The boundary condition (1.2) provides a representative example in the class of boundary conditions related to the Probability Density Function (PDF) methods proposed by S.B. Pope for Computational Fluid Dynamics applications (see Pope [15] and references therein). The PDF methods have been developed for the simulation of turbulent flows. They are based on the Lagrangian stochastic modelling of the fluid motion through a system of Stochastic Differential Equations (SDEs) which describe the dynamics of a generic fluid-particle. These SDEs are nonlinear in the sense of McKean and involve singular interaction kernels. We refer to [5] for a short survey on mathematical issues related to the Lagrangian stochastic models and the well-posedness of (1.1) when $\mathcal{D} = \mathbb{R}^d$ (i.e. $K = 0$).

For the Lagrangian modelling of near-wall turbulent flows, Dreeben and Pope [7] suggested a reflection procedure of the velocities at the boundaries according to a prescribed boundary layer model. Here we formalize those ideas and construct a first example of confined Lagrangian models satisfying (1.2).

The paper is organized as follows. In Section 2, we state our main results on the well-posedness of (1.1). We further show that the trace at the boundary $\gamma(\rho)$ of the time-marginal densities $(\rho_t; t \in (0,T])$ of the solution of (1.1) satisfies the so-called specular boundary condition (see (2.5) below). This implies the mean no-permeability condition (1.2). The rest of the paper is devoted to the proofs. As a preliminary step, in Section 3, we construct a confined version of the Brownian motion and its primitive (i.e. $B = 0$). In Section 4, we combine arguments from Sznitman [17], on the well-posedness of McKean nonlinear SDEs with reflection of Skorokhod type, with the approach used in [5] to the proof of the well-posedness of (1.1). Section 5 is devoted to the study of the trace $\gamma(\rho)$.

Notation

- For any point $x \in \mathbb{R}^d$, we write $x = (x', x^{(d)})$ where x' denotes the $(d-1)^{\text{th}}$ first coordinates and $x^{(d)}$ denotes the dth coordinate. The surface measure $d\sigma_{\mathcal{D}}$ related to $\partial\mathcal{D}$ is $d\sigma_{\mathcal{D}} = dx'$.

- For all $t > 0$, $Q_t = (0,t) \times \mathcal{D} \times \mathbb{R}^d$, $\Sigma_t = (0,t) \times \partial\mathcal{D} \times \mathbb{R}^d$.
- We denote by $\mathbb{D}([0,T];\mathbb{R}^q)$ the space of all $\mathbb{R}^q$-valued càdlàg functions equipped with the Skorokhod topology. For $z \in \mathbb{D}([0,T];\mathbb{R}^q)$, $\triangle z_t$ and z_{t^-} denote, respectively the jump and the left-hand limit of z at time t.
- For any metric space E, we denote by $\mathcal{M}(E)$ the set of probability measures equipped with the weak topology.
- For any point $(t,z,\nu) \in (0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, $g_d(t;z,\nu;\zeta,\mu)$ denotes the transition density of the d-dimensional Brownian motion's primitive and its Brownian motion $(z + \nu t + \int_0^t W_s ds, \nu + W_t)$, starting at (z,ν). A straightforward computation leads to the explicit expression:

$$g_d(t;z,\nu;\zeta,\mu) = \left(\frac{\sqrt{3}}{\pi t^2}\right)^d \exp\left(-\frac{6|\zeta - z - t\nu|^2}{t^3} + \frac{6((\zeta - z - t\nu)\cdot(\mu-\nu))}{t^2} - \frac{2|\mu-\nu|^2}{t}\right). \tag{1.5}$$

2 Main results

From now on, we assume that the distribution μ_0 of (X_0, U_0) and the kernel b in (1.1) satisfy the following hypotheses (H).

$(H\text{-}i)$ The initial measure μ_0 has its support in $\mathcal{D}\times\mathbb{R}^d$ and $\int_{\mathcal{D}\times\mathbb{R}^d} \left(|x| + |u|^2\right) \mu_0(dx,du) < +\infty$.

$(H\text{-}ii)$ $b : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is uniformly bounded and continuous.

2.1 On the well-posedness of (1.1)

Let $\mathcal{E}$ be the sample space

$$\mathcal{E} := \mathcal{C}([0,T];\overline{\mathcal{D}}) \times \mathbb{D}([0,T];\mathbb{R}^d) \times \mathbb{D}([0,T];\mathbb{R}^d).$$

We equip $\mathcal{E}$ with the Skorokhod topology, so that $\mathcal{E}$ is a closed subset of $\mathbb{D}([0,T];\mathbb{R}^{3d})$ and further it is a Polish space (see Jacod and Shiryaev [10]). We denote by $(x_t, u_t, k_t; t \in [0,T])$ the canonical process on $\mathcal{E}$, and by $(\mathcal{B}_t; t \in [0,T])$ the associated canonical filtration. The martingale problem related to (1.1) is then formulated as follows.

Definition 2.1. *A probability measure $\mathbb{P} \in \mathcal{M}(\mathcal{E})$ is a solution to the martingale problem (MP) if the following hold.*

(i) $\mathbb{P} \circ (x_0, u_0, k_0)^{-1} = \mu_0 \otimes \delta_0$, where δ_0 denotes the Dirac mass at the origin on $\mathbb{R}^d$.

(ii) For all $t \in (0,T]$, $\mathbb{P} \circ (x_t, u_t)^{-1}$ admits a positive Lebesgue density ρ_t.

(iii) For all $f \in \mathcal{C}_b^2(\mathbb{R}^{2d})$, the process

$$\begin{aligned} &f(x_t, u_t - k_t) - f(x_0, u_0) - \int_0^t (u_s \cdot \nabla_x f(x_s, u_s - k_s))\, ds \\ &\quad - \int_0^t \left[(B[x_s, u_s; \rho_s] \cdot \nabla_u f(x_s, u_s - k_s)) + \frac{1}{2}\triangle_u f(x_s, u_s - k_s) \right] ds \end{aligned} \tag{2.1}$$

is a continuous $\mathbb{P}$-martingale w.r.t. $(\mathcal{B}_t; t \in [0,T])$.

(iv) $\mathbb{P}$*-a.s., the set* $\{t \in [0,T] \text{ s.t. } x_t^{(d)} = 0\}$ *is at most countable, and*

$$k_t = -2 \sum_{0<s\leq t} (u_{s^-} \cdot n_{\mathcal{D}}) n_{\mathcal{D}} \mathbb{1}_{\{x_s \in \partial\mathcal{D}\}}, \ \forall\, t \in [0,T].$$

Theorem 2.2. *Assume* (H)*. Then there exists a unique solution to the martingale problem (MP).*

Under the solution $\mathbb{P}$ of (MP), one may observe that the canonical process $(x_t, u_t, k_t; t \in [0,T])$ satisfies (1.1).

Section 4 is devoted to the proof of Theorem 2.2. As in [5], the existence of a solution is based on a classical particle approximation method of a smoothed problem, here in the framework of the Skorokhod topology.

2.2 On the mean no-permeability condition

First, we prove the regularity of the time-marginal densities $(\rho_t; t \in (0,T])$ of the solution to (MP), and the existence of the trace $\gamma(\rho)$ at the boundary Σ_T. We give some related integrability properties.

Theorem 2.3. *Assume* (H)*. Let* $\mathbb{P}$ *be the solution to (MP).*

(a) The time-marginal densities (ρ_t) *of* $\mathbb{P}$ *are Hölder-continuous: for* $x, x_0 \in \overline{\mathcal{D}}$*, there exist some positive constants* β*,* α*,* α_1 *and* C *such that for a.e.* $0 < t_0 < t \leq T$*,* $u \in \mathbb{R}^d$*,*

$$|\rho_t(x,u) - \rho_t(x_0,u)| \leq C t_0^{-\alpha} (1 \vee (t-t_0)^{\alpha_1} |x - x_0|^{\beta}.$$

(b) We have $\sum_{n\in\mathbb{N}} \mathbb{P}(\tau_n \leq T) < +\infty$ *and the trace function* $\gamma(\rho)$ *defined by*

$$\gamma(\rho)(t,x,u) := \lim_{\delta\to 0^+} \rho_t((x',\delta),u), \text{ for all } x = (x',0) \in \partial\mathcal{D}, \ \text{ for a.e. } (t,u) \in (0,T) \times \mathbb{R}^d, \tag{2.2}$$

satisfies, for all bounded measurable functions f *on* Σ_T*,*

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}} \Big(f(\tau_n, x_{\tau_n}, u_{\tau_n}) - f(\tau_n, x_{\tau_n}, u_{\tau_n^-}) \Big) \mathbb{1}_{\{\tau_n \leq T\}}\right] \\ &= -\int_{\Sigma_T} (u \cdot n_{\mathcal{D}})\, \gamma(\rho)(s,(x',0),u) f(s,(x',0),u) ds\, dx' du. \end{aligned} \tag{2.3}$$

(c) The following properties hold for the trace function $\gamma(\rho)$*:*

$$\int_{\mathbb{R}^d} |(u \cdot n_{\mathcal{D}})| \gamma(\rho)(t,x,u)\, du < +\infty, \ dt \otimes d\sigma_{\mathcal{D}}\text{-a.e. on } (0,T) \times \partial\mathcal{D}, \tag{2.4a}$$

$$\int_{\mathbb{R}^d} \gamma(\rho)(t,x,u)\, du > 0, \ dt \otimes d\sigma_{\mathcal{D}}\text{-a.e. on } (0,T) \times \partial\mathcal{D}. \tag{2.4b}$$

In view of Theorem 2.3, the conditional expectation of the normal velocity component at the boundary can be explicited as: for $dt \otimes d\sigma_{\mathcal{D}}$-a.e (t,x) in $(0,T) \times \partial\mathcal{D}$,

$$\mathbb{E}_{\mathbb{P}}\left[(U_t \cdot n_{\mathcal{D}})/X_t = x\right] = \frac{\displaystyle\int_{\mathbb{R}^d} (u \cdot n_{\mathcal{D}}) \gamma(\rho)(t,x,u) du}{\displaystyle\int_{\mathbb{R}^d} \gamma(\rho)(t,x,u) du}.$$

We show that $\gamma(\rho)$ satisfies the so-called specular boundary condition arising in the kinetic theory of gases (see, *e.g.*, Cercignani [6]). The boundary condition (1.2) is then established with the following.

Corollary 2.4. *The trace function $\gamma(\rho)$ defined in (2.2) satisfies the specular boundary condition:*

$$\gamma(\rho)(t,x,u) = \gamma(\rho)(t,x,u-2(u\cdot n_{\mathcal{D}})n_{\mathcal{D}}),\ dt\otimes d\sigma_{\mathcal{D}}\otimes du\text{-a.e. on } \Sigma_T. \tag{2.5}$$

Moreover, for a.e. (t,x) in $(0,T)\times\partial\mathcal{D}$,

$$\int_{\mathbb{R}^d}(u\cdot n_{\mathcal{D}})\gamma(\rho)(t,x,u)du = 0, \tag{2.6}$$

and the mean no-permeability condition (1.2) is fulfilled.

Proof. In (2.3), choosing f of the form

$$f(s,x,u) = \phi(s,x,u) + \phi(s,x,u-2(u\cdot n_{\mathcal{D}})n_{\mathcal{D}}),$$

we observe that $f(\tau_n, x_{\tau_n}, u_{\tau_n}) = f(\tau_n, x_{\tau_n}, u_{\tau_n^-})$, $\mathbb{P}$-a.s and

$$\int_{\Sigma_T}(u\cdot n_{\mathcal{D}})\left(\phi(s,x,u)+\phi(s,x,u-2(u\cdot n_{\mathcal{D}}))\right)\gamma(\rho)(s,x,u)\,ds\,d\sigma_{\mathcal{D}}(x)du = 0.$$

Define $\psi(x;\cdot):\mathbb{R}^d\longrightarrow\mathbb{R}^d$ as

$$\psi(x;u) = u-2(u\cdot n_{\mathcal{D}})n_{\mathcal{D}}.$$

Since, for all $x\in\partial\mathcal{D}$, $\psi(x;\cdot)$ is a continuously differentiable involution on $\mathbb{R}^d$ and since its Jacobian determinant is equal to -1, the change of variable $u\mapsto\psi(x;u)$ gives

$$\int_{\Sigma_T}(u\cdot n_{\mathcal{D}})\left(\gamma(\rho)(s,x,u)-\gamma(\rho)(s,x,u-2(u\cdot n_{\mathcal{D}}))\right)\phi(s,x,u)\,ds\,d\sigma_{\mathcal{D}}(x)du = 0,$$

for any bounded measurable function ϕ on Σ_T, from which we obtain (2.5). Moreover we have

$$\begin{aligned}\int_{\mathbb{R}^d}\Big(u\cdot n_{\mathcal{D}}\Big)\gamma(\rho)(t,x,u)\,du &= \int_{\mathbb{R}^d}\Big(u\cdot n_{\mathcal{D}}\Big)\gamma(\rho)(t,x,\psi(x;u))du\\ &= \int_{\mathbb{R}^d}\Big(\psi(x;u)\cdot n_{\mathcal{D}}\Big)\gamma(\rho)(t,x,u)du,\end{aligned}$$

and (2.6) follows by noticing that $(\psi(x;u)\cdot n_{\mathcal{D}}) = -(u\cdot n_{\mathcal{D}})$. □

Theorem 2.3 is proved in Section 5.

3 Preliminaries: the confined Langevin process

Throughout this section, we assume (H-i). We consider the confined equation (1.1) in the case where $b=0$, namely

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t U_s ds,\\ U_t = U_0 + W_t + \displaystyle\sum_{0<s\le t} 2U^{(d)}_{s^-}n_{\mathcal{D}}\mathbb{1}_{\{X^{(d)}_s=0\}},\ \forall\ t\ge 0,\end{cases} \tag{3.1}$$

with (X',U') denoting the $(d-1)^{\text{th}}$ first components of (X,U) and $(X^{(d)},U^{(d)})$ denoting the dth component.

In this section, we show the existence in law and the pathwise uniqueness of the reflected process satisfying (3.1) (Proposition 3.1). Further, we provide some estimates on the semi-group related to the solution to (3.1) (Proposition 3.3).

3.1 Well-posedness of Equation (3.1)

In equation (3.1), (U', X') is a $(d-1)$-dimensional Brownian motion and its associated primitive. Thus, we only need to consider the dth equation of a one-dimensional Brownian motion primitive reflected in $\mathbb{R}^+$ as

$$\begin{cases} X_t^c = X_0^{(d)} + \displaystyle\int_0^t U_s^c ds, \\ U_t^c = U_0^{(d)} + W_t^c - 2 \displaystyle\sum_{0<s\leq t} U_{s^-}^c \mathbb{1}\{X_s^c = 0\}, \ \forall\, t \geq 0. \end{cases} \tag{3.2}$$

On a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0), \mathbb{Q})$, endowed with a one dimensional standard $\mathcal{F}_t$-Brownian motion W^f and such that (X_0, U_0) is a r.v. with law $\mathbb{Q}((X_0, U_0) \in dy\, dv) = \mu_0(dy, dv)$, we consider the free Langevin system (X^f, U^f) satisfying

$$\begin{cases} X_t^f = X_0^{(d)} + \displaystyle\int_0^t U_s^f ds, \\ U_t^f = U_0^{(d)} + W_t^f, \ \forall\, t \geq 0. \end{cases} \tag{3.3}$$

We also consider the family of probability measures $\{\mathbb{Q}_{y,v}\}_{(y,v)\in\mathbb{R}^2}$ on $(\Omega, \mathcal{F})$, defined by

$$\mathbb{Q}_{y,v}(A) = \mathbb{Q}\left(A/(X_0^f, U_0^f) = (y, v)\right), \ \forall A \in \mathcal{F}.$$

As mentioned by Bertoin [3], it is straightforward to check that $|X^f|$ describes the reflected Langevin process in the half line, in the sense that a particle arriving at 0 with velocity $v < 0$ bounces back with velocity $v > 0$. Here, we also need to explicit the construction of U^c in terms of the free Langevin process. McKean [14] has shown that if $(y, v) \neq (0, 0)$ then, $\mathbb{Q}_{y,v}$-almost surely, the paths $t \mapsto (X_t^f, U_t^f)$ never cross $(0, 0)$. Thus, under the assumption (H-i), the sequence of passage times

$$\tau_n^f = \inf\{t > \tau_{n-1}^f \text{ s.t. } X_t^f = 0\}, \text{ for } n \geq 1, \ \tau_0^f = 0, \tag{3.4}$$

is well-defined. The law of the passage times is explicited by Lachal [12, Corollary 3]: for all $n \geq 1$, and $(y, v) \in \mathbb{R} \times \mathbb{R}$ with $y \neq 0$,

$$\begin{aligned} \mathbb{Q}_{y,v}\left(\tau_n^f \in dt\right) = \Bigg(\int_0^t \int_0^{+\infty} & \frac{1}{(2\pi)^{3/2}\sqrt{s}} \exp(-\frac{z^2}{s}) h(t-s; y, v, z) \\ & \times \left(\int_0^{+\infty} \gamma \sinh\left(\frac{\pi\gamma}{2}\right) \frac{\left[2\cosh(\frac{\pi\gamma}{3}) - 1\right]}{\left(2\cosh(\frac{\pi\gamma}{3})\right)^n} K_{i\gamma/2}(\frac{z^2}{s}) d\gamma \right) dz\, ds \Bigg) dt, \end{aligned} \tag{3.5}$$

where $h(t; y, v, z) = \frac{2\sqrt{3}}{\pi t^2} \exp\{-\frac{6y^2}{t^3} - \frac{6yv}{t^2} - \frac{2(v^2+z^2)}{t}\} \cosh(\frac{2z}{t^2}(3y + tv))$ and $K_{i\gamma}$ denotes the modified Bessel function. Since $K_{i\gamma/2}(z)$ is nonnegative for $z \geq 0$, (3.5) gives that for all $t_0 > 0$,

$$\mathbb{Q}_{y,v}\left(\tau_n^f \leq t_0\right) \leq \frac{1}{2^{n-1}} \mathbb{Q}_{y,v}\left(\tau_1^f \leq t_0\right) \leq \frac{1}{2^{n-1}}, \ \forall\, n \geq 1. \tag{3.6}$$

Therefore, $\mathbb{Q}_{y,v}\left(\tau_n^f \leq t_0\right)$ tends to 0 as n grows to infinity, and the sequence $\{\tau_n^f\}_{n\in\mathbb{N}}$ grows to infinity. Defining the sign function as

$$sgn(x) = \begin{cases} 1 \text{ if } x > 0, \\ -1 \text{ elsewhere}, \end{cases}$$

the right-hand limit process $t \mapsto \mathcal{S}_t^f = sgn(X_t^f)_+$ is well defined.

Proposition 3.1. *Assume* $(H\text{-}i)$. *On the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0), \mathbb{Q})$, *there exists a one dimensional standard* $\mathcal{F}_t$*-Brownian motion* W^c *such that*

$$(X_t^c = |X_t^f|, U_t^c = U_t^f \mathcal{S}_t^f; 0 \leq t \leq T) \tag{3.7}$$

is a solution to (3.2). *In particular,* $|U_t^c| = |U_0 + W_t^f|$, *and the paths* $t \mapsto -2\sum_{0<s\leq t} U_{s^-}^c \mathbb{1}_{\{X_s^c=0\}}$ *are positive and non-decreasing. The sequence of hitting times*

$$\tau_n = \inf\{t > \tau_{n-1} \ s.t. \ X_t^c = 0\}, \ for \ n \geq 1, \ \tau_0 = 0, \tag{3.8}$$

is well-defined and grows to infinity. The pathwise uniqueness holds for the solution of (3.2).

Proof. We consider the càdlàg $\mathcal{F}_t$-adapted process U^c and its primitive X^c:

$$\begin{cases} X_t^c = X_0^f + \displaystyle\int_0^t U_s^c ds, \\ U_t^c = U_t^f \mathcal{S}_t^f, \ \forall \ t \geq 0. \end{cases}$$

Since $\mathcal{S}^f$ is a pure jump process, by the integration by parts formula,

$$U_t^c = U_t^f \mathcal{S}_t^f = U_0^c + \int_0^t \mathcal{S}_{s^-}^f \, dU_s^f + \sum_{0<s\leq t} U_s^f \triangle \mathcal{S}_s^f, \ \forall \ t \geq 0.$$

By Lévy's characterization, $(W_t^c := \int_0^t \mathcal{S}_{s^-}^f dU_s^f, t \geq 0)$ is a standard $\mathcal{F}_t$-Brownian motion. Moreover, on the set $\{t \geq 0 \text{ s.t. } X_t^f = 0\}$, if $U_t^f > 0$ then $\mathcal{S}_{t^-}^f = -1$ and $\triangle \mathcal{S}_t^f = 2$, while if $U_t^f < 0$, $\mathcal{S}_{t^-}^f = 1$ and $\triangle \mathcal{S}_t^f = -2$. Then,

$$\sum_{0<s\leq t} U_s^f \triangle \mathcal{S}_s^f = -2 \sum_{0<s\leq t} U_{s-}^c \mathbb{1}_{\{X_s^f=0\}}.$$

The set $\{t \geq 0 \text{ s.t. } X_t^f = 0\}$ being countable, we may replace $\mathcal{S}_t^f$ by $sgn(X_t^f)$ in the dynamics of $|X^f|$. Then,

$$|X_t^f| = |X_0^f| + \int_0^t sgn(X_s^f) \ U_s^f ds = |X_0^f| + \int_0^t U_s^c ds.$$

Then $\mathbb{Q}$-a.s., $X^c = |X^f|$, and $\{\tau_n\}_{n\in\mathbb{N}} = \{\tau_n^f\}_{n\in\mathbb{N}}$. Consequently, (X^c, U^c) satisfies (3.2).

The uniqueness result is a consequence of $(H\text{-}i)$. Consider $(\widetilde{X}, \widetilde{U})$, another solution to (3.2) defined on the same probability space, endowed with the same Brownian motion. Since $\widetilde{X}_0 > 0$, we can define the first passage time at zero $\widetilde{\tau}_1$ of $\widetilde{X}$, and we observe that $\widetilde{\tau}_1 = \tau_1$ due to the continuity of X^c and $\widetilde{X}$. It follows that $U_{\tau_1\wedge\widetilde{\tau}_1}^c = \widetilde{U}_{\tau_1\wedge\widetilde{\tau}_1}$, so that (X^c, U^c) and $(\widetilde{X}, \widetilde{U})$ are equal up to τ_1. By induction, one checks that this assertion holds true up to τ_n for all $n \in \mathbb{N}$. As τ_n tends to $+\infty$ $\mathbb{Q}$-a.s., (X^c, U^c) and $(\widetilde{X}, \widetilde{U})$ are equal on $\mathbb{R}^+$. □

Remark 3.2. *The explicit construction in Proposition 3.1 has a straightforward extension for the Langevin process with bounded drift: for any* $\mathbb{R}^d$*-valued bounded measurable function on* Q_T, $\beta(t,x,u) = (\beta', \beta^{(d)})(t,x,u)$, *from the unique weak* $\mathbb{R}^{2d}$*-valued solution of*

$$\begin{cases} Y_t = X_0 + \displaystyle\int_0^t V_s ds, \\ V_t = U_0 + \displaystyle\int_0^t \widetilde{\beta}(s, Y_s, V_s) ds + \widetilde{W}_t, \ \forall \ t \in [0,T], \end{cases} \tag{3.9}$$

with $\widetilde{\beta}(t,x,u) := (\beta', sgn(x^{(d)})\beta^{(d)})(t,(x',|x^{(d)}|),(u', sgn(x^{(d)})u^{(d)}))$, *one deduces that*

$$(\mathcal{X}_t, \mathcal{U}_t; t \in [0,T]) = ((Y_t', |Y_t^{(d)}|), (V_t', sgn(Y_{t^+}^{(d)})V_t^{(d)}); t \in [0,T])$$

is the weak solution in $\mathcal{D} \times \mathbb{R}^d$ *to*

$$\begin{cases} \mathcal{X}_t = X_0 + \displaystyle\int_0^t \mathcal{U}_s ds, \\ \mathcal{U}_t = U_0 + \displaystyle\int_0^t \beta(s, \mathcal{X}_s, \mathcal{U}_s) ds + W_t - \sum_{0<s\leq t} 2\left(\mathcal{U}_{s^-} \cdot n_{\mathcal{D}}(\mathcal{X}_s)\right) n_{\mathcal{D}}(\mathcal{X}_s) \mathbb{1}_{\{\mathcal{X}_s \in \partial\mathcal{D}\}}, \ \forall\ t \in [0,T]. \end{cases}$$

3.2 On the semi-group of the confined Langevin process

Following the approach of Yamada and Watanabe (see, *e.g.*, Karatzas and Schreve [11]), the pathwise uniqueness of the solution to (3.1) implies its uniqueness in law. According to Ethier and Kurtz [8], the uniqueness in law yields to the strong Markov property for the process (X,U), valued in $\overline{\mathcal{D}} \times \mathbb{R}^d$, solution of (3.1), and it is straightforward to compute the density $\Gamma(t;y,v;x,u)$ of (X_t,U_t) under the probability measure $\mathbb{Q}_{y,v}$, for $(y,v) \in \mathcal{D} \times \mathbb{R}^d$. For any $x = (x', x^{(d)}) \in \mathbb{R}^{d-1} \times [0,+\infty)$, $u = (u', u^{(d)}) \in \mathbb{R}^d$, $y = (y', y^{(d)}) \in \mathbb{R}^{d-1} \times (0,+\infty)$ and $v = (v', v^{(d)}) \in \mathbb{R}^d$, we have

$$\Gamma(t;y,v;x,u) = g_{d-1}(t;y',v';x',u')g^c(t;y^{(d)},v^{(d)};x^{(d)},u^{(d)}), \tag{3.10}$$

where $g_{d-1}(t;y',v';x',u')$ is the density law of the $(d-1)$-dimensional Brownian motion and its primitive starting from (y',v'), explicited in (1.5), and $g^c(t;y^{(d)},v^{(d)};x^{(d)},u^{(d)})$ is the transition density of the confined d^{th} component (X^c,U^c) satisfying (3.2).

In view of (3.7), we get that for all $f \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R})$,

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}_{y^{(d)},v^{(d)}}}\left[f\left(X_t^c,\ U_t^c\right)\right] \\ &= \mathbb{E}_{\mathbb{Q}_{y^{(d)},v^{(d)}}}\left[f\left(X_t^f, U_t^f\right)\mathbb{1}_{\{X_t^f>0\}}\right] + \mathbb{E}_{\mathbb{Q}_{y^{(d)},v^{(d)}}}\left[f\left(-X_t^f, -U_t^f\right)\mathbb{1}_{\{X_t^f<0\}}\right], \end{aligned}$$

as $\{X_t^f = 0\}$ is $\mathbb{Q}_{y^{(d)},v^{(d)}}$-negligible. Therefore,

$$g^c(t;y^{(d)},v^{(d)};\zeta,\nu) = g_1(t;y^{(d)},v^{(d)};\zeta,\nu), +g_1(t;y^{(d)},v^{(d)};-\zeta,-\nu), \tag{3.11}$$

for all $t>0$, a.e. (ζ,ν) in $[0,+\infty) \times \mathbb{R}$ and $(y^{(d)},v^{(d)}) \in (0,+\infty) \times \mathbb{R}$.

Let us define the semi-group $(S_t; t>0)$ associated to the transition Γ by

$$S_t(f)(y,v) = \mathbb{E}_{\mathbb{Q}_{y,v}}\left[f(X_t,U_t)\right] = \int_{\mathcal{D}\times\mathbb{R}^d} \Gamma(t;y,v;x,u)f(x,u)dx\,du. \tag{3.12}$$

For $q \in \mathbb{N}^*$, let $\mathcal{A}_q$ be the second order differential operator defined on $C^{1,2}(\mathbb{R}^q \times \mathbb{R}^q)$ by

$$\mathcal{A}_q(f)(y,v) = (v \cdot \nabla_y f(y,v)) + \frac{1}{2}\triangle_v f(y,v).$$

Proposition 3.3. *(i) For all* $t>0$ *and* $f \in \mathcal{C}_c(\mathcal{D} \times \mathbb{R}^d)$, *the function* $G_{t,f}$ *defined by*

$$G_{t,f}(s,y,v) = S_{t-s}(f)(y,v), \ \textit{for}\ (s,y,v) \in [0,t) \times \mathcal{D} \times \mathbb{R}^d, \tag{3.13}$$

is a classical solution to the following Cauchy problem:

$$\begin{cases} \partial_s G_{t,f} + \mathcal{A}_d(G_{t,f}) = 0, \ \textit{in}\ [0,t) \times \mathcal{D} \times \mathbb{R}^d, \\ G_{t,f}(s,y,v) = G_{t,f}(s,y,v - 2(v \cdot n_{\mathcal{D}}(y))n_{\mathcal{D}}(y)), \ \textit{in}\ [0,t) \times \partial\mathcal{D} \times \mathbb{R}^d, \\ \lim\limits_{s\to t^-} G_{t,f}(s,y,v) = f(y,v), \ \textit{in}\ \mathcal{D} \times \mathbb{R}^d. \end{cases} \tag{3.14}$$

(ii) There exists a constant $C > 0$ such that, for all $t > 0$,

$$\sup_{(y,v)\in\mathcal{D}\times\mathbb{R}^d} \int_{\mathcal{D}\times\mathbb{R}^d} |\nabla_v \Gamma(t; y, v; x, u)|\, dx\, du \leq \frac{C}{\sqrt{t}}. \tag{3.15}$$

Proof. Since the transition probability density function Γ in (3.10) is the fundamental solution of the differential operator $\partial_t - \mathcal{A}_d$, we only need to check that the specular boundary condition in (3.14) holds true. Owing to the smoothness of $y^{(d)} \mapsto g^c(t; y^{(d)}, v^{(d)}; x^{(d)}, u^{(d)})$, one also has

$$\lim_{y^{(d)}\to 0^+} G_{t,f}(s, y, v) = \int_{\mathcal{D}\times\mathbb{R}^d} f(x, u) g_{d-1}(t - s; y', v'; x', u') g^c(t - s; 0, v^{(d)}; x^{(d)}, u^{(d)}) dx\, du.$$

Hence, for all $t > 0$ and $(s, y, v) \in [0, t) \times \partial\mathcal{D} \times \mathbb{R}^d$,

$$G_{t,f}(s, y, v) = \int_{\mathcal{D}\times\mathbb{R}^d} f(x, u) g_{d-1}(t - s; y', v'; x', u') g^c(t - s; 0, v^{(d)}; x^{(d)}, u^{(d)}) dx\, du. \tag{3.16}$$

In view of (1.5) and (3.11), one observes that $g_1(t; 0, v^{(d)}; -x^{(d)}, -u^{(d)}) = g_1(t; 0, -v^{(d)}; x^{(d)}, u^{(d)})$ and

$$\begin{aligned} g^c(t; 0, v^{(d)}; x^{(d)}, u^{(d)}) &= g_1(t; 0, v^{(d)}; x^{(d)}, u^{(d)}) + g_1(t; 0, -v^{(d)}; x^{(d)}, u^{(d)}) \\ &= g^c(t; 0, -v^{(d)}; x^{(d)}, u^{(d)}). \end{aligned}$$

Coming back to (3.16), we deduce that $G_{t,f}$ satisfies the specular boundary condition in (3.14).

The estimate (3.15) is obtained by a straightforward computation on the explicit expression of $\nabla_v \Gamma(t; y, v; x, u)$ in terms of $\partial_v g_1$ in (1.5). □

4 Proof of Theorem 2.2

In this section, the hypotheses (H) are implicitly assumed in all the stated propositions and lemmas.

We construct a solution to the martingale problem (MP) by means of the convergence of a smoothed and confined interacting particle system $\{(X^{i,\epsilon,N}, U^{i,\epsilon,N}, K^{i,\epsilon,N})\}_{i=1,\dots,N}$ defined by

$$\begin{cases} X_t^{i,\epsilon,N} = X_0^i + \displaystyle\int_0^t U_s^{i,\epsilon,N} ds, \\ U_t^{i,\epsilon,N} = U_0^i + \displaystyle\int_0^t \frac{\sum_{j=1}^N b(U_s^{j,\epsilon,N}, U_s^{i,\epsilon,N}) \phi_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N})}{\sum_{j=1}^N \left(\phi_\epsilon(X_s^{i,\epsilon,N} - X_s^{j,\epsilon,N}) + \epsilon\right)} ds + W_t^i + K_t^{i,\epsilon,N}, \\ K_t^{i,\epsilon,N} = -2 \displaystyle\sum_{0<s\leq t} \left(U_{s^-}^{i,\epsilon,N} \cdot n_{\mathcal{D}}\right) n_{\mathcal{D}} \mathbb{1}_{\left\{X_s^{i,\epsilon,N} \in \partial\mathcal{D}\right\}}, \ i = 1, \dots, N, \end{cases} \tag{4.1}$$

where $\{(X_0^i, U_0^i, W^i)\}_{i\geq 1}$ are independent copies of (X_0, U_0, W) on a given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0), \mathbb{Q})$. The sequence $\{\phi_\epsilon; \epsilon > 0\}$ is a family of mollifiers defined by $\phi_\epsilon(x) := \epsilon^{-d}\phi(\frac{x}{\epsilon})$, for some $\phi \in \mathcal{C}_c^1(\mathcal{D})$ such that $\phi \geq 0$ and $\int_{\mathbb{R}^d} \phi(x)dx = 1$. For $i = 1, \dots, N$, the processes $(X^{i,\epsilon,N}, U^{i,\epsilon,N}, K^{i,\epsilon,N})$ are valued in $\mathcal{E}$. The existence and uniqueness in law for the linear equation (4.1) derives from the Girsanov's Theorem and Proposition 3.1.

For a fixed $\epsilon > 0$, when $N \to +\infty$, the particle system (4.1) is aimed to propagate the chaos with a limit law given by the unique solution $\mathbb{P}^\epsilon$ of the following martingale problem.

Definition 4.1. *A probability measure $\mathbb{P}^\epsilon \in \mathcal{M}(\mathcal{E})$ is a solution to the martingale problem (MP_ϵ) if the following hold.*

(i) $\mathbb{P}^\epsilon \circ (x_0, u_0, k_0)^{-1} = \mu_0 \otimes \delta_0$.

(ii) For all $t \in (0,T]$, $\mathbb{P}^\epsilon \circ (x_t, u_t)^{-1}$ *admits a Lebesgue density* ρ_t^ϵ.

(iii) For all $f \in \mathcal{C}_b^2(\mathbb{R}^2)$, *the process*

$$\begin{aligned} &f(x_t, u_t - k_t) - f(x_0, u_0) - \int_0^t (u_s \cdot \nabla_x f(x_s, u_s - k_s))\, ds \\ &- \int_0^t \left[(B_\epsilon[x_s, u_s; \rho_s^\epsilon] \cdot \nabla_u f(x_s, u_s - k_s)) + \frac{1}{2} \triangle_u f(x_s, u_s - k_s) \right] ds \end{aligned} \tag{4.2}$$

is a continuous $\mathbb{P}^\epsilon$*-martingale w.r.t.* $(\mathcal{B}_t; t \in [0,T])$.

(iv) $\mathbb{P}^\epsilon$*-a.s., the set* $\{t \in [0,T] \text{ s.t. } x_t^{(d)} = 0\}$ *is at most countable, and*

$$k_t = -2 \sum_{0<s\le t} (u_{s^-} \cdot n_{\mathcal{D}}) n_{\mathcal{D}} \mathbb{1}_{\{x_s \in \partial\mathcal{D}\}}, \ \forall\ t \in [0,T].$$

The kernel B_ϵ is defined for $(x, u, \gamma) \in \mathcal{D} \times \mathbb{R}^d \times L^1(\mathcal{D} \times \mathbb{R}^d)$, by

$$B_\epsilon[x, u; \gamma] = \frac{\displaystyle\int_{\mathcal{D}\times\mathbb{R}^d} b(v, u)\phi_\epsilon(x - y)\gamma(y, v)\, dy\, dv}{\displaystyle\int_{\mathcal{D}\times\mathbb{R}^d} \phi_\epsilon(x - y)\gamma(y, v) dy dv + \epsilon}. \tag{4.3}$$

In what follows, we also refer to B_ϵ for the kernel $(x, u, m) \mapsto B_\epsilon[x, u; m]$, for $(x, u) \in \mathcal{D} \times \mathbb{R}^d$ and $m \in \mathcal{M}(\mathcal{D} \times \mathbb{R}^d)$, by substituting $\gamma(y, v)dydv$ for $m(dy, dv)$ in definition (4.3). Note that, on $\mathcal{D} \times \mathbb{R}^d \times L^1(\mathcal{D} \times \mathbb{R}^d)$, B_ϵ is a mollified version of the kernel B defined in (1.4).

Theorem 2.2 is a consequence of the following proposition.

Proposition 4.2. *Let* $\bar{\mu}^{\epsilon,N} := \frac{1}{N} \sum_{i=1}^N \delta_{\{X^{i,\epsilon,N}, U^{i,\epsilon,N}, K^{i,\epsilon,N}\}}$ *be the empirical measure associated to (4.1). Then,*

(i) for all $\epsilon > 0$, *the sequence* $(\bar{\mu}^{\epsilon,N}, N \ge 1)$ *converges weakly to the unique solution* $\mathbb{P}^\epsilon$ *of the martingale problem (MP*$_\epsilon$*);*

(ii) when ϵ *tends to* 0, $\mathbb{P}^\epsilon$ *converges to the unique solution of the martingale problem (MP).*

As shown in Sznitman [18], *(i)* ensures that the laws $\mathbb{P}^{\epsilon,N}$ of the particles $\{(X^{i,\epsilon,N}, U^{i,\epsilon,N}, K^{i,\epsilon,N}), 1 \le i \le N\}$ are $\mathbb{P}^\epsilon$-chaotic: for all integers $k \ge 2$ and all finite families $(\psi_l, 1 \le l \le k)$ of functions in $\mathcal{C}_b(\mathcal{C}([0,T]; \overline{\mathcal{D}}) \times \mathbb{D}([0,T]; \mathbb{R}^d) \times \mathbb{D}([0,T]; \mathbb{R}^d))$,

$$\langle \mathbb{P}^{\epsilon,N}, \psi_1 \otimes \cdots \psi_k \otimes 1 \cdots \rangle \to \prod_{l=1}^k \langle \mathbb{P}^\epsilon, \psi_l \rangle, \text{ when } N \to +\infty.$$

The proof of Proposition 4.2 is organized as follows. First, the uniqueness results for the martingale problems (MP) and (MP_ϵ) are stated in Proposition 4.3. Second we prove *(i)* in Section 4.2. Finally, we show the convergence result *(ii)* with Proposition 4.10. The proofs adapt the convergence techniques for martingale problems, from the *free* McKean Lagrangian case in [5] to the present case of *a confined* McKean Lagrangian diffusion with jumps.

4.1 Uniqueness results for the martingale problems (MP) and(MP_ϵ)

It is enough to prove uniqueness results for the time-marginals $(\rho_t; t \in (0,T])$ and $(\rho_t^\epsilon; t \in (0,T])$ related to (MP) and (MP_ϵ) respectively. Indeed, this will ensure that any two weak solutions to (MP) or (MP_ϵ) are equipped with the same drift B or B_ϵ respectively. Owing to the boundedness of b, the uniqueness results for the martingale problems will follow by a change of probability measure. We identify the mild equations solved by the time-marginals, and prove uniqueness results by adapting the step-proofs of [5].

Consider $(S_t^*; t \in (0,T])$, the adjoint of $(S_t; t \in (0,T])$ in (3.12), defined by

$$S_t^*(\mu)(x,u) = \int_{\mathcal{D}\times\mathbb{R}^d} \Gamma(t;y,v;x,u)\mu(dy,dv). \tag{4.4}$$

In view of (3.10), for all $t \in (0,T]$, S_t^* is a linear operator from $\mathcal{M}(\mathcal{D}\times\mathbb{R}^d)$ to $L^1(\mathcal{D}\times\mathbb{R}^d)$. In addition, we define the operator $(S_t'; t \in (0,T])$ by

$$S_t'(f)(x,u) = \int_{\mathcal{D}\times\mathbb{R}^d} \left(\nabla_v \Gamma(t;y,v;x,u)\cdot f(y,v)\right) dy\, dv. \tag{4.5}$$

According to (3.15), for any $t>0$, $S'_{t-\cdot}$ defines a linear mapping from $L^\infty((0,t); L^1(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d))$ to $L^1(Q_t)$ such that for all $\gamma \in L^\infty((0,t); L^1(\mathcal{D}\times\mathbb{R}^d;\mathbb{R}^d))$,

$$\int_{Q_t} \left|S'_{t-s}(\gamma(s))(y,v)\right| ds\, dy\, dv \le \int_0^t \frac{C}{\sqrt{t-s}} \|\gamma(s)\|_{L^1(\mathcal{D}\times\mathbb{R}^d)} ds. \tag{4.6}$$

Proposition 4.3. *(i) The time-marginals $(\rho_t; t\in(0,T])$ and $(\rho_t^\epsilon; t\in(0,T])$ of $\mathbb{P}$ and $\mathbb{P}^\epsilon$, solutions to the martingale problem (MP) and (MP_ϵ) respectively, satisfy the following mild equations in $L^1(\mathcal{D}\times\mathbb{R}^d)$:*

$$\forall\ t\in(0,T]\,,\ \rho_t = S_t^*(\mu_0) + \int_0^t S'_{t-s}(\rho_s(\cdot)B[\cdot;\rho_s])ds, \tag{4.7}$$

$$\forall\ t\in(0,T]\,,\ \rho_t^\epsilon = S_t^*(\mu_0) + \int_0^t S'_{t-s}(\rho_s^\epsilon(\cdot)B_\epsilon[\cdot;\rho_s^\epsilon])ds. \tag{4.8}$$

(ii) The mild equations (4.7) and (4.8) have at most one solution.

Proof. We prove only *(i)*, the proof of *(ii)* being a straightforward replication of the proof of Lemma 4.5 in [5].

For a fixed $t\in(0,T]$ and $f\in\mathcal{C}_c\left(\mathcal{D}\times\mathbb{R}^d\right)$, let $G_{t,f}$ be the classical solution of (3.14) defined in (3.13). As mentioned above, for $\mathbb{P}$ solving (MP), (x,u,k) satisfies (1.1). Therefore,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[G_{t,f}(t,x_t,u_t)\right] =& \mathbb{E}_{\mathbb{P}}\left[G_{t,f}(0,x_0,u_0)\right] + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\left(B\left[x_s,u_s;\rho_s\right]\cdot\nabla_u G_{t,f}(s,x_s,u_s)\right)\right] ds \\ &+ \mathbb{E}_{\mathbb{P}}\left[\int_0^t \left(\partial_s + \mathcal{A}_d\right)(G_{t,f})(s,x_s,u_s)ds\right] \\ &+ \mathbb{E}_{\mathbb{P}}\left[\sum_{0<s\le t}\left(G_{t,f}(s,x_s,u_s) - G_{t,f}(s,x_s,u_{s^-})\right)\mathbb{1}_{\{x_s\in\partial\mathcal{D}\}}\right].\end{aligned}$$

According to Proposition 3.3-*(i)*, the equality above writes

$$\begin{aligned}\int_{\mathcal{D}\times\mathbb{R}^d} f(x,u)\rho_t(x,u)dx\,du =& \int_{\mathcal{D}\times\mathbb{R}^d} G_{t,f}(0,x,u)\mu_0(dx,du) \\ &+ \int_{Q_t}\left(B\left[x,u;\rho_s\right]\cdot\nabla_u G_{t,f}(s,x,u)\right)\rho_s(x,u)ds\,dx\,du.\end{aligned}$$

Noticing that $\int_{\mathcal{D}\times\mathbb{R}^d}\Gamma(t;y,v;x,u)dy\,dv=1$, we have

$$\int_{\mathcal{D}\times\mathbb{R}^d} G_{t,f}(0,x,u)\mu_0(dx,du)=\int_{\mathcal{D}\times\mathbb{R}^d} S_t^*(\mu_0)(x,u)f(x,u)\,dx\,du,$$

with $S_t^*(\mu_0)\in L^1(\mathcal{D}\times\mathbb{R}^d)$. Similarly, owing to the estimate (4.6), it holds that

$$\int_{Q_t}\left(B\left[x,u;\rho_s\right]\cdot\nabla_u G_{t,f}(s,x,u)\right)\rho_s(x,u)ds\,dx\,du=\int_{Q_t} f(x,u)S'_{t-s}(\rho_s(\cdot)B[\cdot;\rho_s])(x,u)ds\,dx\,du.$$

Thus we deduce that, for all $t\in(0,T]$ and $f\in\mathcal{C}_c(\mathcal{D}\times\mathbb{R}^d)$,

$$\int_{\mathcal{D}\times\mathbb{R}^d} f(x,u)\left(\rho_t(x,u)-S_t^*(\mu_0)(x,u)-\int_0^t S'_{t-s}(\rho_s(\cdot)B\left[\cdot;\rho_s\right])(x,u)ds\right)dx\,du=0.$$

With Proposition 3.3-*(ii)*, we conclude that $(\rho_t;t\in(0,T])$ satisfy the mild equation (4.7).

In the case where the drift coefficient is B_ϵ, duplicating the same arguments, it is straightforward to establish (4.8). □

4.2 Existence result for (MP_ϵ)

In this section, we restrict ourselves to the case $d=1$ and $\mathcal{D}=(0,+\infty)$, the proof can be readily extended to the case $\mathcal{D}=\mathbb{R}^{d-1}\times(0,+\infty)$. When $d=1$, the particle system (4.1) writes

$$\begin{cases} X_t^{i,\epsilon,N}=X_0^i+\displaystyle\int_0^t U_s^{i,\epsilon,N}ds,\\ U_t^{i,\epsilon,N}=U_0^i+\displaystyle\int_0^t\frac{\displaystyle\sum_{j=1}^N b(U_s^{j,\epsilon,N},U_s^{i,\epsilon,N})\phi_\epsilon(X_s^{i,\epsilon,N}-X_s^{j,\epsilon,N})}{\displaystyle\sum_{j=1}^N\left(\phi_\epsilon(X_s^{i,\epsilon,N}-X_s^{j,\epsilon,N})+\epsilon\right)}ds+W_t^i+K_t^{i,\epsilon,N},\\ K_t^{i,\epsilon,N}=-2\displaystyle\sum_{0<s\le t}U_{s^-}^{i,\epsilon,N}\mathbb{1}_{\{X_s^{i,\epsilon,N}=0\}},\ i=1,\dots,N.\end{cases}\tag{4.9}$$

The proof of Proposition 4.2-*(i)* proceeds in two steps.

Step 1. We prove a tightness result for the sequence of probability measures $\{\pi^{\epsilon,N}\}_{N\ge1}$ induced by $\bar\mu^{\epsilon,N}$ on $\mathcal{M}(\mathcal{E})$, by using Aldous's Tightness criterion.

Step 2. We check that all limit points of $\{\pi^{\epsilon,N}\}_{N\ge1}$ have full measure on the set of probability measures satisfying the properties *(i)-(iv)* of (MP_ϵ) in Definition 4.1.

We then deduce the existence of a probability measure $\mathbb{P}^\epsilon$ solution to (MP_ϵ). The uniqueness result in Proposition 4.3 implies that all converging subsequences of $\{\pi^{\epsilon,N}\}_{N\ge1}$ tend to $\delta_{\{\mathbb{P}^\epsilon\}}$. It follows that the entire sequence $\{\pi^{\epsilon,N}\}_{N\ge1}$ converges to $\delta_{\{\mathbb{P}^\epsilon\}}$, and enables us to conclude on Proposition 4.2-*(i)*.

Step 1. Tightness result As shown in Sznitman [18], the exchangeability of the particle system (4.9) induces the equivalence between the tightness property of $\{\pi^{\epsilon,N}\}_{N\ge1}$ and the tightness of the sequence $\{\mathbb{Q}\circ(X^{1,\epsilon,N},U^{1,\epsilon,N},K^{1,\epsilon,N})^{-1}\}_{N\ge1}$.

Lemma 4.4. *$\{\mathbb{Q}\circ\left(X^{1,\epsilon,N},U^{1,\epsilon,N},K^{1,\epsilon,N}\right)^{-1}\}_{N\ge1}$ is tight on $\mathcal{E}$.*

Proof. We apply the Aldous criterion to the system $\{X^{1,\epsilon,N}, U^{1,\epsilon,N}, K^{1,\epsilon,N}\}_{N\geq 1}$. For the sake of completeness, let us recall this criterion.

Theorem 4.5. *(see, e.g., Billingsley [4].) Let $\{Y^n\}_{n\in\mathbb{N}}$ be a sequence of r.v. defined on a probability space $(\Omega, \mathcal{F}, P)$ and valued in $\mathbb{D}([0,T];\mathbb{R}^q)$. The sequence $\{P\circ(Y^n)^{-1}\}_{n\in\mathbb{N}}$ is tight on $\mathbb{D}([0,T];\mathbb{R}^q)$ if the following hold.*

(C1) For all $t\geq 0$, $P\circ(Y^n_t)^{-1}$ is tight on $\mathbb{R}^q$.

(C2) For all $\varepsilon>0$, $\eta>0$, there exist $\delta_0>0$ and $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ and for all discrete-valued $\sigma(Y^n_s; s\in[0,T])$-stopping times β such that $0\leq\beta+\delta_0\leq T$,

$$\sup_{\delta\in[0,\delta_0]} P\left(\left|Y^n_{\delta+\beta}-Y^n_\beta\right|\geq\eta\right)\leq\varepsilon.$$

We check *(C1)*. With $K^{i,\epsilon,N}_t$ obtained from the evolution equation of $U^{i,\epsilon,N}$, we easily get that

$$\begin{aligned}&\sup_{N\geq1}\mathbb{E}_{\mathbb{Q}}[|X^{1,\epsilon,N}_t|]+\sup_{N\geq1}\mathbb{E}_{\mathbb{Q}}[|K^{1,\epsilon,N}_t|]\\&\leq\mathbb{E}_{\mathbb{Q}}\left(|X^1_0|+|U^1_0|+|W^1_t|\right)+(1+T)\sup_{N\geq1}\mathbb{E}_{\mathbb{Q}}\left[\sup_{\theta\in[0,T]}|U^{1,\epsilon,N}_\theta|\right]+T\|b\|_\infty.\end{aligned}\tag{4.10}$$

Applying the Itô formula to $|U^{1,\epsilon,N}_t|^2$, the jump term vanishes and we get

$$|U^{1,\epsilon,N}_t|^2=|U^{1,\epsilon,N}_0|^2+2\int_0^T\left(D^{1,\epsilon,N}_s\cdot U^{1,\epsilon,N}_s\right)ds+2\int_0^T U^{1,\epsilon,N}_s dW^1_s+t,$$

with $(D^{1,\epsilon,N}_t; t\in[0,T])$ defined by

$$D^{1,\epsilon,N}_t=\frac{\sum_{j=1}^N b(U^{j,\epsilon,N}_t,U^{1,\epsilon,N}_t)\phi_\epsilon(X^{1,\epsilon,N}_t-X^{j,\epsilon,N}_t)}{\sum_{j=1}^N\left(\phi_\epsilon(X^{1,\epsilon,N}_t-X^{j,\epsilon,N}_t)+\epsilon\right)}.$$

By classical arguments and thanks to (H), this allows to show that there exists a positive constant C depending on T, $\mathbb{E}[|U_0|^2]$ and $\|b\|_\infty$ such that

$$\sup_{N\geq1}\mathbb{E}_{\mathbb{Q}}[\sup_{\theta\in[0,T]}|U^{1,\epsilon,N}_\theta|^2]\leq C.\tag{4.11}$$

Then *(C1)* is fulfilled.

Now we check *(C2)*. For a given $N\geq2$, let β be a $\sigma\left(X^{1,\epsilon,N},U^{1,\epsilon,N},K^{1,\epsilon,N}\right)$-stopping time with discrete values such that $\beta+\delta_0\leq T$. Then we observe that

$$\lim_{\delta_0\to0^+}\sup_{\delta\in[0,\delta_0]}\mathbb{Q}\left(\left|X^{1,\epsilon,N}_{\delta+\beta}-X^{1,\epsilon,N}_\beta\right|\geq\eta\right)\leq\lim_{\delta_0\to0^+}\frac{\delta_0}{\eta}\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]}\left|U^{1,\epsilon,N}_t\right|\right]\leq\lim_{\delta_0\to0^+}\frac{\delta_0 C}{\eta},$$

for a constant $C>0$, which does not depend on N and ϵ. Since b is bounded, we introduce the probability measure $\widetilde{\mathbb{Q}}$ defined by $d\widetilde{\mathbb{Q}}=Z^{1,\epsilon,N}_T d\mathbb{Q}$, where the $\mathcal{F}_t$-martingale $(Z^{1,\epsilon,N}_t; t\in[0,T])$ is given by

$$Z^{1,\epsilon,N}_t=\exp\left(-\int_0^t D^{1,\epsilon,N}_s\,dW^1_s-\int_0^t\left|D^{1,\epsilon,N}_s\right|^2ds\right),$$

Girsanov's Theorem implies that $(W_t^{1,\epsilon,N} := W_t^1 + \int_0^t D_s^{1,\epsilon,N} ds; t \in [0,T])$ is a Brownian motion on $(\Omega, \mathcal{F}_T, \widetilde{\mathbb{Q}})$. Proposition 3.1 ensures that

$$\widetilde{\mathbb{Q}} \circ (X^{1,\epsilon,N}, U^{1,\epsilon,N}, K^{1,\epsilon,N})^{-1} = \mathbb{Q} \circ (X^c, U^c, -2 \sum_{0<s\leq \cdot} U_{s^-}^c \mathbb{1}_{\{X_s^c=0\}})^{-1},$$

where (X^c, U^c) is the solution of (3.2). For the jump component $K^{1,\epsilon,N}$, using the change of probability $d\widetilde{\mathbb{Q}} = Z_T^{1,\epsilon,N} d\mathbb{Q}$, as $\beta + \delta_0 \leq T$, we have

$$\mathbb{Q}\left(\left|K_{\delta+\beta}^{1,\epsilon,N} - K_\beta^{1,\epsilon,N}\right| \geq \eta\right) \leq \sqrt{\frac{1}{\eta}} \exp\left(T\|b\|_\infty^2/2\right) \sqrt{\mathbb{E}_{\mathbb{Q}}\left[\left|\sum_{\beta<s\leq\beta+\delta} -2U_{s^-}^c \mathbb{1}_{\{X_s^c=0\}}\right|\right]}.$$

According to Proposition 3.1, $s \mapsto -2U_{s^-}^c \mathbb{1}_{\{X_s^c=0\}}$ is non-negative. Moreover,

$$\sup_{\delta\in[0,\delta_0]} \mathbb{E}_{\mathbb{Q}}\left[\sum_{\beta<s\leq\beta+\delta} -2U_{s^-}^c \mathbb{1}_{\{X_s^c=0\}}\right] \leq \sqrt{T} + 2\mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T]} |U_t^c|\right] < +\infty.$$

Owing to the right continuity of $t \mapsto \sum_{0<s\leq t} -2U_{s^-}^c \mathbb{1}_{\{X_s^c=0\}}$, and by the Dominated Convergence Theorem, one immediately gets that

$$\lim_{\delta_0\to 0^+} \mathbb{E}_{\mathbb{Q}}\left[\sum_{\beta<s\leq\beta+\delta_0} -2U_{s^-}^c \mathbb{1}_{\{X_s^c=0\}}\right] = 0.$$

Consequently, $\sup_{\delta\in[0,\delta_0]} \mathbb{Q}\left(\left|K_{\delta+\beta}^{1,\epsilon,N} - K_\beta^{1,\epsilon,N}\right| \geq \eta\right)$ tends to 0 when δ_0 goes to 0. For the velocity component, notice that $\mathbb{Q}(|U_{\delta+\beta}^{1,\epsilon,N} - U_\beta^{1,\epsilon,N}| \geq 3\eta)$ is bounded by

$$\mathbb{Q}\left(\left|K_{\delta+\beta}^{1,\epsilon,N} - K_\beta^{1,\epsilon,N}\right| \geq \eta\right) + \frac{1}{\eta}\mathbb{E}_{\mathbb{Q}}\left[\left|W_{\delta+\beta}^1 - W_\beta^1\right|\right] + \frac{1}{\eta}\mathbb{E}_{\mathbb{Q}}\left[\left|\int_\beta^{\delta+\beta} D_s^{1,\epsilon,N} ds\right|\right].$$

We already studied the first term. The boundedness of b provides that

$$\mathbb{E}_{\mathbb{Q}}\left[\left|\int_\beta^{\delta+\beta} D_s^{1,\epsilon,N} ds\right|\right] \leq \delta\|b\|_\infty,$$

and, since β is discrete valued,

$$\mathbb{E}_{\mathbb{Q}}\left[\left|W_{\delta+\beta}^1 - W_\beta^1\right|\right] \leq \mathbb{E}_{\mathbb{Q}}\left[\sup_{t\in[0,T-\delta]} \left|W_{\delta+t}^1 - W_t^1\right|\right] \leq \sqrt{\frac{2}{\pi}\delta}.$$

These estimates imply that

$$\lim_{\delta_0\to 0^+} \sup_{\delta\in[0,\delta_0]} \mathbb{Q}\left(\left|U_{\delta+\beta}^{1,\epsilon,N} - U_\beta^{1,\epsilon,N}\right| \geq 3\eta\right) = 0,$$

which ends the proof. □

Step 2. Identification of limit points. Let $\pi^{\epsilon,\infty}$ denote the limit of a converging subsequence of $\left\{\pi^{\epsilon,N}\right\}_{N\geq 1}$ that we still index by N for simplicity. Below, we show that all elements of the support of $\pi^{\epsilon,\infty}$ satisfy (MP_ϵ)-*(i)* to (MP_ϵ)-*(iv)*. First, since (X_0^i, U_0^i) are i.i.d. with law μ_0, (MP_ϵ)-*(i)* is clearly satisfied, for $\pi^{\epsilon,\infty}$-a.e. $m \in \mathcal{M}(\mathcal{E})$. Next we prove

Lemma 4.6. *For $\pi^{\epsilon,\infty}$-almost all $m \in \mathcal{M}(\mathcal{E})$, $m((x_0,u_0) \in dx\,du) = \mu_0(dx,du)$ and*

$$\begin{cases} x_t = x_0 + \displaystyle\int_0^t u_s ds, \\ u_t = u_0 + \displaystyle\int_0^t B_\epsilon[x_s,u_s;m_s]\,ds + w_t + k_t, \\ m_t \text{ is the law of } (x_t,u_t) \text{ under } m, \text{ for all } t \in [0,T], \end{cases} \tag{4.12}$$

where $(w_t; t \in [0,T])$ is a Wiener process.

Proof. We preliminary check that $\pi^{\epsilon,\infty}$ has full measure on the set of all $m \in \mathcal{M}(\mathcal{E})$ such that the process $t \mapsto (x_t, u_t - k_t)$ is m-a.s. continuous. Since $\mathcal{C}([0,T];\mathbb{R}^2)$ is closed for the Skorokhod topology, the set

$$D = \{(x,u,k) \in \mathcal{E} \text{ s.t. } t \mapsto (x_t, u_t - k_t) \text{ is continuous}\}$$

is closed in $\mathcal{E}$. Applying twice the Portemanteau Theorem (see, *e.g.*, Billingsley [4]), we get first that

$$m(D) \geq \limsup_{n\to+\infty} m_n(D),$$

for all sequences m_n of $\mathcal{M}(\mathcal{E})$ converging weakly to m. Thus $\{m \in \mathcal{M}(\mathcal{E}) \text{ s.t. } m(D) = 1\}$ is a closed subset of $\mathcal{M}(\mathcal{E})$. And next

$$\pi^{\epsilon,\infty}(\{m \in \mathcal{M}(\mathcal{E}) \text{ s.t. } m(D) = 1\}) \geq \lim_{N\to+\infty} \mathbb{Q}\left(\bar{\mu}^{\epsilon,N}(D) = 1\right).$$

Since $\mathbb{Q}$-a.s., $\bar{\mu}^{\epsilon,N}$ has full measure on D, we get our claim.

For $f \in \mathcal{C}_b^2(\mathbb{R}^2)$ and for $\pi^{\epsilon,\infty}$-a.e. $m \in \mathcal{M}(\mathcal{E})$, the process

$$\begin{aligned} t \mapsto & f(x_t, u_t - k_t) - f(x_0,u_0) - \int_0^t \left(u_s \partial_x f(x_s, u_s - k_s) + \frac{1}{2}\partial_u^2 f(x_s, u_s - k_s)\right) ds \\ & - \int_0^t \left(B_\epsilon[x_s,u_s;m_s]\partial_u f(x_s,u_s-k_s)\right) ds \end{aligned} \tag{4.13}$$

is m-a.s. continuous, and we prove now that it is a martingale. To this aim, for all $0 \leq t_1 \leq \cdots \leq t_q \leq s < t \leq T$ and all finite families of functions $\{\psi_i\}_{1\leq i\leq q}$ in $\mathcal{C}_b(\mathbb{R}^3)$, we want to show that

$$\mathbb{E}_{\pi^{\epsilon,\infty}}\left[\left|F_{t_1,t_2,\ldots,t_q,s,t}(m)\right|\right] = 0, \tag{4.14}$$

where $F_{t_1,t_2,\ldots,t_q,s,t} : \mathcal{M}(\mathcal{E}) \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned} & F_{t_1,t_2,\ldots,t_q,s,t}(m) \\ & = \mathbb{E}_m\left[\prod_{i=1}^q \psi_i(x_{t_i},u_{t_i},k_{t_i})\Big(f(x_t,u_t-k_t) - f(x_s,u_s-k_s) \right. \\ & \qquad \left. - \int_s^t \left(u_\theta \partial_x f(x_\theta, u_\theta - k_\theta) + \frac{1}{2}\partial_u^2 f(x_\theta, u_\theta - k_\theta)\right) d\theta\Big)\right] \\ & \quad - \mathbb{E}_m\left[\prod_{i=1}^q \psi_i(x_{t_i},u_{t_i},k_{t_i}) \int_s^t B_\epsilon\left[x_\theta,u_\theta;m_\theta\right] \partial_u f(x_\theta,u_\theta-k_\theta)\, d\theta\right] \\ & =: \mathbb{E}_m\left[\Phi_{t_1,t_2,\ldots,t_q,s,t}\right] + \mathbb{E}_m\left[\Psi_{t_1,t_2,\ldots,t_q,s,t}(m)\right]. \end{aligned} \tag{4.15}$$

Let us quote a result from Graham and Méléard [9].

Lemma 4.7. *Let w denote the canonical process in $\mathbb{D}([0,T];\mathbb{R}^p)$. Let $\mathcal{P} \in \mathcal{M}(\mathcal{M}(\mathbb{D}([0,T];\mathbb{R}^p)))$. Then the set*

$$\mathbf{D}_{\mathcal{P}} = \{\theta \in [0,T] \ s.t. \ \mathcal{P}\left(\{m \in \mathcal{M}\left(\mathbb{D}([0,T];\mathbb{R}^p)\right) \ s.t. \ m\left(|\triangle w_\theta| > 0\right) > 0\}\right) > 0\}$$

is at most countable.

For $t_1 \leq \cdots \leq t_q \leq s \leq t \leq T$, let us consider a bounded approximation $\Phi^{\kappa}_{t_1,t_2,\ldots,t_q,s,t}$ of $\Phi_{t_1,t_2,\ldots,t_q,s,t}$, substituting u_θ with $-\kappa \vee (u_\theta \wedge \kappa)$ in (4.15); namely

$$\begin{aligned}\Phi^{\kappa}_{t_1,t_2,\cdots,t_n,s,t} := \prod_{i=1}^{q} \psi_i(x_{t_i}, u_{t_i}, k_{t_i}) \Big(& f(x_t, u_t - k_t) - f(x_s, u_s - k_s) \\ & - \int_s^t \left(-\kappa \vee (u_\theta \wedge \kappa) \partial_x f(x_\theta, u_\theta - k_\theta) + \frac{1}{2} \partial_u^2 f(x_\theta, u_\theta - k_\theta) \right) d\theta \Big).\end{aligned}$$

According to Lemma 4.7, there exists a countable set of times $\mathbf{D}_{\pi^{\epsilon,\infty}}$ such that, for all $0 \leq t_1 \leq \cdots \leq t_q \leq s < t \leq T$ outside $\mathbf{D}_{\pi^{\epsilon,\infty}}$, the mappings $\omega \mapsto \Phi_{t_1,t_2,\cdots,t_q,s,t}(\omega)$ and $\omega \mapsto \Psi_{t_1,t_2,\ldots,t_q,s,t}(\widetilde{m},\omega)$ are m-a.s. continuous on the sample space $\mathcal{E}$, for $\pi^{\epsilon,\infty}$-a.e. m and $\widetilde{m}$. Outside $\mathbf{D}_{\pi^{\epsilon,\infty}}$, it is clear that $\omega \mapsto \Phi^{\kappa}_{t_1,t_2,\cdots,t_q,s,t}$ is continuous, and so $m \mapsto \mathbb{E}_m\left[\Phi^{\kappa}_{t_1,t_2,\cdots,t_q,s,t}\right]$ is continuous on $\mathcal{M}(\mathcal{E})$. The continuity of $m \mapsto \mathbb{E}_m\left[\Psi_{t_1,t_2,\cdots,t_q,s,t}(m)\right]$ is obtained as a simple extension of the unconfined situation treated in the proof of [5, Proposition 5.6]. Then, by Fatou's Lemma and the weak convergence of $\pi^{\epsilon,N}$, we get

$$\begin{aligned}&\mathbb{E}_{\pi^{\epsilon,\infty}} \left|\mathbb{E}_m\left(\Phi_{t_1,t_2,\cdots,t_q,s,t} + \Psi_{t_1,t_2,\cdots,t_q,s,t}(m)\right)\right| \\ &\leq \liminf_{\kappa\to+\infty} \lim_{N\to+\infty} \mathbb{E}_{\pi^{\epsilon,N}} \left|\mathbb{E}_m\left(\Phi^{\kappa}_{t_1,t_2,\cdots,t_q,s,t} + \Psi_{t_1,t_2,\cdots,t_q,s,t}(m)\right)\right|.\end{aligned} \tag{4.16}$$

But,

$$\begin{aligned}&\lim_{N\to+\infty} \mathbb{E}_{\mathbb{Q}} \left|\mathbb{E}_{\bar{\mu}^{\epsilon,N}}\left(\Phi^{\kappa}_{t_1,t_2,\cdots,t_q,s,t} + \Psi_{t_1,t_2,\cdots,t_q,s,t}(\bar{\mu}^{\epsilon,N})\right)\right| \\ &\leq \lim_{N\to+\infty} \mathbb{E}_{\mathbb{Q}} \left|F_{t_1,t_2,\cdots,t_q,s,t}(\bar{\mu}^{\epsilon,N})\right| + \prod_{i=1}^{q} \|\psi_i\|_\infty^2 \|\partial_x f\|_\infty \lim_{N\to+\infty} \frac{2}{\kappa N} \sum_{i=1}^{N} \mathbb{E}_{\mathbb{Q}}[\sup_{\theta\in[0,T]} |U^{i,\epsilon,N}_\theta|^2].\end{aligned}$$

Under $\mathbb{Q}$, $F_{t_1,t_2,\ldots,t_q,s,t}(\bar{\mu}^{\epsilon,N})$ writes

$$\frac{1}{N} \sum_{j=1}^{N} \prod_{i=1}^{q} \psi_i(X^{j,\epsilon,N}_{t_i}, U^{j,\epsilon,N}_{t_i}, K^{j,\epsilon,N}_{t_i}) \int_s^t \partial_u f(X^{j,\epsilon,N}_\theta, U^{j,\epsilon,N}_\theta - K^{j,\epsilon,N}_\theta) dW^j_\theta.$$

Owing to the independence of the Brownian family $\{W^j\}_{j\in\mathbb{N}^*}$ and the exchangeability of the particle system, it holds that

$$\mathbb{E}_{\mathbb{Q}}\left[\left(F_{t_1,t_2,\ldots,t_q,s,t}(\bar{\mu}^{\epsilon,N})\right)^2\right] \leq (t-s)\|\partial_u f\|_\infty^2 \frac{1}{N} \prod_{i=1}^{q} \|\psi_i\|_\infty^2.$$

Consequently, $\mathbb{E}_{\mathbb{Q}}[|F_{t_1,t_2,\ldots,t_q,s,t}(\bar{\mu}^{\epsilon,N})|]$ tends to 0 when N goes to $+\infty$. Furthermore, using the momentum estimate (4.11), one obtains that

$$\lim_{\kappa\to+\infty} \lim_{N\to+\infty} \frac{2}{\kappa N} \sum_{i=1}^{N} \mathbb{E}_{\mathbb{Q}}\left[\sup_{\theta\in[0,T]} \left|U^{i,\epsilon,N}_\theta\right|^2\right] \leq \lim_{\kappa\to+\infty} \frac{2}{\kappa} \left(\sup_{N\geq 1} \mathbb{E}_{\mathbb{Q}}\left[\sup_{\theta\in[0,T]} \left|U^{1,\epsilon,N}_\theta\right|^2\right]\right) = 0.$$

Coming back to (4.16), we get that for all $0 \le t_1 \le \cdots \le t_q \le s \le t \le T$ outside $\mathbf{D}_{\pi^{\epsilon,\infty}}$,

$$\mathbb{E}_{\pi^{\epsilon,\infty}} \left| \mathbb{E}_m \left[\Phi_{t_1,t_2,\cdots,t_q,s,t} + \Psi_{t_1,t_2,\cdots,t_q,s,t}(m) \right] \right| = 0.$$

Now, any t_i in $[0,T]$ can be approximated with a decreasing sequence $\{t_i^l\}_{l\in\mathbb{N}}$ outside $\mathbf{D}_{\pi^{\epsilon,\infty}}$. Owing to the right-continuity of $(t_1, t_2, \cdots, t_q, s, t) \mapsto F_{t_1,t_2,\cdots,t_q,s,t}(m)$ for all $m \in \mathcal{E}$, and Fatou's Lemma, (4.14) holds true, and we conclude on our claim. □

We complete *Step* 2 by identifying $(k_t; t \in [0,T])$.

Lemma 4.8. *For $\pi^{\epsilon,\infty}$-a.e. $m \in \mathcal{M}(\mathcal{E})$, $(x_t, u_t, k_t; t \in [0,T])$ satisfies m-a.s.:*

(a) For all jump times $t \in [0,T]$ of u, $\triangle u_t = -2u_{t^-}$.

(b) k is a non-decreasing function, and $k_t = \int_0^t \mathbb{1}_{\{x_s = 0\}} dk_s$, $\forall\, t \in [0,T]$.

(c) The set $\{x_t \in [0,T]$ s.t. $x_t = 0\}$ is at most countable.

As a consequence of Lemma 4.8, $(k_t; t \in [0,T])$ in (4.12) is a pure jump process, which stands for the cumulative jump part of the velocity $(u_t; t \in [0,T])$ on the null set of $t \mapsto x_t$; namely $k_t = -2\sum_{0<s\le t} u_{s^-} \mathbb{1}_{\{x_s=0\}}$ for all t. The existence of the time-marginal densities of m follows by applying Girsanov's Theorem. This ends *Step 2*.

Proof of Lemma 4.8. Let us introduce the set $\mathcal{H}$ which consists in all elements (x, u, k) of $\mathcal{E}$ such that u and k satisfies the properties *(a)* and *(b)*. We observe the following.

Lemma 4.9. *Under the Skorokhod topology, $\mathcal{H}$ is a closed subset of $\mathcal{E}$.*

We postpone the proof of this statement at the end of this section. With the observation that $\pi^{\epsilon,N}(\mathcal{M}(\mathcal{H})) = 1$ for all $N \ge 1$, the Portemanteau Theorem gives

$$\pi^{\epsilon,\infty}\left(\mathcal{M}(\mathcal{H})\right) \ge \limsup_{N\to+\infty} \pi^{\epsilon,N}\left(\mathcal{M}(\mathcal{H})\right) = 1.$$

Consequently, for $\pi^{\epsilon,\infty}$-a.e. $m \in \mathcal{M}(\mathcal{E})$, $(x_t, u_t, k_t; t \in [0,T])$ satisfies *(a)* and *(b)*. To prove *(c)*, we fix $m \in \mathcal{M}(\mathcal{H})$. Since $(w_t; t \in [0,T])$ in (4.12) is a $\mathcal{B}_t$-Brownian motion and b is bounded, we define $\widetilde{m}$ by $d\widetilde{m} = z_T^\epsilon dm$ with $(z_t^\epsilon; t \in [0,T])$ given by

$$z_t^\epsilon = \exp\left(-\int_0^t B_\epsilon[x_s, u_s; m_s] dw_s - \frac{1}{2}\int_0^t |B_\epsilon[x_s, u_s; m_s]|^2 ds \right).$$

By Girsanov Theorem, $(\widetilde{w}_t^\epsilon := \int_0^t B_\epsilon[x_s, u_s; m_s]\, ds + w_t; t \in [0,T])$ is a Wiener process under $\widetilde{m}$, and

$$\begin{cases} x_t = x_0 + \displaystyle\int_0^t u_s ds, \\ u_t = u_0 + \widetilde{w}_t^\epsilon + k_t, \ \forall\, t \in [0,T]. \end{cases} \tag{4.17}$$

By Assumption $(H\text{-}i)$, $x_0 > 0$ $\widetilde{m}$-a.s., and thus the first hitting time τ_1 at 0 is positive. In particular, $u_{\tau_1^-} < 0$. Thus we can define $\tau_2 = \inf\{t > \tau_1 \text{ s.t. } x_t = 0\}$, and according to *(a)* and *(b)*, we observe that $k_{\tau_1} = -2u_{\tau_1^-}$. Repeating this argument, we prove that the sequence $\{\tau_n; n \in \mathbb{N}\}$ exists, and further that $(x, u, k, \widetilde{w}^\epsilon)$ under $\widetilde{m}$ is a solution in law of (3.2). By Proposition 3.1, we conclude that τ_n grows to infinity, and thus that *(c)* holds true. □

Proof of Lemma 4.9. Let $\{\zeta^n = (x^n, u^n, k^n); n \in \mathbb{N}\}$ be a sequence in $\mathcal{H}$, converging to $\zeta = (x, u, k)$ in $\mathcal{E}$. According to the definition of the Skorokhod topology (see, *e.g.*, Jacod and Shiryaev [10]), there exists a sequence of continuous increasing functions $\{\lambda_n\}_{n\in\mathbb{N}}$ defined on $[0, T]$ such that for all n, $\lambda_n(0) = 0$, $\lambda_n(T) = T$,

$$\begin{aligned} &\lim_{n\to+\infty} \sup_{t\in[0,T]} |\lambda_n(t) - t| = 0, \quad \lim_{n\to+\infty} \sup_{t\in[0,T]} |\zeta^n(\lambda_n(t)) - \zeta(t)| = 0, \\ &\text{and for all } t \in [0, T], \quad \lim_{n\to+\infty} |\Delta\zeta^n(\lambda_n(t)) - \Delta\zeta(t)| = 0. \end{aligned} \tag{4.18}$$

According to (4.18), $\triangle u$ and k are given by

$$\begin{aligned} &\triangle u(t) = \lim_{n\to+\infty} \triangle u^n(\lambda_n(t)) = -\lim_{n\to+\infty} 2u^n(\lambda_n(t)^-) = -2u(t^-), \ \forall\, t \in [0, T], \\ &k(t) - k(s) = \lim_{n\to+\infty} (k^n(\lambda_n(t)) - k^n(\lambda_n(s))) \geq 0, \ \forall\, 0 \leq s < t \leq T. \end{aligned}$$

To conclude that (x, u, k) belongs to $\mathcal{H}$, it remains to show that $k(t) = \int_0^t \mathbb{1}_{\{x(s)\,=\,0\}} dk(s)$, for all $t \in [0, T]$. Since $x \geq 0$ on $[0, T]$, we prove that $\int_0^T x(s)\, dk(s) = 0$. Let us check that

$$\lim_{n\to+\infty} \int_0^T x^n(s) dk^n(s) = \int_0^T x(s)\, dk(s). \tag{4.19}$$

As $\int_0^T x^n(s)dk^n(s) = 0$ for all $n \in \mathbb{N}$, this will give our claim. Using the change of variable $s \mapsto \lambda_n^{-1}(s)$, and since x and x^n are continuous,

$$\begin{aligned} &\left| \int_0^T x^n(s)dk^n(s) - \int_0^T x(s)\, dk(s) \right| \\ &\leq \left| \int_0^T (x^n(\lambda^n(s)) - x(s))\, d(k^n \circ \lambda_n)(s) \right| + \left| \int_0^T x(s)\, d(k^n \circ \lambda_n)(s) - \int_0^T x(s)dk(s) \right| \\ &\leq \max_{t\in[0,T]} |x^n(\lambda_n(t)) - x(t)|\, |k^n(T)| + \left| \int_0^T x(s)d(k^n \circ \lambda_n)(s) - \int_0^T x(s)dk(s) \right|. \end{aligned}$$

According to (4.18), $\max_{t\in[0,T]} |x^n(\lambda_n(t)) - x(t)|$ tends to 0 as n tends to $+\infty$, the measure $d(k^n \circ \lambda_n)$ converges weakly to dk and (4.19) follows. □

4.3 Existence result for (*MP*)

Now we construct a solution to the martingale problem (*MP*). We only sketch the main steps, which combine arguments in [5, Section 5] with those in Section 4.2.

Proposition 4.10. *The solution $\mathbb{P}^\epsilon$ to the martingale problem (MP_ϵ) converges to the solution of the martingale problem (MP).*

Proof. We mimic the proof of Lemma 4.4 to check that $\mathbb{P}^\epsilon$ is tight on $\mathcal{E}$. Let $\mathbb{P}^\epsilon$ denote a converging subsequence, and $\mathbb{P}$ its limit. Following the proof-steps of [5, Proposition 5.6], we further verify that $\mathbb{P}$ satisfies Conditions *(i)* and *(ii)* of (*MP*). From *(iii)*, *(iv)* follows simply by replicating the proof of Lemma 4.8.

For the martingale property *(iii)*, replicating the main steps of the proof of Lemma 4.6, we observe that *(iii)* follows by proving that

$$\begin{aligned} &\lim_{\epsilon\to0^+} \mathbb{E}_{\mathbb{P}^\epsilon} \left[\prod_{i=1}^n \psi_i(x_{t_i}, u_{t_i}, k_{t_i}) \int_s^t \left(B_\epsilon \left[x_\theta, u_\theta; \rho_\theta^\epsilon \right] \cdot \nabla_u f(x_\theta, u_\theta - k_\theta) \right) d\theta \right] \\ &= \mathbb{E}_{\mathbb{P}} \left[\prod_{i=1}^n \psi_i(x_{t_i}, u_{t_i}, k_{t_i}) \int_s^t \left(B \left[x_\theta, u_\theta; \rho_\theta \right] \cdot \nabla_u f(x_\theta, u_\theta - k_\theta) \right) d\theta \right], \end{aligned}$$

for all $0 \le t_1 \le \cdots \le t_q \le s < t \le T$ outside $\mathbf{D}_{\mathbb{P}} := \{\theta \in [0,T] \text{ s.t. } \mathbb{P}(|\triangle u_\theta| + |\triangle k_\theta| > 0)\}$. As outlined in [5], the above convergence can be proved by adapting some arguments from Stroock and Varadhan [16], as long as we show that

$$\lim_{|h|,|\delta|\to 0} \limsup_{\epsilon\to 0^+} \int_{\mathcal{D}\times\mathbb{R}^d} |\rho_t^\epsilon(x+h,u+\delta) - \rho_t^\epsilon(x,u)|\, dx\, du = 0, \ \forall\, t \in (0,T]. \tag{4.20}$$

Following the proof of [5, Lemma 5.9], (4.20) is obtained with the mild equation (4.8) and the property that

$$(y,v) \mapsto \int_{\mathcal{D}\times\mathbb{R}} |\nabla_v \Gamma(\theta; y, v; x+h, u+\delta) - \nabla_v \Gamma(\theta; y, v; x, u)|\, dx\, du$$

is bounded and continuous for all $\theta > 0$, $h \in \mathcal{D}$ and $\delta \in \mathbb{R}^d$ (this assertion is deduced from Proposition 3.3-*(ii)*). □

5 Proof of Theorem 2.3

Let $\mathbb{P}$ be a solution to the martingale problem (MP) and let $(\rho_t;\ t \in (0,T])$ be the related time-marginal densities.

Proof of *(a)*. According to Remark 3.2 and since we have the uniqueness in law for the solution of (3.9), for all $0 < t_0 < t \le T$, a.e. $(x,u) \in \overline{\mathcal{D}} \times \mathbb{R}^d$,

$$\rho_t(x,u) = \Big(\widetilde{\rho}_t(x,u) + \widetilde{\rho}_t((x', -x^{(d)}), (u', -u^{(d)}))\Big)\, \mathbb{1}_{\{x^{(d)} \ge 0\}},$$

where $(\widetilde{\rho}_t;\ t \in (0,T])$ are the time-marginal densities of the free Langevin process (3.9) with drift $\widetilde{\beta}$ constructed from $\beta(t,x,u) := B[x,u;\rho_t]$. Moreover, $(\widetilde{\rho}_t;\ t \in (0,T])$ is solution to the linear mild equation

$$\begin{aligned}\widetilde{\rho}_t(x,u) = &\int_{\mathbb{R}^{2d}} g_d(t-t_0; y, v; x, u)\widetilde{\rho}_{t_0}(y,v)dydv \\ &+ \int_{t_0}^t \int_{\mathbb{R}^{2d}} (\widetilde{\beta}(s,y,v)\cdot \nabla_v g_d(t-s; y, v; x, u))\widetilde{\rho}_s(y,v)dydvds.\end{aligned}$$

Then, for all $0 < t_0 < t \le T$, a.e. $u \in \mathbb{R}^d$, $x, x_0 \in \mathbb{R}^{2d}$,

$$\begin{aligned}&|\widetilde{\rho}_t(x,u) - \widetilde{\rho}_t(x_0,u)| \\ &\le \int_{\mathbb{R}^{2d}} |g_d(t-t_0; y, v; x, u) - g_d(t-t_0; y, v; x_0, u)|\, \widetilde{\rho}_{t_0}(y,v)dy\, dv \\ &\quad + \|b\|_\infty \int_{t_0}^t \int_{\mathbb{R}^{2d}} |\nabla_v g_d(t-s; y, v; x, u) - \nabla_v g_d(t-s; y, v; x_0, u)|\, \widetilde{\rho}_s(y,v)dy\, dv\, ds.\end{aligned} \tag{5.1}$$

The proof of the Hölder-continuity is then based on the following regularity result on the kernel g_d.

Lemma 5.1. *For $p > 4d+2$, let $h \in L^p(\mathbb{R}^{2d})$ and $H \in L^p((0,T)\times\mathbb{R}^{2d})$. Then, for all $(x,x_0) \in \mathbb{R}^{2d}$, and for all $0 < t_0 < t \le T$, it holds that*

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} |g_d(t-t_0; y, v; x, u) - g_d(t-t_0; y, v; x_0, u)|\, |h(y,v)|\, dy\, dv \le c|x-x_0|^{\frac{1}{3}-\frac{4d+1}{3p}} \|h\|_{L^p(\mathbb{R}^{2d})}, \\ &\int_{(t_0,t)\times\mathbb{R}^{2d}} |\nabla_v g_d(t-s; y, v; x, u) - \nabla_v g_d(t-s; y, v; x_0, u)|\, |H(s,y,v)|\, dy\, dv\, ds \\ &\le c|x-x_0|^{\frac{1}{3}-\frac{4d+2}{3p}} \|H\|_{L^p((t_0,t)\times\mathbb{R}^{2d})}.\end{aligned}$$

The proof of Lemma 5.1 is a straightforward consequence of Proposition 5.2 (using the step-proof of Theorem 1.2) in Manfredini and Polidoro [13].

To apply Lemma 5.1, we exhibit an L^p estimate on $\widetilde{\rho}_t$, for $t > 0$: for the solution $(\mathbb{P}, Y, V, \widetilde{W})$ to (3.9), we consider $\widetilde{\mathbb{P}}$ defined by $d\widetilde{\mathbb{P}} = Z_T d\mathbb{P}$ with

$$Z_t = \exp\left(- \int_0^t (\widetilde{\beta}(s, Y_s, V_s) \cdot d\widetilde{W}_s) - \frac{1}{2} \int_0^t \left| \widetilde{\beta}(s, Y_s, V_s) \right|^2 ds \right).$$

Then for all bounded measurable functions F, and for all $l > 1$,

$$\left| \int_{\mathbb{R}^{2d}} F(x,u) \widetilde{\rho}_t(x,u) dx\, du \right| = \left| \mathbb{E}_{\widetilde{\mathbb{P}}} \left[Z_T^{-1} F(Y_t, V_t) \right] \right| \leq C \left[\mathbb{E}_{\widetilde{\mathbb{P}}} \left| F(Y_t, V_t) \right|^l \right]^{\frac{1}{l}},$$

for a constant $C > 0$ depending only on $\|b\|_\infty$ and l. Moreover,

$$\mathbb{E}_{\widetilde{\mathbb{P}}} \left[\left| F(Y_t, V_t) \right|^l \right] = \int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} g_d(t; y, v; x, u) \mu_0(dy, dv) \right) |F(x,u)|^l dx\, du,$$

and for $F \in L^{lq}(\mathbb{R}^{2d})$, for (r, q) such that $\frac{1}{r} + \frac{1}{q} = 1$, we obtain that

$$\left| \int_{\mathbb{R}^{2d}} F(x,u) \widetilde{\rho}_t(x,u) dx\, du \right| \leq C \left\| \int_{\mathbb{R}^{2d}} g_d(t; y, v; \cdot, \cdot) \mu_0(dy, dv) \right\|_{L^r(\mathbb{R}^{2d})}^{\frac{1}{l}} \|F\|_{L^{lq}(\mathbb{R}^{2d})}.$$

Note that for all $r > 1$, for all $t > 0$ and $(y, v) \in \mathbb{R}^{2d}$, $\|g_d(t; y, v; \cdot, \cdot)\|_{L^r(\mathbb{R}^{2d})} \leq C(r) t^{-2d(1-1/r)}$. This gives

$$\left| \int_{\mathbb{R}^{2d}} F(x,u) \widetilde{\rho}_t(x,u) dx\, du \right| \leq C t^{-\frac{2d}{lq}} \|F\|_{L^{lq}(\mathbb{R}^{2d})}.$$

Choosing $lq = p/(p-1)$ and $l = 2$, we get $\|\widetilde{\rho}_t\|_{L^p(\mathbb{R}^{2d})} \leq C t^{-\frac{2d(p-1)}{p}}$.

By Lemma 5.1, for $x, x_0 \in \mathbb{R}^{2d}$, and $p > 4d + 2$, there exists a constant $C > 0$ such that, for a.e. $0 < t_0 < t \leq T$, $u \in \mathbb{R}^d$,

$$\begin{aligned}
&|\widetilde{\rho}_t(x,u) - \widetilde{\rho}_t(x_0,u)| \\
&\leq C \left(|x - x_0|^{\frac{1}{3} - \frac{4d+1}{3p}} \|\widetilde{\rho}_{t_0}\|_{L^p(\mathbb{R}^{2d})} + |x - x_0|^{\frac{1}{3} - \frac{4d+2}{3p}} \|\widetilde{\rho}\|_{L^p((t_0,t) \times \mathbb{R}^{2d})} \right) \\
&\leq C t_0^{-\frac{2d(p-1)}{p}} \left(|x - x_0|^{\frac{1}{3} - \frac{4d+1}{3p}} + |x - x_0|^{\frac{1}{3} - \frac{4d+2}{3p}} (t - t_0)^{\frac{1}{p}} \right)
\end{aligned}$$

and *(a)* follows.

Proof of *(b)*. We start by proving that the measure $\sum_{n \in \mathbb{N}} \mathbb{P} \circ \left(\tau_n, x_{\tau_n}, u_{\tau_n^-} \right)^{-1}$ is finite on the set $\{(t, x, u) \in (0, T) \times \partial\mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}) > 0\}$, namely:

$$\sum_{n \in \mathbb{N}} \mathbb{P}(\tau_n \leq T) < +\infty. \tag{5.2}$$

We proceed as in the proof of Lemma 4.8 by removing the drift term B thanks to the change of probability measure $d\widetilde{\mathbb{P}} = z_T d\mathbb{P}$, with $(z_t; t \in [0, T])$ defined as

$$z_t = \exp\left(- \int_0^t (B[x_s, u_s; \rho_s] \cdot dw_s) - \frac{1}{2} \int_0^t |B[x_s, u_s; \rho_s]|^2 ds \right).$$

With $\widetilde{w}$ defined similarly to $\widetilde{w}^\epsilon$ in (4.17), $(x, u, \widetilde{w})$ solves (3.1) under $\widetilde{\mathbb{P}}$ and the law of τ_n is given by (3.5). Then

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \le T\}}\right] = \mathbb{E}_{\widetilde{\mathbb{P}}}\left[z_T^{-1}\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \le T\}}\right] \le C\sqrt{\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left(\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \le T\}}\right)^2\right]},$$

for some finite constant $C > 0$. Since the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is non-decreasing, we also get

$$\left(\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \le T\}}\right)^2 \le 2\sum_{n\in\mathbb{N}}\left(\sum_{m=0}^{n} \mathbb{1}_{\{\tau_m \le T\}}\right)\mathbb{1}_{\{\tau_n \le T\}} \le 2\sum_{n\in\mathbb{N}}(n+1)\mathbb{1}_{\{\tau_n \le T\}}.$$

Using (3.6), one observes that

$$\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left(\sum_{n\in\mathbb{N}} \mathbb{1}_{\{\tau_n \le T\}}\right)^2\right] \le 2\sum_{n\in\mathbb{N}}(n+1)\widetilde{\mathbb{P}}\left(\tau_n \le T\right) \le 2\sum_{n\in\mathbb{N}}\frac{(n+1)}{2^{n-1}}.$$

We deduce (5.2).

For (2.2), one may observe that, for any $f \in \mathcal{C}_c^\infty(Q_T)$,

$$\begin{aligned}&\int_{(0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^d} u^{(d)}\gamma(\rho)(t,(x',0),u)f(t,(x',0),u)dt\,dx'\,du\\ &= \lim_{\delta\to 0^+}\frac{1}{\delta}\int_0^\delta\int_{(0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^d} u^{(d)}\rho_t((x',x^{(d)}),u)f(t,(x',x^{(d)}),u)dt\,dx'\,du\,dx^{(d)}.\end{aligned} \tag{5.3}$$

For a fixed $\delta > 0$, we set $\beta_\delta(y) := 1 - (\frac{y}{\delta}\wedge 1)$, a.e. differentiable on $(0,+\infty)$. The Itô's formula induces

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}}\left(f(\tau_n,x_{\tau_n},u_{\tau_n}) - f(\tau_n,x_{\tau_n},u_{\tau_n^-})\right)\mathbb{1}_{\{\tau_n \le T\}}\right] - \frac{1}{\delta}\int_0^T \mathbb{E}_{\mathbb{P}}\left[u_{s^-}^{(d)}\mathbb{1}_{\{x_s^{(d)}\le\delta\}}f(s,x_s,u_{s^-})\right]ds\\ &= \mathbb{E}_{\mathbb{P}}\left[\beta_\delta(x_T^{(d)})f(t,x_T,u_T) - \beta_\delta(x_0^{(d)})f(0,x_0,u_0) - \int_0^T \beta_\delta(x_s^{(d)})\left(\partial_s f + \mathcal{A}_{\rho_s}(f)\right)(s,x_s,u_{s^-})ds\right].\end{aligned}$$

Recalling that $\{s\in[0,T]\,;\, x_s^{(d)} = 0\}$ is at most countable $\mathbb{P}$-a.s., we obtain that

$$\begin{aligned}&\lim_{\delta\to 0^+}\frac{1}{\delta}\int_0^\delta\left(\int_{(0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^d} u^{(d)}\rho(t,(x',x^{(d)}),u)f(t,x,u)dt\,dx'\,du\right)dx^{(d)}\\ &= \mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}}\left(f(\tau_n,x_{\tau_n},u_{\tau_n}) - f(\tau_n,x_{\tau_n},u_{\tau_n^-})\right)\mathbb{1}_{\{\tau_n \le T\}}\right],\end{aligned}$$

and coming back to (5.3), for any $f \in \mathcal{C}_c(\Sigma_T)$,

$$\begin{aligned}&\int_{(0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}^d} u^{(d)}\gamma(\rho)(t,(x',0),u)f(t,(x',0),u)dt\,dx'\,du\\ &= \mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}}\left(f(\tau_n,x_{\tau_n},u_{\tau_n}) - f(\tau_n,x_{\tau_n},u_{\tau_n^-})\right)\mathbb{1}_{\{\tau_n \le T\}}\right].\end{aligned} \tag{5.4}$$

Thanks to (5.2), the relation above can be extended to any bounded measurable function f on Σ_T. This ends the proof of *(b)*.

Proof of *(c)*. According to (5.4) and (5.2), we immediately deduce that

$$\|u^{(d)}\gamma(\rho)\|_{L^1(\Sigma_T)} = \sup_{\substack{f\in C_c(\Sigma_T);\\ \sup_{\zeta\in\Sigma_T}|f(\zeta)|=1}} \int_{\Sigma_T} u^{(d)}\gamma(\rho)(t,(x',0),u)f(t,(x',0),u)dt\,dx'\,du < +\infty,$$

and (2.4a) follows.

To prove (2.4b), let us first observe that when $b=0$, the result is obvious since

$$\gamma(\rho)(t,(x',0),u) = \int_{\mathcal{D}\times\mathbb{R}^d} \Gamma(t;y,v;(x',0),u))\mu_0(dy,dv) > 0,$$

with $\Gamma(t;y,v;(x',0),u))$ given in (3.10). Next, for the case with drift, we observe that it suffices to show that for all set $A\in\mathcal{B}((0,T)\times\mathbb{R}^{d-1})$ such that A has positive $dt\otimes dx'$-finite measure,

$$\int_{A\times\{0\le(u\cdot n_{\mathcal{D}})\}} \gamma(\rho)(t,(x',0),u)dt\,dx'du > 0. \tag{5.5}$$

Fix such a set A. Since $\gamma(\rho)(t,x,u)$ is nonnegative,

$$\int_{A\times\{0\le(u\cdot n_{\mathcal{D}})\}} \gamma(\rho)(t,(x',0),u)dt\,dx'du \ge \int_{A\times\{0\le(u\cdot n_{\mathcal{D}})<1\}} (u\cdot n_{\mathcal{D}})\gamma(\rho)(t,(x',0),u)dt\,dx'du.$$

According to (2.3), it follows that

$$\int_{A\times\{0\le(u\cdot n_{\mathcal{D}})\}} \gamma(\rho)(t,(x',0),u)dtdx'du \ge \sum_{n\in\mathbb{N}} \mathbb{P}\left((\tau_n,x'_{\tau_n})\in A, 0<(u_{\tau_n^-}\cdot n_{\mathcal{D}})<1\right).$$

As $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent measures, we only need to check that

$$\sum_{n\in\mathbb{N}} \widetilde{\mathbb{P}}\left((\tau_n,x'_{\tau_n},u^{(d)}_{\tau_n})\in A\times(0,1)\right) > 0.$$

Equivalently, using (2.3)

$$\begin{aligned}&\sum_{n\in\mathbb{N}} \widetilde{\mathbb{P}}\left((\tau_n,x'_{\tau_n})\in A, 0<u^{(d)}_{\tau_n}<1\right)\\ &= \int_{\mathcal{D}\times\mathbb{R}^d}\left(\int_A\int_{\mathbb{R}^{d-1}}\int_0^1 u^{(d)}\Gamma(t;y,v;(x',0),(u',u^{(d)}))du^{(d)}du'dx'\right)\mu_0(dy,dv).\end{aligned}$$

The result follows, observing that $\Gamma(t;y,v;(x',0),(u',u^{(d)}))>0$.

6 Conclusion and perspectives

In this work, we have constructed a confined conditional McKean Lagrangian process $((X_t,U_t);t\in[0,T])$ satisfying the mean no-permeability condition

$$\mathbb{E}\left[(U_t\cdot n_{\mathcal{D}})/X_t=x\right]=0,\ \text{for } (t,x)\in(0,T]\times\partial\mathcal{D}.$$

This study is motivated by the application of Lagrangian stochastic models to the downscaling problem in Computational Fluid Dynamic (CFD).

In [1],[2], the authors construct a PDF method applied to the downscaling problem in meteorology. The goal is to compute a finer scale wind prediction from a coarse one given in

a bounded domain $\mathcal{D}$. To this aim, the authors propose a Lagrangian stochastic model for the atmospheric flow description and construct a particle algorithm to solve this fine resolution. This Lagrangian model is confined in $\mathcal{D}$ and must satisfy a Dirichlet condition of the type

$$\mathbb{E}\left[U_t/X_t = x\right] = V_{\text{coarse}}(t,x), \text{ for } (t,x) \in (0,T) \times \partial\mathcal{D}, \tag{6.1}$$

where V_{coarse} is a given velocity field. This application in CFD motivates at least two future extensions of the present work: the case of a more general domain $\mathcal{D}$, and the case of the non homogeneous boundary condition (6.1).

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