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Is a Brownian motion skew?

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Abstract

We study the asymptotic behavior of the maximum likelihood estimator corresponding to the observation of a trajectory of a Skew Brownian motion, through a uniform time discretization. We characterize the speed of convergence and the limiting distribution when the step size goes to zero, which in this case are non-classical, under the null hypothesis of the Skew Brownian motion being an usual Brownian motion. This allows to design a test on the skewness parameter. We show that numerical simulations that can be easily performed to estimate the skewness parameter, and provide an application in Biology.

Keywords: Skew Brownian motion, statistical estimation, maximum likelihood.

1 Introduction

The Skew Brownian Motion (SBm) has attracted interest within other facts, due to its relations with diffusions with discontinuous coefficients or to media with permeable barriers, being the first example of the solution of a stochastic

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differential equation with the local time of the solution as drift [6]: the SBm $X = \{X_t \colon 0 \le t \le T\}$ can be defined as the strong solution of the stochastic differential equation

$$X_t = x + B_t + \theta \ell_t^x, \tag{1}$$

where $B = \{B_t \colon 0 \le t \le T\}$ is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the initial condition is $x \ge 0$ (the case $x < 0$ is symmetrical), $\theta \in [-1, 1]$ is the *skewness parameter*, and $\ell^x = \{\ell_t^x \colon 0 \le t \le T\}$ is the local time at level zero of the (unknown) solution X of the equation departing from x, defined by

$$\ell_t^x = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{(-\epsilon,\epsilon)}(X_s) ds. \tag{2}$$

In case $x = 0$ we denote $\ell_t^0 = \ell_t$, being this case particularly interesting due to some explicit calculations that can be carried out (see Section 3).

In the literature the skewness parameter is sometimes defined as $p = (\theta + 1)/2$; this second parametrization being more convenient for an alternative construction of the SBm: depart from the reflected Brownian motion and choose, independently with probability $p \in [0, 1]$, whether each particular excursion of the reflected Brownian motion remains positive.

In the special case $\theta = 1$ ($p = 1$), the solution to (1) is the reflected Brownian motion. The case $\theta = 0$ ($p = 1/2$) corresponds to the the standard Brownian motion.

Recently, several papers have considered the SBm in modelling or simulation issues, as well as some optimization problems. See the review by A. Lejay [13] for references on the subject, as well as a survey of the various possible constructions and applications of the SBm.

In this paper we are interested in the statistical estimation of θ, the skewness parameter, when we observe a trajectory of the process through an equally spaced time grid. From the statistical point of view we find this problem interesting because it is in certain sense intermediate between the classical problem of drift estimation in a diffusion, where the measures generated by the trajectories of the process for different values of the parameter are equivalent [11, 17], and the estimation of the variance (the volatility in financial terms) of a diffusion (see for instance [4], or [9] and the references therein), where the probability measures generated by the trajectories are singular for different values of the parameter. At the best of our knowledge, the only estimator of θ is the one constructed by O. Bardou and M. Martinez [1], where they assume that the SBm is reflected at levels 1 and -1 to ensure ergodicity, considering a different scheme of observation of the trajectory.

Our main result states that the maximum likelihood estimator (MLE) corresponding to the observation of a discretization of one trajectory of the process, with the corresponding normalization, satisfies the so called Local Asymptotically Mixed Normality (LAMN) property at the point $\theta = 0$. With this result and the identification of the limiting distribution of the scaled MLE estimator, one may construct some hypothesis test to determine whether or not the Brownian is skew. This fact suggests certain asymptotic properties of the MLE, as exposed for instance in the classical book of Ibragimov and Has'minskii [7]. Nevertheless, as our results in terms of convergence of statistical experiments are not exactly the ones needed in the hypothesis of general LAMN theorems, we follow a direct approach to construct the estimator and to study its asymptotic properties. This approach, that can be followed in rare occasions, has the advantage of clarifying the proof of the asymptotic properties and providing insight in the corresponding numerical computations.

The rest of the paper is organized as follows. Section 2 describes the maximum likelihood methodology and the convergence results. In Section 3 we describe the limit distribution. Sections 4 presents the statistical Test and some numerical simulations on the likelihood function. Section 5 presents an application to diffusion of species in two different habitats, and Section 6 our conclusions. Finally, in the Appendix we provide the theorems taken from [8] used in the proof of our main results in Section 2.

2 The maximum likelihood estimator

Consider the SBm X with parameter $\theta \in (-1, 1)$ defined in (1) and the sampling scheme denoted by $X_i := X_{iT/n}$ $(i = 0, \ldots, n)$, and $\Delta = T/n$. In this section we derive the asymptotic behaviour of maximum likelihood estimator θ_n of the parameter θ when we observe the sample $X_1, \ldots, X_n$. The transition density of the SBm of parameter $\theta \in [-1, 1]$ is given by:

$$q_\theta(t, x, y) = p(t, y - x) + \operatorname{sgn}(y)\theta p(t, |x| + |y|),$$

where

$$p(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\Big(- \frac{x^2}{2t}\Big)$$

is the density of a Gaussian random variable with variance t and mean 0. The likelihood of the sample is given by

$$Z_n(\theta) = \prod_{i=0}^{n-1} \frac{q_\theta(\Delta, X_i, X_{i+1})}{q_0(\Delta, X_i, X_{i+1})}.$$

Observing that for any $x, y \in \mathbb{R}$ we have

$$\frac{p(\Delta, |x| + |y|)}{q_0(\Delta, x, y)} = \exp\left(-\frac{|xy| + xy}{\Delta}\right) = \exp\left(\frac{-2(xy)^+}{\Delta}\right) \leq 1, \tag{3}$$

(where $z^+ = (|z| + z)/2 = \max(z, 0)$), we can write

$$Z_n(\theta) = \prod_{\substack{X_i>0 \\ X_{i+1}<0}} (1-\theta) \prod_{\substack{X_i<0 \\ X_{i+1}>0}} (1+\theta) \prod_{\substack{X_i<0 \\ X_{i+1}<0}} \left(1 - \theta e^{\frac{-2X_iX_{i+1}}{\Delta}}\right)$$
$$\times \prod_{\substack{X_i>0 \\ X_{i+1}>0}} \left(1 + \theta e^{\frac{-2X_iX_{i+1}}{\Delta}}\right) = \prod_{i=0}^{n-1} \left(1 + h(\sqrt{n}X_i, \sqrt{n}X_{i+1})\right),$$

where

$$h(x, y) = \operatorname{sgn}(x + y) \exp\left(-(2/T)(x(x + y))^+\right),$$

to see that $Z_n(\theta)$ is a polynomial of degree n, with n real roots. Remember that we assume $X_0 = x \geq 0$. In case the trajectory we observe does not hit the zero level, we obtain

$$Z_n(\theta) = \prod_{i=0}^{n-1} \left(1 + \theta e^{\frac{-2X_iX_{i+1}}{\Delta}}\right)$$

and $Z_n(\theta)$ is increasing in θ. In this case our maximum likelihood estimator is $\theta_n = 1$. In the case that the trajectory crosses the zero level, we see that the polynomial has roots $\theta = \pm 1$ (for large enough n), and no roots inside this interval. As $Z_n(0) = 1$, this gives a unique maximum at the point θ_n in the interval $(-1, 1)$.

Our main result is the weak convergence of the MLE to a distribution that we characterize. Three main differences can be noted in respect to the classical statistical situation: (i) the convergence of the estimator is more slowly ($n^{1/4}$) than in the classical case; (ii) the limit is not Gaussian, but a mixture of Gaussian random variables; (iii) the convergence is stable, stronger than the usual convergence in distribution, but natural in this context, known as local asymptotic mixed normality (LAMN) in the literature (see for instance [12]). We also have to take into account, in accordance to our previous discussions on the existence and the value of the MLE, that the event that the trajectory hits the level zero is crucial in the results we obtain (in fact, if the trajectory does not hit this level, the MLE remains constant for all n). Consider then the events

$$A_n = \{\omega\colon \inf_{1\leq i\leq n} X_i < 0\}, \qquad A = \{\omega\colon \inf_{0\leq t\leq T} X_t(\omega) < 0\}. \tag{4}$$

As X is continuous, $\mathbf{1}_{A_n} \to \mathbf{1}_A$ a.s. ($\mathbf{1}_B$ stands for the indicator of the set B). We now review the stable convergence (see [10]), and introduce the conditional stable convergence that will take place in our case.

Definition 1. Consider a sequence of random variables $Y, Y_1, Y_2, \dots$ defined on a the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a σ-algebra $\mathcal{G} \subset \mathcal{F}$.

We say that the sequence of random variables $Y_1, Y_2 \dots$ converge $\mathcal{G}$-stably in distribution to Y, and denote

$$Y_n \xrightarrow[n\to\infty]{\mathcal{G}\text{-stably}} Y$$

when

$$\mathbb{E}\left(Zf(Y_n)\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left(Zf(Y)\right)$$

for any bounded $\mathcal{G}$ measurable random variable Z, and any bounded and continuous function f.

Furthermore, consider a sequence of sets $A, A_1, A_2, \dots$. We say that the sequence of random variables $Y_1, Y_2 \dots$ conditional on A_n converge $\mathcal{G}$-stably in distribution to Y conditional on A, and denote

$$Y_n \mid A_n \xrightarrow[n\to\infty]{\mathcal{G}\text{-stably}} Y \mid A,$$

when

$$\mathbb{E}\left(Zf(Y_n) \mid A_n\right) \xrightarrow[n\to\infty]{} \mathbb{E}\left(Zf(Y) \mid A\right)$$

for any bounded $\mathcal{G}$ measurable random variable Z, and any bounded and continuous function f.

We are now in position of presenting our main result. Indeed, this theorem will be an immediate sequel of Theorem 2 below.

Theorem 1. *Consider a Skew Brownian motion defined in* (1) *with the sampling scheme described in the beginning of Section 2 and the events A_n and A defined in* (4)*. Then for the maximum likelihood estimator θ_n we have the convergence*

$$n^{1/4}\theta_n \mid A_n \xrightarrow[n\to\infty]{\mathcal{F}\text{-stably}} \frac{W(\ell_T^x)}{\ell_T^x} \mid A,$$

under the Brownian motion distribution (that is when $\theta = 0$), where $W = \{W_t \colon t \geq 0\}$ is a standard Brownian motion independent of B. In particular, when $x = 0$, we have

$$n^{1/4}\theta_n \xrightarrow[n\to\infty]{\mathcal{F}\text{-stably}} \frac{W(\ell_T^x)}{\ell_T^x} \tag{5}$$

2.1 Some results on derivatives of the log-likelihood

In order to study the asymptotic behaviour of θ_n, the MLE, we consider the *log-likelihood*, defined by

$$L_n(\theta) = \log \prod_{i=0}^{n-1} q_\theta(\Delta, X_i, X_{i+1}) \tag{6}$$

and introduce its scaled (for notational convenience) k-th derivatives, for $k \geq 1$, by

$$L_n^{(k)}(\theta) = \frac{1}{(k-1)!} \frac{\partial^k}{\partial \theta^k} L_n(\theta),$$

that are computed as

$$L_n^{(k)}(\theta) = (-1)^{k-1} \sum_{i=0}^{n-1} \frac{\operatorname{sgn}(X_{i+1})^k p(\Delta, |X_i| + |X_{i+1}|)^k}{q_\theta(\Delta, X_i, X_{i+1})^k}. \tag{7}$$

An analytical development of $L_n^{(1)}(\theta)$ holds around 0:

$$L_n^{(1)}(\theta) = \sum_{k=0}^{+\infty} \theta^k L_n^{(k+1)}(0). \tag{8}$$

Condition (3) implies that $|L_n^{(k)}(0)| \leq n$ and thus the series in (8) is absolutely convergent for $|\theta| < 1$.

Introduce, for $k = 1, 2, \ldots$, the sequence of functions

$$h_k(x, y) = \left[\operatorname{sgn}(x+y) \exp\left(-(2/T)(x(x+y))^+\right)\right]^k.$$

We can then rewrite $L_n^{(k)}(0)$, for $k = 1, 2, \ldots$, as:

$$L_n^{(k)}(0) = (-1)^{k-1} \sum_{i=0}^{n-1} h_k(\sqrt{n} X_i, \sqrt{n}(X_{i+1} - X_i)).$$

We then see that the study of the limit behaviour of this type of sums, presented in the next proposition, can be directly obtained from results obtained by J. Jacod [8]. (For convenience, we present Jacod's results from [8] in an Appendix).

Proposition 1. *Assume that $\theta = 0$ in (1), i.e. let X be a Brownian motion on $[0, T]$ departing from x, and let ℓ^x denote its local time at zero.*

(a) *Assume that* $k = 2, 4, \ldots$. *Denote*

$$\mu_k = -2 \int_0^\infty \left[1 + \frac{1}{2k-1} \exp\left(\frac{2k(k-1)x^2}{(2k-1)^2} \right) \right] \Phi(-x)\, dx. \tag{9}$$

Then

$$\frac{L_n^{(k)}(0)}{n^{1/2}} \xrightarrow[n\to\infty]{prob.} \mu_k \ell_T^x. \tag{10}$$

(b) *Assume that* $k = 1, 3, \ldots$. *Denote*

$$\mu_k = 2 \int_0^\infty \left[1 + \frac{1}{4k-1} \exp\left(\frac{4k(2k-1)x^2}{(4k-1)^2} \right) \right] \Phi(-x)\, dx.$$

Then, there exists a Brownian motion W *independent from* B *such that*

$$\frac{L_n^{(k)}(0)}{n^{1/4}} \xrightarrow[n\to\infty]{\mathcal{F}\text{-stably}} \mu_k W(\ell_T^x). \tag{11}$$

Remark 1. Observe that on the event

$$\{\omega \colon \inf_{0\le t\le T} X_t(\omega) > 0\}$$

we have $\ell_T^x(\omega) = W\left(\ell_T^x(\omega)\right) = 0$. In this situation, as all the information about the relevant parameter θ is produced when the process hits the level zero, no statistical inference can be carried out. Observe that in case $x = 0$ we have

$$\mathbb{P}\left(\left\{ \omega \colon \inf_{0\le t\le T} X_t > 0 \right\} \right) = 0.$$

Remark 2. Indeed, the results of J. Jacod could be applied to multi-dimensional statistics. This way, we obtain the joint $\mathcal{F}$-stable convergence of any vector $n^{-1/4}(L_n^{(1)}(0), \ldots, L_n^{(2k+1)}(0))$ for any integer k, and then the joint $\mathcal{F}$-stable convergence of

$$\begin{aligned}(n^{-1/4}L_n^{(1)}(0), n^{-1/2}L_n^{(2)}(0), \ldots, n^{-1/4}L_n^{(2k+1)}(0), n^{-1/2}L_n^{(2k+2)}(0)) \\ \xrightarrow[n\to\infty]{\mathcal{F}\text{-stably}} (\mu_1 W(\ell_T^x), \mu_2 \ell_T^x, \ldots, \mu_{2k+1} W(\ell_T^x), \mu_{2+2}\ell_T^x).\end{aligned}$$

Proof. We apply Theorem 3 in the Appendix. Observe that

$$h_k(x, y) \le \exp\left(-(x(x+y))^+\right) \le \exp\left(|y| - |x \wedge x^2|\right).$$

We then have that (26) holds with $a = 1$, $\hat{h}(x) = \exp\left(-|x \wedge x^2|\right)$ and $r = 0$, then it holds for any $r > 0$. In consequence, by the aftermentioned Theorem,

the convergence in (27) holds for $h = h_k$ with $k = 2, 4, \dots$. It rests to compute the constant in (28). We have

$$\begin{aligned}
c(h_k) &= \iint_{\mathbb{R}^2} h_k(x,y)p(1,y)\,dx\,dy = 2\int_0^\infty dx \int_{-\infty}^{-x} p(1,y)\,dy \\
&\quad + \frac{2}{\sqrt{2\pi}}\int_0^\infty dx \int_{-x}^\infty \exp\left(-\frac{1}{2}y^2 - 2kxy - 2kx^2\right) dy \\
&= 2\int_0^\infty \Phi(-x)\,dx \\
&\quad + \frac{2}{\sqrt{2\pi}}\int_0^\infty dx \exp\left(2k(k-1)x^2\right) \int_{-x}^\infty \exp\left(-\frac{1}{2}(y+2kx)^2\right) dy \\
&= 2\int_0^\infty \Phi(-x)\,dx + 2\int_0^\infty \exp\left(2k(k-1)x^2\right)\Phi(-(2k-1)x)\,dx \\
&= 2\int_0^\infty \Phi(-x)\,dx + \frac{2}{2k-1}\int_0^\infty \exp\left(\frac{2k(k-1)x^2}{(2k-1)^2}\right)\Phi(-x)\,dx. \quad (12)
\end{aligned}$$

Taking into account that $\mu_k = -c(h_k)$ we conclude that (10) holds with μ_k given in (9).

To prove (b) we rely on Theorem 4 in the appendix. Observe then that $c(h_k) = 0$ for odd k due to the property

$$h_k(-x,-y) = -h_k(x,y) \quad \text{for odd } k.$$

In view of the fact that (26) holds for all h_k with $r = 4$, taking into account that $(h_k)^2 = h_{2k}$, we conclude that $\mu_k = c\,(h_{2k})$. In view of the the computations in (12) with $2k$ instead of k we conclude (11), and the proof of the proposition. □

Remark 3. On the event A_n, the discrete path has crossed the origin. Hence, the continuous path did so at a random time τ. Using the strong Markov property, this implies that one may consider a path starting from 0 for any time $t \geq \tau$. The local time of the Brownian motion is equal in distribution to the maximum of the Brownian motion. Hence, on A_n and A, $\ell_t^x > 0$ for any time $t \geq \tau$.

Corollary 1. *In the conditions of Proposition 1, for $k = 0, 1, 2, \dots$, we have:*

$$\frac{L^{(2k+1)}(0)}{L^{(2)}(0)} \xrightarrow[n\to\infty]{prob.} 0, \qquad \frac{L^{(2k+2)}(0)}{L^{(2)}(0)} \xrightarrow[n\to\infty]{prob.} \frac{\mu_{2k+2}}{\mu_2}\mathbf{1}_A, \quad (13)$$

$$n^{1/4}\frac{L^{(2k+1)}(0)}{L^{(2)}(0)} \mid A_n \xrightarrow[n\to\infty]{\mathcal{F}\text{-stably}} \frac{\mu_{2k+1}W(\ell_T^x)}{\mu_2\ell_T^x} \mid A, \quad (14)$$

In particular, as $\mu_2 = -\mu_1$, we obtain

$$-n^{1/4}\frac{L^{(1)}(0)}{L^{(2)}(0)} \mid A_n \xrightarrow[n\to\infty]{\mathcal{F}\text{-stably}} \frac{W(\ell_T^x)}{\ell_T^x} \mid A, \tag{15}$$

Proof. We begin by the second part in (13). As $\ell_T^x > 0$ on the set A, we have

$$\frac{L^{(2k+2)}(0)}{L^{(2)}(0)}\mathbf{1}_A = \frac{n^{-1/2}L^{(2k+2)}(0)}{n^{-1/2}L^{(2)}(0)}\mathbf{1}_A \xrightarrow[n\to\infty]{\text{prob.}} \frac{\mu_{2k+2}\ell_T^x}{\mu_2\ell_T^x}\mathbf{1}_A = \frac{\mu_{2k+2}}{\mu_2}\mathbf{1}_A.$$

Assume now that $\omega \in A^c$. We have $a(\omega) = \inf_{0\le t\le T} X_t(\omega) > 0$ a.s. on A^c. Now

$$\begin{aligned}|L_n^{(2k+2)}(0)| &\le (2k+1)! \sup_{1\le i\le n} e^{(-4kn/T)X_iX_{i+1}}|L_n^{(2)}(0)| \\ &\le (2k+1)! e^{(-(4k-2)n/T)a(\omega)^2}|L_n^{(2)}(0)|,\end{aligned}$$

what gives the second part of (13) on the set A^c. We postpone by now the proof of the first part of (13).

Let us then verify (14). We first prove that

$$\left(\frac{L_n^{(2)}(0)}{n^{1/2}}, \frac{L_n^{(2k+1)}(0)}{n^{1/4}}, \mathbf{1}_{A_n}\right) \xrightarrow[n\to\infty]{\mathcal{F}\text{-stably}} \left(\mu_2\ell_T^x, \mu_{2k+1}W(\ell_T^x), \mathbf{1}_A\right).$$

This amounts to prove that, for $Z \ge 0$, $\mathcal{F}$-measurable and bounded, and real λ, μ and ν, we have

$$\begin{aligned}\delta_n := \mathbb{E}Z\exp\left(i\left\{\lambda\frac{L_n^{(2)}(0)}{n^{1/2}} + \mu\frac{L_n^{(2k+1)}(0)}{n^{1/4}} + \nu\mathbf{1}_{A_n}\right\}\right) \\ \xrightarrow[n\to\infty]{} \mathbb{E}Z\exp\left(i\left\{\lambda\mu_2\ell_T^x + \mu\mu_{2k+1}W(\ell_T^x) + \nu\mathbf{1}_A\right\}\right) =: \delta.\end{aligned} \tag{16}$$

We know that

$$\left(\frac{L_n^{(2)}(0)}{n^{1/2}}, \mathbf{1}_{A_n}\right) \xrightarrow[n\to\infty]{\text{prob.}} \left(\mu_2\ell_T^x, \mathbf{1}_A\right),$$

as $\mathbf{1}_{A_n} \to \mathbf{1}_A$ a.s. We then have

$$
\begin{aligned}
|\delta_n - \delta| \leq & \left| \mathbb{E}Z \exp\left(i \left\{ \lambda \frac{L_n^{(2)}(0)}{n^{1/2}} + \mu \frac{L_n^{(2k+1)}(0)}{n^{1/4}} + \nu \mathbf{1}_{A_n} \right\} \right) \right. \\
& \left. - \mathbb{E}Z \exp\left(i \left\{ \lambda \mu_2 \ell_T^x + \mu \frac{L_n^{(2k+1)}(0)}{n^{1/4}} + \nu \mathbf{1}_A \right\} \right) \right| \\
& + \left| \mathbb{E}Z \exp\left(i \left\{ \lambda \mu_2 \ell_T^x + \mu \frac{L_n^{(2k+1)}(0)}{n^{1/4}} + \nu \mathbf{1}_A \right\} \right) \right. \\
& \left. - \mathbb{E}Z \exp\left(i \left\{ \lambda \mu_2 \ell_T^x + \mu \mu_{2k+1} W(\ell_T^x) + \nu \mathbf{1}_A \right\} \right) \right| \\
\leq & \, \mathbb{E}Z \left| \exp\left(i \left\{ \lambda \frac{L_n^{(2)}(0)}{n^{1/2}} + \nu \mathbf{1}_{A_n} \right\} \right) - \exp\left(i \left\{ \lambda \mu_2 \ell_T^x + \nu \mathbf{1}_A \right\} \right) \right| \\
& + \mathbb{E}Z \left| \exp\left(i \mu \frac{L_n^{(2k+1)}(0)}{n^{1/4}} \right) - \exp\left(i \mu \mu_{2k+1} W(\ell_T^x) \right) \right| \to 0,
\end{aligned}
$$

concluding the proof of (16). The proof of (14) follows with the help of the continuous and bounded function $f_K(t) = t\mathbf{1}_{\{|t| \leq K\}} + K\mathbf{1}_{\{t > K\}} - K\mathbf{1}_{\{t < -K\}}$. We have

$$
f_K\left(n^{1/4} \frac{L^{(2k+1)}(0)}{L^{(2)}(0)} \mathbf{1}_{A_n} \right) \xrightarrow[n \to \infty]{\mathcal{F}\text{-stably}} f_K\left(\frac{\mu_{2k+1} W(\ell_T^x)}{\mu_2 \ell_T^x} \mathbf{1}_A \right)
$$

for all $K > 0$, and, as the limit is bounded in probability, we obtain (14).

In what respects the first part of (13) the computation on the set A^c is similar to the previous one. In the set A, we have

$$
\frac{L^{(2k+1)}(0)}{L^{(2)}(0)} \mathbf{1}_A = n^{-1/4} \left(\frac{n^{-1/4} L^{(2k+1)}(0)}{n^{-1/2} L^{(2)}(0)} \mathbf{1}_A \right) \xrightarrow[n \to \infty]{\text{prob.}} 0,
$$

as the expression within brackets has weak limit. □

2.2 A simple estimator

The MLE is the point θ_n at which $\theta \mapsto L_n(\theta)$ reaches its maximum, i.e. θ_n is the (unique in our case) root of the equation

$$
L_n^{(1)}(\theta_n) = 0.
$$

Let us set

$$
\alpha_n = -n^{1/4} \frac{L_n^{(1)}(0)}{L_n^{(2)}(0)}. \tag{17}
$$

From Corollary 1 formula (15), α_n is known to converge $\mathcal{F}$-stably as $n \to \infty$.

Below, we will see that $n^{1/4}\theta_n$ and $n^{1/4}\alpha_n$ have the same limit, which yields Theorem 1.

The value $\alpha_n/n^{1/4}$, which is pretty simple to compute from the data, specially in contrast to θ_n that requires a numerical solver to be computed, can be used as an estimator of the skewness parameter.

Let us also remark that α_n is chosen so that the first two terms in the Taylor series (8) of $L_n^{(1)}(\alpha_n/n^{1/4})$ cancel out.

2.3 Asymptotic development of the MLE

We then prove a theorem and a theorem and a proposition which enclose Theorem 1.

Theorem 2. *For any integer $p \geq 0$, there exists a vector $(d_n^{(1)}, \dots, d_n^{(p+1)})$ of random variables given the recursive relation $d_n^{(1)} = 1$ and*

$$d_n^{(m+1)} = -\sum_{k=1}^{m+1} \frac{L_n^{(k+1)}(0)}{L_n^{(2)}(0)} \sum_{\substack{1 \leq i_1, \dots, i_k \leq m \\ i_1 + \dots + i_k = m+1}} d_n^{(i_1)} \cdots d_n^{(i_k)} \tag{18}$$

that converges $\mathcal{F}$-stably conditioning to A to a vector $(d^{(1)}, \dots, d^{(p+1)})$ depending only on ℓ_T^x and $W(\ell_T^x)$. Besides, for any $\epsilon > 0$, there exists some integer n_0 large enough and some K such that

$$\mathbb{P}\left[n^{\frac{p}{4}+\frac{1}{2}}|\theta_n - \Theta_n| \geq K\right] \leq \epsilon \text{ for any } n \geq n_0,$$

where

$$\Theta_n = \frac{\alpha_n}{n^{1/4}} + d_n^{(2)} \frac{\alpha_n^2}{n^{1/2}} + \dots + d_n^{(p+1)} \frac{\alpha_n^{p+1}}{n^{(p+1)/4}}.$$

In addition, $d_n^{(2)}$ converges to 0 and $n^{1/4} d_n^{(2)}$ is bounded.

We prove this theorem after the next proposition, which will be stated in the following framework: Using the result of Remark 2, we consider the asymptotic behavior of the vector

$$\left(n^{-1/4} L_n^{(1)}(0), n^{-1/2} L_n^{(2)}(0), \dots, n^{-1/2} L_n^{(2k)}(0)\right)$$

for some $k \geq 1$. We may then consider a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ such that this sequence is equal in distribution to a sequence converging almost surely to $(\mu_1 W(\ell_T^x), \mu_2 \ell_T^x, \dots, \mu_{2k} \ell_T^x)$. We now consider some point in this probability space such that $\ell_T > 0$. If the starting point is 0, then the event $\{\ell_T > 0\}$ is of full measure.

Proposition 2. *On the probability space* $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ *above, the random sequences* $d_n^{(i)}$ *given by* (18) *are convergent and bounded in* n*. Besides, for* $m = 1, 2, 3, \dots,$

$$\theta_n = \frac{\alpha_n}{n^{1/4}} + d_n^{(2)} \frac{\alpha_n^2}{n^{2/4}} + d_n^{(3)} \frac{\alpha_n^3}{n^{3/4}} + \cdots + d_n^{(m)} \frac{\alpha_n^m}{n^{m/4}} + \mathrm{O}\left(\frac{1}{n^{(m+1)/4}}\right)$$

almost surely in the event $\{\ell_T^x > 0\}$*.*

Let us start by a simple lemma to get a control over the finite Taylor expansion of $L_n^{(1)}(\theta)$.

Lemma 1. *For any* θ *and any integer* $m \geq 1$*, we have that for a random constant* C *such that*

$$\left| L_n^{(1)}(\theta) - \sum_{k=0}^{m} L_n^{(k+1)}(0)\theta^k \right| \leq \sup_{|\xi| \leq |\theta|} |L_n^{(m+2)}(\xi)| \cdot |\theta|^{m+1} \leq C n^{1/2} \frac{|\theta|^{m+1}}{(1-|\theta|)^{m+2}}. \tag{19}$$

Proof. With (3) and for $\theta \in (-1, 1)$,

$$1 - |\theta| \leq \frac{q_\theta(\Delta, X_i, X_{i+1})}{q_0(\Delta(X_i, X_{i+1}))} \leq 1 + |\theta|. \tag{20}$$

with (7), since $L_n^{(k)}(0)/n^{1/2}$ converges in probability (either to $\mu_k \ell_T^x$ or to 0 depending if k is even or odd), there exists a random constant C

$$|L_n^{(k)}(\theta)| \leq \frac{n^{1/2} C}{(1-|\theta|)^k}.$$

Hence

$$\left| L_n^{(1)}(\theta) - \sum_{k=0}^{m} L_n^{(k+1)}(0)\theta^k \right| \leq \sup_{|\xi| \leq |\theta|} |L_n^{(m+2)}(\xi)| \cdot |\theta|^{m+1}.$$

With (10) and (11), this gives (19) because $L_n^{(k)}(0)/n^{1/2}$ is bounded in n. □

Lemma 2. *For* n *large enough, the function* $L_n^{(1)}(\theta)$ *is invertible. Besides, the function* $(L_n^{(1)}(\theta))^{-1}$ *is Lipschitz in* θ *with a constant* $8/n^{1/2}\mu_2 \ell_T^x$ *on the event* A_n*.*

Proof. With (20),

$$-L_n^{(2)}(\theta) \geq \left(\sum_{i=1}^{n-1} \frac{p(\Delta, |X_i| + |X_{i+1}|)^2}{q_0(\Delta, X_i, X_{i+1})^2} \right) \frac{1}{(1+|\theta|)^2} \geq \frac{L_n^{(2)}(0)}{4}.$$

Since $L_n^{(2)}(0) < 0$ for n large enough as $n^{-1/2}L_n^{(2)}(\theta)$ converges in probability to some negative random variable (see (10)), we get that $L_n^{(1)}(\theta)$ is one-to-one. With the formula $\partial_\theta(L^{(1)}(\theta))^{-1} = 1/L_n^{(2)}(L_n^{(1)}(\theta))$, $(L_n^{(1)}(\theta))^{-1}$ is Lipschitz in θ with constant $4/L_n^{(2)}(0)$. □

The idea of the proof is then the following: We construct an of estimator Θ_n such that for some constant C and $p \geq 0$,

$$\sup_{n\in\mathbb{N}} n^{p/4}|L_n^{(1)}(\Theta_n)| \leq C.$$

Since $L_n^{(1)}(\theta_n) = 0$,

$$\begin{aligned}|\Theta_n - \theta_n| &= |(L_n^{(1)})^{-1}(L_n^{(1)}(\Theta_n)) - (L_n^{(1)})^{-1}(L_n^{(1)}(\theta_n))| \\ &\leq \frac{8}{n^{1/2}\mu_2\ell_T^x}|L_n^{(1)}(\Theta_n)| \leq \frac{8C}{n^{p/4+1/2}\mu_2\ell_T^x}. \quad (21)\end{aligned}$$

Proof of Proposition 2. For the sake of simplicity, let us set $q = n^{1/4}$.

Set $\Theta_n = \alpha_n q + \beta_n q^2 + \gamma_n q^3 + \xi_n q^4$ for some β_n, γ_n and ξ_n to be carefully chosen. Here, we consider only the first terms in the development of θ_n. It is easily to convince oneself that this method may be applied to any order and that the involved terms $\beta_n, \gamma_n, \xi_n, \ldots$ may be computed recursively and gives rise to (18).

With (19) and $m = 4$, there exists a constant C such that

$$\begin{aligned}&|L^{(1)}(\Theta_n) - L_n^{(1)}(0) - L_n^{(2)}(0)\Theta_n \\ &\qquad - L_n^{(3)}(0)\Theta_n^2 - L_n^{(4)}(0)\Theta_n^3 - L_n^{(5)}(0)\Theta_n^4| \leq Cn^{1/2}\frac{|\Theta_n|^5}{(1-|\Theta_n|)^6}. \quad (22)\end{aligned}$$

Remark that $L_n^{(1)}(0) - L_n^{(2)}(0)\alpha_n q = 0$. In order to get rid of the terms in q^2, set

$$\beta_n = \frac{-L_n^{(3)}(0)}{L_n^{(2)}(0)}\alpha_n^2$$

Since α_n converges and $n^{1/4}L_n^{(3)}(0)/L_n^{(2)}(0)$ also converges stably, then $n^{1/4}\beta_n$ converges stably. Also, β_n converges to 0.

In order to get rid of the terms in q^3, set

$$\gamma_n = -\frac{L_n^{(4)}(0)}{L_n^{(2)}(0)}\alpha_n^3 - \frac{L_n^{(3)}(0)}{L_n^{(2)}(0)}\alpha_n\beta_n.$$

From Corollary 1, γ_n converges stably since α_n and β_n converges stably.

In order to get rid of the terms in q^4, set

$$\xi_n = -2\frac{L_n^{(3)}(0)}{L_n^{(2)}(0)}(\alpha_n\gamma_n + \beta_n^2) - 4\frac{L_n^{(4)}(0)}{L_n^{(2)}(0)}\alpha_n^2\beta_n.$$

Again, ξ_n converges thanks to Corollary 1.

Hence

$$L_n^{(1)}(\Theta_n) = \sum_{r=5}^{20} q^r B_n^{(r)} + R_n(\Theta_n),$$

where $R_n(\Theta_n) \leq n^{1/2}|\Theta_n|^5/(1-|\Theta_n|)^6$ and $B_n^{(r)}$ are terms that depend linearly on $L_n^{(k)}(0)$ and on the power of the α_n, β_n, γ_n and ξ_n. Since the $L_n^{(k)}(0)/n^{1/2}$ are bounded, we obtain that the $n^{3/4}B_n^{(r)}$ are bounded.

In addition, $n^{1/4}\Theta_n$ is bounded in n, so that $n^{3/4}R_n(\Theta_n)$ is bounded in n. With (21), this proves that for some constant K,

$$|\Theta_n - \theta_n| \leq \frac{K}{n^{5/4}}.$$

This result may be generalized to any order. Finally, let us note that $\beta_n = \alpha_n^2 d_n^{(2)}$ with $d_n^{(2)} = -L_n^{(3)}(0)/L_n^{(2)}(0)$. With (13), $d_n^{(2)}$ converges in probability to 0 and $n^{1/4}d_n^{(2)}$ is bounded in n. $\gamma_n = \alpha_n^3 d_n^{(3)}$ and $\xi_n = \alpha_n^4 d_n^{(4)}$ where $d_n^{(3)}$ and $d_n^{(4)}$ are bounded in n. □

Proof of Theorem 1. Let us consider the event $\{\ell_T^x > 0\}$. It corresponds to the event A as the local time of the Brownian motion is positive just after having hit 0. Since on this event, the local time has a density (see Lemma 3 below) which is derived from the one of the first hitting time of a point x, for each $\epsilon > 0$, one may find a set $\Omega(\epsilon)$ as well as some values $0 < a' < b'$ and c' such that $\omega \in \Omega(\epsilon)$ implies that $\ell_T^x \in (a', b')$ and $|W(\ell_T(x))/\ell_T^x| \leq c'$ and

$$\mathbb{P}[\Omega(\epsilon)|\{\ell_T^x > 0\}] \geq 1 - \epsilon/2.$$

From the joint convergence of the $n^{1/4}L_n^{(2k+1)}(0)$ to $\mu_{2k+1}W(\ell_T^x)$ and the joint convergence of the $n^{1/2}L_n^{(2k)}(0)$ to $\mu_{2k}\ell_T^x$, we get for any $\epsilon > 0$, there exists $0 < k < a'$ and $K > b'$ as well as a measurable set $\Omega'(\epsilon, n) \subset \Omega(\epsilon)$ such that

$$L_n^{(k)}(0) \leq K\sqrt{n} \text{ on } \Omega'(\epsilon, n)$$
$$L_n^{(2)}(0) \geq k\sqrt{n} \text{ on } \Omega'(\epsilon, n)$$
$$\text{and } \forall n \geq n_0,\ \mathbb{P}[\Omega'(n, \epsilon)|\{\ell_T^x > 0\}] \geq 1 - \epsilon.$$

In the proof of Proposition 2, we constructed some estimator Θ_n such that for some $p \geq 0$, $n^p L_n^{(1)}(\Theta_n)$ is bounded by some constant depending the upper bounds of the $n^{1/2} L_n^{(k)}(0)$. Besides, we use the Lipschitz constant of $(L_n^{(1)})^{-1}$ which depends on the lower bound of $n^{1/2} L_n^{(2)}(0)$. Thus, on $\Omega'(\epsilon, n)$, we obtain that $|\theta_n - \Theta_n| \leq C/n^{p+1/2}$, where C depends only on K and k, assuming that $n \geq n_0$. This means that

$$n^{p+1/2}|\theta_n - \Theta_n| \leq C.$$

Thus, for any $\epsilon > 0$, there exists n_0 large enough such that

$$\forall n \geq n_0,\ \mathbb{P}[n^{p+1/2}|\theta_n - \Theta_n| \geq C] \leq \epsilon.$$

which yields the result. □

2.4 The contrast function

In order to study the maximum likelihood, it is also possible to consider the contrast function

$$Z_n(\theta) = \frac{\exp(L_n(\theta))}{\exp(L_n(0))}.$$

Using the asymptotic development of $L_n(u)$ around 0, we get that

$$\log Z_n(\theta) = \theta L_n^{(1)}(0) + \frac{\theta^2}{2} L_n^{(2)}(0) + \mathrm{O}(\theta^3).$$

Thus, with the result of Proposition 1 and taking into account that $\mu_2 = -\mu_1$, we see that

$$\log Z_n(\theta/n^{1/4}) \xrightarrow[n\to\infty]{\mathcal{F}\text{-stably}} \mu_1 \left(\theta W(\ell_T^x) - \frac{\theta^2}{2} \ell_T^x \right). \tag{23}$$

From this convergence we can intuitively check our result in (5), based in the theory of convergence of statistical experiments and the LAMN property in (23). The theory states (under certain stringent conditions that we do not verify) that the maximum likelihood estimator of the pre-limit experiments converges stably to the maximum likelihood estimator of the limit experiment [7]. It is direct, differentiating with respect to θ in the r.h.s. of (23), to obtain, when $\ell_T^x > 0$, that the MLE in the limit experiment is $W(\ell_T^x)/\ell_T^x$. We then obtain (5) in the form

$$n^{1/4}\theta_n \xrightarrow[n\to\infty]{\mathcal{F}\text{-stably}} W(\ell_T^x)/\ell_T^x.$$

3 The limit distribution

As $n^{1/4}\theta_n$ and $n^{1/4}\alpha_n$ converge to $\Upsilon = W(\ell_T^x)/\ell_T^x$, we give the main characteristics of this random variables. To simplify the computations, we assume that $x = 0$ and $T = 1$, so that we write $\ell_1 = \ell_T^x$.

Indeed, this random variable is easy to simulate.

Lemma 3. *The distribution of Υ is symmetric. Besides, its density is*

$$f_\Upsilon(x) = \frac{\mathrm{d}F_\Upsilon(x)}{\mathrm{d}x} = \int_0^{+\infty} \mathrm{d}y \int_0^1 \frac{\sqrt{y}}{2\pi\sqrt{t^3}} \exp\left(\frac{-xy}{2} - \frac{y^2}{2t}\right) \mathrm{d}t. \tag{24}$$

and it is equal in distribution to

$$\Upsilon = \frac{G(H)}{H} \text{ with } H = \frac{1}{2}(U + \sqrt{V + U^2}), \tag{25}$$

where $G(H)$, U and V are independent random variables, $G(H)$, $G(H) \sim \mathcal{N}(0, H)$, $U \sim \mathcal{N}(0, 1)$ and $V \sim \exp(1/2)$.

Proof. It is well known that the local time ℓ_1 at time 1 is equal in distribution to the supremum of the Brownian motion $\sup_{r\in[0,1]} B_r$ on $[0, 1]$. It follows that

$$F_{\ell_1}(y) = \mathbb{P}_0[\ell_1 < y] = \mathbb{P}_0[\sup_{r\in[0,1]} B_r < y] = \mathbb{P}_0[\tau_y > 1],$$

where $\tau_y = \inf\{t > 0 \,|\, B_t = y\}$. The density $\upsilon(t; y)$ of τ_y is equal to

$$\upsilon(t; y) = \frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{y^2}{2t}\right),$$

so that

$$F_{\ell_1}(y) = 1 - \int_0^1 \frac{1}{\sqrt{2\pi t^3}} \exp\left(-\frac{y^2}{2t}\right) \mathrm{d}t,$$

and the density $f_{\ell_1}(y)$ of ℓ_1 is then equal to

$$f_{\ell_1}(y) = \int_0^1 \frac{2y}{\sqrt{2\pi t^3}} \exp\left(-\frac{y^2}{2t}\right) \mathrm{d}t.$$

Thus, conditioning with respect to the value of ℓ_1,

$$F_\Upsilon(x) = \mathbb{P}[\Upsilon < x] = \int_0^{+\infty} \mathbb{P}[W(y) < xy] f(y) \,\mathrm{d}y$$

and this leads to (24).

Expression (25) follows from the equality in distribution of ℓ_1 and $\frac{1}{2}(U + \sqrt{V + U^2})$. This expression has been used in order to simulate the reflected Brownian motion [15, 16]. □

The variance of Υ is 3.16. We see in Figure 1 that the density of Υ is close to that of the normal distribution, yet narrower.

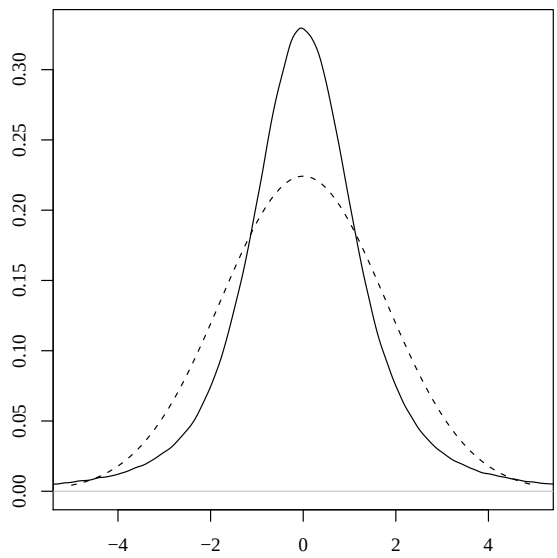


Figure 1: Density of Υ (solid) and density of the normal distribution with variance $\mathrm{Var}(\Upsilon)$ (dashed).

4 Numerical tests and observations on the likelihood

Numerical tests are easy to perform, as all the formulae are easy to implement.

4.1 On the coefficient α_n

Several tests can be performed on $\alpha_n = -n^{1/4} L_n^{(1)}(0)/L_n^{(2)}(0)$, mainly to see whether it is reasonable to use it instead of the MLE θ_n.

First, one can check that θ_n and $\alpha_n/n^{1/4}$ are pretty close, by setting $\theta_n = \mathrm{argmax}_{\theta\in(-1,1)} L_n(\theta)$ and computing it using a numerical procedure. In Table 1, one can check that the error of $|\theta_n - \alpha_n/n^{1/4}|$ is of order $1/n^{3/4}$, so that $\alpha_n/n^{1/4}$ is a pretty good approximation of θ_n, and is much more faster to compute.

Second, one can check the variance of α_n, as well as the adequacy of α_n with the distribution of $\Upsilon = W(\ell_1)/\ell_1$. For this, we have used a set of 10,000

n	mean	$n^{1/2}\times$ mean	$n^{3/4}\times$ mean	std dev	quant. 90 %
100	0.026	0.26	0.8	0.057	0.082
200	0.028	0.40	1.5	0.083	0.057
500	0.013	0.29	1.3	0.055	0.026
1,000	0.013	0.41	2.3	0.040	0.033
2,000	0.006	0.26	1.8	0.025	0.015
5,000	0.006	0.42	3.5	0.041	0.006
10,000	0.002	0.20	2.0	0.005	0.003

Table 1: Statistics of $|\theta_n - \alpha_n/n^{1/4}|$ over 100 paths.

n	$D_n(\alpha_n, \Upsilon)$	p-value	$D_n(\alpha_n, G)$	p-value
100	0.020	0.30	0.082	$<2\times10^{-16}$
250	0.017	0.005	0.079	$<2\times10^{-16}$
500	0.024	1×10^{-5}	0.083	$<2\times10^{-16}$
750	0.029	9×10^{-8}	0.087	$<2\times10^{-16}$
1,000	0.012	0.098	0.072	$<2\times10^{-16}$
2,500	0.023	4×10^{-5}	0.085	$<2\times10^{-16}$
5,000	0.021	2×10^{-4}	0.079	$<2\times10^{-16}$

Table 2: Kolmogorov-Smirnov test on α_n against $\Upsilon\sqrt{\mathrm{Var}(\alpha_n)/\mathrm{Var}(\Upsilon)}$ and the normal distribution G with variance $\mathrm{Var}(\alpha_n)$ over 10,000 paths.

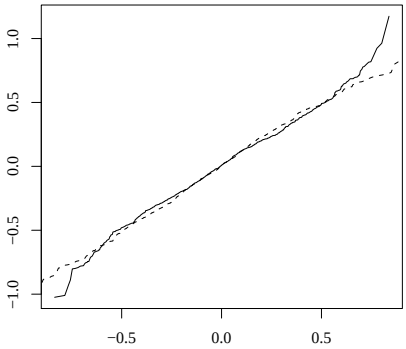
simulations of Υ, and we have renormalized Υ to get the same variance as α_n. Using a Kolmogorov-Smirnov test, we can see in Table 2 that even for a low value of n (*e.g.* $n = 1000$), we get a good adequation with the distribution of Υ. Yet, for $n = 100$, the distribution of $\alpha_n/n^{1/4}$ or θ_n (by keeping only the values in $(-1, 1)$, which means 88% of the values of α_n with $n = 100$ and 96% for $n = 1{,}000$) is in fact close to the Gaussian distribution.

However, the variance of α_n is dependent on n and is not stable with n.

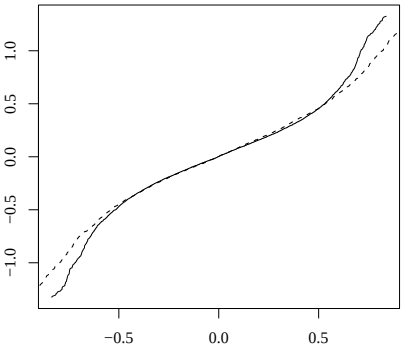
In addition, for small values of n, there are some values of α_n such that $\alpha_n/n^{1/4}$ is outside $[-1, 1]$.

4.2 On the order of convergence

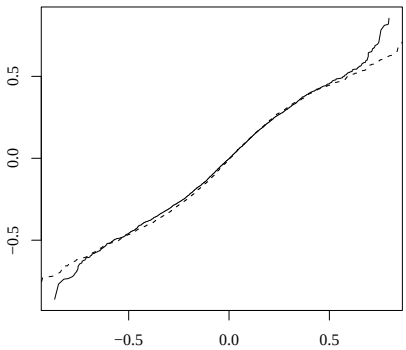
One could wonder if the rate of convergence of θ_n is really of order $-1/4$. Numerical simulations show that the rate of convergence, for n in the range 100 to 100,000 is of order δ with $\delta \approx -0.18$, which is smaller than -0.25. This value is found using a regression on the logarithm standard deviation



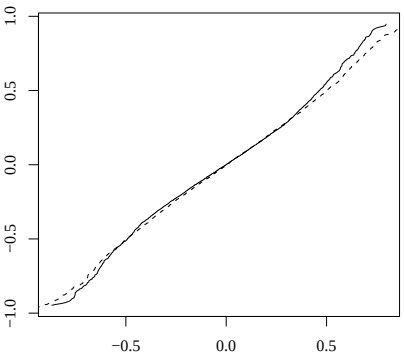
(a) θ_n and α_n against normal, $n = 100$



(b) θ_n and α_n against Υ, $n = 100$



(c) θ_n and α_n against normal, $n = 1000$



(d) θ_n and α_n against Υ, $n = 1000$

Figure 2: Quantile-Quantile plot of θ_n (solid) and $\alpha_n/n^{1/4}$ (dashed) against the normal distribution (left) and the distribution of Υ (right) with variance $\mathrm{Var}(\theta_n)$ for 10,000 samples.

of 500 samples of θ_n (See Figure 3).

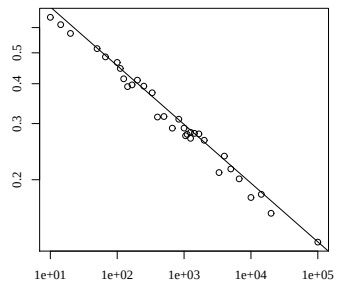


Figure 3: Logarithmic regression on the standard deviation of θ_n

Indeed, one can note that the variance of α_n also depends on n, and in the range from 50 to 600,000, a numerical study of 10,000 samples of α_n shows that $\mathrm{Var}(\alpha_n)$ seems to be equal to Cn^β with $\beta \approx 0.08$. This has to be taken into account in order to design some test of hypotheses.

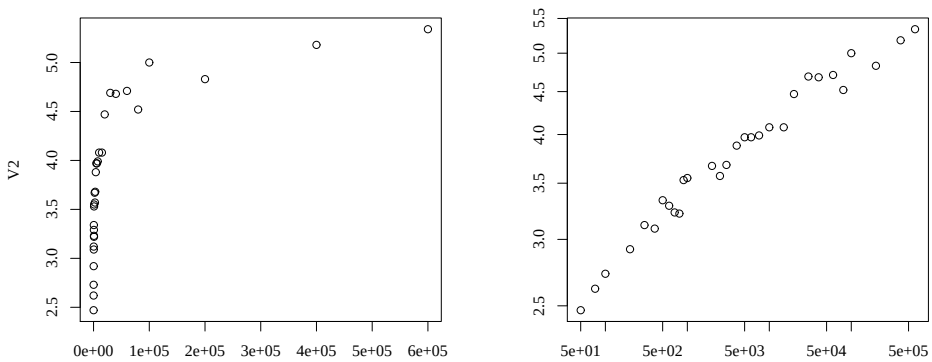


Figure 4: Variance of α_n as a function of n in the linear and logarithmic scale.

4.3 A hypothesis test

It is then possible to develop a hypothesis test of $\theta = 0$ against $\theta \neq 0$. For this, let us compute

$$\mathbb{P}\left[|\theta_n| \geq \frac{K}{n^{1/4}}\right] \approx \mathbb{P}\left[|\Upsilon| \geq \frac{K}{cn^{1/4}}\right].$$

Of course, the second type error cannot be computed, and we do not know the asymptotic behavior of $L_n(\theta)$ when $\theta \neq 0$. However, it is rather easy to perform simulation and thus to get some numerical information about the MLE θ_n and α_n. For example, we see in Figure 5 an approximation of the density of $\alpha_n/n^{1/4}$ for $\theta = 0.5$ compared to an approximation of the density of $\alpha_n/n^{1/4}$ for the Brownian motion with $n = 1{,}000$. We can note that the histogram of $\alpha_n/n^{1/4}$ has its peak on 0.5.

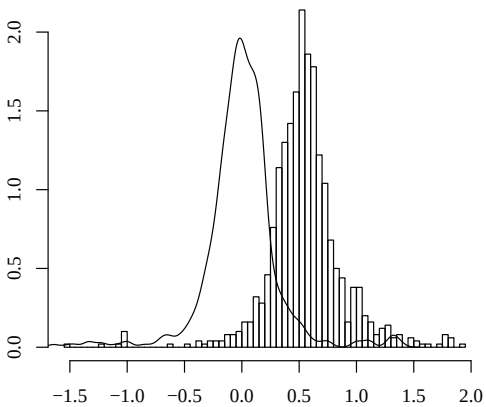


Figure 5: Histogram of the density of $\alpha_n/n^{1/4}$ for $n = 1{,}000$ realizations of the SBm with $\theta = 0.5$ against the an approximation of the density of $\alpha_n/n^{1/4}$ for the Brownian motion.

5 An example of application: diffusion of species

As endowed in the introduction, the SBm is a fundamental tool when one has to model a permeable barrier. In addition, it appears when one writes down

the processes generated by diffusion equations with discontinuous coefficients in a one dimensional media: this issue is presented in the survey article [13] with references to the articles where the SBm arised and covering various fields, such as ecology, finance, astrophysics, geophysics, ...

We present here a possible application to ecology of our hypothesis test, which can be surely applied to other fields.

5.1 Has a boundary between two habitats an effect?

Diffusions are commonly used in ecology to explain the spread of a specie, at the level of individual cells (See for example the book [2]) or the level of an animal in a wild environment.

Several authors have proposed the use of biased diffusions to model the behavior of a specie at the boundary between two habitats [3, 19]), when the species diffuse with different species at speed in each habitat.

Now, consider a situation where the dispersion of a specie in two different habitats is well modelled by a diffusion process, and that the measurement of the diffusion coefficient give the same value. Does it means that the boundary has no effect on the displacement of the individuals?

Let us apply this in a one-dimensional world, where one habitat is $[0, +\infty)$ and the other is $(-\infty, 0]$. We assume that we may track the position of an individual, whose displacement in each of the habitat is given by $x + \sigma B_t$.

Then, we may apply our hypothesis test to determine whether or not the position shall be modelled by

$$(\mathrm{H}_0) \quad X_t = x + \sigma B_t$$

or by

$$(\mathrm{H}_1) \quad X_t = x + \sigma B_t + \theta \ell_t^0(X).$$

Under Hypothesis (H_0), the boundary has no effect and is not seen. Under Hypothesis (H_1), the individual is more likely to go in one of the two habitat, depending on the sign of θ.

5.2 What is the underlying operator?

Now, let us consider that we have a measurement of the diffusion coefficients that gives two different values a_+ on $\mathbb{R}_+$ and a_- on $\mathbb{R}_-$.

One may then wonder which differential operator shall be used to model the diffusive behavior. For $a = a_+ \mathbf{1}_{[0,+\infty)} + a_- \mathbf{1}_{(-\infty,0)}$, is it

$$L = \frac{1}{2}\nabla(a\nabla\cdot) \text{ or } A = \frac{1}{2}a\triangle?$$

On $(0,+\infty)$ and $(-\infty,0)$, there is no difference between these two operators, which means that the local dynamic of the particle/individual is not affected by the choice of L and A. However, the difference arises at 0: the process X generated by L is solution to

$$X_t = x + \int_0^t \sqrt{a(X_s)}\,\mathrm{d}B_s + \frac{a_+ - a_-}{a_+ + a_-}\ell_t^0(X)$$

while the process Y generated by A is solution to

$$Y_t = x + \int_0^t \sqrt{a(X_s)}\,\mathrm{d}B_s,$$

for a Brownian motion B (See for example [13, 14]). From the analytical point of view: the domain $\mathrm{Dom}(A)$ of A contains the functions of class $\mathcal{C}^2(\mathbb{R})$ which are bounded with bounded, first and second order derivatives. The domain $\mathrm{Dom}(L)$ of L contains functions of class $\mathcal{C}^2(\mathbb{R}\setminus\{0\})$ with bounded first and second order derivatives which are furthermore continuous at 0, and such that $a_+\nabla f(0+) = a_-\nabla f(0-)$. This condition is called the *flux condition*. In many physical situations, it is assumed that the flux $a\nabla u$ is continuous and this is why divergence-form operators of type L arise.

Remark 4. Both L and A can be embedded in a single class of operators of type $\frac{\rho}{2}\nabla(a\nabla\cdot)$. If ρ and a are constant on $(0,+\infty)$ and $(-\infty,0)$, then we may use the following characterization: let us consider

$$C = \frac{1}{2}\nabla(a\nabla\cdot) \text{ with } a = a_+\mathbf{1}_{[0,+\infty)} + a_-\mathbf{1}_{(-\infty,0)}$$

and

$$\mathrm{Dom}(C) = \left\{ f \in \mathcal{C}^2(\mathbb{R}\setminus\{0\}) \,\middle|\, \begin{array}{l} f, f', f'' \text{ are bounded on } \mathbb{R}\setminus\{0\} \\ f(0-) = f(0+) \\ (1+\lambda)f'(0+) = (1-\lambda)f'(0-),\ \lambda\in(-1,1) \end{array} \right\}.$$

This class of operators is then specified by three parameters, $a_+ > 0$, $a_- > 0$ and $\lambda \in (-1,1)$. The operator A corresponds to $\lambda = 0$, while L corresponds to $\lambda = (a_+ - a_-)/(a_+ + a_-)$.

For $\Phi(x) = \int_0^x \mathrm{d}x/\sqrt{a(x)}$, $\widehat{X} = \Phi(X)$ is solution to the SDE [13, 14]

$$\widehat{X}_t = \Phi(x) + B_t + \frac{\sqrt{a_+} - \sqrt{a_-}}{\sqrt{a_+} + \sqrt{a_-}}\ell_t^0(\widehat{X}),$$

while $\widehat{Y} = \Phi(Y)$ is solution to the SDE

$$\widehat{Y}_t = \Phi(x) + B_t + \frac{\sqrt{a_-} - \sqrt{a_+}}{\sqrt{a_+} + \sqrt{a_-}} \ell_t^0(\widehat{Y}).$$

We then see that both $\widehat{X}$ and $\widehat{Y}$ are Skew Brownian motions, but the coefficients in front of their local time have opposite signs.

Even if we have not studied the asymptotic behavior of the MLE for the SBm with skewness parameter different from 0, numerical experiments back the following hypotheses test:

1. Given an observed X, estimate the diffusion coefficient for the process on each side of 0.

2. Apply the function Φ to the observed process.

3. Compute the MLE θ_n of the Skewness parameter. If $a_+ > a_-$ (resp. $a_+ < a_-$) and $\theta_n > 0$ then decided that the infinitesimal generator of X is L (resp. A). Otherwise, decide that it is A (resp. L).

6 Conclusion

In this article, we have studied the behavior of the maximum likelihood for the Skew Brownian motion when the parameter to estimate is 0.

In particular, we have shown that the rate of convergence of the estimator θ_n is $n^{1/4}$ and not $n^{1/2}$ as in the classical case. This should not be surprising: indeed, away from 0, the Skew Brownian motion behaves like a Brownian motion, and only its dynamic close to 0 allows one to see the difference between a Skew Brownian motion with a parameter $\theta \neq 0$ and a Brownian motion. It is also not surprising that the local time enters in the limit distribution.

The case $\theta \neq 0$ remains open. One needs to prove results similar to the one of J. Jacod [8], when the Brownian motion is replaced by the Skew Brownian motion (its distribution with respect to the Wiener measure is singular). Of course, one cannot expect the limit law to be symmetric. Yet, it is pretty easy to simulate the Skew Brownian motion and to estimate the maximum likelihood, so that numerical studies are easy to perform.

7 Appendix

In this Appendix we provide the theorems given in [8] used for the proofs of the main results in Section 2. We slightly change the notation and present

the results in the particular cases that are relevant to us in the present work.

Denote by $X = \{X_t \colon 0 \le t \le T\}$ a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Introduce a Borel function $h \colon \mathbb{R}^2 \to \mathbb{R}$ such that there exist $a \in \mathbb{R}$ and $\hat{h} \colon \mathbb{R} \to \mathbb{R}$ such that

$$h(x,y) \le e^{a|y|}\hat{h}(x) \quad \text{and} \quad \int |x|^r |\hat{h}(x)|\, dx < \infty. \tag{26}$$

Theorem 3 (**Theorem 1.1 p. 508 in** [8]**).** *Consider h as above, satisfying* (26) *with $r = 0$. Then*

$$\frac{1}{n^{1/2}} \sum_{i=0}^{n-1} h(\sqrt{n}X_{i/n}, \sqrt{n}(X_{(i+1)/n} - X_{i/n})) \xrightarrow[n\to\infty]{prob.} c(h)\ell_T^x, \tag{27}$$

where

$$c(h) = \iint_{\mathbb{R}^2} h(x,y)p(1,y)\, dx\, dy, \tag{28}$$

and ℓ^x denotes the local time of X at level zero.

Remark 5. It must be noticed that the convergence in (27), as stated in [8], is stronger, in the sense that both terms in (27) are processes (i.e. depend on t) and the convergence is locally uniformly in time, in probability. Recall that a sequence $(Z^n)_{n\ge1}$ of processes is said to *converges locally uniformly in time, in probability*, to a limiting processes Z if for any $t \in \mathbb{R}^+$ the sequence $\sup_{s\le t} |Z_s^n - Z_s|$ goes to 0 in probability.

Theorem 4 (**Theorem 1.2 p. 511 in** [8]**).** *Consider h as above, satisfying* (26) *with some $r > 3$, and assume that $c(h) = 0$ (see* (28)*). Then*

$$\frac{1}{n^{1/4}} \sum_{i=0}^{n-1} h(\sqrt{n}X_{i/n}, \sqrt{n}(X_{(i+1)/n} - X_{i/n})) \xrightarrow[n\to\infty]{\mathcal{F}\text{-stable in dist.}} c\left(h^2\right) W(\ell_T^x), \tag{29}$$

where $W = \{W_t \colon t \ge 0\}$ is a Brownian motion independent of X, and ℓ^x is the local time of X at level zero. The constant $c\left(h^2\right)$ is given in (28) *for the function h^2.*

Remark 6. As in the previous remark, the Theorem stated in [8] is stronger, now in the sense that both terms in (29) are processes, and the processes converge stably in distribution in the Skorokhod space of càdlàg functions.

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