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# Napier's ideal construction of the logarithms* 

Denis Roegel

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## 1 Introduction

Today John Napier (1550-1617) is most renowned as the inventor of logarithms. ${ }^{1}$ He had conceived the general principles of logarithms in 1594 or before and he spent the next twenty years in developing their theory [108, p. 63], [33, pp. 103-104]. His description of logarithms, Mirifici Logarithmorum Canonis Descriptio, was published in Latin in Edinburgh in 1614 [131, 161] and was considered "one of the very greatest scientific discoveries that the world has seen" [83]. Several mathematicians had anticipated properties of the correspondence between an arithmetic and a geometric progression, but only Napier and Jost Bürgi (1552-1632) constructed tables for the purpose of simplifying the calculations. Bürgi's work was however only published in incomplete form in 1620, six years after Napier published the Descriptio [26]. ${ }^{2}$

Napier's work was quickly translated in English by the mathematician and cartographer Edward Wright ${ }^{3}$ (1561-1615) [145, 179] and published posthumously in 1616 [132, 162]. A second edition appeared in 1618. Wright was a friend of Henry Briggs (1561-1630) and this in turn may have led Briggs to visit Napier in 1615 and 1616 and further develop the decimal logarithms.

[^0]Briggs' first table of logarithms, Logarithmorum chilias prima, appeared in 1617 [20] and contained the logarithms in base 10 of the first 1000 (chilias) integers to 14 places. It was followed by his Arithmetica logarithmica in 1624 [21] and his Trigonometria Britannica in 1633 [22].

The details of Napier's construction of the logarithms were published posthumously in 1619 in his Mirifici Logarithmorum Canonis Constructio [133, 134, 136]. Both the Descriptio and the Constructio have had several editions. ${ }^{4}$

It should be stressed that Napier's logarithms were not the logarithms known today as Neperian (or natural, or hyperbolic) logarithms, but a related construction made at a time when the integral and differential calculus did not yet exist. Napier's construction was wholly geometrical and based on the theory of proportions.

We have recomputed the tables published in 1614 and 1616 according to Napier's rules, ${ }^{5}$ using the GNU mpfr library [54]. The computations have been done as if Napier had made no error.

[^1]
## 2 Napier's construction

Napier was interested in simplifying ${ }^{6}$ computations and he introduced a new notion of numbers which he initially called "artificial numbers." This name is used in his Mirifici logarithmorum canonis constructio, published in 1619, but written long before his descriptio of 1614. In the latter work, Napier introduced the word logarithm, from the Greek roots logos (ratio) and arithmos (number). A number of authors have interpreted this name as meaning the "number of ratios," ${ }^{7}$ but G. A. Gibson and W. R. Thomas believe that the word logarithm merely signifies "ratio-number" or "number connected with ratio" [188, p. 195]. Moreover, Thomas believes that Napier found the expression "number connected with ratio" in Archimedes' Sand Reckoner, for it is known that Napier was a good Greek scholar and a number of princeps editions of Archimedes' book were available in England by the early 17th century $[188,86]$.

In any case, Napier meant to provide an arithmetic measure of a (geometric) ratio, or at least to define a continuous correspondence between two progressions. Napier took as origin the value $10^{7}$ and defined its logarithm to be 0 . Any smaller value $x$ was given a logarithm corresponding to the ratio between $10^{7}$ and $x$.

### 2.1 First ideas

It is likely that Napier first had the idea of a correspondence between a geometric series and an arithmetic one. Such correspondences have been exhibited before, in particular by Archimedes, Nicolas Chuquet in 1484, or not long before Napier by Stifel [184]. According to Cantor, Napier's words for negative numbers are the same as those of Stifel, which leads him to think that he was acquainted with Stifel's work [35, p. 703]. Bürgi was also

[^2]indirectly influenced by Stifel and, in fact, the most extensive correspondence between an arithmetic and a geometric progression was probably that of Bürgi [26]. But, as we will see, Napier could impossibly go the whole way of an explicit correspondence, and he needed another definition.

### 2.2 The definition of the logarithm

A precise definition of the logarithm was given by Napier as follows. ${ }^{8}$ He considered two lines (figure 1), with two points moving from left to right at different speeds. On the first line, the point $\alpha$ is moving arithmetically from left to right, that is, with a constant speed equal to $10^{7}$. The figure shows the positions of the point at instant $T, 2 T$, etc., where $T$ is some unit of time. The first interval is traversed in $10^{7} T$. In modern terms, $x_{\alpha}(t)=10^{7} t$.


Figure 1: The progressions $\alpha$ and $\beta$.
On the second line (which is of length $10^{7}$ ), the point $\beta$ is moving geometrically, in the sense that the distances left to traverse at the same instants $T, 2 T$, etc. form a geometrical sequence. Let us analyze Napier's definition in modern terms. If $d_{i}$ is the remaining distance at instant $t_{i}$, we have $\frac{d_{i+1}}{d_{i}}=c$, where $c$ is some constant. If $x_{\beta}(t)$ is the distance traversed since the beginning, we have

$$
\frac{10^{7}-x_{\beta}((n+1) T)}{10^{7}-x_{\beta}(n T)}=c
$$

If we set $y_{\beta}(t)=10^{7}-x_{\beta}(t)$, we have $y_{\beta}((n+1) T)=c y_{\beta}(n T)$ and therefore $y_{\beta}(n T)=c^{n} y_{\beta}(0)$. It follows that $y_{\beta}(t)=c^{t / T} y_{\beta}(0)=A a^{t}$, for $t$ equal to multiples of $T$.

Initially $(t=0)$, point $\beta$ is at the left of the line, and therefore $A=10^{7}$. It follows that

$$
x_{\beta}(t)=10^{7}\left(1-a^{t}\right)=10^{7}\left(1-e^{t \ln a}\right)
$$

Again, this is normally only defined for $t=k T$.

[^3]Napier set the initial speed to $10^{7}$ and this determines the motion entirely. Indeed, from $y_{\beta}^{\prime}(t)=-10^{7} a^{t} \ln a$, we easily obtain $a=e^{-1}$. The initial speed therefore determines $a$. Moreover $x_{\beta}(t)$ does not depend on $T$, which means that Napier's construction defines a correspondence where the unit $T$ plays no role. The motion defined by Napier therefore corresponds to the equation

$$
x_{\beta}(t)=10^{7}\left(1-e^{-t}\right)
$$

but it should be remembered that Napier nowhere uses these notations, and that there were no exponentials or derivatives in his construction. One particular consequence (which was Napier's assumption) of this definition is that the speed of $\beta$ is equal to the distance left to traverse:

$$
x_{\beta}^{\prime}(t)=10^{7} a^{t}=10^{7}-x_{\beta}(t)=y_{\beta}(t) .
$$

Another consequence is that point $\beta$ travels for equal time increments between any two numbers that are equally proportioned, and this will be used later when computing the logarithms of the second table constructed by Napier.

Now, Napier's definition of the logarithm, which we denote $\lambda_{n}$, is the following: if at some time, $\beta$ is at position $x$ and $\alpha$ at position $y$, then $\lambda_{n}\left(10^{7}-x\right)=y$. For instance, at the beginning, $x=y=0$ and $\lambda_{n}\left(10^{7}\right)=0$.

Using a kinematic approach and the theory of proportions, ${ }^{9}$ Napier was thus able to define a correspondence between two continua, those of lines $\alpha$ and $\beta$.

The modern expression for Napier's logarithms can easily be obtained. To $y_{\beta}(t)$ Napier associates $x_{\alpha}(t)$, or in other words, to $x=10^{7} e^{-t}$, Napier associates $y=\lambda_{n}(x)=10^{7} t$, hence

$$
\begin{equation*}
\lambda_{n}(x)=10^{7}\left(\ln \left(10^{7}\right)-\ln x\right) . \tag{1}
\end{equation*}
$$

In particular, $\frac{\lambda_{n}(x)}{10^{7}}-\ln \left(10^{7}\right)$ is the modern logarithm in base $\frac{1}{e}$, but, contrary to what has been written by various authors, $\lambda_{n}(x)$ is not the logarithm in base $\frac{1}{e}$. However, if instead of the factor $10^{7}$, we take 1 , then $\lambda_{n}(x)$ is indeed the modern logarithm of base $\frac{1}{e} \cdot{ }^{10}$

[^4]But we can also write $\frac{\lambda_{n}(x)}{10^{7}}=-\ln \left(\frac{x}{10^{7}}\right)$ and we can view $10^{7}$ as a scaling factor, both for $\lambda_{n}(x)$ and for $x$. In other words, if we scale both the sines and the logarithms in Napier's tables down by $10^{7}$, we obtain a correspondence which is essentially the logarithm of base $\frac{1}{e}$.

Although the logarithms are precisely defined by Napier's procedure, their computation is not obvious. This, per se, is already particularly interesting, as Napier managed to define a numerical concept, and to separate this concept from its actual computation.

Next, Napier defines geometric divisions which he will use to approximate the values of the logarithms.

### 2.3 Geometric divisions

Napier then constructed three geometric sequences. These sequences all start with $10^{7}$ and then use different multiplicative factors. These tables are given in sections 10.1, 10.2, and 10.3. ${ }^{11}$

The first sequence $a_{i}$ uses the ratio $r_{1}=0.9999999$ such that $\frac{a_{i+1}}{a_{i}}=r_{1}$ and $a_{0}=10^{7}$. Napier computes 100 terms of the sequence using exactly 7 decimal places, until $a_{100}=9999900.0004950$ [136, p. 13]. The values of $a_{i}$ are given in section 10.1. Napier's last value is correct.

The second sequence $b_{i}$ also starts with $10^{7}$, but uses the ratio $r_{2}=$ 0.99999. We have $\frac{b_{i+1}}{b_{i}}=r_{2}$. Napier computes 50 terms until he reaches $b_{50}=$ 9995001.224804 using exactly 6 decimal places. Napier's value was actually $b_{50}^{\prime}=9995001.222927$ [136, p. 14] and he must have made a computation error, as the error is much larger than the one due to rounding. ${ }^{12}$ This error, together with the confidence Napier displays for his calculations, certainly indicates that he alone did the computations. ${ }^{13}$

The first and second sequences are related in that $b_{1} \approx a_{100}$, which follows from $0.9999999^{100} \approx 0.99999$. However, as we will see, Napier does not identify $a_{100}$ and $b_{1}$ nor $\lambda_{n}\left(a_{100}\right)$ and $\lambda_{n}\left(b_{1}\right)$.

[^5]The third sequence $c$ is actually made of 69 subsequences. Each of these subsequences uses the ratio $r_{3}=0.9995$. The first of these subsequences is $c_{0,0}, c_{1,0}, c_{2,0}, \ldots, c_{20,0}$ and it starts with $c_{0,0}=10^{7}$. The second subsequence is $c_{0,1}, c_{1,1}, c_{2,1}, \ldots, c_{20,1}$. The 69 th subsequence is $c_{0,68}, c_{1,68}, c_{2,68}, \ldots, c_{20,68}$. For $1 \leq i \leq 20$ and $0 \leq j \leq 68$, we have $\frac{c_{i, j}}{c_{i-1, j}}=r_{3}$. In particular, $c_{1,0}=$ $9995000 \approx b_{50}$ because $0.99999^{50} \approx 0.9995$.

The relationship between one subsequence and the next one is by a constant ratio $r_{4}=0.99$ and $\frac{c_{0, j+1}}{c_{0, j}}=r_{4}$. Moreover, $c_{20, i} \approx c_{0, i+1}$, because $0.9995^{20} \approx 0.99$.

Napier computed the first column of the third table using 5 decimal places. Then, he used these values to compute the other values of the rows with 4 decimal places. Napier found $c_{20,0}=9900473.57808$ [136, p. 14] (exact: 9900473.578023 ), ${ }^{14} c_{20,1}=9801468.8423$ [136, p. 15] (exact: 9801468.842243), $c_{20,2}=9703454.1539$ [136, p. 16] (exact: 9703454.153821), $\ldots, c_{0,68}=$ 5048858.8900 [136, p. 16] (exact: 5048858.887871), $c_{20,68}=4998609.4034$ [136, p. 16] (exact: 4998609.401853 ).

Napier has therefore divided the interval between $10^{7}$ and about $5 \cdot 10^{6}$ into $69 \times 20=1380$ subintervals with a constant ratio, although there are some junctions between the 69 subsequences. Then, each of these 1380 intervals could itself be divided into 50 intervals corresponding to a ratio of 0.99999 , and these intervals could in turn be divided into 100 intervals whose ratio is 0.9999999 . These sequences therefore provide an approximation of a subdivision of the interval between $10^{7}$ and $5 \cdot 10^{6}$ into many small intervals corresponding to the ratio 0.9999999 . If the logarithm of 0.9999999 could be computed, these sequences would provide a means to compute approximations of the logarithms of the other values in the sequence.

The values of the third table are now dense enough to be used in conjunction with the sines of angles from $90^{\circ}$ to $30^{\circ}$, whose sine is 5000000 when the sinus totus is $10^{7}$.

Ideally, Napier would have divided the whole interval [ $10^{7}-5 \cdot 10^{6}$ ] using the ratio $r_{1}$, but this was an impossible task. Using the ratio 0.9999999 a total of $100 \times 50 \times 20 \times 69=6900000$ times, he would have obtained as his last value 5015760.517 . He would in fact have needed to go a little further to be below $5 \cdot 10^{6}$. But if he had proceeded this way, by iterative divisions, he would have introduced important rounding errors. On the contrary, with his three sequences, Napier was able to keep the rounding errors to a minimum,

[^6]since he never had more than 100 multiplications in a row, and in fact mostly only 20 .

### 2.4 Approximation of the logarithms

Napier's next task is not to compute mere approximations of the logarithms of each value in the sequences, but interval approximations, ${ }^{15}$ that is, bounds for each logarithm, taking in particular into account the fact that the various sequences are not seamless. By using interval approximations, Napier knows exactly the accuracy of his computations.

### 2.4.1 Computing the first logarithm



Figure 2: The bounds of $\lambda_{n}(C B)$.
Napier first tries to compute the logarithm of $a_{1}=9999999$. He considers the configuration of figure 2 where $A$ and $B$ are the endpoints of $\beta$ and $C$ is some point in between. $E$ is the origin of line $\alpha$ and $F$ is the point of $\alpha$ corresponding to $C$. The $\beta$ line is extended to the left to $D$ such that the time $T$ taken to go from $D$ to $A$ is equal to the time taken to go from $A$ to $C$. This, as was explained above, entails that

$$
\begin{equation*}
\frac{C B}{A B}=\frac{A B}{D B} . \tag{2}
\end{equation*}
$$

Since $E F$ is the logarithm of $C B$, and since the speed of $\beta$ is equal to the distance between $\beta$ and $B$ and since this distance decreases, the speed of $\beta$ also decreases, and naturally $A C<E F$. Similarly, $D A>E F$. We now have two bounds for the logarithm of $C B$ :

$$
A C<\lambda_{n}(C B)<D A
$$

[^7]Using equation (2), we can compute these bounds:

$$
\begin{aligned}
A C & =A B-C B=10^{7}-C B \\
D A & =D B-A B=A B\left(\frac{D B}{A B}-1\right)=A B\left(\frac{A B}{C B}-1\right)=A B \times \frac{A B-C B}{C B} \\
& =A B \times \frac{A C}{C B}
\end{aligned}
$$

In practice, this gives

$$
\begin{equation*}
10^{7}-C B<\lambda_{n}(C B)<10^{7}\left(\frac{10^{7}-C B}{C B}\right) \tag{3}
\end{equation*}
$$

Taking $C B=9999999$, we have:

$$
1<\lambda_{n}(9999999)<\frac{10^{7}}{9999999} \approx 1+10^{-7}
$$

In the Constructio, Napier gives the upper bound 1.00000010000001 [136, p. 21].

Napier decided to take the average of the two bounds and set $\lambda_{n}(9999999) \approx$ 1.00000005 [136, p. 21]. The exact value of $\lambda_{n}(9999999)$ is

$$
10^{7}\left(\ln \left(10^{7}\right)-\ln (9999999)\right)=1.000000050000003 \ldots
$$

Napier's approximation was therefore excellent. ${ }^{16}$
Napier can now compute all the other logarithms in the first sequence $a_{i}$, since he has approximations of the first two values of the sequence:

$$
\begin{gathered}
\lambda_{n}\left(a_{0}\right)=0 \\
1<\lambda_{n}\left(a_{1}\right)<\frac{10^{7}}{9999999}
\end{gathered}
$$

We also have $\lambda_{n}\left(a_{i+1}\right)-\lambda_{n}\left(a_{i}\right)=\lambda_{n}\left(a_{1}\right)-\lambda_{n}\left(a_{0}\right)=\lambda_{n}\left(a_{1}\right)$. Hence $\lambda_{n}\left(a_{i}\right)=$ $i \lambda_{n}\left(a_{1}\right)$ and

$$
i<\lambda_{n}\left(a_{i}\right)<i \times \frac{10^{7}}{9999999}
$$

Napier takes as approximation of $\lambda_{n}\left(a_{i}\right)$ :

$$
\lambda_{n}\left(a_{i}\right) \approx i \times 1.00000005
$$

[^8]and fills the table in section 10.1. In order to distinguish the theoretical values of the logarithms from the values corresponding to Napier's process, we will denote the latter with $l_{1}\left(a_{i}\right)$ for the first table, $l_{2}\left(b_{i}\right)$ for the second table, and $l_{3}\left(c_{i, j}\right)$ for the third table. Moreover, if necessary, we will use $l_{i}^{\prime}\left(a_{j}\right)$ for Napier's actual table values when they differ from the recomputed ones. ${ }^{17}$ We will always have
\[

$$
\begin{aligned}
& \lambda_{n}\left(a_{i}\right) \approx l_{1}\left(a_{i}\right) \\
& \lambda_{n}\left(b_{i}\right) \approx l_{2}\left(b_{i}\right) \\
& \lambda_{n}\left(c_{i, j}\right) \approx l_{3}\left(c_{i, j}\right)
\end{aligned}
$$
\]

Given the excellent approximation of $\lambda_{n}\left(a_{1}\right)$, all the values of the first table are also excellent approximations. In our reconstruction, we have given the approximations Napier would have obtained, had his computations been correct and not rounded. In other words, our only approximation was to take the approximation of the first logarithm as an arithmetic mean of the bounds.

The last value of the first table is $\lambda_{n}\left(a_{100}\right)$ and its bounds by the above reasoning are (with an exact computation):

$$
100<\lambda_{n}\left(a_{100}\right)<100.000010000001 \ldots
$$

### 2.4.2 Computing the first logarithm of the second table

Next, Napier tackles the computation of the logarithm of $b_{1}=9999900$. Napier already knows the logarithm of $a_{100} \approx b_{1}$, but he will not consider the logarithms of $a_{100}$ and $b_{1}$ to be equal. Instead, what Napier does is to consider the difference between $\lambda_{n}\left(b_{1}\right)$ and $\lambda_{n}\left(a_{100}\right)$ and he shows that this difference is bounded. He proceeds as follows. First, he observes that if $\frac{a}{b}=\frac{c}{d}$, then $\lambda_{n}(a)-\lambda_{n}(b)=\lambda_{n}(c)-\lambda_{n}(d)$ which is a consequence of Napier's definition. Then, he considers figure 3. Let $A B=10^{7}$ and $C B$ and $D B$ be the two logarithms whose difference will be considered. Points $E$ and $F$ are defined such that

$$
\begin{aligned}
& \frac{E A}{A B}=\frac{C D}{D B} \\
& \frac{A F}{A B}=\frac{C D}{C B}
\end{aligned}
$$

[^9]

Figure 3: Bounding the difference of two logarithms.

It follows that

$$
\begin{aligned}
& \frac{E B}{A B}=\frac{\frac{(A B)(C D)}{D B}+A B}{A B}=\frac{A B\left(\frac{C D}{D B}+1\right)}{A B}=\frac{C D+D B}{D B}=\frac{C B}{D B} \\
& \frac{F B}{A B}=\frac{A B-F A}{A B}=\frac{A B-A B \cdot \frac{C D}{C B}}{A B}=\frac{C B-C D}{C B}=\frac{D B}{C B}
\end{aligned}
$$

and therefore

$$
\lambda_{n}(D B)-\lambda_{n}(C B)=\lambda_{n}(F B)-\lambda_{n}(A B)=\lambda_{n}(F B)
$$

As seen above, since $\frac{F B}{A B}=\frac{A B}{E B}$, we have

$$
A F<\lambda_{n}(F B)<E A
$$

and therefore

$$
\begin{equation*}
\frac{(A B)(C D)}{C B}<\lambda_{n}(D B)-\lambda_{n}(C B)<\frac{(A B)(C D)}{D B} \tag{4}
\end{equation*}
$$

Taking $C B=a_{100}$ and $D B=b_{1}$, we have

$$
10^{7} \times \frac{\left(a_{100}-b_{1}\right)}{a_{100}}<\lambda_{n}\left(b_{1}\right)-\lambda_{n}\left(a_{100}\right)<10^{7} \times \frac{\left(a_{100}-b_{1}\right)}{b_{1}}
$$

which numerically gives (with the exact values)

$$
0.00049500333301 \ldots<\lambda_{n}\left(b_{1}\right)-\lambda_{n}\left(a_{100}\right)<0.00049500333303 \ldots
$$

Since the two bounds were very close, Napier took $\lambda_{n}\left(b_{1}\right)-\lambda_{n}\left(a_{100}\right)=$ 0.0004950 , and from the approximation of $\lambda_{n}\left(a_{100}\right)$ found above, he obtained [136, pp. 29-30]:

$$
\begin{gathered}
100+0.0004950 \ldots<\lambda_{n}\left(a_{100}\right)+\left(\lambda_{n}\left(b_{1}\right)-\lambda_{n}\left(a_{100}\right)\right)<100.0000100 \ldots+0.0004950 \ldots \\
100.0004950 \ldots<\lambda_{n}\left(b_{1}\right)<100.0005050 \ldots
\end{gathered}
$$

More accurate bounds are:

$$
100.00049500333301 \ldots<\lambda_{n}\left(b_{1}\right)<100.00050500333403 \ldots
$$

Napier chose to take the average between the two bounds, namely $l_{2}\left(b_{1}\right)=$ 100.0005 , which is an excellent approximation of the real value

$$
\lambda_{n}\left(b_{1}\right)=100.00050000333 \ldots
$$

Napier now computes all the logarithms of the second sequence, using the approximation

$$
\lambda_{n}\left(b_{i}\right) \approx i \times 100.0005
$$

The bounds of the last value, $\lambda_{n}\left(b_{50}\right)$ are

$$
50 \times 100.0004950 \ldots<\lambda_{n}\left(b_{50}\right)<50 \times 100.0005050 \ldots
$$

and Napier chose $l_{2}\left(b_{50}\right)=5000.0250000$ which is also an excellent approximation of the real value

$$
\lambda_{n}\left(b_{50}\right)=5000.0250001 \ldots
$$

### 2.4.3 Computing two logarithms of the third table

In order to complete the third table, Napier had to compute approximations of two logarithms: $\lambda_{n}\left(c_{1,0}\right)=\lambda_{n}(9995000)$ and $\lambda_{n}\left(c_{0,1}\right)=\lambda_{n}(9900000)$.

We might first think of using the fact that $b_{50}$ is very close to $c_{1,0}$, and consider the difference $\lambda_{n}\left(c_{1,0}\right)-\lambda_{n}\left(b_{50}\right)$. Its bounds can be found using equation (4):

$$
10^{7} \frac{b_{50}-c_{1,0}}{b_{50}}<\lambda_{n}\left(c_{1,0}\right)-\lambda_{n}\left(b_{50}\right)<10^{7} \frac{b_{50}-c_{1,0}}{c_{1,0}}
$$

that is, using the exact value of $b_{50}$ :

$$
1.225416581 \ldots<\lambda_{n}\left(c_{1,0}\right)-\lambda_{n}\left(b_{50}\right)<1.225416731 \ldots
$$

But in fact, Napier obtains even better bounds, in that he goes back to the first table, which is the most dense one. Napier first computes $x$ such that $\frac{x}{10^{7}}=\frac{c_{1,0}}{b_{50}}$ which gives $x=9999998.774583418771 \ldots$.. Napier had actually found $x=9999998.7764614$ [136, p. 31]. ${ }^{18}$ Then, we have $\lambda_{n}\left(c_{1,0}\right)-\lambda_{n}\left(b_{50}\right)=\lambda_{n}(x)-\lambda_{n}\left(10^{7}\right)=\lambda_{n}(x)$. We can now express the bounds of $\lambda_{n}(x)-\lambda_{n}\left(a_{1}\right)$, the closest value to $x$ in the first table being $a_{1}=9999999$ :

[^10]\[

$$
\begin{gathered}
10^{7} \frac{\left(a_{1}-x\right)}{a_{1}}<\lambda_{n}(x)-\lambda_{n}\left(a_{1}\right)<10^{7} \frac{\left(a_{1}-x\right)}{x} \\
0.2254166037 \ldots<\lambda_{n}(x)-\lambda_{n}\left(a_{1}\right)<0.2254166088 \ldots \\
1<\lambda_{n}\left(a_{1}\right)<1.000000100 \ldots
\end{gathered}
$$
\]

Hence

$$
1.2254166037 \ldots<\lambda_{n}(x)<1.2254167088 \ldots
$$

the exact value being $1.225416656 \ldots$
These bounds are more accurate than the ones obtained using the second table only, but the first bounds were also very good.

Napier had actually found

$$
1.2235386 \ldots<\lambda_{n}(x)<1.2235387 \ldots
$$

and the difference with the real bounds is a consequence of Napier's error on $b_{50}$ which has translated in an error of about 0.002 on $x$.

From this, we obtain (using the exact values of the bounds)
$5000.0247501 \ldots+1.2254166 \ldots<\lambda_{n}\left(c_{1,0}\right)<5000.0252501 \ldots+1.2254167 \ldots$
that is (using the exact values)

$$
5001.2501667 \ldots<\lambda_{n}\left(c_{1,0}\right)<5001.2506668 \ldots
$$

The average of the bounds is 5001.2504168 , but Napier found 5001.2485387 (see [136, p. 31]). We take $l_{3}\left(c_{1,0}\right)=5001.2504168$.

The difference between Napier's value and the exact one is still due to the error on $b_{50}$. This error will propagate and increase when building the third table.

The exact value of $\lambda_{n}\left(c_{1,0}\right)$ is $5001.25041682 \ldots$ and Napier's (ideal) ${ }^{19}$ approximation is again excellent.

Napier can then compute approximations of $c_{i, 0}$ for $i \leq 20$. For $c_{20,0}$, the exact bounds are

$$
100025.003335 \ldots<\lambda_{n}\left(c_{20,0}\right)<100025.013337 \ldots
$$

and Napier uses the value 100024.9707740 [136, p. 32] instead of the better $100025.008 \ldots$, still as a consequence of the error on $b_{50}$. However, rounded to one decimal, both values are identical (see [136, p. 34]).

[^11]Next, Napier observes that $c_{20,0} \approx c_{0,1}$. Similarly, Napier bounds the difference $\lambda_{n}\left(c_{0,1}\right)-\lambda_{n}\left(c_{20,0}\right)$. If he had used the first column of the third table, he would have obtained:

$$
10^{7} \frac{\left(c_{20,0}-c_{0,1}\right)}{c_{20,0}}<\lambda_{n}\left(c_{0,1}\right)-\lambda_{n}\left(c_{20,0}\right)<10^{7} \frac{\left(c_{20,0}-c_{0,1}\right)}{c_{0,1}}
$$

that is, using the exact value of $c_{20,0}$ :

$$
478.33875779 \ldots<\lambda_{n}\left(c_{0,1}\right)-\lambda_{n}\left(c_{20,0}\right)<478.36163968 \ldots
$$

But, what Napier did, was to use the second table instead of the first column of the third table. Napier found $x$ such that $\frac{x}{10^{7}}=\frac{c_{0,1}}{c_{20,0}}$, which gives $x=$ $9999521.66124220 \ldots$... Napier had actually found $x=9999521.6611850[136$, p. 33]. This value is outside of the range of the first table, but it is within the range of the second table. Then, we have $\lambda_{n}\left(c_{0,1}\right)-\lambda_{n}\left(c_{20,0}\right)=\lambda_{n}(x)-$ $\lambda_{n}\left(10^{7}\right)=\lambda_{n}(x)$. We can now express the bounds of $\lambda_{n}\left(b_{5}\right)-\lambda_{n}(x)$, the closest value to $x$ in the first table being $b_{5}=9999500.009999900 \ldots$ :

$$
\begin{gathered}
10^{7} \frac{\left(x-b_{5}\right)}{x}<\lambda_{n}\left(b_{5}\right)-\lambda_{n}(x)<10^{7} \frac{\left(x-b_{5}\right)}{b_{5}} \\
21.6522780<\lambda_{n}\left(b_{5}\right)-\lambda_{n}(x)<21.6523249 \ldots \\
-21.6523250<\lambda_{n}(x)-\lambda_{n}\left(b_{5}\right)<-21.6522780 \\
500.0024750<\lambda_{n}\left(b_{5}\right)<500.0025251
\end{gathered}
$$

Hence

$$
478.3501501<\lambda_{n}(x)<478.3502471
$$

These bounds are much more accurate than the ones obtained using the third table only.

Napier had actually found

$$
478.3502290<\lambda_{n}(x)<478.3502812 .
$$

From this, we obtain (using the exact values of the bounds)
$100025.003335 \ldots+478.350150 \ldots<\lambda_{n}\left(c_{0,1}\right)<100025.013337 \ldots+478.350247 \ldots$
that is (using exact values)

$$
100503.353485<\lambda_{n}\left(c_{0,1}\right)<100503.363585
$$

The average of the two bounds is 100503.358535 and the exact value of $\lambda_{n}\left(c_{0,1}\right)$ is $100503.35853501 \ldots$... Napier, instead, found 100503.3210291 [136, p. 33] and the difference between these values is still only due to the error on $b_{50}$.

Having now approximations for $\lambda_{n}\left(c_{20,0}\right)$ and $\lambda_{n}\left(c_{0,1}\right)$, Napier can obtain bounds for all the values in the third table. First, he obtains bounds for all $\lambda_{n}\left(c_{0, j}\right)$, because $\lambda_{n}\left(c_{0, j}\right)-\lambda_{n}\left(c_{0,0}\right)=j \times\left(\lambda_{n}\left(c_{0,1}\right)-\lambda_{n}\left(c_{0,0}\right)\right)$, hence

$$
\lambda_{n}\left(c_{0, j}\right)=j \times \lambda_{n}\left(c_{0,1}\right)
$$

If necessary, the bounds on $\lambda_{n}\left(c_{0,1}\right)$ can be used to find bounds for $\lambda_{n}\left(c_{0, j}\right)$.
Finally, we have

$$
\begin{aligned}
\lambda_{n}\left(c_{i, j}\right)-\lambda_{n}\left(c_{0, j}\right) & =i \times\left(\lambda_{n}\left(c_{1, j}\right)-\lambda_{n}\left(c_{0, j}\right)\right) \\
& =i \times\left(\lambda_{n}\left(c_{1,0}\right)-\lambda_{n}\left(c_{0,0}\right)\right) \\
& =i \times \lambda_{n}\left(c_{1,0}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{n}\left(c_{i, j}\right) & =\lambda_{n}\left(c_{0, j}\right)+i \times \lambda_{n}\left(c_{1,1}\right) \\
& =j \times \lambda_{n}\left(c_{0,1}\right)+i \times \lambda_{n}\left(c_{1,0}\right)
\end{aligned}
$$

All of the logarithms of the third table are computed as a mere linear combination of the logarithms $\lambda_{n}\left(c_{1,0}\right)$ and $\lambda_{n}\left(c_{0,1}\right)$.

### 2.5 Finding the logarithms of all sines

As we have shown, the sole purpose of the first and second tables was to build the third table.

Now, the third table will be used to find the logarithms of all the integers between 0 and $10^{7}$, or rather, of all those which are sine values, the radius being taken equal to $10^{7}$.

We will use the following notation to denote such sine values, $n$ standing for Napier:

$$
\sin _{n} x=10^{7} \sin x
$$

At that time, the sine was not considered as a ratio, but as the length of a semi-chord in a circle of a certain radius. For Napier, the sine of $90^{\circ}$ is $10^{7}$.

Napier, of course, did not have an exponent notation.

### 2.5.1 The source of the sines

First, Napier needed the sines and cosines for every minute between $0^{\circ}$ and $45^{\circ}$. Napier's Constructio is somewhat misleading on this matter. For those who would like to construct such a table, Napier suggests to use a table such as Reinhold's table or a more accurate one [136, p. 45]. The table he had in mind must be the one published in 1554 [150]. Reinhold's tables contain a table of tangents and a table of sine, the latter for every minute. It appears that the sine values used by Napier agree with those of Reinhold from $0^{\circ}$ to $89^{\circ}$, except for a few typos. The value of the sinus total is also $10^{7}$ in Reinhold's tables. However, there are a number of differences from $89^{\circ}$ to $90^{\circ}$, showing that Napier necessarily used at least another different source.

It is also easy to see that Napier cannot have used Rheticus' 1551 canon [151], as it only gives the trigonometric functions every 10 minutes. He could however have used Rheticus' Opus palatinum (1596) [152] which gave the sines with a radius of $10^{10}$ every 10 seconds, and Napier would merely have had to cut off three digits and round. The comparison with Rheticus' Opus palatinum shows a few differences, suggesting that this was probably not Napier's source. Moreover, Napier had probably started his work before any copy of the Opus palatinum reached him.

The actual source was found in 1990 by Glowatzki and Göttsche who were investigating Regiomontanus' tables [69]. They concluded that Napier either used the table published by Fincke in 1583 [48], or the one published by Lansberge in 1591 [192]. We have not consulted Fincke's table, but we have found that the values are the same in Lansberge's 1591 edition ${ }^{20}$ and in Napier's tables.

Both Fincke and Lansberge's tables go back to those of Regiomontanus (1436-1476), and perhaps even to those of Bianchini (1410-c1469) [168, p. 421]. ${ }^{21}$

In any case, whatever the source, Napier had to obtain the logarithm for every sine. There were a total of $90 \times 60=5400$ different sine values and some of these values could be obtained from others, as we will see later.

Computing the initial tables and all logarithms took Napier 20 years.

[^12]
### 2.5.2 Computing a logarithm within the range of the third table

We have seen earlier, when computing the first logarithm, that

$$
10^{7}-n<\lambda_{n}(n)<10^{7} \frac{\left(10^{7}-n\right)}{n}
$$

Setting $x=10^{7}-n$, Napier's procedure amounts to approximate $\lambda_{n}(n)$ by $\frac{1}{2}\left(x+\frac{x}{1-\frac{x}{10^{7}}}\right) \approx x+\frac{x^{2}}{2 \cdot 10^{7}}$.

Therefore, if $\frac{x^{2}}{2 \cdot 10^{7}}$ can be neglected, a good approximation of $\lambda_{n}(n)$ is $x=10^{7}-n$. Napier uses this approximation when $n \geq 9996700$, that is, when $\frac{x^{2}}{2 \cdot 10^{7}} \leq 0.5445 \ldots$. Napier chose this limit, because he found, using equation (3), that $3300<\lambda_{n}(9996700)<3301$ [136, p. 36], that is, the smallest value such that its logarithm is bounded by two consecutive integers. The correct bounds are actually

$$
3300<\lambda_{n}(9996700)<3301.0893 \ldots
$$

and Napier was slightly wrong, but in any case, Napier takes the lower bound as an approximation of $\lambda_{n}(n)$ when $n \geq 9996700$, because he considers that the error will then be less than 0.5 . An exact computation shows that the error is smaller than 0.5 only when $n>9996837$. But even if this computation was slightly incorrect (assuming that the limit of this case in the Constructio is the limit used in the construction of the table), the problem actually only concerns two sine values of the table of log. sines, namely those of $88^{\circ} 32^{\prime}$ and $88^{\circ} 33^{\prime}$.

When $5 \cdot 10^{6} \leq n<9996700$ and $n$ is not in the third table, Napier proceeds as follows. First, $i, j$ are found such that $c_{i+1, j}<n<c_{i, j}$ or such that $c_{0, j+1}<n<c_{20, j}$. We then have two numbers $x, y$, such that $x<n<y$ and we know $\lambda_{n}(x)$ and $\lambda_{n}(y)$.

We can now bound either $\lambda_{n}(x)-\lambda_{n}(n)$ or $\lambda_{n}(n)-\lambda_{n}(y)$. In the former case

$$
-10^{7}\left(\frac{n-x}{x}\right)<\lambda_{n}(n)-\lambda_{n}(x)<-10^{7}\left(\frac{n-x}{n}\right)
$$

and in the latter case

$$
10^{7}\left(\frac{y-n}{y}\right)<\lambda_{n}(n)-\lambda_{n}(y)<10^{7}\left(\frac{y-n}{n}\right)
$$

We also have bounds for $\lambda_{n}(x)$ and $\lambda_{n}(y)$ and by adding them, we would obtain bounds for $\lambda_{n}(n)$. We can however forget about the bounds and use only approximations of $\lambda_{n}(n)-\lambda_{n}(x)$ or $\lambda_{n}(n)-\lambda_{n}(y)$ and of $\lambda_{n}(x)$ or $\lambda_{n}(y)$.

For example, if $n=5 \cdot 10^{6}$, we can take $x=c_{20,68}=4998609.401853$, $y=c_{19,68}=5001109.956832$ and find bounds for $\lambda_{n}(n)-\lambda_{n}(y)$, since $n$ is closest to $y$. We have $\lambda_{n}(y)=6929252.1$. Therefore,

$$
\begin{gathered}
10^{7} \frac{1109.956832}{5001109.956832}<\lambda_{n}(n)-\lambda_{n}(y)<10^{7} \frac{1109.956832}{5 \cdot 10^{6}} \\
2219.420 \ldots<\lambda_{n}(n)-\lambda_{n}(y)<2219.913 \ldots \\
\lambda_{n}(n) \approx 6929252.1+2219.7=6931471.8
\end{gathered}
$$

If $n=8 \cdot 10^{6}$, we can take $x=c_{5,22}=7996285.161399, y=c_{4,22}=$ 8000285.304051 and find bounds for $\lambda_{n}(n)-\lambda_{n}(y)$, since $n$ is closest to $y$. We have $\lambda_{n}(y)=2231078.9$. Therefore,

$$
\begin{gathered}
10^{7} \frac{285.304051}{8000285.304051}<\lambda_{n}(n)-\lambda_{n}(y)<10^{7} \frac{285.304051}{8 \cdot 10^{6}} \\
356.61 \ldots<\lambda_{n}(n)-\lambda_{n}(y)<356.63 \ldots \\
\lambda_{n}(n) \approx 2231078.9+356.6=2231435.5
\end{gathered}
$$

Napier actually found $\lambda_{n}(5000000)=6931469.22$ and $\lambda_{n}(8000000)=$ 2231434.68. The exact values are 6931471.805599 and 2231435.513142 , but as the above computations show, Napier could have found much more accurate values, had he not made an error in his second table and had he computed correctly the first logarithms of the third table.

### 2.5.3 Napier's short table

In his Constructio [133, 136], Napier gives a table which sums up the values of the differences of the logarithms of two numbers, if these numbers correspond to a given ratio. The table will be used to find the logarithms of numbers outside the range $\left[10^{7}-5 \cdot 10^{6}\right]$. Napier had shown earlier that if $\frac{a}{b}=\frac{c}{d}$, then $\lambda_{n}(a)-\lambda_{n}(b)=\lambda_{n}(c)-\lambda_{n}(d)$. The difference $\lambda_{n}(p)-\lambda_{n}(q)$ therefore only depends on $\frac{p}{q}$, not on the actual values of $p$ and $q$. It is also important to realize that $\lambda_{n}(p)-\lambda_{n}(q)=-10^{7} \ln (p / q)=\lambda_{n}(p / q)-10^{7} \ln \left(10^{7}\right)$ and therefore that $\lambda_{n}(p)-\lambda_{n}(q) \neq \lambda_{n}(p / q)$.

We now set $\rho(p / q)=\lambda_{n}(q)-\lambda_{n}(p)$. In order to fill his table, Napier uses the value of $\lambda_{n}(5000000)$ computed earlier. He then obtains $\rho(2)=$ $\lambda_{n}(5000000)-\lambda_{n}(10000000)=\lambda_{n}(5000000)=6931469.22$. This is the difference corresponding to a ratio of 2 . Then, Napier computes $\lambda_{n}(x / 4)-$ $\lambda_{n}(x / 2)=\lambda_{n}(x / 2)-\lambda_{n}(x)=\rho(2)$ hence $\rho(4)=\lambda_{n}(x / 4)-\lambda_{n}(x)=2\left(\lambda_{n}(x / 2)-\right.$ $\left.\lambda_{n}(x)\right)=2 \rho(2)$. Similarly, he obtains $\rho(8)=3 \rho(2)=20794407.66$.

Then, Napier uses the value of $\lambda_{n}(8000000)$, also computed earlier, and writes $\lambda_{n}(1000000)-\lambda_{n}(8000000)=\rho(8)$, hence

$$
\lambda_{n}(1000000)=\lambda_{n}(8000000)+\rho(8)=23025842.34
$$

Finally, $\rho(10)=\lambda_{n}(1000000)-\lambda_{n}(10000000)=\lambda_{n}(1000000)$. By iterating this process, Napier computes $\rho(20), \rho(40)$, etc., until $\rho(10000000)$. In general, we have $\rho(a b)=\rho(a)+\rho(b)$ and $\rho\left(a^{b}\right)=b \rho(a)$.

Napier can now complete his short table (figure 4).
It can be noticed that $\rho(10)=10^{7} \ln 10$, and so, in a way, Napier accidentally computed $\ln 10$, but also $\ln 2, \ln 4, \ln 8$, etc. How accurate his calculations were can be judged by the accuracy of $\ln 2$ and $\ln 10$ which have respectively five and six correct places.

Some authors have wondered about the value of $\lambda_{n}(1) .{ }^{22}$ The short table readily gives an approximation of $\lambda_{n}(1)$, since $\lambda_{n}(1)=\lambda_{n}(1)-\lambda_{n}\left(10^{7}\right)=$ $\rho\left(10^{7}\right)$, but in Napier's table the small differences at the beginning of the table accumulate and produce greater differences for $\lambda_{n}(1)$. Using the modern expression seen previously, we find of course $\lambda_{n}(1)=161180956.509 \ldots$..

Using our table, we find therefore $\lambda_{n}(1)=161180955.81$, but Napier had $\lambda_{n}(1)=161180896.38$ [136, p. 39]. However, this value was of little importance in the construction of his logarithms.

### 2.5.4 Computing a logarithm outside the range of the third table

If $n<5 \cdot 10^{6}$, we can determine $k=2,4,8,10, \ldots$, such that $5 \cdot 10^{6} \leq k n \leq 10^{7}$. There are at most two such values, and we can choose the one we like. Then, we know that $\lambda_{n}(n)-\lambda_{n}(k n)=\rho(k)$ and $\lambda_{n}(n)=\lambda_{n}(k n)+\rho(k)$.

For example, let us compute $\lambda_{n}$ (87265), where 87265 is an approximation of $\sin _{n} 30^{\prime}$. The exact value is $\sin _{n} 30^{\prime}=87265.354 \ldots$ In our case, we can take $k=100$. We first find an approximation of $\lambda_{n}(8726500)$ using the technique seen above. Setting $n=8726500$, we have:

$$
c_{12,13}<n<c_{11,13}
$$

with $c_{11,13}=8727067.052058, c_{12,13}=8722703.518532$, and $\lambda_{n}\left(c_{11,13}\right)=$

[^13]| Given Proportions <br> of Sines. | Corresponding <br> $i$ | Differences of <br> Logarithm. <br> $d_{i}$ | Given Proportions <br> of Sines. | Corresponding <br> Differences of <br> Logarithm. |
| :---: | :---: | :---: | :--- | :---: |
| 2 | to one | 6931469.22 | 8000 | to one |
| 4 | $"$ | 13862938.44 | 10000 | $"$ |
| 8 | $"$ | 20794407.66 | 20000 | $"$ |
| 10 | $"$ | 23025842.34 | 40000 | $"$ |
| 20 | $"$ | 29957311.56 | 80000 | $"$ |
| 40 | $"$ | 36888780.78 | 100000 | $"$ |
| 80 | $"$ | 43820250.00 | 200000 | $"$ |
| 100 | $"$ | 46051684.68 | 400000 | $"$ |
| 200 | $"$ | 52983153.90 | 800000 | $"$ |
| 400 | $"$ | 59914623.12 | 1000000 | $"$ |
| 800 | $"$ | 66846092.34 | 2000000 | $"$ |
| 1000 | $"$ | 69077527.02 | 4000000 | $"$ |
| 2000 | $"$ | 76008996.24 | 8000000 | $"$ |
| 4000 | $"$ | 82940465.46 | 10000000 | $"$ |


| Given Proportions of Sines. <br> i | Corresponding Differences of Logarithm. $d_{i}$ | Given Proportions of Sines. i | Corresponding Differences of Logarithm. $d_{i}$ |
| :---: | :---: | :---: | :---: |
| 2 to one | 6931471.77 | 8000 to one | 89871967.80 |
| 4 | 13862943.54 | 10000 | 92103403.32 |
| 8 | 20794415.31 | 20000 | 99034875.09 |
| 10 | 23025850.83 | 40000 | 105966346.86 |
| 20 | 29957322.60 | 80000 | 112897818.63 |
| 40 | 36888794.37 | 100000 | 115129254.15 |
| 80 | 43820266.14 | 200000 | 122060725.92 |
| 100 | 46051701.66 | 400000 | 128992197.69 |
| 200 | 52983173.43 | 800000 | 135923669.46 |
| 400 | 59914645.20 | 1000000 | 138155104.98 |
| 800 | 66846116.97 | 2000000 | 145086576.75 |
| 1000 | 69077552.49 | 4000000 | 152018048.52 |
| 2000 | 76009024.26 | 8000000 | 158949520.29 |
| 4000 | 82940496.03 | 10000000 | 161180955.81 |

Figure 4: Napier's short table (above) and with ideal values (below). The first table was copied from the Constructio, whereas the second is the table Napier should have obtained, had he made no error and no rounding. This table differs from the exact values (figure 5), because Napier takes the average of the bounds.

| Given Proportions of Sines. <br> $i$ |  | Corresponding Differences of Logarithm. $d_{i}$ | Given P of | portions nes. | Corresponding Differences of Logarithm. $d_{i}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | to one | 6931471.81 | 8000 | to one | 89871968.21 |
| 4 | " | 13862943.61 | 10000 | " | 92103403.72 |
| 8 | " | 20794415.42 | 20000 | " | 99034875.53 |
| 10 | " | 23025850.93 | 40000 | " | 105966347.33 |
| 20 | " | 29957322.74 | 80000 | " | 112897819.14 |
| 40 | " | 36888794.54 | 100000 |  | 115129254.65 |
| 80 | " | 43820266.35 | 200000 | " | 122060726.46 |
| 100 | " | 46051701.86 | 400000 | " | 128992198.26 |
| 200 | " | 52983173.67 | 800000 | " | 135923670.07 |
| 400 | " | 59914645.47 | 1000000 | " | 138155105.58 |
| 800 | " | 66846117.28 | 2000000 | " | 145086577.39 |
| 1000 | " | 69077552.79 | 4000000 | " | 152018049.19 |
| 2000 |  | 76009024.60 | 8000000 | " | 158949521.00 |
| 4000 | " | 82940496.40 | 10000000 |  | 161180956.51 |

Figure 5: Exact values of the ratios, rounded to two places.
1361557.4. Therefore

$$
\begin{aligned}
\lambda_{n}(n) & \approx \lambda_{n}\left(c_{11,13}\right)+10^{7}\left(\frac{c_{11,13}-n}{n}\right) \\
& \approx 1361557.4+649.80 \\
& \approx 1362207.20
\end{aligned}
$$

Then

$$
\begin{aligned}
\lambda_{n}(87265) & =\lambda_{n}(100 \times 87265)+\rho(100) \\
& \approx 1362207.20+46051701.66 \\
& \approx 47413908.86
\end{aligned}
$$

Napier found $\lambda_{n}(87265)=47413852$, but he did not compute $\lambda_{n}(87265)$ that way. However, if he had used this method, he would still have had an error, firstly as a consequemce of the incorrect value of $c_{11,13}$ in his third table, itself as a consequence of the error on $b_{50}$, and secondly because of the error on $\rho(100)$. Moreover, Napier would have had to do 5400 such computations.

It is however easier to compute some of the logarithms from other logarithms. Napier actually used the following equation [136, p. 42]:

$$
\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} \alpha\right)=\lambda_{n}\left(\sin _{n} \frac{\alpha}{2}\right)+\lambda_{n}\left(\sin _{n}\left(90-\frac{\alpha}{2}\right)\right) .
$$

which is an immediate consequence of $\sin \alpha=2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$.

With this formula, it is sufficient to compute the logarithms of the sines between $45^{\circ}$ and $90^{\circ}$ and therefore to make use of only half of the third table. The previous equation also yields

$$
\begin{equation*}
\lambda_{n}\left(\sin _{n} \alpha\right)=\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} 2 \alpha\right)-\lambda_{n}\left(\sin _{n}(90-\alpha)\right) \tag{5}
\end{equation*}
$$

and if $22^{\circ} 30^{\prime} \leq \alpha<45^{\circ}, \lambda_{n}\left(\sin _{n} \alpha\right)$ is expressed using already known values. The same process can be iterated, and once the values of the logarithms of $\sin _{n} \alpha$ for $\alpha \geq 22^{\circ} 30^{\prime}$ are known, the values of the logarithms of $\sin _{n} \alpha$ for $11^{\circ} 15^{\prime} \leq \alpha<22^{\circ} 30^{\prime}$ can be obtained, and so on [136, p. 43].

As an example, take $\alpha=12^{\circ} 30^{\prime}$. Using equation (5), we find

$$
\begin{aligned}
\lambda_{n}\left(\sin _{n} 12^{\circ} 30^{\prime}\right) & =\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} 25^{\circ}\right)-\lambda_{n}\left(\sin _{n}\left(77^{\circ} 30^{\prime}\right)\right) \\
& =\lambda_{n}(5000000)+\lambda_{n}(4226183)-\lambda_{n}(9762960) \\
& =6931469+8612856-239895 \\
& =15304430
\end{aligned}
$$

and this is the value in Napier's table. In general, however, the resulting value is not exact, because all the errors add up. The first approach, in which a value is transposed to a standard interval, would be better, assuming the basic tables have been correctly computed, and the values of $\rho$ are known with sufficient accuracy.

Figure 6 shows a more complete computation, for $\lambda_{n}\left(\sin _{n} 30^{\prime}\right)$ which can be obtained from values computed earlier. From this calculation, it appears that only some of the results coincide with those of Napier. For instance, using equation (5), Napier should have found $\lambda_{n}\left(\sin _{n} 32^{\circ}\right)=6350304$ and not 6350305. Three out of seven computations show a discrepancy, if we start with other values of the tables, and Napier's value of $\lambda_{n}(5000000)$. The most likely explanation is that Napier did the computation with more decimal places, but rounded the results.

## 3 Napier's 1614 tables

Napier's tables span 90 pages, one page covering half a degree (figure 7). The sines are arranged semi-quadrantically and on the same line we have (approximations of $) \sin _{n}(\alpha), \lambda_{n}(\sin \alpha), \lambda_{n}(\sin \alpha)-\lambda_{n}(\sin (90-\alpha)), \lambda_{n}(\sin (90-\alpha))$, $\sin (90-\alpha)$, in that order.

For instance, for $\alpha=0^{\circ} 30^{\prime}$, we have:

$$
\begin{aligned}
& \lambda_{n}\left(\sin _{n} 32^{\circ}\right)=\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} 64^{\circ}\right)-\lambda_{n}\left(\sin _{n}\left(58^{\circ}\right)\right) \\
& =\lambda_{n}(5000000)+\lambda_{n}(8987946)-\lambda_{n}(8480481) \\
& =6931469+1067014-1648179 \\
& =6350304 \text { (table: 6350305) } \\
& \lambda_{n}\left(\sin _{n} 16^{\circ}\right)=\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} 32^{\circ}\right)-\lambda_{n}\left(\sin _{n}\left(74^{\circ}\right)\right) \\
& =\lambda_{n}(5000000)+\lambda_{n}\left(\sin _{n} 32^{\circ}\right)-\lambda_{n}(9612617) \\
& =6931469+6350305-395086 \\
& =12886688 \text { (table: 12886689) } \\
& \lambda_{n}\left(\sin _{n} 8^{\circ}\right)=\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} 16^{\circ}\right)-\lambda_{n}\left(\sin _{n}\left(82^{\circ}\right)\right) \\
& =\lambda_{n}(5000000)+\lambda_{n}\left(\sin _{n} 16^{\circ}\right)-\lambda_{n}(9902681) \\
& =6931469+12886689-97796 \\
& =19720362 \text { (table: 19720362) } \\
& \lambda_{n}\left(\sin _{n} 4^{\circ}\right)=\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} 8^{\circ}\right)-\lambda_{n}\left(\sin _{n}\left(86^{\circ}\right)\right) \\
& =\lambda_{n}(5000000)+\lambda_{n}\left(\sin _{n} 8^{\circ}\right)-\lambda_{n}(9975640) \\
& =6931469+19720362-24390 \\
& =26627441 \text { (table: 26627442) } \\
& \lambda_{n}\left(\sin _{n} 2^{\circ}\right)=\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} 4^{\circ}\right)-\lambda_{n}\left(\sin _{n}\left(88^{\circ}\right)\right) \\
& =\lambda_{n}(5000000)+\lambda_{n}\left(\sin _{n} 4^{\circ}\right)-\lambda_{n}(9993908) \\
& =6931469+26627442-6094 \\
& =33552817 \text { (table: 33552817) } \\
& \lambda_{n}\left(\sin _{n} 1^{\circ}\right)=\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} 2^{\circ}\right)-\lambda_{n}\left(\sin _{n}\left(89^{\circ}\right)\right) \\
& =\lambda_{n}(5000000)+\lambda_{n}\left(\sin _{n} 2^{\circ}\right)-\lambda_{n}(9998477) \\
& =6931469+33552817-1523 \\
& =40482763 \text { (table: 40482764) } \\
& \lambda_{n}\left(\sin _{n} 30^{\prime}\right)=\lambda_{n}\left(\frac{1}{2} \sin _{n} 90\right)+\lambda_{n}\left(\sin _{n} 1^{\circ}\right)-\lambda_{n}\left(\sin _{n}\left(89^{\circ} 30^{\prime}\right)\right) \\
& =\lambda_{n}(5000000)+\lambda_{n}\left(\sin _{n} 1^{\circ}\right)-\lambda_{n}(9999619) \\
& =6931469+40482764-381 \\
& =47413852 \text { (table: 47413852) }
\end{aligned}
$$

Figure 6: Computation of $\lambda_{n}\left(\sin _{n} 30^{\prime}\right)$ using values computed previously. In each case, the table values and not the exact values have been used to compute a new value.

$$
\begin{aligned}
\sin _{n}(\alpha) & =87265 \\
\lambda_{n}\left(\sin _{n} \alpha\right) & =47413852 \\
\lambda_{n}\left(\sin _{n} \alpha\right)-\lambda_{n}\left(\sin _{n}(90-\alpha)\right) & =47413471 \\
\lambda_{n}\left(\sin _{n}(90-\alpha)\right) & =381 \\
\sin _{n}(90-\alpha) & =9999619
\end{aligned}
$$

If we define $\sigma_{n}(x)=\lambda_{n}\left(\sin _{n} x\right)$ and $\tau_{n}(x)=$ differential $=\sigma_{n}(x)-\sigma_{n}(90-$ $x), \tau_{n}(x)$ then has a very simple expression:

$$
\begin{aligned}
\tau_{n}(x) & =\lambda_{n}\left(\sin _{n} x\right)-\lambda_{n}\left(\cos _{n} x\right) \\
& =10^{7}\left(\ln \left(10^{7}\right)-\ln \left(\sin _{n} x\right)\right)-10^{7}\left(\ln \left(10^{7}\right)-\ln \left(\cos _{n} x\right)\right) \\
& =-10^{7} \ln \left(\tan _{n} x\right)
\end{aligned}
$$

$\tau_{n}$ is positive from $0^{\circ}$ to $45^{\circ}$ and negative afterwards, which is indicated by the ' $+\mid-$ ' signs at the top of the middle column.

## 4 Wright's 1616 tables

When he translated the Descriptio, Wright has actually reset the tables [132]. The most conspicuous change is probably the translation of the heading ( Gr . becoming Deg.), but in fact Wright reduced all the sines and logarithms by one figure [18, p. 14]. For instance, for $0^{\circ} 30^{\prime}$, Napier had $\sin _{n}\left(0^{\circ} 30^{\prime}\right)=87265$ and $\lambda_{n}\left(\sin _{n}\left(0^{\circ} 30^{\prime}\right)\right)=47413852$ (figure 7 ), whereas Wright had $\sin _{w}\left(0^{\circ} 30^{\prime}\right)=$ 8726 and $\lambda_{w}\left(\sin _{w}\left(0^{\circ} 30^{\prime}\right)\right)=4741385$ (figure 8). According to Oughtred, this change is made to make the interpolation easier [67, pp. 179]. This feature was reproduced in our reconstruction of the 1616 table.

Another difference was the use of a decimal point in a portion of the table (figure 8).

The change introduced by Wright naturally also changes the modern expression of the logarithm in these tables. By definition, we have:

$$
\lambda_{w}(x)=\frac{1}{10} \lambda_{n}(10 x)=10^{6}\left(\ln \left(10^{7}\right)-\ln (10 x)\right)
$$

and in particular

$$
\lambda_{w}(1)=13815510.557 \ldots
$$

In his article, Bower shows how values such as $\lambda_{n}(1)$ and $\lambda_{w}(1)$ can be computed using the methods set forth in Napier's descriptio or in its translation [18, pp. 15-16].

| $\begin{aligned} & \text { Ge } \\ & 0 \\ & 0 \end{aligned}$ | Sinus 11 Lognvitami \| Differentia | gatithmil\| Sinus |
| :---: | :---: | :---: |
| $\left.\begin{aligned} & 1 \\ & 1 \\ & 2 \end{aligned} \right\rvert\,$ |  | 0 10000000 60 <br> 1 10000000 59 <br> 2 9999998 58$\|$89 |
| $\begin{aligned} & 3 \\ & 4 \\ & 5 \\ & \hline \end{aligned}$ |  |  |
| $\begin{aligned} & 6 \\ & 7 \\ & 8 \end{aligned}$ |  | 16 9999986 54 <br> 22 999980 53 <br> 28 9999974 52 |
| $\left.\begin{array}{\|l\|} 8 \\ 40 \\ 41 \end{array} \right\rvert\,$ | 26180 59453453 59453418 <br> 29088 58399857 58399814 <br> 31997 5744659 57446707 |  |
| $\begin{aligned} & 12 \\ & \vdots \\ & \vdots \\ & x \\ & \times 4 \\ & \hline \end{aligned}$ |  | 62 9999940 48 <br> 73 9999928 47 <br> 84 9999917 46 |
| $\begin{aligned} & 75 \\ & 16 \\ & 37 \\ & \hline \end{aligned}$ | $\left.\begin{array}{l\|l\|l\|} 43632 \\ 46541 \\ 49450 & 54345225 & 54345129 \\ 53699843 & 53699734 \\ 53093600 & 53093577 \end{array} \right\rvert\,$ | 96 9999905 45 <br> 109 9999892 44 <br> 123 9999878 43 |
| $\begin{aligned} & 18 \\ & 99 \\ & 20 \end{aligned}$ |  | $\begin{array}{l\|l\|l\|l} 138 & 9999863 & \mid 42 \\ 154 & 9999847 & 41 \\ 170 & 99998 ; 1 & 40 \end{array}$ |
| $\begin{aligned} & 21 \\ & 22 \\ & 23 \end{aligned}$ |  |  |
| $\begin{aligned} & 24 \\ & 25 \\ & 26 \end{aligned}$ |  |  |
| $\begin{aligned} & 27 \\ & 28 \\ & 29 \\ & \hline \end{aligned}$ | 785939   <br> 81448   <br> 84357 $484643:$ 48467122 <br> 48103763 48103431  <br> 47752859 47752503 $\|$ |  |
| $301$ | $87265\|\|47413852\| 474 \times 3471$ |  |

Figure 7: Excerpt of Napier's table, from the 1620 reprint. This table is almost identical to the 1614 table, but the current table was reset. Compare this page with Wright's version (figure 8).


Figure 8: Excerpt of Wright's translation. Compare this page with the original version (figure 7).

The expression of the differential remains similar:

$$
\begin{aligned}
\tau_{w}(x) & =\lambda_{w}\left(\sin _{w} x\right)-\lambda_{w}\left(\cos _{w} x\right) \\
& =10^{6}\left(\ln \left(10 \cos _{w} x\right)-\ln \left(10 \sin _{w} x\right)\right)=-10^{6} \ln \tan _{w} x
\end{aligned}
$$

and therefore the 1614 values of the differential could also be divided by 10 . However, this process seems to have erred for the first values of the table. ${ }^{23}$

The second edition of Wright's translation, published in 1618, also contained an anonymous appendix, probably written by William Oughtred, where for the first time the so-called radix method for computing logarithms was used [67]. This method was described in more detail by Briggs in his Arithmetica logarithmica [21].

## 5 Decimal fractions

Decimal fractions as we know them today are used by Napier, but they were a very recent introduction. Stevin made a decisive step forward in 1585 in his work De Thiende [35, pp. 615-617], [169]. Fractions had been considered before, and so had the concept of position. Several mathematicians either came close to it, ${ }^{24}$ or even propounded or used a decimal fraction notation, ${ }^{25}$ but Stevin was the first to devote a book to that specific matter and to introduce a notation in which the fractional digits were at the same level as the integer digits, and in which a specific notation indicated the weight of each fractional digit. Stevin therefore had an (admittedly clumsy) index notation for the fractional digits, although he did not use a full index notation for the integer part. The integer part was considered as having index 0 , and this 0 could be viewed as a position marker equivalent to our fractional dot. Stevin wrote 318(0)9(1)3(2)7(3) for our 318.937. Stevin extended the four fundamental operations to these numbers and proved the validity of his rules. The very late introduction of these concepts, although a position concept had existed in Sumerian mathematics and decimal numeration can be traced back to Egypt in the fourth millenium B.C., is partly due to the resistance to the introduction of so-called Hindu-Arabic numerals, which were not very popular before 1500 .

[^14]Dots had been used in the notation of numbers before 1500, but these dots were not decimal dots. They were usually used to separate groups of digits. Some writers have also written two numbers next to each other, one for the integer part and one for the fractional part, with some separating symbol (for instance ' $\mid$ '), but without grasping the full significance of this juxtaposition. Clavius, for instance, used a point as decimal separator in 1593, but still wrote decimal fractions as common fractions in 1608, and his grasp of the decimal notation is open to doubt [169, p. 177]. Jost Bürgi, however, put a small zero under the last integral figure in his unpublished Coss (called the Arithmetica by Cantor) completed around the end of the 16th century, and Kepler ascribed the new kind of decimal notation to Bürgi. Pitiscus also used decimal points in his Trigonometria published in 1608 and 1612, however, his use of them lacked consistency. ${ }^{26}$ It was occasional, not systematic [169, p. 181].

Napier was in fact the main introducer of decimal fractions into common practice and his new mathematical instrument became the best vehicle of the decimal idea. The Descriptio actually only contains rare instances of decimal fractions. There are no decimal fractions in Napier's 1614 table, ${ }^{27}$ but within the tables of Wright's 1616 translation, decimal fractions occur for angles between 89 and 90 degrees. ${ }^{28}$ For instance, the sine of $89^{\circ} 30^{\prime}$ degrees is given as 999961.9 and its logarithm as 38.1 (figure 8). But decimal fractions appear in full force in the Constructio which was written a number of years before the Descriptio and subtends it. Several examples of values found in the Constructio, and in particular of the values contained in the three tables of progressions and the short table, have already been given earlier in this document.

And one of the first sentences of the Constructio is the following [136, p. 8]: "In numbers distinguished thus by a period in their midst, whatever is written after the period is a fraction, the denominator of which is unity with as many cyphers after it as there are figures after the period."

[^15]
## 6 Computing with Napier's logarithms

### 6.1 Basic computations

Napier's logarithms can be used for basic multiplications and divisions, which are transformed into additions and subtractions. It is easy to see that

$$
\begin{aligned}
\lambda_{n}(a b) & =10^{7}\left(\ln \left(10^{7}\right)-\ln a-\ln b\right) \\
& =\lambda_{n}(a)+\lambda_{n}(b)-10^{7} \ln \left(10^{7}\right) \\
& =\lambda_{n}(a)+\lambda_{n}(b)-\lambda_{n}(1) \\
\lambda_{n}(a / b) & =10^{7}\left(\ln \left(10^{7}\right)-\ln a+\ln b\right) \\
& =\lambda_{n}(a)-\lambda_{n}(b)+10^{7} \ln \left(10^{7}\right) \\
& =\lambda_{n}(a)-\lambda_{n}(b)+\lambda_{n}(1)
\end{aligned}
$$

The previous computations normally involve the addition or subtraction of a large number, $\lambda_{n}(1)=161180896.38$ in Napier's Constructio.

However, in many cases, one can dispense with this constant. For instance, if we wish to compute $\lambda_{n}(a b / c)$, we have

$$
\begin{aligned}
\lambda_{n}(a b / c) & =\lambda_{n}(a b)-\lambda_{n}(c)+\lambda_{n}(1) \\
& =\lambda_{n}(a)+\lambda_{n}(b)-\lambda_{n}(1)-\lambda_{n}(c)+\lambda_{n}(1) \\
& =\lambda_{n}(a)+\lambda_{n}(b)-\lambda_{n}(c)
\end{aligned}
$$

These equations are easily transposed to Wright's version of the logarithms:

$$
\begin{aligned}
\lambda_{w}(a b) & =\lambda_{w}(a)+\lambda_{w}(b)-\lambda_{w}(1) \\
\lambda_{w}(a / b) & =\lambda_{w}(a)-\lambda_{w}(b)+\lambda_{w}(1) \\
\lambda_{w}(a b / c) & =\lambda_{w}(a)+\lambda_{w}(b)-\lambda_{w}(c)
\end{aligned}
$$

It may come as a surprise, however, that Napier did not give such simple examples in the Descriptio. Napier's simplest examples appear in chapter 5, at the end of the Book I of the Descriptio. He considers four cases. In the first case, $a, b$, and $c$ are such that $\frac{c}{b}=\frac{b}{a}$, the values of $a$ and $b$ are given and $c$ is sought. Napier shows that $\lambda_{n}(c)=2 \lambda_{n}(b)-\lambda_{n}(a)$, from which $c$ is easily obtained. In the second case, we have the same proportions $\frac{c}{b}=\frac{b}{a}$, but $a$ and $c$ are given, and $b$ is sought. Napier explains that the square root $b=\sqrt{a c}$ is replaced by a division by two: $\lambda_{n}(b)=\frac{1}{2}\left(\lambda_{n}(a)+\lambda_{n}(c)\right)$. In the third case, four numbers $a, b, c$, and $d$ are considered such that $\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$. Knowing the first three, the fourth is sought. We have $\lambda_{n}(d)=\lambda_{n}(b)+\lambda_{n}(c)-\lambda_{n}(a)$. In
the fourth case, we have the same proportions as in the third case, but $a$ and $d$ are given, and $b$ and $c$ are sought. We have $\lambda_{n}(c)=\lambda_{n}(d)+\frac{1}{3}\left(\lambda_{n}(a)-\lambda_{n}(d)\right)$ and $\lambda_{n}(b)=\lambda_{n}(a)+\frac{1}{3}\left(\lambda_{n}(d)-\lambda_{n}(a)\right)$.

### 6.2 Oughtred's radix method

In the appendix of the second edition of Wright's translation, probably authored by William Oughtred, the radix method is given for calculating the logarithm of any number. This method actually involves an extension of Napier's short table. The values in the new table were the values of $\rho_{w}(1)$, $\rho_{w}(2), \rho_{w}(3)$, etc., which are the equivalent to $\rho(1), \rho(2), \rho(3)$, etc., but with one digit less.

In order to compute $\lambda_{w}(n)$, Oughtred's idea is to find $a$ such that $980000<$ an $<1000000$ (Wright's translation has $\lambda_{w}\left(10^{6}\right)=0$ ) and such that $a$ is a product of simple factors such as $1,2,3, \ldots, 10,20, \ldots, 90,100, \ldots, 1.1$, $1.2,1.3, \ldots, 1.01,1.02, \ldots, 1.09$. Then, $\lambda_{w}(n)=\lambda_{w}(a n)+\rho_{w}(a)$ and $\rho_{w}(a)$ can be computed using Oughtred's table and the relation $\rho_{w}(a b)=\rho_{w}(a)+$ $\rho_{w}(b) . \lambda_{w}(a n)$ can be computed either by sight, when an is near 1000000 , by using the closest value in the table, or by interpolation. Interestingly, the appendix also makes use of Oughtred's $\times$ for the multiplication, and abreviations for the sine, tangent, cosine, cotangent, which appear here for one of the first time in formulæ. A full analysis of the appendix was given by Glaisher [67] and we give more details on the radix method in our study of Briggs' Arithmetica logarithmica [160].

### 6.3 Negative numbers

In the Descriptio (book 1, chapter 1), Napier explains that numbers greater than the sinus total have a negative logarithm: the Logarithmes of numbers greater then the whole sine, are lesse then nothing (...) the Logarithmes which are lesse then nothing, we cal Defective, or wanting, setting this marke - before them [132, p. 6].

### 6.4 Scaling notation

In the Descriptio, Napier introduced a primitive exponent notation for manipulating numbers that need to be scaled to fit within his table. For instance, if the logarithm of 137 is sought, he finds 1371564 among the sines. The value in the table is 19866327 (exact: 19866333.98) and Napier writes 19866327-0000, meaning that four figures must be removed from the sine. Napier then manipulates such logarithms which he calls "impure," as if the
number of zeros were exponents. Logarithms without a $-000 \ldots$ or $+000 \ldots$ are called "pure." For Napier, the impure logarithm $23025842+0$ is in fact equal to 0 , and it can be added to an impure logarithm .. $-0 \ldots$ in order to render it pure.

What Napier really does is merely to move the exponent to the mantissa. Napier is manipulating a "decimal characteristic," but transposed in his logarithms. Apart from Glaisher in 1920 [68, p. 165], Napier's notation apparently hasn't much been described, yet it is very easy to understand if we transpose it in modern notations. If $x$ is a pure logarithm, then $x+\underbrace{000 \ldots \ldots 0}_{n}$ is merely a notation for $x-n \rho(10)$. And $x-\underbrace{000 \ldots \ldots 0}_{n}$ is a notation for $x+n \rho(10)$. Given that $\rho(10)=\lambda_{n}(1)-\lambda_{n}(10)$, it is very easy to see that $\lambda_{n}(a)=\lambda_{n}\left(10^{n} a\right)+n \rho(10)$. With the previous example, we have $\lambda_{n}(137.1564)=\lambda_{n}(1371564)+4 \rho(10)=19866327-0000$.

So, $23025842+0$ actually represents $23025842-\rho(10)=0$. With this notation, Napier can add or subtract the zeros as if they were exponents, because he really adds or subtracts $\rho(10)$.

This cumbersome notation was certainly one of the main reasons which explained the move to decimal logarithms in which the characteristics become mere integers. The problem of Napier's logarithms (and also of the natural logarithms) is that their base is not equal to the base of the numeration.

### 6.5 Interpolation

In 1616, Briggs supplied a chapter in which he introduced interpolation methods.

### 6.6 Trigonometric computations

### 6.6.1 First example

The main purpose of Napier's table of logarithms was to simplify trigonometric calculations. In the Descriptio, Napier gives a number of rules for specific triangle problems. For instance, proposition 4 in chapter 2 of book 2 reads: In any Triangle: the summe of the Logarithmes of any angle and side inclosing the same, is equall to the summe of the Logarithmes of the side, and the angle opposite to them. [132, p. 35] When Napier writes "the logarithm of an angle," it implicitely means the "logarithm of the sine of the angle."

In other words, considering the figure 9 given by Napier, the proposition means for instance that

$$
\lambda_{n}\left(\sin _{n} A\right)+\lambda_{n}(c)=\lambda_{n}(a)+\lambda_{n}\left(\sin _{n} C\right)
$$



Figure 9: The application of logarithms to trigonometry.
This follows easily from the law of sines. We have

$$
\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}
$$

Therefore

$$
\ln (\sin A)=\ln (\sin C)+\ln a-\ln c
$$

But $\lambda_{n}(x)=\lambda_{n}(1)-10^{7} \ln x$, hence

$$
-10^{7} \ln (\sin A)=-10^{7} \ln (\sin C)-10^{7} \ln a+10^{7} \ln c
$$

and
$\lambda_{n}\left(\sin _{n} A\right)-\lambda_{n}(1)=\left(\lambda_{n}\left(\sin _{n} C\right)-\lambda_{n}(1)\right)+\left(\lambda_{n}(a)-\lambda_{n}(1)\right)-\left(\lambda_{n}(c)-\lambda_{n}(1)\right)$
which reduces to

$$
\lambda_{n}\left(\sin _{n} A\right)=\lambda_{n}\left(\sin _{n} C\right)+\lambda_{n}(a)-\lambda_{n}(c)
$$

or

$$
\lambda_{n}\left(\sin _{n} A\right)+\lambda_{n}(c)=\lambda_{n}(a)+\lambda_{n}\left(\sin _{n} C\right)
$$

which is Napier's result.
With the previous example, Napier (in 1614) obtains $\lambda_{n}(a)=5454707-00$, $\lambda_{n}\left(\sin _{n} C\right)=8246889, \lambda_{n}(c)=13354921-00$, from which he computes $\lambda_{n}\left(\sin _{n} A\right)=346684$, the resulting logarithm being pure, and this is nearly $\lambda_{n}\left(\sin _{n} 75^{\circ}\right)$, and therefore $A=75^{\circ}$. Napier adds that the result would be $105^{\circ}$ if the angle appeared to be obtuse.

If one checks Napier's table, only the value of $\lambda_{n}\left(\sin _{n} C\right)=8246889$ can be found. For the two other values $a$ and $c$, we find approaching values:

| $\alpha$ | $\sin _{n}(\alpha)$ | $\lambda_{n}\left(\sin _{n}(\alpha)\right)$ |
| :---: | :---: | ---: |
| $15^{\circ} 14^{\prime}$ | 2627505 | 13365493 |
| $15^{\circ} 15^{\prime}$ | 2630312 | 13354817 |
| $35^{\circ} 25^{\prime}$ | 5795183 | 5455577 |
| $35^{\circ} 26^{\prime}$ | 5797553 | 5451488 |

$100 a$ lies between $\sin _{n}\left(35^{\circ} 25^{\prime}\right)$ and $\sin _{n}\left(35^{\circ} 26^{\prime}\right)$, and $100 c$ lies between $\sin _{n}\left(15^{\circ} 14^{\prime}\right)$ and $\sin _{n}\left(15^{\circ} 15^{\prime}\right)$.

It isn't clear how Napier's values were obtained, because his interpolation procedure gives different values:

$$
\begin{gathered}
10^{7} \frac{(2630312-2630200)}{2630312}<\lambda_{n}(2630200)-\lambda_{n}(2630312)<10^{7} \frac{(2630312-2630200)}{2630200} \\
425.804 \ldots<\lambda_{n}(2630200)-\lambda_{n}(2630312)<425.823 \ldots \\
\lambda_{n}(2630200) \approx \lambda_{n}(2630312)+425.8=13354817+425.8=13355243
\end{gathered}
$$

which differs from Napier's 13354921.

$$
\begin{gathered}
10^{7} \frac{(5795500-5795183)}{5795500}<\lambda_{n}(5795183)-\lambda_{n}(5795500)<10^{7} \frac{(5795500-5795183)}{5795183} \\
546.97 \ldots<\lambda_{n}(5795183)-\lambda_{n}(5795500)<547.00 \ldots \\
\lambda_{n}(5795500) \approx \lambda_{n}(5795183)-547=5455577-547=5455030
\end{gathered}
$$

which differs from Napier's 5454701.
Wright's example is exactly the same, but his logarithm values are all ten times smaller, which ensures that Napier's proposition is still valid. ${ }^{29}$

### 6.6.2 Second example

The third proposition of Book 2 in the Descriptio involves the differential, which corresponds to the logarithm of the tangent. The proposition reads thus [132, p. 33]: In a right angled triangle the Logarithme of any legge is equall to the summe of the Differentiall of the opposite angle, and the Logarithme of the leg remaining.

We can take as an example figure 10 which is Napier's example. We are given a triangle with three sides. Like before, if we set $\sigma_{n}(x)=\lambda_{n}\left(\sin _{n} x\right)$

[^16]

Figure 10: The computation of a right-angled triangle (not to scale).
and $\tau_{n}(x)=$ differential $=\sigma_{n}(x)-\sigma_{n}(90-x)$, Napier's proposition then amounts to:

$$
\begin{aligned}
\lambda_{n}(b) & =\lambda_{n}(c)+\tau_{n}(B) \\
& =\lambda_{n}(c)+\sigma_{n}(B)-\sigma_{n}(90-B) \\
& =\lambda_{n}(c)+\lambda_{n}\left(\sin _{n} B\right)-\lambda_{n}\left(\cos _{n} B\right)
\end{aligned}
$$

Since $\lambda_{n}(x)=10^{7}\left(\ln \left(10^{7}\right)-\ln x\right)=\lambda_{n}(1)-10^{7} \ln x$, the above reduces to

$$
\begin{aligned}
\ln b & =\ln c+\ln (\sin B)-\ln (\cos B) \\
& =\ln c+\ln (\tan B)
\end{aligned}
$$

and therefore

$$
b=c \tan B
$$

which is correct.
Napier uses this proposition in order to find the angle $B$. Approximating 137000 by $\sin _{n}\left(47^{\prime}\right)=136714$ and 938400 by $\sin _{n}\left(69^{\circ} 47^{\prime}\right)=9383925$, he has (in 1614):

$$
\begin{aligned}
\tau_{n}(B) & =\lambda_{n}(b)-\lambda_{n}(c) \\
& =42924534-000-635870-000 \\
& =42288664
\end{aligned}
$$

and since the differential is 42304768 for $0^{\circ} 50^{\prime}$ and 42106711 for $0^{\circ} 51^{\prime}$, Napier concludes that $B \approx 0^{\circ} 50^{\prime}$.

Wright has the same triangle, with the same dimensions, but the values of the logarithms are $\lambda_{w}(b)=4292453-00$ and $\lambda_{w}(c)=63587-00$ and he finds $\tau_{w}(B)=4228866$ and from this obtains $B=0^{\circ} 50^{\prime} 11^{\prime \prime}$. It isn't clear how Wright obtained the latter value, which is incorrect. In Wright's table $\tau_{w}\left(0^{\circ} 50^{\prime}\right)=4230477$ and $\tau_{w}\left(0^{\circ} 51^{\prime}\right)=4210571$ and a linear interpolation gives $0^{\circ} 50^{\prime} 5^{\prime \prime}$ or $0^{\circ} 50^{\prime} .08$.

Napier gave other examples, and also considered the use of logarithms for the resolution of spherical triangles. Napier devised a rule called of "circular parts," which was useful for such triangles [87, 104, 128].

## 7 Napier's scientific heritage

### 7.1 Napier's suggestions

Napier suggested several improvements to his tables. First, in the Constructio [136, p. 46], he suggested a finer grain construction of the three fundamental tables, which should ensure a greater accuracy. The new table 1 was to contain 100 steps, the new table 2 also 100 steps (compared to 50 in the initial scheme), and the new table 3 was to contain 100 columns of 35 steps (instead of 69 columns of 20 steps). The total number of intervals would therefore be $35 \cdot 10^{6}$ instead of $6.9 \cdot 10^{6}$.

Then, also in the Constructio, he suggested a better kind of logarithms, in which the logarithm of 1 is 0 and the logarithm of either 10 or $\frac{1}{10}$ is equal to 10 . He then explained how these logarithms can be computed. He gave in particular an example whereby the decimal logarithm of 5 can be computed by repeated square root extractions and constructing a sequence approximating the sought result. Starting with $\log 10=1$ and $\log 1=0$, we compute $\log \sqrt{1 \times 10}=0.5$, then $\log \sqrt{10 \times \sqrt{10}}=0.75$, getting closer to $\log 5$ by using square roots of products of numbers greater and smaller than 5 [136, pp. 51 and $97-100]$, [35, pp. 736-737], [190, pp. 168-169].

### 7.2 Decimal logarithms

Napier's work was taken over and adapted by Briggs who published his first table as soon as 1617. The first to publish tables of decimal logarithms of trigonometric functions was Gunter in 1620 [75]. Briggs' main tables were published in 1624 and 1633, and Adriaan Vlacq published less accurate - but
more extensive - tables in 1628 and 1633. These tables were then used as the basis of almost all later tables until the beginning of the 20th century.

### 7.3 Natural (Neperian) logarithms

These logarithms were introduced by Speidell in 1622 or 1623 [67, pp. 175176] although a table that looked like one of natural logarithms already appeared in the appendix of the second edition of Wright's translation. This table was in fact a table extending Napier's short table for a number of simple ratios. The values in the table were the values of $\rho(1), \rho(2), \rho(3)$, etc., which are equal to $10^{6} \ln (1), 10^{6} \ln (2), 10^{6} \ln (3)$, etc., but this table was never thought as being a table of logarithms. It is only a table for the differences of Napierian logarithms and it should not be historically interpreted otherwise, although Glaisher views them as the first publication of natural logarithms. A full analysis of this appendix which is thought to be by William Oughtred was given by Glaisher [67].

### 7.4 Other tables

1620 also saw the publication of Bürgi's tables, albeit without a description of their use. ${ }^{30}$ Kepler was familiar both with Napier's logarithms and with Bürgi's work, and he also published tables of his own.

At about the same time, Benjamin Ursinus and John Speidell also published tables extending Napier's tables, see in particular Cantor [35, p. 739743]. Speidell's table of logarithms was reproduced by Maseres [110].

### 7.5 Slide rules

A very important consequence of Napier's invention was the development of the slide rule. First came Gunter's scale. Figure 11 shows a logarithmic scale, which was one of the scales found on the scale named after Edmund Gunter (1581-1626) who invented it in 1620. Multiplications or divisions could be done with this rule using a pair of dividers. Around 1622, William Oughtred (1574-1660) had the idea of using two logarithmic scales and putting them side by side (figure 12), which made it possible to dispense with the dividers. Oughtred's initial design used circular scales [144, 200], and pairs of straight scales were only introduced later.

[^17]

Figure 11: A logarithmic scale from 1 to 100.

It is easy to see how these scales are used and why they work. In figure 11, the scale goes from 1 to 100 , and the positions of the numbers are proportional to their logarithm. In other words, the distance between $n$ and 1 is proportional to $\log (n)$. The distance between 1 and 10 is the same as between 10 and 100 , because their logarithms are equidifferent. A given divider opening corresponds to every ratio. Multiplying by 2 corresponds to the distance between 1 and 2 , which is also the distance between 2 and 4 , between 4 and 8 , between 10 and 20, etc. A pair of divider can therefore easily be used to perform multiplications or divisions.

In figure 12, two such scales are put in parallel and the 1 of the first rule is put above some position of the second scale, here $a=2.5$. Since a given ratio corresponds to the same linear distance on both scales, the value $c$ facing a certain value $b$ of the first scale, for instance 6 , is such that $\frac{c}{a}=\frac{b}{1}$, and therefore $c=a b$. The same arrangement can be used for divisions, and dividers are no longer needed.


Figure 12: Two logarithmic scales showing the computation of $2.5 \times 6=15$.
On the history of the slide rule, we refer the reader to Cajori's articles and books $[30,31,34]^{31}$ or Stoll's popular account in the Scientific American [185].

On the construction of the logarithmic lines on such a scale, see also Robertson [154] and Nicholson [142].

A good source for more recent information on the history of slide rules is the Journal of the Oughtred Society.

It is important to remember that many "popular" encyclopædias covering a large subject are bound to contain many errors, even when written by reputed mathematicians. Two interesting articles worth reading in this context are those of Mautz [113] and Miller [123].

[^18]
## 8 Note on the recalculation of the tables

### 8.1 Auxiliary tables

In the tables closing this document, we have computed the $a_{i}, b_{i}$ and $c_{i, j}$ exactly and the logarithms using Napier's averaging method. Hence, we took $l_{1}\left(a_{1}\right)=1.00000005, l_{2}\left(b_{2}\right)=100.0005, l_{3}\left(c_{1,0}\right)=5001.2504168$ and $l_{3}\left(c_{0,1}\right)=100503.358535$.

### 8.2 Main tables

There are three volumes accompanying this study. The first ${ }^{32}$ gives the ideal recomputation of Napier's logarithms, that is the values of $\sin _{n} \alpha$ and $\lambda_{n}(\alpha)$ with an exact computation.

The other two volumes ${ }^{33}$ provide approximations of Napier's tables, assuming that Napier did not make any mistake in the auxiliary tables and assuming he did not round any value. These two volumes have been constructed using Napier's radix table, as well as his method of interpolation. Because of the latter, the values in these two tables do differ from the ideal table.

Differences between Napier's actual table and our tables are due to errors on sines, to rounding, to errors in the auxiliary tables, or to other errors during the interpolation. It seemed pointless to try to mimick all these errors.

It should also be noted that we did not use the trigonometric relations to compute the logarithms for angles under $45^{\circ}$, because Napier would probably not have used them, if he had been aware of the propagation of errors, or if he had had the time to do more exact computations. It would have seemed strange to correct the errors in Napier's auxiliary tables, and at the same time to use an error-prone trigonometric computation, since the combined result of this procedure would still have been far from Napier's actual table.

We have not rounded the sines before taking the logarithms, as it seems that Napier used more accurate values of the sines for the small angles than those which are given in his table. He may have used the rounded values for larger angles.

We might try to produce a more faithful approximation of Napier's table in the future. Such an approximation would in particular take rounding into account in the computation of the sequences $a_{i}, b_{i}$, etc.

[^19]The reconstruction of the 1616 table was obtained by rounding the reconstruction of the 1614 table. Wright didn't recompute any value of the table.

## 9 Acknowledgements

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## 10 Napier's construction tables

### 10.1 First table

|  | Number <br> $i$ | Logarithm <br> $l_{1}\left(a_{i}\right)$ | $i$ | Number <br> $a_{i}$ | Logarithm <br> $l_{1}\left(a_{i}\right)$ |
| ---: | ---: | ---: | :---: | :---: | :---: |
| 0 | 10000000.0000000 | 0.0000000 | 25 | 9999975.0000300 | 25.0000013 |
| 1 | 9999999.0000000 | 1.0000001 | 26 | 9999974.0000325 | 26.0000013 |
| 2 | 9999998.0000001 | 2.0000001 | 27 | 9999973.0000351 | 27.0000014 |
| 3 | 9999997.0000003 | 3.0000002 | 28 | 9999972.0000378 | 28.0000014 |
| 4 | 9999996.0000006 | 4.0000002 | 29 | 9999971.0000406 | 29.0000015 |
| 5 | 9999995.0000010 | 5.0000003 | 30 | 9999970.0000435 | 30.0000015 |
| 6 | 9999994.0000015 | 6.0000003 | 31 | 9999969.0000465 | 31.0000016 |
| 7 | 9999993.0000021 | 7.0000004 | 32 | 9999968.0000496 | 32.0000016 |
| 8 | 9999992.0000028 | 8.0000004 | 33 | 9999967.0000528 | 33.0000017 |
| 9 | 9999991.0000036 | 9.0000005 | 34 | 9999966.0000561 | 34.0000017 |
| 10 | 9999990.0000045 | 10.0000005 | 35 | 9999965.0000595 | 35.0000018 |
| 11 | 9999989.0000055 | 11.0000006 | 36 | 9999964.0000630 | 36.0000018 |
| 12 | 9999988.0000066 | 12.0000006 | 37 | 9999963.0000666 | 37.0000019 |
| 13 | 9999987.0000078 | 13.0000007 | 38 | 9999962.0000703 | 38.0000019 |
| 14 | 9999986.0000091 | 14.0000007 | 39 | 9999961.0000741 | 39.0000020 |
| 15 | 9999985.0000105 | 15.0000008 | 40 | 9999960.0000780 | 40.0000020 |
| 16 | 9999984.0000120 | 16.0000008 | 41 | 9999959.0000820 | 41.0000021 |
| 17 | 9999983.0000136 | 17.0000009 | 42 | 9999958.0000861 | 42.0000021 |
| 18 | 9999982.0000153 | 18.0000009 | 43 | 9999957.0000903 | 43.0000022 |
| 19 | 9999981.0000171 | 19.0000010 | 44 | 9999956.0000946 | 44.0000022 |
| 20 | 9999980.0000190 | 20.0000010 | 45 | 9999955.0000990 | 45.0000023 |
| 21 | 9999979.0000210 | 21.0000011 | 46 | 9999954.0001035 | 46.0000023 |
| 22 | 9999978.0000231 | 22.0000011 | 47 | 9999953.0001081 | 47.0000024 |
| 23 | 9999977.0000253 | 23.0000012 | 48 | 9999952.0001128 | 48.0000024 |
| 24 | 9999976.0000276 | 24.0000012 | 49 | 9999951.0001176 | 49.0000025 |
| 25 | 9999975.0000300 | 25.0000013 | 50 | 9999950.0001225 | 50.0000025 |


|  | Number | Logarithm <br> $l_{1}\left(a_{i}\right)$ | $i$ | Number | Logarithm <br> $l_{1}\left(a_{i}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 50 | 9999950.0001225 | 50.0000025 | 75 | 9999925.0002775 | 75.0000038 |
| 51 | 9999949.0001275 | 51.0000026 | 76 | 9999924.0002850 | 76.0000038 |
| 52 | 9999948.0001326 | 52.0000026 | 77 | 9999923.0002926 | 77.0000039 |
| 53 | 9999947.0001378 | 53.0000027 | 78 | 9999922.0003003 | 78.0000039 |
| 54 | 9999946.0001431 | 54.0000027 | 79 | 9999921.0003081 | 79.0000040 |
| 55 | 9999945.0001485 | 55.0000028 | 80 | 9999920.0003160 | 80.0000040 |
| 56 | 9999944.0001540 | 56.0000028 | 81 | 9999919.0003240 | 81.0000041 |
| 57 | 9999943.0001596 | 57.0000029 | 82 | 9999918.0003321 | 82.0000041 |
| 58 | 9999942.0001653 | 58.0000029 | 83 | 9999917.0003403 | 83.0000042 |
| 59 | 9999941.0001711 | 59.0000030 | 84 | 9999916.0003486 | 84.0000042 |
| 60 | 9999940.0001770 | 60.0000030 | 85 | 9999915.0003570 | 85.0000043 |
| 61 | 9999939.0001830 | 61.0000031 | 86 | 9999914.0003655 | 86.0000043 |
| 62 | 9999938.0001891 | 62.0000031 | 87 | 9999913.0003741 | 87.0000044 |
| 63 | 9999937.0001953 | 63.0000032 | 88 | 9999912.0003828 | 88.0000044 |
| 64 | 9999936.0002016 | 64.0000032 | 89 | 9999911.0003916 | 89.0000045 |
| 65 | 9999935.0002080 | 65.0000033 | 90 | 9999910.0004005 | 90.0000045 |
| 66 | 9999934.0002145 | 66.0000033 | 91 | 9999909.0004095 | 91.0000046 |
| 67 | 9999933.0002211 | 67.0000034 | 92 | 9999908.0004186 | 92.0000046 |
| 68 | 9999932.0002278 | 68.0000034 | 93 | 9999907.0004278 | 93.0000047 |
| 69 | 9999931.0002346 | 69.0000035 | 94 | 9999906.0004371 | 94.0000047 |
| 70 | 9999930.0002415 | 70.0000035 | 95 | 9999905.0004465 | 95.0000048 |
| 71 | 9999929.0002485 | 71.0000036 | 96 | 9999904.0004560 | 96.0000048 |
| 72 | 9999928.0002556 | 72.0000036 | 97 | 9999903.0004656 | 97.0000049 |
| 73 | 9999927.0002628 | 73.0000037 | 98 | 9999902.0004753 | 98.0000049 |
| 74 | 9999926.0002701 | 74.0000037 | 99 | 9999901.0004851 | 99.0000050 |
| 75 | 9999925.0002775 | 75.0000038 | 100 | 9999900.0004950 | 100.0000050 |

### 10.2 Second table

| $i$ | $\begin{gathered} \text { Number } \\ b_{i} \end{gathered}$ | $\begin{gathered} \hline \text { Logarithn } \\ l_{2}\left(b_{i}\right) \\ \hline \end{gathered}$ | $i$ | $\begin{gathered} \text { Numb } \\ b_{i} \end{gathered}$ | $l_{2}\left(b_{i}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 10000000.000000 | 0.0000000 | 25 | 997500.2999 | 2500.0125000 |
| 1 | 9999900.000000 | 0.0005000 | 26 | 9997400.324974 | 2600.0130000 |
| 2 | 9999800.001000 | 00.0010000 | 27 | 9997300.350971 | 2700.0135000 |
| 3 | 9999700.003000 | 00.0015000 | 28 | 9997200.37796 | 2800.0140000 |
| 4 | 9999600.006000 | 00.0020000 | 29 | 9997100.40596 | 2900.0145000 |
| 5 | 9999500.010000 | 00.0025000 | 30 | 9997000.434959 | 3000.0150000 |
| 6 | 9999400.015000 | 00.0030000 | 31 | 9996900.464955 | 3100.0155000 |
| 7 | 9999300.021000 | 00.0035000 | 32 | 9996800.495950 | 3200.0160000 |
| 8 | 9999200.027999 | 00.0040000 | 33 | 9996700.527945 | 3300.0165000 |
| 9 | 9999100.035999 | 900.0045000 | 34 | 9996600.560940 | 3400.0170000 |
| 10 | 9999000.044999 | 1000.0050000 | 35 | 9996500.59493 | 3500.0175000 |
| 11 | 9998900.054998 | 1100.0055000 | 36 | 9996400.629929 | 3600.0180000 |
| 12 | 9998800.065998 | 1200.0060000 | 37 | 9996300.665922 | 3700.0185000 |
| 13 | 9998700.077997 | 1300.0065000 | 38 | 9996200.702916 | 3800.0190000 |
| 14 | 9998600.090996 | 1400.0070000 | 39 | 9996100.740909 | 3900.0195000 |
| 15 | 9998500.104995 | 1500.0075000 | 40 | 9996000.779901 | 4000.0200000 |
| 16 | 9998400.119994 | 1600.0080000 | 41 | 9995900.81989 | 4100.02050 |
| 17 | 9998300.135993 | 1700.0085000 | 42 | 9995800.86088 | 4200.0210000 |
| 18 | 9998200.152992 | 1800.0090000 | 43 | 9995700.902877 | 4300.0215000 |
| 19 | 9998100.170990 | 1900.0095000 | 44 | 9995600.94586 | 4400.0220000 |
| 20 | 9998000.189989 | 2000.0100000 | 45 | 9995500.989858 | 4500.0225000 |
| 21 | 9997900.209987 | 2100.0105000 | 46 | 9995401.034848 | 4600.0230000 |
| 22 | 9997800.230985 | 2200.0110000 | 47 | 9995301.080838 | 700.0235000 |
| 23 | 9997700.252982 | 2300.0115000 | 48 | 9995201.127827 | 4800.0240000 |
| 24 | 9997600.275980 | 2400.0120000 | 49 | 9995101.175816 | 4900.0245000 |
| 25 | 9997500.299977 | 2500.0125000 | 50 | 9995001.224804 | 5000.0250000 |

### 10.3 Third table

| Column 0 |  |  | Column 1 |  |  | Column 2 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | $\begin{gathered} \text { Number } \\ c_{i, 0} \end{gathered}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 0}\right) \end{aligned}$ | $i$ | $\begin{gathered} \text { Number } \\ c_{i, 1} \end{gathered}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 1}\right) \end{aligned}$ | $i$ | $\begin{aligned} & \text { Number } \\ & c_{i, 2} \end{aligned}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 2}\right) \end{gathered}$ |
| 0 | 10000000.000000 | 0.0 | 0 | 9900000.000000 | 100503.4 | 0 | 9801000.000000 | 201006.7 |
| 1 | 9995000.000000 | 5001.3 | 1 | 9895050.000000 | 105504.6 | 1 | 9796099.500000 | 206008.0 |
| 2 | 9990002.500000 | 10002.5 | 2 | 9890102.475000 | 110505.9 | 2 | 9791201.450250 | 211009.2 |
| 3 | 9985007.498750 | 15003.8 | 3 | 9885157.423762 | 115507.1 | 3 | 9786305.849525 | 216010.5 |
| 4 | 9980014.995001 | 20005.0 | 4 | 9880214.845051 | 120508.4 | 4 | 9781412.696600 | 221011.7 |
| 5 | 9975024.987503 | 25006.3 | 5 | 9875274.737628 | 125509.6 | 5 | 9776521.990252 | 226013.0 |
| 6 | 9970037.475009 | 30007.5 | 6 | 9870337.100259 | 130510.9 | 6 | 9771633.729257 | 231014.2 |
| 7 | 9965052.456272 | 35008.8 | 7 | 9865401.931709 | 135512.1 | 7 | 9766747.912392 | 236015.5 |
| 8 | 9960069.930044 | 40010.0 | 8 | 9860469.230743 | 140513.4 | 8 | 9761864.538436 | 241016.7 |
| 9 | 9955089.895079 | 45011.3 | 9 | 9855538.996128 | 145514.6 | 9 | 9756983.606167 | 246018.0 |
| 10 | 9950112.350131 | 50012.5 | 10 | 9850611.226630 | 150515.9 | 10 | 9752105.114364 | 251019.2 |
| 11 | 9945137.293956 | 55013.8 | 11 | 9845685.921017 | 155517.1 | 11 | 9747229.061806 | 256020.5 |
| 12 | 9940164.725309 | 60015.0 | 12 | 9840763.078056 | 160518.4 | 12 | 9742355.447275 | 261021.7 |
| 13 | 9935194.642946 | 65016.3 | 13 | 9835842.696517 | 165519.6 | 13 | 9737484.269552 | 266023.0 |
| 14 | 9930227.045625 | 70017.5 | 14 | 9830924.775169 | 170520.9 | 14 | 9732615.527417 | 271024.2 |
| 15 | 9925261.932102 | 75018.8 | 15 | 9826009.312781 | 175522.1 | 15 | 9727749.219653 | 276025.5 |
| 16 | 9920299.301136 | 80020.0 | 16 | 9821096.308125 | 180523.4 | 16 | 9722885.345044 | 281026.7 |
| 17 | 9915339.151486 | 85021.3 | 17 | 9816185.759971 | 185524.6 | 17 | 9718023.902371 | 286028.0 |
| 18 | 9910381.481910 | 90022.5 | 18 | 9811277.667091 | 190525.9 | 18 | 9713164.890420 | 291029.2 |
| 19 | 9905426.291169 | 95023.8 | 19 | 9806372.028257 | 195527.1 | 19 | 9708308.307975 | 296030.5 |
| 20 | 9900473.578023 | 100025.0 | 20 | 9801468.842243 | 200528.4 | 20 | 9703454.153821 | 301031.7 |


| Column 3 |  |  | Column 4 |  |  | Column 5 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number | Logarithm $l_{3}\left(c_{i 3}\right)$ | $i$ | Number | Logarithm $l_{3}\left(c_{i, 1}\right)$ | $i$ | Number | Logarithm $l_{3}\left(c_{i, 5}\right)$ |
| 0 | 9702990.000000 | 301510.1 | 0 | 9605960.100000 | 402013.4 | 0 | 9509900.499000 | 502516.8 |
| 1 | 9698138.505000 | 306511.3 | 1 | 9601157.119950 | 407014.7 | 1 | 9505145.548750 | 507518.0 |
| 2 | 9693289.435747 | 311512.6 | 2 | 9596356.541390 | 412015.9 | 2 | 9500392.975976 | 512519.3 |
| 3 | 9688442.791030 | 316513.8 | 3 | 9591558.363119 | 417017.2 | 3 | 9495642.779488 | 517520.5 |
| 4 | 9683598.569634 | 321515.1 | 4 | 9586762.583938 | 422018.4 | 4 | 9490894.958098 | 522521.8 |
| 5 | 9678756.770349 | 326516.3 | 5 | 9581969.202646 | 427019.7 | 5 | 9486149.510619 | 527523.0 |
| 6 | 9673917.391964 | 331517.6 | 6 | 9577178.218044 | 432020.9 | 6 | 9481406.435864 | 532524.3 |
| 7 | 9669080.433268 | 336518.8 | 7 | 9572389.628935 | 437022.2 | 7 | 9476665.732646 | 537525.5 |
| 8 | 9664245.893052 | 341520.1 | 8 | 9567603.434121 | 442023.4 | 8 | 9471927.399780 | 542526.8 |
| 9 | 9659413.770105 | 346521.3 | 9 | 9562819.632404 | 447024.7 | 9 | 9467191.436080 | 547528.0 |
| 10 | 9654584.063220 | 351522.6 | 10 | 9558038.222588 | 452025.9 | 10 | 9462457.840362 | 552529.3 |
| 11 | 9649756.771188 | 356523.8 | 11 | 9553259.203476 | 457027.2 | 11 | 9457726.611442 | 557530.5 |
| 12 | 9644931.892803 | 361525.1 | 12 | 9548482.573875 | 462028.4 | 12 | 9452997.748136 | 562531.8 |
| 13 | 9640109.426856 | 366526.3 | 13 | 9543708.332588 | 467029.7 | 13 | 9448271.249262 | 567533.0 |
| 14 | 9635289.372143 | 371527.6 | 14 | 9538936.478421 | 472030.9 | 14 | 9443547.113637 | 572534.3 |
| 15 | 9630471.727457 | 376528.8 | 15 | 9534167.010182 | 477032.2 | 15 | 9438825.340080 | 577535.5 |
| 16 | 9625656.491593 | 381530.1 | 16 | 9529399.926677 | 482033.4 | 16 | 9434105.927410 | 582536.8 |
| 17 | 9620843.663347 | 386531.3 | 17 | 9524635.226714 | 487034.7 | 17 | 9429388.874447 | 587538.0 |
| 18 | 9616033.241516 | 391532.6 | 18 | 9519872.909100 | 492035.9 | 18 | 9424674.180009 | 592539.3 |
| 19 | 9611225.224895 | 396533.8 | 19 | 9515112.972646 | 497037.2 | 19 | 9419961.842919 | 597540.6 |
| 20 | 9606419.612282 | 401535.1 | 20 | 9510355.416160 | 502038.4 | 20 | 9415251.861998 | 602541.8 |


| Column 6 |  |  | Column 7 |  |  | Column 8 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | $\begin{gathered} \text { Number } \\ c_{i, 6} \end{gathered}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 6}\right) \\ \hline \end{gathered}$ | $i$ | $\begin{gathered} \text { Number } \\ c_{i, 7} \\ \hline \end{gathered}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 7}\right) \\ \hline \end{gathered}$ | $i$ | $\begin{gathered} \text { Number } \\ c_{i, 8} \end{gathered}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 8}\right) \\ & \hline \end{aligned}$ |
| 0 | 9414801.494010 | 603020.2 | 0 | 9320653.479070 | 703523.5 | 0 | 9227446.944279 | 804026.9 |
| 1 | 9410094.093263 | 608021.4 | 1 | 9315993.152330 | 708524.8 | 1 | 9222833.220807 | 809028.1 |
| 2 | 9405389.046216 | 613022.7 | 2 | 9311335.155754 | 713526.0 | 2 | 9218221.804197 | 814029.4 |
| 3 | 9400686.351693 | 618023.9 | 3 | 9306679.488176 | 718527.3 | 3 | 9213612.693295 | 819030.6 |
| 4 | 9395986.008517 | 623025.2 | 4 | 9302026.148432 | 723528.5 | 4 | 9209005.886948 | 824031.9 |
| 5 | 9391288.015513 | 628026.4 | 5 | 9297375.135358 | 728529.8 | 5 | 9204401.384004 | 829033.1 |
| 6 | 9386592.371505 | 633027.7 | 6 | 9292726.447790 | 733531.0 | 6 | 9199799.183312 | 834034.4 |
| 7 | 9381899.075320 | 638028.9 | 7 | 9288080.084566 | 738532.3 | 7 | 9195199.283721 | 839035.6 |
| 8 | 9377208.125782 | 643030.2 | 8 | 9283436.044524 | 743533.5 | 8 | 9190601.684079 | 844036.9 |
| 9 | 9372519.521719 | 648031.4 | 9 | 9278794.326502 | 748534.8 | 9 | 9186006.383237 | 849038.1 |
| 10 | 9367833.261958 | 653032.7 | 10 | 9274154.929339 | 753536.0 | 10 | 9181413.380045 | 854039.4 |
| 11 | 9363149.345327 | 658033.9 | 11 | 9269517.851874 | 758537.3 | 11 | 9176822.673355 | 859040.6 |
| 12 | 9358467.770655 | 663035.2 | 12 | 9264883.092948 | 763538.5 | 12 | 9172234.262019 | 864041.9 |
| 13 | 9353788.536769 | 668036.4 | 13 | 9260250.651402 | 768539.8 | 13 | 9167648.144888 | 869043.1 |
| 14 | 9349111.642501 | 673037.7 | 14 | 9255620.526076 | 773541.0 | 14 | 9163064.320815 | 874044.4 |
| 15 | 9344437.086680 | 678038.9 | 15 | 9250992.715813 | 778542.3 | 15 | 9158482.788655 | 879045.6 |
| 16 | 9339764.868136 | 683040.2 | 16 | 9246367.219455 | 783543.5 | 16 | 9153903.547260 | 884046.9 |
| 17 | 9335094.985702 | 688041.4 | 17 | 9241744.035845 | 788544.8 | 17 | 9149326.595487 | 889048.1 |
| 18 | 9330427.438209 | 693042.7 | 18 | 9237123.163827 | 793546.0 | 18 | 9144751.932189 | 894049.4 |
| 19 | 9325762.224490 | 698043.9 | 19 | 9232504.602245 | 798547.3 | 19 | 9140179.556223 | 899050.6 |
| 20 | 9321099.343378 | 703045.2 | 20 | 9227888.349944 | 803548.5 | 20 | 9135609.466445 | 904051.9 |


| Column 9 |  |  | Column 10 |  |  |  | Column 11 |  |  |
| ---: | :---: | :---: | ---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Number | Logarithm <br> $i$ | $c_{i, 9}$ | $l_{3}\left(c_{i, 9}\right)$ | $i$ | Number | Logarithm <br> $l_{3,10}$ |  |  |
| $l_{3}\left(c_{i, 10}\right)$ | $i$ | $c_{i, 11}$ | $l_{3}\left(c_{i, 11}\right)$ |  |  |  |  |  |  |
| 0 | 9135172.474836 | 904530.2 | 0 | 9043820.750088 | 1005033.6 | 0 | 8953382.542587 | 1105536.9 |  |
| 1 | 9130604.888599 | 909531.5 | 1 | 9039298.839713 | 1010034.8 | 1 | 8948905.851316 | 1110538.2 |  |
| 2 | 9126039.586155 | 914532.7 | 2 | 9034779.190293 | 1015036.1 | 2 | 8944431.398390 | 1115539.4 |  |
| 3 | 9121476.566362 | 919534.0 | 3 | 9030261.800698 | 1020037.3 | 3 | 8939959.182691 | 1120540.7 |  |
| 4 | 9116915.828078 | 924535.2 | 4 | 9025746.669798 | 1025038.6 | 4 | 8935489.203100 | 1125541.9 |  |
| 5 | 9112357.370164 | 929536.5 | 5 | 9021233.796463 | 1030039.8 | 5 | 8931021.458498 | 1130543.2 |  |
| 6 | 9107801.191479 | 934537.7 | 6 | 9016723.179565 | 1035041.1 | 6 | 8926555.947769 | 1135544.4 |  |
| 7 | 9103247.290884 | 939539.0 | 7 | 9012214.817975 | 1040042.3 | 7 | 8922092.669795 | 1140545.7 |  |
| 8 | 9098695.667238 | 944540.2 | 8 | 9007708.710566 | 1045043.6 | 8 | 8917631.623460 | 1145546.9 |  |
| 9 | 9094146.319405 | 949541.5 | 9 | 9003204.856210 | 1050044.8 | 9 | 8913172.807648 | 1150548.2 |  |
| 10 | 9089599.246245 | 954542.7 | 10 | 8998703.253782 | 1055046.1 | 10 | 8908716.221245 | 1155549.4 |  |
| 11 | 9085054.446622 | 959544.0 | 11 | 8994203.902155 | 1060047.3 | 11 | 8904261.863134 | 1160550.7 |  |
| 12 | 9080511.919398 | 964545.2 | 12 | 8989706.800204 | 1065048.6 | 12 | 8899809.732202 | 1165551.9 |  |
| 13 | 9075971.663439 | 969546.5 | 13 | 8985211.946804 | 1070049.8 | 13 | 8895359.827336 | 1170553.2 |  |
| 14 | 9071433.677607 | 974547.7 | 14 | 8980719.340831 | 1075051.1 | 14 | 8890912.147423 | 1175554.4 |  |
| 15 | 9066897.960768 | 979549.0 | 15 | 8976228.981160 | 1080052.3 | 15 | 8886466.691349 | 1180555.7 |  |
| 16 | 9062364.511788 | 984550.2 | 16 | 8971740.866670 | 1085053.6 | 16 | 8882023.458003 | 1185557.0 |  |
| 17 | 9057833.329532 | 989551.5 | 17 | 8967254.996237 | 1090054.8 | 17 | 8877582.446274 | 1190558.2 |  |
| 18 | 9053304.412867 | 994552.7 | 18 | 8962771.368738 | 1095056.1 | 18 | 8873143.655051 | 1195559.5 |  |
| 19 | 9048777.760661 | 999554.0 | 19 | 8958289.983054 | 1100057.3 | 19 | 8868707.083224 | 1200560.7 |  |
| 20 | 9044253.371780 | 1004555.2 | 20 | 8953810.838063 | 1105058.6 | 20 | 8864272.729682 | 1205562.0 |  |


| Column 12 |  |  | Column 13 |  |  | Column 14 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 12}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 12}\right) \\ \hline \end{gathered}$ | $i$ | Number <br> $c_{i, 13}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 13}\right) \\ \hline \end{gathered}$ | $i$ | Number <br> $c_{i, 14}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 14}\right) \\ \hline \end{gathered}$ |
| 0 | 8863848.717161 | 1206040.3 | 0 | 8775210.229990 | 1306543.7 | 0 | 8687458.127690 | 1407047.0 |
| 1 | 8859416.792803 | 1211041.6 | 1 | 8770822.624875 | 1311544.9 | 1 | 8683114.398626 | 1412048.3 |
| 2 | 8854987.084406 | 1216042.8 | 2 | 8766437.213562 | 1316546.2 | 2 | 8678772.841427 | 1417049.5 |
| 3 | 8850559.590864 | 1221044.1 | 3 | 8762053.994955 | 1321547.4 | 3 | 8674433.455006 | 1422050.8 |
| 4 | 8846134.311069 | 1226045.3 | 4 | 8757672.967958 | 1326548.7 | 4 | 8670096.238278 | 1427052.0 |
| 5 | 8841711.243913 | 1231046.6 | 5 | 8753294.131474 | 1331549.9 | 5 | 8665761.190159 | 1432053.3 |
| 6 | 8837290.388291 | 1236047.8 | 6 | 8748917.484408 | 1336551.2 | 6 | 8661428.309564 | 1437054.5 |
| 7 | 8832871.743097 | 1241049.1 | 7 | 8744543.025666 | 1341552.4 | 7 | 8657097.595409 | 1442055.8 |
| 8 | 8828455.307225 | 1246050.3 | 8 | 8740170.754153 | 1346553.7 | 8 | 8652769.046612 | 1447057.0 |
| 9 | 8824041.079572 | 1251051.6 | 9 | 8735800.668776 | 1351554.9 | 9 | 8648442.662088 | 1452058.3 |
| 10 | 8819629.059032 | 1256052.8 | 10 | 8731432.768442 | 1356556.2 | 10 | 8644118.440757 | 1457059.5 |
| 11 | 8815219.244503 | 1261054.1 | 11 | 8727067.052058 | 1361557.4 | 11 | 8639796.381537 | 1462060.8 |
| 12 | 8810811.634880 | 1266055.3 | 12 | 8722703.518532 | 1366558.7 | 12 | 8635476.483346 | 1467062.0 |
| 13 | 8806406.229063 | 1271056.6 | 13 | 8718342.166772 | 1371559.9 | 13 | 8631158.745105 | 1472063.3 |
| 14 | 8802003.025948 | 1276057.8 | 14 | 8713982.995689 | 1376561.2 | 14 | 8626843.165732 | 1477064.5 |
| 15 | 8797602.024435 | 1281059.1 | 15 | 8709626.004191 | 1381562.4 | 15 | 8622529.744149 | 1482065.8 |
| 16 | 8793203.223423 | 1286060.3 | 16 | 8705271.191189 | 1386563.7 | 16 | 8618218.479277 | 1487067.0 |
| 17 | 8788806.621811 | 1291061.6 | 17 | 8700918.555593 | 1391564.9 | 17 | 8613909.370037 | 1492068.3 |
| 18 | 8784412.218501 | 1296062.8 | 18 | 8696568.096316 | 1396566.2 | 18 | 8609602.415352 | 1497069.5 |
| 19 | 8780020.012391 | 1301064.1 | 19 | 8692219.812267 | 1401567.4 | 19 | 8605297.614145 | 1502070.8 |
| 20 | 8775630.002385 | 1306065.3 | 20 | 8687873.702361 | 1406568.7 | 20 | 8600994.965338 | 1507072.0 |


| Column 15 |  |  | Column 16 |  |  | Column 17 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 15}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 15}\right) \\ \hline \end{gathered}$ | $i$ | Number <br> $c_{i, 16}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 16}\right) \\ \hline \end{gathered}$ | $i$ | Number $c_{i, 17}$ | Logarithm $l_{3}\left(c_{i, 17}\right)$ |
| 0 | 8600583.546413 | 1507550.4 | 0 | 8514577.710949 | 1608053.7 | 0 | 8429431.933839 | 1708557.1 |
| 1 | 8596283.254640 | 1512551.6 | 1 | 8510320.422093 | 1613055.0 | 1 | 8425217.217872 | 1713558.3 |
| 2 | 8591985.113012 | 1517552.9 | 2 | 8506065.261882 | 1618056.2 | 2 | 8421004.609263 | 1718559.6 |
| 3 | 8587689.120456 | 1522554.1 | 3 | 8501812.229251 | 1623057.5 | 3 | 8416794.106959 | 1723560.8 |
| 4 | 8583395.275896 | 1527555.4 | 4 | 8497561.323137 | 1628058.7 | 4 | 8412585.709905 | 1728562.1 |
| 5 | 8579103.578258 | 1532556.6 | 5 | 8493312.542475 | 1633060.0 | 5 | 8408379.417050 | 1733563.3 |
| 6 | 8574814.026469 | 1537557.9 | 6 | 8489065.886204 | 1638061.2 | 6 | 8404175.227342 | 1738564.6 |
| 7 | 8570526.619455 | 1542559.1 | 7 | 8484821.353261 | 1643062.5 | 7 | 8399973.139728 | 1743565.8 |
| 8 | 8566241.356146 | 1547560.4 | 8 | 8480578.942584 | 1648063.7 | 8 | 8395773.153158 | 1748567.1 |
| 9 | 8561958.235468 | 1552561.6 | 9 | 8476338.653113 | 1653065.0 | 9 | 8391575.266582 | 1753568.3 |
| 10 | 8557677.256350 | 1557562.9 | 10 | 8472100.483786 | 1658066.2 | 10 | 8387379.478948 | 1758569.6 |
| 11 | 8553398.417722 | 1562564.1 | 11 | 8467864.433544 | 1663067.5 | 11 | 8383185.789209 | 1763570.8 |
| 12 | 8549121.718513 | 1567565.4 | 12 | 8463630.501328 | 1668068.7 | 12 | 8378994.196314 | 1768572.1 |
| 13 | 8544847.157653 | 1572566.6 | 13 | 8459398.686077 | 1673070.0 | 13 | 8374804.699216 | 1773573.4 |
| 14 | 8540574.734075 | 1577567.9 | 14 | 8455168.986734 | 1678071.2 | 14 | 8370617.296867 | 1778574.6 |
| 15 | 8536304.446708 | 1582569.1 | 15 | 8450941.402241 | 1683072.5 | 15 | 8366431.988218 | 1783575.9 |
| 16 | 8532036.294484 | 1587570.4 | 16 | 8446715.931539 | 1688073.7 | 16 | 8362248.772224 | 1788577.1 |
| 17 | 8527770.276337 | 1592571.6 | 17 | 8442492.573574 | 1693075.0 | 17 | 8358067.647838 | 1793578.4 |
| 18 | 8523506.391199 | 1597572.9 | 18 | 8438271.327287 | 1698076.2 | 18 | 8353888.614014 | 1798579.6 |
| 19 | 8519244.638003 | 1602574.1 | 19 | 8434052.191623 | 1703077.5 | 19 | 8349711.669707 | 1803580.9 |
| 20 | 8514985.015684 | 1607575.4 | 20 | 8429835.165527 | 1708078.7 | 20 | 8345536.813872 | 1808582.1 |


| Column 18 |  |  | Column 19 |  |  | Column 20 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 18}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 18}\right) \\ \hline \end{gathered}$ | $i$ | Number <br> $c_{i, 19}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 19}\right) \\ \hline \end{gathered}$ | $i$ | Number <br> $c_{i, 20}$ | Logarithm $l_{3}\left(c_{i, 20}\right)$ |
| 0 | 8345137.614501 | 1809060.5 | 0 | 8261686.238356 | 1909563.8 | 0 | 8179069.375972 | 2010067.2 |
| 1 | 8340965.045694 | 1814061.7 | 1 | 8257555.395237 | 1914565.1 | 1 | 8174979.841284 | 2015068.4 |
| 2 | 8336794.563171 | 1819063.0 | 2 | 8253426.617539 | 1919566.3 | 2 | 8170892.351364 | 2020069.7 |
| 3 | 8332626.165889 | 1824064.2 | 3 | 8249299.904230 | 1924567.6 | 3 | 8166806.905188 | 2025070.9 |
| 4 | 8328459.852806 | 1829065.5 | 4 | 8245175.254278 | 1929568.8 | 4 | 8162723.501735 | 2030072.2 |
| 5 | 8324295.622880 | 1834066.7 | 5 | 8241052.666651 | 1934570.1 | 5 | 8158642.139985 | 2035073.4 |
| 6 | 8320133.475068 | 1839068.0 | 6 | 8236932.140318 | 1939571.3 | 6 | 8154562.818915 | 2040074.7 |
| 7 | 8315973.408331 | 1844069.2 | 7 | 8232813.674248 | 1944572.6 | 7 | 8150485.537505 | 2045075.9 |
| 8 | 8311815.421627 | 1849070.5 | 8 | 8228697.267410 | 1949573.8 | 8 | 8146410.294736 | 2050077.2 |
| 9 | 8307659.513916 | 1854071.7 | 9 | 8224582.918777 | 1954575.1 | 9 | 8142337.089589 | 2055078.4 |
| 10 | 8303505.684159 | 1859073.0 | 10 | 8220470.627317 | 1959576.3 | 10 | 8138265.921044 | 2060079.7 |
| 11 | 8299353.931317 | 1864074.2 | 11 | 8216360.392004 | 1964577.6 | 11 | 8134196.788084 | 2065080.9 |
| 12 | 8295204.254351 | 1869075.5 | 12 | 8212252.211808 | 1969578.8 | 12 | 8130129.689690 | 2070082.2 |
| 13 | 8291056.652224 | 1874076.7 | 13 | 8208146.085702 | 1974580.1 | 13 | 8126064.624845 | 2075083.4 |
| 14 | 8286911.123898 | 1879078.0 | 14 | 8204042.012659 | 1979581.3 | 14 | 8122001.592532 | 2080084.7 |
| 15 | 8282767.668336 | 1884079.2 | 15 | 8199939.991653 | 1984582.6 | 15 | 8117940.591736 | 2085085.9 |
| 16 | 8278626.284502 | 1889080.5 | 16 | 8195840.021657 | 1989583.8 | 16 | 8113881.621440 | 2090087.2 |
| 17 | 8274486.971360 | 1894081.7 | 17 | 8191742.101646 | 1994585.1 | 17 | 8109824.680629 | 2095088.4 |
| 18 | 8270349.727874 | 1899083.0 | 18 | 8187646.230595 | 1999586.3 | 18 | 8105769.768289 | 2100089.7 |
| 19 | 8266214.553010 | 1904084.2 | 19 | 8183552.407480 | 2004587.6 | 19 | 8101716.883405 | 2105090.9 |
| 20 | 8262081.445733 | 1909085.5 | 20 | 8179460.631276 | 2009588.8 | 20 | 8097666.024963 | 2110092.2 |


| Column 21 |  |  | Column 22 |  |  | Column 23 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 21}$ | Logarithm $l_{3}\left(c_{i, 21}\right)$ | $i$ | Number <br> $c_{i, 22}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 22}\right) \end{aligned}$ | $i$ | Number $c_{i, 23}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 23}\right) \\ & \hline \end{aligned}$ |
| 0 | 8097278.682213 | 2110570.5 | 0 | 8016305.895390 | 2211073.9 | 0 | 7936142.836437 | 2311577.2 |
| 1 | 8093230.042871 | 2115571.8 | 1 | 8012297.742443 | 2216075.1 | 1 | 7932174.765018 | 2316578.5 |
| 2 | 8089183.427850 | 2120573.0 | 2 | 8008291.593572 | 2221076.4 | 2 | 7928208.677636 | 2321579.7 |
| 3 | 8085138.836136 | 2125574.3 | 3 | 8004287.447775 | 2226077.6 | 3 | 7924244.573297 | 2326581.0 |
| 4 | 8081096.266718 | 2130575.5 | 4 | 8000285.304051 | 2231078.9 | 4 | 7920282.451010 | 2331582.2 |
| 5 | 8077055.718585 | 2135576.8 | 5 | 7996285.161399 | 2236080.1 | 5 | 7916322.309785 | 2336583.5 |
| 6 | 8073017.190725 | 2140578.0 | 6 | 7992287.018818 | 2241081.4 | 6 | 7912364.148630 | 2341584.7 |
| 7 | 8068980.682130 | 2145579.3 | 7 | 7988290.875309 | 2246082.6 | 7 | 7908407.966556 | 2346586.0 |
| 8 | 8064946.191789 | 2150580.5 | 8 | 7984296.729871 | 2251083.9 | 8 | 7904453.762572 | 2351587.2 |
| 9 | 8060913.718693 | 2155581.8 | 9 | 7980304.581506 | 2256085.1 | 9 | 7900501.535691 | 2356588.5 |
| 10 | 8056883.261834 | 2160583.0 | 10 | 7976314.429215 | 2261086.4 | 10 | 7896551.284923 | 2361589.8 |
| 11 | 8052854.820203 | 2165584.3 | 11 | 7972326.272001 | 2266087.6 | 11 | 7892603.009281 | 2366591.0 |
| 12 | 8048828.392793 | 2170585.5 | 12 | 7968340.108865 | 2271088.9 | 12 | 7888656.707776 | 2371592.3 |
| 13 | 8044803.978596 | 2175586.8 | 13 | 7964355.938810 | 2276090.1 | 13 | 7884712.379422 | 2376593.5 |
| 14 | 8040781.576607 | 2180588.0 | 14 | 7960373.760841 | 2281091.4 | 14 | 7880770.023233 | 2381594.8 |
| 15 | 8036761.185819 | 2185589.3 | 15 | 7956393.573961 | 2286092.6 | 15 | 7876829.638221 | 2386596.0 |
| 16 | 8032742.805226 | 2190590.5 | 16 | 7952415.377174 | 2291093.9 | 16 | 7872891.223402 | 2391597.3 |
| 17 | 8028726.433823 | 2195591.8 | 17 | 7948439.169485 | 2296095.1 | 17 | 7868954.777790 | 2396598.5 |
| 18 | 8024712.070606 | 2200593.0 | 18 | 7944464.949900 | 2301096.4 | 18 | 7865020.300401 | 2401599.8 |
| 19 | 8020699.714571 | 2205594.3 | 19 | 7940492.717425 | 2306097.6 | 19 | 7861087.790251 | 2406601.0 |
| 20 | 8016689.364714 | 2210595.5 | 20 | 7936522.471067 | 2311098.9 | 20 | 7857157.246356 | 2411602.3 |


| Column 24 |  |  | Column 25 |  |  | Column 26 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 24}$ | Logarithm $l_{3}\left(c_{i, 24}\right)$ | $i$ | Number $c_{i, 25}$ | Logarithm $l_{3}\left(c_{i, 25}\right)$ | $i$ | Number <br> $c_{i, 26}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 26}\right) \\ \hline \end{gathered}$ |
| 0 | 7856781.408072 | 2412080.6 | 0 | 7778213.593991 | 2512584.0 | 0 | 7700431.458052 | 2613087.3 |
| 1 | 7852853.017368 | 2417081.9 | 1 | 7774324.487194 | 2517585.2 | 1 | 7696581.242323 | 2618088.6 |
| 2 | 7848926.590859 | 2422083.1 | 2 | 7770437.324951 | 2522586.5 | 2 | 7692732.951701 | 2623089.8 |
| 3 | 7845002.127564 | 2427084.4 | 3 | 7766552.106288 | 2527587.7 | 3 | 7688886.585226 | 2628091.1 |
| 4 | 7841079.626500 | 2432085.6 | 4 | 7762668.830235 | 2532589.0 | 4 | 7685042.141933 | 2633092.3 |
| 5 | 7837159.086687 | 2437086.9 | 5 | 7758787.495820 | 2537590.2 | 5 | 7681199.620862 | 2638093.6 |
| 6 | 7833240.507144 | 2442088.1 | 6 | 7754908.102072 | 2542591.5 | 6 | 7677359.021052 | 2643094.8 |
| 7 | 7829323.886890 | 2447089.4 | 7 | 7751030.648021 | 2547592.7 | 7 | 7673520.341541 | 2648096.1 |
| 8 | 7825409.224947 | 2452090.6 | 8 | 7747155.132697 | 2552594.0 | 8 | 7669683.581370 | 2653097.3 |
| 9 | 7821496.520334 | 2457091.9 | 9 | 7743281.555131 | 2557595.2 | 9 | 7665848.739580 | 2658098.6 |
| 10 | 7817585.772074 | 2462093.1 | 10 | 7739409.914353 | 2562596.5 | 10 | 7662015.815210 | 2663099.8 |
| 11 | 7813676.979188 | 2467094.4 | 11 | 7735540.209396 | 2567597.7 | 11 | 7658184.807302 | 2668101.1 |
| 12 | 7809770.140698 | 2472095.6 | 12 | 7731672.439291 | 2572599.0 | 12 | 7654355.714898 | 2673102.3 |
| 13 | 7805865.255628 | 2477096.9 | 13 | 7727806.603072 | 2577600.2 | 13 | 7650528.537041 | 2678103.6 |
| 14 | 7801962.323000 | 2482098.1 | 14 | 7723942.699770 | 2582601.5 | 14 | 7646703.272773 | 2683104.8 |
| 15 | 7798061.341839 | 2487099.4 | 15 | 7720080.728420 | 2587602.7 | 15 | 7642879.921136 | 2688106.1 |
| 16 | 7794162.311168 | 2492100.6 | 16 | 7716220.688056 | 2592604.0 | 16 | 7639058.481176 | 2693107.3 |
| 17 | 7790265.230012 | 2497101.9 | 17 | 7712362.577712 | 2597605.2 | 17 | 7635238.951935 | 2698108.6 |
| 18 | 7786370.097397 | 2502103.1 | 18 | 7708506.396423 | 2602606.5 | 18 | 7631421.332459 | 2703109.8 |
| 19 | 7782476.912349 | 2507104.4 | 19 | 7704652.143225 | 2607607.7 | 19 | 7627605.621793 | 2708111.1 |
| 20 | 7778585.673892 | 2512105.6 | 20 | 7700799.817153 | 2612609.0 | 20 | 7623791.818982 | 2713112.3 |


| Column 27 |  |  | Column 28 |  |  | Column 29 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 27}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 27}\right) \\ \hline \end{gathered}$ | $i$ | Number $c_{i, 28}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 28}\right) \\ \hline \end{gathered}$ | $i$ | Number <br> $c_{i, 29}$ | Logarithm $l_{3}\left(c_{i, 29}\right)$ |
| 0 | 7623427.143471 | 2713590.7 | 0 | 7547192.872036 | 2814094.0 | 0 | 7471720.943316 | 2914597.4 |
| 1 | 7619615.429899 | 2718591.9 | 1 | 7543419.275600 | 2819095.3 | 1 | 7467985.082844 | 2919598.6 |
| 2 | 7615805.622184 | 2723593.2 | 2 | 7539647.565963 | 2824096.5 | 2 | 7464251.090303 | 2924599.9 |
| 3 | 7611997.719373 | 2728594.4 | 3 | 7535877.742180 | 2829097.8 | 3 | 7460518.964758 | 2929601.1 |
| 4 | 7608191.720514 | 2733595.7 | 4 | 7532109.803308 | 2834099.0 | 4 | 7456788.705275 | 2934602.4 |
| 5 | 7604387.624653 | 2738596.9 | 5 | 7528343.748407 | 2839100.3 | 5 | 7453060.310923 | 2939603.6 |
| 6 | 7600585.430841 | 2743598.2 | 6 | 7524579.576533 | 2844101.5 | 6 | 7449333.780767 | 2944604.9 |
| 7 | 7596785.138126 | 2748599.4 | 7 | 7520817.286744 | 2849102.8 | 7 | 7445609.113877 | 2949606.2 |
| 8 | 7592986.745557 | 2753600.7 | 8 | 7517056.878101 | 2854104.0 | 8 | 7441886.309320 | 2954607.4 |
| 9 | 7589190.252184 | 2758601.9 | 9 | 7513298.349662 | 2859105.3 | 9 | 7438165.366165 | 2959608.7 |
| 10 | 7585395.657058 | 2763603.2 | 10 | 7509541.700487 | 2864106.5 | 10 | 7434446.283482 | 2964609.9 |
| 11 | 7581602.959229 | 2768604.4 | 11 | 7505786.929637 | 2869107.8 | 11 | 7430729.060340 | 2969611.2 |
| 12 | 7577812.157749 | 2773605.7 | 12 | 7502034.036172 | 2874109.0 | 12 | 7427013.695810 | 2974612.4 |
| 13 | 7574023.251671 | 2778606.9 | 13 | 7498283.019154 | 2879110.3 | 13 | 7423300.188962 | 2979613.7 |
| 14 | 7570236.240045 | 2783608.2 | 14 | 7494533.877644 | 2884111.5 | 14 | 7419588.538868 | 2984614.9 |
| 15 | 7566451.121925 | 2788609.4 | 15 | 7490786.610706 | 2889112.8 | 15 | 7415878.744598 | 2989616.2 |
| 16 | 7562667.896364 | 2793610.7 | 16 | 7487041.217400 | 2894114.0 | 16 | 7412170.805226 | 2994617.4 |
| 17 | 7558886.562416 | 2798611.9 | 17 | 7483297.696791 | 2899115.3 | 17 | 7408464.719824 | 2999618.7 |
| 18 | 7555107.119134 | 2803613.2 | 18 | 7479556.047943 | 2904116.5 | 18 | 7404760.487464 | 3004619.9 |
| 19 | 7551329.565575 | 2808614.4 | 19 | 7475816.269919 | 2909117.8 | 19 | 7401058.107220 | 3009621.2 |
| 20 | 7547553.900792 | 2813615.7 | 20 | 7472078.361784 | 2914119.0 | 20 | 7397357.578166 | 3014622.4 |


| Column 30 |  |  | Column 31 |  |  | Column 32 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 30}$ | Logarithm $l_{3}\left(c_{i, 30}\right)$ | $i$ | Number <br> $c_{i, 31}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 31}\right) \\ \hline \end{gathered}$ | $i$ | Number <br> $c_{i, 32}$ | Logarithm $l_{3}\left(c_{i, 32}\right)$ |
| 0 | 7397003.733883 | 3015100.8 | 0 | 7323033.696544 | 3115604.1 | 0 | 7249803.359579 | 3216107.5 |
| 1 | 7393305.232016 | 3020102.0 | 1 | 7319372.179696 | 3120605.4 | 1 | 7246178.457899 | 3221108.7 |
| 2 | 7389608.579400 | 3025103.3 | 2 | 7315712.493606 | 3125606.6 | 2 | 7242555.368670 | 3226110.0 |
| 3 | 7385913.775110 | 3030104.5 | 3 | 7312054.637359 | 3130607.9 | 3 | 7238934.090985 | 3231111.2 |
| 4 | 7382220.818223 | 3035105.8 | 4 | 7308398.610040 | 3135609.1 | 4 | 7235314.623940 | 3236112.5 |
| 5 | 7378529.707813 | 3040107.0 | 5 | 7304744.410735 | 3140610.4 | 5 | 7231696.966628 | 3241113.7 |
| 6 | 7374840.442960 | 3045108.3 | 6 | 7301092.038530 | 3145611.6 | 6 | 7228081.118145 | 3246115.0 |
| 7 | 7371153.022738 | 3050109.5 | 7 | 7297441.492511 | 3150612.9 | 7 | 7224467.077586 | 3251116.2 |
| 8 | 7367467.446227 | 3055110.8 | 8 | 7293792.771764 | 3155614.1 | 8 | 7220854.844047 | 3256117.5 |
| 9 | 7363783.712504 | 3060112.0 | 9 | 7290145.875379 | 3160615.4 | 9 | 7217244.416625 | 3261118.7 |
| 10 | 7360101.820647 | 3065113.3 | 10 | 7286500.802441 | 3165616.6 | 10 | 7213635.794416 | 3266120.0 |
| 11 | 7356421.769737 | 3070114.5 | 11 | 7282857.552040 | 3170617.9 | 11 | 7210028.976519 | 3271121.2 |
| 12 | 7352743.558852 | 3075115.8 | 12 | 7279216.123264 | 3175619.1 | 12 | 7206423.962031 | 3276122.5 |
| 13 | 7349067.187073 | 3080117.0 | 13 | 7275576.515202 | 3180620.4 | 13 | 7202820.750050 | 3281123.7 |
| 14 | 7345392.653479 | 3085118.3 | 14 | 7271938.726944 | 3185621.6 | 14 | 7199219.339675 | 3286125.0 |
| 15 | 7341719.957152 | 3090119.5 | 15 | 7268302.757581 | 3190622.9 | 15 | 7195619.730005 | 3291126.2 |
| 16 | 7338049.097174 | 3095120.8 | 16 | 7264668.606202 | 3195624.1 | 16 | 7192021.920140 | 3296127.5 |
| 17 | 7334380.072625 | 3100122.0 | 17 | 7261036.271899 | 3200625.4 | 17 | 7188425.909180 | 3301128.7 |
| 18 | 7330712.882589 | 3105123.3 | 18 | 7257405.753763 | 3205626.6 | 18 | 7184831.696225 | 3306130.0 |
| 19 | 7327047.526148 | 3110124.5 | 19 | 7253777.050886 | 3210627.9 | 19 | 7181239.280377 | 3311131.2 |
| 20 | 7323384.002385 | 3115125.8 | 20 | 7250150.162361 | 3215629.1 | 20 | 7177648.660737 | 3316132.5 |


| Column 33 |  |  | Column 34 |  |  | Column 35 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 33}$ | Logarithm $l_{3}\left(c_{i, 33}\right)$ | $i$ | Number <br> $c_{i, 34}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 34}\right) \\ \hline \end{gathered}$ | $i$ | Number $c_{i, 35}$ | Logarithm $l_{3}\left(c_{i, 35}\right)$ |
| 0 | 7177305.325983 | 3316610.8 | 0 | 7105532.272723 | 3417114.2 | 0 | 7034476.949996 | 3517617.5 |
| 1 | 7173716.673320 | 3321612.1 | 1 | 7101979.506587 | 3422115.4 | 1 | 7030959.711521 | 3522618.8 |
| 2 | 7170129.814983 | 3326613.3 | 2 | 7098428.516833 | 3427116.7 | 2 | 7027444.231665 | 3527620.0 |
| 3 | 7166544.750076 | 3331614.6 | 3 | 7094879.302575 | 3432117.9 | 3 | 7023930.509549 | 3532621.3 |
| 4 | 7162961.477701 | 3336615.8 | 4 | 7091331.862924 | 3437119.2 | 4 | 7020418.544294 | 3537622.6 |
| 5 | 7159379.996962 | 3341617.1 | 5 | 7087786.196992 | 3442120.4 | 5 | 7016908.335022 | 3542623.8 |
| 6 | 7155800.306963 | 3346618.3 | 6 | 7084242.303894 | 3447121.7 | 6 | 7013399.880855 | 3547625.1 |
| 7 | 7152222.406810 | 3351619.6 | 7 | 7080700.182742 | 3452122.9 | 7 | 7009893.180914 | 3552626.3 |
| 8 | 7148646.295606 | 3356620.8 | 8 | 7077159.832650 | 3457124.2 | 8 | 7006388.234324 | 3557627.6 |
| 9 | 7145071.972459 | 3361622.1 | 9 | 7073621.252734 | 3462125.4 | 9 | 7002885.040207 | 3562628.8 |
| 10 | 7141499.436472 | 3366623.3 | 10 | 7070084.442108 | 3467126.7 | 10 | 6999383.597687 | 3567630.1 |
| 11 | 7137928.686754 | 3371624.6 | 11 | 7066549.399887 | 3472127.9 | 11 | 6995883.905888 | 3572631.3 |
| 12 | 7134359.722411 | 3376625.8 | 12 | 7063016.125187 | 3477129.2 | 12 | 6992385.963935 | 3577632.6 |
| 13 | 7130792.542550 | 3381627.1 | 13 | 7059484.617124 | 3482130.4 | 13 | 6988889.770953 | 3582633.8 |
| 14 | 7127227.146278 | 3386628.3 | 14 | 7055954.874815 | 3487131.7 | 14 | 6985395.326067 | 3587635.1 |
| 15 | 7123663.532705 | 3391629.6 | 15 | 7052426.897378 | 3492132.9 | 15 | 6981902.628404 | 3592636.3 |
| 16 | 7120101.700939 | 3396630.8 | 16 | 7048900.683929 | 3497134.2 | 16 | 6978411.677090 | 3597637.6 |
| 17 | 7116541.650088 | 3401632.1 | 17 | 7045376.233587 | 3502135.4 | 17 | 6974922.471252 | 3602638.8 |
| 18 | 7112983.379263 | 3406633.3 | 18 | 7041853.545471 | 3507136.7 | 18 | 6971435.010016 | 3607640.1 |
| 19 | 7109426.887574 | 3411634.6 | 19 | 7038332.618698 | 3512137.9 | 19 | 6967949.292511 | 3612641.3 |
| 20 | 7105872.174130 | 3416635.8 | 20 | 7034813.452389 | 3517139.2 | 20 | 6964465.317865 | 3617642.6 |


| Column 36 |  |  | Column 37 |  |  | Column 38 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 36}$ | Logarithm $l_{3}\left(c_{i, 36}\right)$ | $i$ | Number $c_{i, 37}$ | Logarithm $l_{3}\left(c_{i, 37}\right)$ | $i$ | Number <br> $c_{i, 38}$ | Logarithm $l_{3}\left(c_{i, 38}\right)$ |
| 0 | 6964132.180496 | 3618120.9 | 0 | 6894490.858691 | 3718624.3 | 0 | 6825545.950104 | 3819127.6 |
| 1 | 6960650.114405 | 3623122.2 | 1 | 6891043.613261 | 3723625.5 | 1 | 6822133.177129 | 3824128.9 |
| 2 | 6957169.789348 | 3628123.4 | 2 | 6887598.091455 | 3728626.8 | 2 | 6818722.110540 | 3829130.1 |
| 3 | 6953691.204454 | 3633124.7 | 3 | 6884154.292409 | 3733628.0 | 3 | 6815312.749485 | 3834131.4 |
| 4 | 6950214.358851 | 3638125.9 | 4 | 6880712.215263 | 3738629.3 | 4 | 6811905.093110 | 3839132.6 |
| 5 | 6946739.251672 | 3643127.2 | 5 | 6877271.859155 | 3743630.5 | 5 | 6808499.140564 | 3844133.9 |
| 6 | 6943265.882046 | 3648128.4 | 6 | 6873833.223226 | 3748631.8 | 6 | 6805094.890993 | 3849135.1 |
| 7 | 6939794.249105 | 3653129.7 | 7 | 6870396.306614 | 3753633.0 | 7 | 6801692.343548 | 3854136.4 |
| 8 | 6936324.351981 | 3658130.9 | 8 | 6866961.108461 | 3758634.3 | 8 | 6798291.497376 | 3859137.6 |
| 9 | 6932856.189805 | 3663132.2 | 9 | 6863527.627907 | 3763635.5 | 9 | 6794892.351627 | 3864138.9 |
| 10 | 6929389.761710 | 3668133.4 | 10 | 6860095.864093 | 3768636.8 | 10 | 6791494.905452 | 3869140.1 |
| 11 | 6925925.066829 | 3673134.7 | 11 | 6856665.816161 | 3773638.0 | 11 | 6788099.157999 | 3874141.4 |
| 12 | 6922462.104295 | 3678135.9 | 12 | 6853237.483252 | 3778639.3 | 12 | 6784705.108420 | 3879142.6 |
| 13 | 6919000.873243 | 3683137.2 | 13 | 6849810.864511 | 3783640.5 | 13 | 6781312.755866 | 3884143.9 |
| 14 | 6915541.372807 | 3688138.4 | 14 | 6846385.959079 | 3788641.8 | 14 | 6777922.099488 | 3889145.1 |
| 15 | 6912083.602120 | 3693139.7 | 15 | 6842962.766099 | 3793643.0 | 15 | 6774533.138438 | 3894146.4 |
| 16 | 6908627.560319 | 3698140.9 | 16 | 6839541.284716 | 3798644.3 | 16 | 6771145.871869 | 3899147.6 |
| 17 | 6905173.246539 | 3703142.2 | 17 | 6836121.514074 | 3803645.5 | 17 | 6767760.298933 | 3904148.9 |
| 18 | 6901720.659916 | 3708143.4 | 18 | 6832703.453317 | 3808646.8 | 18 | 6764376.418783 | 3909150.1 |
| 19 | 6898269.799586 | 3713144.7 | 19 | 6829287.101590 | 3813648.0 | 19 | 6760994.230574 | 3914151.4 |
| 20 | 6894820.664686 | 3718145.9 | 20 | 6825872.458039 | 3818649.3 | 20 | 6757613.733459 | 3919152.6 |


| Column 39 |  |  | Column 40 |  |  | Column 41 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 39}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 39}\right) \end{aligned}$ | $i$ | Number $c_{i, 40}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 40}\right) \end{aligned}$ | $i$ | $\begin{gathered} \text { Number } \\ c_{i, 41} \\ \hline \end{gathered}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 41}\right) \\ & \hline \end{aligned}$ |
| 0 | 6757290.490603 | 3919631.0 | 0 | 6689717.585697 | 4020134.3 | 0 | 6622820.409840 | 4120637.7 |
| 1 | 6753911.845358 | 3924632.2 | 1 | 6686372.726904 | 4025135.6 | 1 | 6619508.999635 | 4125639.0 |
| 2 | 6750534.889435 | 3929633.5 | 2 | 6683029.540541 | 4030136.8 | 2 | 6616199.245135 | 4130640.2 |
| 3 | 6747159.621990 | 3934634.7 | 3 | 6679688.025770 | 4035138.1 | 3 | 6612891.145513 | 4135641.5 |
| 4 | 6743786.042179 | 3939636.0 | 4 | 6676348.181757 | 4040139.3 | 4 | 6609584.699940 | 4140642.7 |
| 5 | 6740414.149158 | 3944637.2 | 5 | 6673010.007666 | 4045140.6 | 5 | 6606279.907590 | 4145644.0 |
| 6 | 6737043.942083 | 3949638.5 | 6 | 6669673.502663 | 4050141.8 | 6 | 6602976.767636 | 4150645.2 |
| 7 | 6733675.420112 | 3954639.7 | 7 | 6666338.665911 | 4055143.1 | 7 | 6599675.279252 | 4155646.5 |
| 8 | 6730308.582402 | 3959641.0 | 8 | 6663005.496578 | 4060144.3 | 8 | 6596375.441613 | 4160647.7 |
| 9 | 6726943.428111 | 3964642.2 | 9 | 6659673.993830 | 4065145.6 | 9 | 6593077.253892 | 4165649.0 |
| 10 | 6723579.956397 | 3969643.5 | 10 | 6656344.156833 | 4070146.8 | 10 | 6589780.715265 | 4170650.2 |
| 11 | 6720218.166419 | 3974644.7 | 11 | 6653015.984755 | 4075148.1 | 11 | 6586485.824907 | 4175651.5 |
| 12 | 6716858.057336 | 3979646.0 | 12 | 6649689.476762 | 4080149.3 | 12 | 6583192.581995 | 4180652.7 |
| 13 | 6713499.628307 | 3984647.2 | 13 | 6646364.632024 | 4085150.6 | 13 | 6579900.985704 | 4185654.0 |
| 14 | 6710142.878493 | 3989648.5 | 14 | 6643041.449708 | 4090151.8 | 14 | 6576611.035211 | 4190655.2 |
| 15 | 6706787.807054 | 3994649.7 | 15 | 6639719.928983 | 4095153.1 | 15 | 6573322.729693 | 4195656.5 |
| 16 | 6703434.413150 | 3999651.0 | 16 | 6636400.069019 | 4100154.3 | 16 | 6570036.068328 | 4200657.7 |
| 17 | 6700082.695944 | 4004652.2 | 17 | 6633081.868984 | 4105155.6 | 17 | 6566751.050294 | 4205659.0 |
| 18 | 6696732.654596 | 4009653.5 | 18 | 6629765.328050 | 4110156.8 | 18 | 6563467.674769 | 4210660.2 |
| 19 | 6693384.288268 | 4014654.7 | 19 | 6626450.445386 | 4115158.1 | 19 | 6560185.940932 | 4215661.5 |
| 20 | 6690037.596124 | 4019656.0 | 20 | 6623137.220163 | 4120159.3 | 20 | 6556905.847961 | 4220662.7 |


| Column 42 |  |  | Column 43 |  |  | Column 44 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number <br> $c_{i, 42}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 42}\right) \end{aligned}$ | $i$ | Number <br> $c_{i, 43}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 43}\right) \end{aligned}$ | $i$ | Number $c_{i, 44}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 44}\right) \end{aligned}$ |
| 0 | 6556592.205741 | 4221141.1 | 0 | 6491026.283684 | 4321644.4 | 0 | 6426116.020847 | 4422147.8 |
| 1 | 6553313.909639 | 4226142.3 | 1 | 6487780.770542 | 4326645.7 | 1 | 6422902.962837 | 4427149.0 |
| 2 | 6550037.252684 | 4231143.6 | 2 | 6484536.880157 | 4331646.9 | 2 | 6419691.511355 | 4432150.3 |
| 3 | 6546762.234057 | 4236144.8 | 3 | 6481294.611717 | 4336648.2 | 3 | 6416481.665600 | 4437151.5 |
| 4 | 6543488.852940 | 4241146.1 | 4 | 6478053.964411 | 4341649.4 | 4 | 6413273.424767 | 4442152.8 |
| 5 | 6540217.108514 | 4246147.3 | 5 | 6474814.937429 | 4346650.7 | 5 | 6410066.788054 | 4447154.0 |
| 6 | 6536946.999960 | 4251148.6 | 6 | 6471577.529960 | 4351651.9 | 6 | 6406861.754660 | 4452155.3 |
| 7 | 6533678.526460 | 4256149.8 | 7 | 6468341.741195 | 4356653.2 | 7 | 6403658.323783 | 4457156.5 |
| 8 | 6530411.687196 | 4261151.1 | 8 | 6465107.570324 | 4361654.4 | 8 | 6400456.494621 | 4462157.8 |
| 9 | 6527146.481353 | 4266152.3 | 9 | 6461875.016539 | 4366655.7 | 9 | 6397256.266374 | 4467159.0 |
| 10 | 6523882.908112 | 4271153.6 | 10 | 6458644.079031 | 4371656.9 | 10 | 6394057.638241 | 4472160.3 |
| 11 | 6520620.966658 | 4276154.8 | 11 | 6455414.756992 | 4376658.2 | 11 | 6390860.609422 | 4477161.5 |
| 12 | 6517360.656175 | 4281156.1 | 12 | 6452187.049613 | 4381659.4 | 12 | 6387665.179117 | 4482162.8 |
| 13 | 6514101.975847 | 4286157.3 | 13 | 6448960.956088 | 4386660.7 | 13 | 6384471.346527 | 4487164.0 |
| 14 | 6510844.924859 | 4291158.6 | 14 | 6445736.475610 | 4391661.9 | 14 | 6381279.110854 | 4492165.3 |
| 15 | 6507589.502396 | 4296159.8 | 15 | 6442513.607372 | 4396663.2 | 15 | 6378088.471299 | 4497166.5 |
| 16 | 6504335.707645 | 4301161.1 | 16 | 6439292.350569 | 4401664.4 | 16 | 6374899.427063 | 4502167.8 |
| 17 | 6501083.539791 | 4306162.3 | 17 | 6436072.704393 | 4406665.7 | 17 | 6371711.977349 | 4507169.0 |
| 18 | 6497832.998021 | 4311163.6 | 18 | 6432854.668041 | 4411666.9 | 18 | 6368526.121361 | 4512170.3 |
| 19 | 6494584.081522 | 4316164.8 | 19 | 6429638.240707 | 4416668.2 | 19 | 6365341.858300 | 4517171.5 |
| 20 | 6491336.789482 | 4321166.1 | 20 | 6426423.421587 | 4421669.4 | 20 | 6362159.187371 | 4522172.8 |


| Column 45 |  |  | Column 46 |  |  | Column 47 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number <br> $c_{i, 45}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 45}\right) \\ \hline \end{gathered}$ | $i$ | Number <br> $c_{i, 46}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 46}\right) \\ & \hline \end{aligned}$ | $i$ | Number $c_{i, 47}$ | Logarithm $l_{3}\left(c_{i, 47}\right)$ |
| 0 | 6361854.860639 | 4522651.1 | 0 | 6298236.312032 | 4623154.5 | 0 | 6235253.948912 | 4723657.9 |
| 1 | 6358673.933208 | 4527652.4 | 1 | 6295087.193876 | 4628155.7 | 1 | 6232136.321938 | 4728659.1 |
| 2 | 6355494.596242 | 4532653.6 | 2 | 6291939.650279 | 4633157.0 | 2 | 6229020.253777 | 4733660.4 |
| 3 | 6352316.848944 | 4537654.9 | 3 | 6288793.680454 | 4638158.2 | 3 | 6225905.743650 | 4738661.6 |
| 4 | 6349140.690519 | 4542656.1 | 4 | 6285649.283614 | 4643159.5 | 4 | 6222792.790778 | 4743662.9 |
| 5 | 6345966.120174 | 4547657.4 | 5 | 6282506.458972 | 4648160.7 | 5 | 6219681.394382 | 4748664.1 |
| 6 | 6342793.137114 | 4552658.6 | 6 | 6279365.205743 | 4653162.0 | 6 | 6216571.553685 | 4753665.4 |
| 7 | 6339621.740545 | 4557659.9 | 7 | 6276225.523140 | 4658163.2 | 7 | 6213463.267908 | 4758666.6 |
| 8 | 6336451.929675 | 4562661.1 | 8 | 6273087.410378 | 4663164.5 | 8 | 6210356.536274 | 4763667.9 |
| 9 | 6333283.703710 | 4567662.4 | 9 | 6269950.866673 | 4668165.7 | 9 | 6207251.358006 | 4768669.1 |
| 10 | 6330117.061858 | 4572663.6 | 10 | 6266815.891240 | 4673167.0 | 10 | 6204147.732327 | 4773670.4 |
| 11 | 6326952.003327 | 4577664.9 | 11 | 6263682.483294 | 4678168.2 | 11 | 6201045.658461 | 4778671.6 |
| 12 | 6323788.527326 | 4582666.1 | 12 | 6260550.642052 | 4683169.5 | 12 | 6197945.135632 | 4783672.9 |
| 13 | 6320626.633062 | 4587667.4 | 13 | 6257420.366731 | 4688170.7 | 13 | 6194846.163064 | 4788674.1 |
| 14 | 6317466.319746 | 4592668.6 | 14 | 6254291.656548 | 4693172.0 | 14 | 6191748.739983 | 4793675.4 |
| 15 | 6314307.586586 | 4597669.9 | 15 | 6251164.510720 | 4698173.2 | 15 | 6188652.865613 | 4798676.6 |
| 16 | 6311150.432792 | 4602671.1 | 16 | 6248038.928464 | 4703174.5 | 16 | 6185558.539180 | 4803677.9 |
| 17 | 6307994.857576 | 4607672.4 | 17 | 6244914.909000 | 4708175.7 | 17 | 6182465.759910 | 4808679.1 |
| 18 | 6304840.860147 | 4612673.6 | 18 | 6241792.451546 | 4713177.0 | 18 | 6179374.527030 | 4813680.4 |
| 19 | 6301688.439717 | 4617674.9 | 19 | 6238671.555320 | 4718178.3 | 19 | 6176284.839767 | 4818681.6 |
| 20 | 6298537.595497 | 4622676.1 | 20 | 6235552.219542 | 4723179.5 | 20 | 6173196.697347 | 4823682.9 |


| Column 48 |  |  | Column 49 |  |  | Column 50 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number <br> $c_{i, 48}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 48}\right) \end{gathered}$ | $i$ | Number $c_{i, 49}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 49}\right) \end{aligned}$ | $i$ | $\begin{gathered} \text { Number } \\ c_{i, 50} \end{gathered}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 50}\right) \end{aligned}$ |
| 0 | 6172901.409423 | 4824161.2 | 0 | 6111172.395329 | 4924664.6 | 0 | 6050060.671375 | 5025167.9 |
| 1 | 6169814.958718 | 4829162.5 | 1 | 6108116.809131 | 4929665.8 | 1 | 6047035.641040 | 5030169.2 |
| 2 | 6166730.051239 | 4834163.7 | 2 | 6105062.750726 | 4934667.1 | 2 | 6044012.123219 | 5035170.4 |
| 3 | 6163646.686213 | 4839165.0 | 3 | 6102010.219351 | 4939668.3 | 3 | 6040990.117158 | 5040171.7 |
| 4 | 6160564.862870 | 4844166.2 | 4 | 6098959.214241 | 4944669.6 | 4 | 6037969.622099 | 5045172.9 |
| 5 | 6157484.580439 | 4849167.5 | 5 | 6095909.734634 | 4949670.8 | 5 | 6034950.637288 | 5050174.2 |
| 6 | 6154405.838148 | 4854168.7 | 6 | 6092861.779767 | 4954672.1 | 6 | 6031933.161969 | 5055175.4 |
| 7 | 6151328.635229 | 4859170.0 | 7 | 6089815.348877 | 4959673.3 | 7 | 6028917.195388 | 5060176.7 |
| 8 | 6148252.970912 | 4864171.2 | 8 | 6086770.441203 | 4964674.6 | 8 | 6025902.736791 | 5065177.9 |
| 9 | 6145178.844426 | 4869172.5 | 9 | 6083727.055982 | 4969675.8 | 9 | 6022889.785422 | 5070179.2 |
| 10 | 6142106.255004 | 4874173.7 | 10 | 6080685.192454 | 4974677.1 | 10 | 6019878.340529 | 5075180.4 |
| 11 | 6139035.201877 | 4879175.0 | 11 | 6077644.849858 | 4979678.3 | 11 | 6016868.401359 | 5080181.7 |
| 12 | 6135965.684276 | 4884176.2 | 12 | 6074606.027433 | 4984679.6 | 12 | 6013859.967159 | 5085182.9 |
| 13 | 6132897.701433 | 4889177.5 | 13 | 6071568.724419 | 4989680.8 | 13 | 6010853.037175 | 5090184.2 |
| 14 | 6129831.252583 | 4894178.7 | 14 | 6068532.940057 | 4994682.1 | 14 | 6007847.610656 | 5095185.4 |
| 15 | 6126766.336956 | 4899180.0 | 15 | 6065498.673587 | 4999683.3 | 15 | 6004843.686851 | 5100186.7 |
| 16 | 6123702.953788 | 4904181.2 | 16 | 6062465.924250 | 5004684.6 | 16 | 6001841.265008 | 5105187.9 |
| 17 | 6120641.102311 | 4909182.5 | 17 | 6059434.691288 | 5009685.8 | 17 | 5998840.344375 | 5110189.2 |
| 18 | 6117580.781760 | 4914183.7 | 18 | 6056404.973942 | 5014687.1 | 18 | 5995840.924203 | 5115190.4 |
| 19 | 6114521.991369 | 4919185.0 | 19 | 6053376.771455 | 5019688.3 | 19 | 5992843.003741 | 5120191.7 |
| 20 | 6111464.730373 | 4924186.2 | 20 | 6050350.083070 | 5024689.6 | 20 | 5989846.582239 | 5125192.9 |


| Column 51 |  |  | Column 52 |  |  | Column 53 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number <br> $c_{i, 51}$ | Logarithm $l_{3}\left(c_{i, 51}\right)$ | $i$ | Number $c_{i, 52}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 52}\right) \\ \hline \end{gathered}$ | $i$ | Number $c_{i, 53}$ | Logarithm $l_{3}\left(c_{i, 53}\right)$ |
| 0 | 5989560.064662 | 5125671.3 | 0 | 5929664.464015 | 5226174.6 | 0 | 5870367.819375 | 5326678.0 |
| 1 | 5986565.284629 | 5130672.5 | 1 | 5926699.631783 | 5231175.9 | 1 | 5867432.635465 | 5331679.3 |
| 2 | 5983572.001987 | 5135673.8 | 2 | 5923736.281967 | 5236177.1 | 2 | 5864498.919147 | 5336680.5 |
| 3 | 5980580.215986 | 5140675.0 | 3 | 5920774.413826 | 5241178.4 | 3 | 5861566.669688 | 5341681.8 |
| 4 | 5977589.925878 | 5145676.3 | 4 | 5917814.026619 | 5246179.6 | 4 | 5858635.886353 | 5346683.0 |
| 5 | 5974601.130915 | 5150677.5 | 5 | 5914855.119606 | 5251180.9 | 5 | 5855706.568410 | 5351684.3 |
| 6 | 5971613.830350 | 5155678.8 | 6 | 5911897.692046 | 5256182.1 | 6 | 5852778.715126 | 5356685.5 |
| 7 | 5968628.023434 | 5160680.0 | 7 | 5908941.743200 | 5261183.4 | 7 | 5849852.325768 | 5361686.8 |
| 8 | 5965643.709423 | 5165681.3 | 8 | 5905987.272328 | 5266184.6 | 8 | 5846927.399605 | 5366688.0 |
| 9 | 5962660.887568 | 5170682.5 | 9 | 5903034.278692 | 5271185.9 | 9 | 5844003.935905 | 5371689.3 |
| 10 | 5959679.557124 | 5175683.8 | 10 | 5900082.761553 | 5276187.1 | 10 | 5841081.933937 | 5376690.5 |
| 11 | 5956699.717346 | 5180685.0 | 11 | 5897132.720172 | 5281188.4 | 11 | 5838161.392970 | 5381691.8 |
| 12 | 5953721.367487 | 5185686.3 | 12 | 5894184.153812 | 5286189.6 | 12 | 5835242.312274 | 5386693.0 |
| 13 | 5950744.506803 | 5190687.5 | 13 | 5891237.061735 | 5291190.9 | 13 | 5832324.691118 | 5391694.3 |
| 14 | 5947769.134550 | 5195688.8 | 14 | 5888291.443204 | 5296192.1 | 14 | 5829408.528772 | 5396695.5 |
| 15 | 5944795.249983 | 5200690.0 | 15 | 5885347.297483 | 5301193.4 | 15 | 5826493.824508 | 5401696.8 |
| 16 | 5941822.852358 | 5205691.3 | 16 | 5882404.623834 | 5306194.7 | 16 | 5823580.577596 | 5406698.0 |
| 17 | 5938851.940931 | 5210692.5 | 17 | 5879463.421522 | 5311195.9 | 17 | 5820668.787307 | 5411699.3 |
| 18 | 5935882.514961 | 5215693.8 | 18 | 5876523.689811 | 5316197.2 | 18 | 5817758.452913 | 5416700.5 |
| 19 | 5932914.573703 | 5220695.0 | 19 | 5873585.427966 | 5321198.4 | 19 | 5814849.573687 | 5421701.8 |
| 20 | 5929948.116417 | 5225696.3 | 20 | 5870648.635252 | 5326199.7 | 20 | 5811942.148900 | 5426703.0 |


| Column 54 |  |  | Column 55 |  |  | Column 56 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 54}$ | Logarithm $l_{3}\left(c_{i, 54}\right)$ | $i$ | Number $c_{i, 55}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 55}\right) \end{gathered}$ | $i$ | Number $c_{i, 56}$ | Logarithm <br> $l_{3}\left(c_{i, 56}\right)$ |
| 0 | 5811664.141181 | 5427181.4 | 0 | 5753547.499769 | 5527684.7 | 0 | 5696012.024772 | 5628188.1 |
| 1 | 5808758.309111 | 5432182.6 | 1 | 5750670.726019 | 5532686.0 | 1 | 5693164.018759 | 5633189.3 |
| 2 | 5805853.929956 | 5437183.9 | 2 | 5747795.390656 | 5537687.2 | 2 | 5690317.436750 | 5638190.6 |
| 3 | 5802951.002991 | 5442185.1 | 3 | 5744921.492961 | 5542688.5 | 3 | 5687472.278031 | 5643191.8 |
| 4 | 5800049.527489 | 5447186.4 | 4 | 5742049.032215 | 5547689.7 | 4 | 5684628.541892 | 5648193.1 |
| 5 | 5797149.502726 | 5452187.6 | 5 | 5739178.007698 | 5552691.0 | 5 | 5681786.227621 | 5653194.3 |
| 6 | 5794250.927974 | 5457188.9 | 6 | 5736308.418695 | 5557692.2 | 6 | 5678945.334508 | 5658195.6 |
| 7 | 5791353.802510 | 5462190.1 | 7 | 5733440.264485 | 5562693.5 | 7 | 5676105.861840 | 5663196.8 |
| 8 | 5788458.125609 | 5467191.4 | 8 | 5730573.544353 | 5567694.7 | 8 | 5673267.808910 | 5668198.1 |
| 9 | 5785563.896546 | 5472192.6 | 9 | 5727708.257581 | 5572696.0 | 9 | 5670431.175005 | 5673199.3 |
| 10 | 5782671.114598 | 5477193.9 | 10 | 5724844.403452 | 5577697.2 | 10 | 5667595.959418 | 5678200.6 |
| 11 | 5779779.779041 | 5482195.1 | 11 | 5721981.981250 | 5582698.5 | 11 | 5664762.161438 | 5683201.8 |
| 12 | 5776889.889151 | 5487196.4 | 12 | 5719120.990260 | 5587699.7 | 12 | 5661929.780357 | 5688203.1 |
| 13 | 5774001.444207 | 5492197.6 | 13 | 5716261.429765 | 5592701.0 | 13 | 5659098.815467 | 5693204.3 |
| 14 | 5771114.443485 | 5497198.9 | 14 | 5713403.299050 | 5597702.2 | 14 | 5656269.266059 | 5698205.6 |
| 15 | 5768228.886263 | 5502200.1 | 15 | 5710546.597400 | 5602703.5 | 15 | 5653441.131426 | 5703206.8 |
| 16 | 5765344.771820 | 5507201.4 | 16 | 5707691.324101 | 5607704.7 | 16 | 5650614.410860 | 5708208.1 |
| 17 | 5762462.099434 | 5512202.6 | 17 | 5704837.478439 | 5612706.0 | 17 | 5647789.103655 | 5713209.3 |
| 18 | 5759580.868384 | 5517203.9 | 18 | 5701985.059700 | 5617707.2 | 18 | 5644965.209103 | 5718210.6 |
| 19 | 5756701.077950 | 5522205.1 | 19 | 5699134.067170 | 5622708.5 | 19 | 5642142.726499 | 5723211.8 |
| 20 | 5753822.727411 | 5527206.4 | 20 | 5696284.500137 | 5627709.7 | 20 | 5639321.655135 | 5728213.1 |


| Column 57 |  |  | Column 58 |  |  | Column 59 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 57}$ | Logarithm $l_{3}\left(c_{i, 57}\right)$ | $i$ | Number $c_{i, 58}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 58}\right) \end{gathered}$ | $i$ | $\begin{gathered} \text { Number } \\ c_{i, 59} \end{gathered}$ | Logarithm $l_{3}\left(c_{i, 59}\right)$ |
| 0 | 5639051.904524 | 5728691.4 | 0 | 5582661.385479 | 5829194.8 | 0 | 5526834.771624 | 5929698.2 |
| 1 | 5636232.378572 | 5733692.7 | 1 | 5579870.054786 | 5834196.0 | 1 | 5524071.354238 | 5934699.4 |
| 2 | 5633414.262382 | 5738693.9 | 2 | 5577080.119759 | 5839197.3 | 2 | 5521309.318561 | 5939700.7 |
| 3 | 5630597.555251 | 5743695.2 | 3 | 5574291.579699 | 5844198.5 | 3 | 5518548.663902 | 5944701.9 |
| 4 | 5627782.256474 | 5748696.4 | 4 | 5571504.433909 | 5849199.8 | 4 | 5515789.389570 | 5949703.2 |
| 5 | 5624968.365345 | 5753697.7 | 5 | 5568718.681692 | 5854201.0 | 5 | 5513031.494875 | 5954704.4 |
| 6 | 5622155.881163 | 5758698.9 | 6 | 5565934.322351 | 5859202.3 | 6 | 5510274.979127 | 5959705.7 |
| 7 | 5619344.803222 | 5763700.2 | 7 | 5563151.355190 | 5864203.5 | 7 | 5507519.841638 | 5964706.9 |
| 8 | 5616535.130820 | 5768701.4 | 8 | 5560369.779512 | 5869204.8 | 8 | 5504766.081717 | 5969708.2 |
| 9 | 5613726.863255 | 5773702.7 | 9 | 5557589.594622 | 5874206.0 | 9 | 5502013.698676 | 5974709.4 |
| 10 | 5610919.999823 | 5778703.9 | 10 | 5554810.799825 | 5879207.3 | 10 | 5499262.691827 | 5979710.7 |
| 11 | 5608114.539823 | 5783705.2 | 11 | 5552033.394425 | 5884208.5 | 11 | 5496513.060481 | 5984711.9 |
| 12 | 5605310.482554 | 5788706.4 | 12 | 5549257.377728 | 5889209.8 | 12 | 5493764.803951 | 5989713.2 |
| 13 | 5602507.827312 | 5793707.7 | 13 | 5546482.749039 | 5894211.1 | 13 | 5491017.921549 | 5994714.4 |
| 14 | 5599706.573399 | 5798708.9 | 14 | 5543709.507665 | 5899212.3 | 14 | 5488272.412588 | 5999715.7 |
| 15 | 5596906.720112 | 5803710.2 | 15 | 5540937.652911 | 5904213.6 | 15 | 5485528.276382 | 6004716.9 |
| 16 | 5594108.266752 | 5808711.4 | 16 | 5538167.184084 | 5909214.8 | 16 | 5482785.512244 | 6009718.2 |
| 17 | 5591311.212618 | 5813712.7 | 17 | 5535398.100492 | 5914216.1 | 17 | 5480044.119487 | 6014719.4 |
| 18 | 5588515.557012 | 5818713.9 | 18 | 5532630.401442 | 5919217.3 | 18 | 5477304.097428 | 6019720.7 |
| 19 | 5585721.299234 | 5823715.2 | 19 | 5529864.086241 | 5924218.6 | 19 | 5474565.445379 | 6024721.9 |
| 20 | 5582928.438584 | 5828716.4 | 20 | 5527099.154198 | 5929219.8 | 20 | 5471828.162656 | 6029723.2 |


| Column 60 |  |  | Column 61 |  |  | Column 62 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 60}$ | Logarithm $l_{3}\left(c_{i, 60}\right)$ | $i$ | Number $c_{i, 61}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 61}\right) \end{aligned}$ | $i$ | Number $c_{i, 62}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 62}\right) \end{gathered}$ |
| 0 | 5471566.423908 | 6030201.5 | 0 | 5416850.759669 | 6130704.9 | 0 | 5362682.252072 | 6231208.2 |
| 1 | 5468830.640696 | 6035202.8 | 1 | 5414142.334289 | 6135706.1 | 1 | 5360000.910946 | 6236209.5 |
| 2 | 5466096.225375 | 6040204.0 | 2 | 5411435.263122 | 6140707.4 | 2 | 5357320.910490 | 6241210.7 |
| 3 | 5463363.177263 | 6045205.3 | 3 | 5408729.545490 | 6145708.6 | 3 | 5354642.250035 | 6246212.0 |
| 4 | 5460631.495674 | 6050206.5 | 4 | 5406025.180717 | 6150709.9 | 4 | 5351964.928910 | 6251213.2 |
| 5 | 5457901.179926 | 6055207.8 | 5 | 5403322.168127 | 6155711.1 | 5 | 5349288.946446 | 6256214.5 |
| 6 | 5455172.229336 | 6060209.0 | 6 | 5400620.507043 | 6160712.4 | 6 | 5346614.301972 | 6261215.7 |
| 7 | 5452444.643222 | 6065210.3 | 7 | 5397920.196789 | 6165713.6 | 7 | 5343940.994821 | 6266217.0 |
| 8 | 5449718.420900 | 6070211.5 | 8 | 5395221.236691 | 6170714.9 | 8 | 5341269.024324 | 6271218.2 |
| 9 | 5446993.561689 | 6075212.8 | 9 | 5392523.626073 | 6175716.1 | 9 | 5338598.389812 | 6276219.5 |
| 10 | 5444270.064909 | 6080214.0 | 10 | 5389827.364260 | 6180717.4 | 10 | 5335929.090617 | 6281220.7 |
| 11 | 5441547.929876 | 6085215.3 | 11 | 5387132.450577 | 6185718.6 | 11 | 5333261.126072 | 6286222.0 |
| 12 | 5438827.155911 | 6090216.5 | 12 | 5384438.884352 | 6190719.9 | 12 | 5330594.495509 | 6291223.2 |
| 13 | 5436107.742333 | 6095217.8 | 13 | 5381746.664910 | 6195721.1 | 13 | 5327929.198261 | 6296224.5 |
| 14 | 5433389.688462 | 6100219.0 | 14 | 5379055.791577 | 6200722.4 | 14 | 5325265.233662 | 6301225.7 |
| 15 | 5430672.993618 | 6105220.3 | 15 | 5376366.263682 | 6205723.6 | 15 | 5322602.601045 | 6306227.0 |
| 16 | 5427957.657121 | 6110221.5 | 16 | 5373678.080550 | 6210724.9 | 16 | 5319941.299744 | 6311228.2 |
| 17 | 5425243.678293 | 6115222.8 | 17 | 5370991.241510 | 6215726.1 | 17 | 5317281.329094 | 6316229.5 |
| 18 | 5422531.056453 | 6120224.0 | 18 | 5368305.745889 | 6220727.4 | 18 | 5314622.688430 | 6321230.7 |
| 19 | 5419819.790925 | 6125225.3 | 19 | 5365621.593016 | 6225728.6 | 19 | 5311965.377086 | 6326232.0 |
| 20 | 5417109.881030 | 6130226.5 | 20 | 5362938.782219 | 6230729.9 | 20 | 5309309.394397 | 6331233.2 |


| Column 63 |  |  | Column 64 |  |  | Column 65 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | $\begin{gathered} \text { Number } \\ c_{i, 63} \end{gathered}$ | Logarithm $l_{3}\left(c_{i, 63}\right)$ | $i$ | Number $c_{i, 64}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 64}\right) \end{gathered}$ | $i$ | $\begin{gathered} \text { Number } \\ c_{i, 65} \end{gathered}$ | $\begin{aligned} & \text { Logarithm } \\ & l_{3}\left(c_{i, 65}\right) \\ & \hline \end{aligned}$ |
| 0 | 5309055.429551 | 6331711.6 | 0 | 5255964.875256 | 6432214.9 | 0 | 5203405.226503 | 6532718.3 |
| 1 | 5306400.901836 | 6336712.8 | 1 | 5253336.892818 | 6437216.2 | 1 | 5200803.523890 | 6537719.6 |
| 2 | 5303747.701385 | 6341714.1 | 2 | 5250710.224372 | 6442217.4 | 2 | 5198203.122128 | 6542720.8 |
| 3 | 5301095.827535 | 6346715.3 | 3 | 5248084.869259 | 6447218.7 | 3 | 5195604.020567 | 6547722.1 |
| 4 | 5298445.279621 | 6351716.6 | 4 | 5245460.826825 | 6452219.9 | 4 | 5193006.218557 | 6552723.3 |
| 5 | 5295796.056981 | 6356717.8 | 5 | 5242838.096411 | 6457221.2 | 5 | 5190409.715447 | 6557724.6 |
| 6 | 5293148.158953 | 6361719.1 | 6 | 5240216.677363 | 6462222.4 | 6 | 5187814.510590 | 6562725.8 |
| 7 | 5290501.584873 | 6366720.3 | 7 | 5237596.569024 | 6467223.7 | 7 | 5185220.603334 | 6567727.1 |
| 8 | 5287856.334081 | 6371721.6 | 8 | 5234977.770740 | 6472224.9 | 8 | 5182627.993033 | 6572728.3 |
| 9 | 5285212.405914 | 6376722.8 | 9 | 5232360.281855 | 6477226.2 | 9 | 5180036.679036 | 6577729.6 |
| 10 | 5282569.799711 | 6381724.1 | 10 | 5229744.101714 | 6482227.5 | 10 | 5177446.660697 | 6582730.8 |
| 11 | 5279928.514811 | 6386725.3 | 11 | 5227129.229663 | 6487228.7 | 11 | 5174857.937366 | 6587732.1 |
| 12 | 5277288.550554 | 6391726.6 | 12 | 5224515.665048 | 6492230.0 | 12 | 5172270.508397 | 6592733.3 |
| 13 | 5274649.906278 | 6396727.8 | 13 | 5221903.407215 | 6497231.2 | 13 | 5169684.373143 | 6597734.6 |
| 14 | 5272012.581325 | 6401729.1 | 14 | 5219292.455512 | 6502232.5 | 14 | 5167099.530957 | 6602735.8 |
| 15 | 5269376.575034 | 6406730.3 | 15 | 5216682.809284 | 6507233.7 | 15 | 5164515.981191 | 6607737.1 |
| 16 | 5266741.886747 | 6411731.6 | 16 | 5214074.467879 | 6512235.0 | 16 | 5161933.723201 | 6612738.3 |
| 17 | 5264108.515804 | 6416732.8 | 17 | 5211467.430646 | 6517236.2 | 17 | 5159352.756339 | 6617739.6 |
| 18 | 5261476.461546 | 6421734.1 | 18 | 5208861.696930 | 6522237.5 | 18 | 5156773.079961 | 6622740.8 |
| 19 | 5258845.723315 | 6426735.3 | 19 | 5206257.266082 | 6527238.7 | 19 | 5154194.693421 | 6627742.1 |
| 20 | 5256216.300453 | 6431736.6 | 20 | 5203654.137449 | 6532240.0 | 20 | 5151617.596074 | 6632743.3 |


| Column 66 |  |  | Column 67 |  |  | Column 68 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $i$ | Number $c_{i, 66}$ | Logarithm $l_{3}\left(c_{i, 66}\right)$ | $i$ | $\begin{gathered} \text { Number } \\ c_{i, 67} \end{gathered}$ | $\begin{gathered} \text { Logarithm } \\ l_{3}\left(c_{i, 67}\right) \\ \hline \end{gathered}$ | $i$ | Number $c_{i, 68}$ | Logarithm $l_{3}\left(c_{i, 68}\right)$ |
| 0 | 5151371.174238 | 6633221.7 | 0 | 5099857.462496 | 6733725.0 | 0 | 5048858.887871 | 6834228.4 |
| 1 | 5148795.488651 | 6638222.9 | 1 | 5097307.533764 | 6738726.3 | 1 | 5046334.458427 | 6839229.6 |
| 2 | 5146221.090907 | 6643224.2 | 2 | 5094758.879998 | 6743727.5 | 2 | 5043811.291198 | 6844230.9 |
| 3 | 5143647.980361 | 6648225.4 | 3 | 5092211.500558 | 6748728.8 | 3 | 5041289.385552 | 6849232.1 |
| 4 | 5141076.156371 | 6653226.7 | 4 | 5089665.394807 | 6753730.0 | 4 | 5038768.740859 | 6854233.4 |
| 5 | 5138505.618293 | 6658227.9 | 5 | 5087120.562110 | 6758731.3 | 5 | 5036249.356489 | 6859234.6 |
| 6 | 5135936.365484 | 6663229.2 | 6 | 5084577.001829 | 6763732.5 | 6 | 5033731.231811 | 6864235.9 |
| 7 | 5133368.397301 | 6668230.4 | 7 | 5082034.713328 | 6768733.8 | 7 | 5031214.366195 | 6869237.1 |
| 8 | 5130801.713102 | 6673231.7 | 8 | 5079493.695971 | 6773735.0 | 8 | 5028698.759011 | 6874238.4 |
| 9 | 5128236.312246 | 6678232.9 | 9 | 5076953.949123 | 6778736.3 | 9 | 5026184.409632 | 6879239.6 |
| 10 | 5125672.194090 | 6683234.2 | 10 | 5074415.472149 | 6783737.5 | 10 | 5023671.317427 | 6884240.9 |
| 11 | 5123109.357993 | 6688235.4 | 11 | 5071878.264413 | 6788738.8 | 11 | 5021159.481768 | 6889242.1 |
| 12 | 5120547.803314 | 6693236.7 | 12 | 5069342.325280 | 6793740.0 | 12 | 5018648.902028 | 6894243.4 |
| 13 | 5117987.529412 | 6698237.9 | 13 | 5066807.654118 | 6798741.3 | 13 | 5016139.577577 | 6899244.6 |
| 14 | 5115428.535647 | 6703239.2 | 14 | 5064274.250291 | 6803742.5 | 14 | 5013631.507788 | 6904245.9 |
| 15 | 5112870.821379 | 6708240.4 | 15 | 5061742.113166 | 6808743.8 | 15 | 5011124.692034 | 6909247.1 |
| 16 | 5110314.385969 | 6713241.7 | 16 | 5059211.242109 | 6813745.0 | 16 | 5008619.129688 | 6914248.4 |
| 17 | 5107759.228776 | 6718242.9 | 17 | 5056681.636488 | 6818746.3 | 17 | 5006114.820123 | 6919249.6 |
| 18 | 5105205.349161 | 6723244.2 | 18 | 5054153.295670 | 6823747.5 | 18 | 5003611.762713 | 6924250.9 |
| 19 | 5102652.746487 | 6728245.4 | 19 | 5051626.219022 | 6828748.8 | 19 | 5001109.956832 | 6929252.1 |
| 20 | 5100101.420113 | 6733246.7 | 20 | 5049100.405912 | 6833750.0 | 20 | 4998609.401853 | 6934253.4 |


[^0]:    *This document is part of the LOCOMAT project, the LORIA Collection of Mathematical Tables: http://www.loria.fr/~roegel/locomat.html.
    ${ }^{1}$ Among his many activities and interests, Napier also devoted a lot of time to a commentary of Saint John's Revelation, which was published in 1593. One author went so far as writing that Napier "invented logarithms in order to speed up his calculations of the Number of the Beast." [40]
    ${ }^{2}$ It is possible that Napier knew of some of Bürgi's work on the computation of sines, through Ursus' Fundamentum astronomicum (1588) [149].
    ${ }^{3}$ In 1599 , prior to Napier, logarithms had actually been used implicitely by Wright, but without Wright realizing that he had done so, and without using them to simplify calculations [33]. Therefore, Wright can not (and did not) lay claim on a prior discovery. See Wedemeyer's article [201] for additional information.

[^1]:    ${ }^{4}$ Latin editions of the Descriptio were published in 1614 [131], 1619, 1620 [134] (google books, id: QZ4_AAAAcAAJ), 1658, 1807 [110], and 1857. English editions were published in 1616, 1618, and 1857 (by Filipowski) [135]. Latin editions of the Constructio were published in 1619 [133], 1620 [134] (google books, id: VukHAQAAIAAJ, UJ4_AAAAcAAJ), and 1658. Some of the copies of the edition published in 1620 in Lyon carry the year 1619. The 1658 edition is a remainder of the 1619-1620 editions, with title page and preliminary matter reset [108]. An English edition of the Constructio was published in 1889 [136]. In addition, Archibald cites a 1899 edition of the two Latin texts [4, p. 187], and he probably refers to Gravelaar's description of Napier's works [71], but as a matter of fact, the latter only contains excerpts of the Descriptio and Constructio. More recently, W. F. Hawkins also gave English translations of both works [78] in his thesis. Then, Jean Peyroux published a French translation of the Descriptio in 1993 [96]. Finally, Ian Bruce made new translations which are available on the web at http://www. 17 centurymaths.com. For extensive details on the various editions of Napier's works known by 1889, see Macdonald [136, pp. 101-169].
    ${ }^{5}$ Throughout this article, we consider an idealization of Napier's calculations, assuming that some values were computed exactly. In fact, Napier's values were rounded, and the present results would be slightly different in a presentation of Napier's actual construction. A very detailed analysis of these problems was given by Fischer [49]. We might provide a table based on Napier's rounding methods in the future.

[^2]:    ${ }^{6}$ It seems that Napier had heard from the method of prosthaphaeresis for the simplification of trigonometric computations through John Craig who had obtained it from Paul Wittich in Frankfurt at the end of the 1570s [62, pp. 11-12]. The same Wittich also influenced Tycho Brahe and Jost Bürgi, who came close to discovering logarithms [166]. This connection with John Craig has been misinterpreted as meaning that Longomontanus invented logarithms which would then have been copied by Napier [33, p. 99-101]. On the unfairness of several authors towards Napier, see Cajori [33, pp. 107-108]. Perhaps the extreme case of unfairness is that of Jacomy-Régnier who claims that Napier had his calculating machine made by Bürgi himself and that in return the shy Bürgi told him of his invention of logarithms [88, p. 53].
    ${ }^{7}$ For example, Carslaw is misled by the apparent meaning of "logarithm," although he observes that Napier's construction does not absolutely agree with this meaning [37, p. 82].

[^3]:    ${ }^{8}$ We have borrowed some of the notations from Michael Lexa's introduction [103].

[^4]:    ${ }^{9}$ Many works have been concerned with proportions before Napier, and we can mention in particular the Portuguese Alvarus Thomas who in 1509 used a geometric progression to divide a line [187, 33].
    ${ }^{10}$ For an overview of various bases which have been ascribed to Napier's logarithms, see Matzka [111, 112], Cantor [35, p. 736], Tropfke [190, p. 150], and Mautz [113, pp. 6-7]. Many other authors addressed this question, and the reader should consult the references cited at the end of this article.

[^5]:    ${ }^{11}$ Some of our notations for the sequences appear to be identical, or nearly identical, to those of Carslaw [37, pp. 79-81], but this is a mere coincidence. For some reason, Carslaw did not name the values of the third table, except those of the first line and first column.
    ${ }^{12}$ Even without the details of Napier's calculations, and even without Napier's manuscript, it is probably possible to pinpoint the error more precisely, assuming that only one error was made, but we have not made any attempt, and we are not aware of any other analysis of the error. We should stress here that if we had used Napier's rounding, we would not have obtained $b_{50}=9995001.224804$ but $b_{50}=9995001.224826$, as explained by Fischer [49]. The rounded computation will be given in a future document.
    ${ }^{13}$ In § 60 of the Constructio, Napier observes some inconsistencies in the table and suggests a way to improve it, without noticing that at least part of the problem lies in a computation error.

[^6]:    ${ }^{14}$ Again, in this work, we consider ideal computations and we postpone the examination of rounded calculations to a separate work. Using Napier's rounding, he should actually have obtained $c_{20,0}=9900473.57811$ and the other values of the table should also slightly be different.

[^7]:    ${ }^{15}$ What Napier actually does is interval arithmetic. This field was only developed extensively in the 20th century, starting with Young's seminal work [204]. For a popular introduction to interval arithmetic, see Hayes' article in the American Scientist [79].

[^8]:    ${ }^{16}$ Some authors write incorrectly that Napier took the logarithm of 9999999 to be 1. This is for instance the case of Delambre [42, vol. 1, p. xxxv]. The logarithm of 9999999 may have been 1 in Napier's first experiments, but not in the actual Descriptio.

[^9]:    ${ }^{17} l^{\prime}$ slightly differs from $l$ as a consequence of Napier's errors or rounding. $l$ differs from $\lambda_{n}$ because Napier took the arithmetic mean when approximating the values of the logarithms.

[^10]:    ${ }^{18}$ The difference is essentially due to the error on $b_{50}$, when constructing the sequence $b_{i}$.

[^11]:    ${ }^{19}$ Ideal, if Napier hadn't made any error.

[^12]:    ${ }^{20}$ One should be careful, because the 1604 edition of Lansberge does not have the same sine table as the 1591 edition. Only the 1591 edition agrees with Napier.
    ${ }^{21}$ Detailed accounts of the history of trigonometric tables can be found in particular in the works of Braunmühl [199], Zeller [205], and van Brummelen [191]. See also Lüneburg's section on sine tables [105, pp. 148-162].

[^13]:    ${ }^{22}$ Bower shows for instance how $\lambda_{n}(1)$ can be computed using the methods set forth in the 1614 and 1616 editions of the Descriptio. Because of Napier's slight errors in his tables, and because of other approximations, the use of Napier's original tables would give a slightly different value, as explained by Bower [18]. See also Sommerville [178].

[^14]:    ${ }^{23}$ For instance, Wright gives $\lambda_{w}\left(\sin _{w} 2^{\prime}\right)=7449419, \lambda_{w}\left(\sin _{w} 89^{\circ} 58^{\prime}\right)=.2$ and the differential as 7449421 . The three first lines of the table show such idiosyncrasies, but perhaps they are only printing errors.
    ${ }^{24}$ One of the earliest example is John of Murs in his Quadripartitum numerorum completed in 1343.
    ${ }^{25}$ For instance, Viète used decimal fractions in his Universalium Inspectionum (1579).

[^15]:    ${ }^{26}$ Cantor was one of the authors who wrote that Pitiscus' 1608 and 1612 tables exhibited the use of the dot as a separator of a decimal part [35, pp. 617-619]. This, however, is not true. Even a cursory examination of these two tables shows that they contain indeed dots, but these dots are separating groups of digits, and if some of them separate the decimal part, they can of course not be ascribed that sole meaning.
    ${ }^{27}$ It should be noted that Maseres' reprint of the Descriptio of course retypeset it, and the layout is slightly different. Maseres' main change was to group the figures by the introduction of commas, which do not appear in the original version [110]. Maseres gives for instance $\sin _{m} 43^{\circ} 30^{\prime}=6,883,546$ and $\lambda_{m}\left(\sin _{m} 43^{\circ} 30^{\prime}\right)=7,734,510$.
    ${ }^{28}$ Wright must have used decimal points only when the logarithms were too small, and when he felt that rounding the values to an integer would produce a too great loss of accuracy.

[^16]:    ${ }^{29}$ So, Wright has $\lambda_{w}(a)=545471-0, \lambda_{w}\left(\sin _{w} C\right)=824689, \lambda_{w}(c)=1335492-0$, from which he deduces $\lambda_{w}\left(\sin _{w} A\right)=34668$. Wright's translation mistakenly writes 34668-0.

[^17]:    ${ }^{30}$ For an analysis of Bürgi's tables, see our reconstruction [166]. Although Bürgi's tables can be viewed as tables of logarithms, Bürgi did not reach the abstract notion developed by Napier, and should not be considered as a co-inventor of logarithms.

[^18]:    ${ }^{31}$ In his 1909 book, Cajori first attributes the invention of the slide rule to Wingate, but then corrects himself in an addenda.

[^19]:    ${ }^{32}$ Volume napier 1614 idealdoc.
    ${ }^{33}$ Volumes napier1614doc and napier1616doc.

[^20]:    ${ }^{34}$ Note on the titles of the works: Original titles come with many idiosyncrasies and features (line splitting, size, fonts, etc.) which can often not be reproduced in a list of references. It has therefore seemed pointless to capitalize works according to conventions which not only have no relation with the original work, but also do not restore the title entirely. In the following list of references, most title words (except in German) will therefore be left uncapitalized. The names of the authors have also been homogenized and initials expanded, as much as possible.

    The reader should keep in mind that this list is not meant as a facsimile of the original works. The original style information could no doubt have been added as a note, but we have not done it here.

