Incremental moments and Hölder exponents of multifractional multistable processes

Ronan Le Guével, Jacques Lévy Véhel

▶ **To cite this version:**

Incremental moments and Hölder exponents
of multifractional multistable processes

R. Le Guével

Université Nantes, Laboratoire de Mathématiques Jean Leray UMR CNRS 6629

2 Rue de la Houssinière - BP 92208 - F-44322 Nantes Cedex 3, France

ronan.leguevel@univ-nantes.fr

and

J. Lévy Véhel

Regularity team, INRIA Saclay, Parc Orsay Université

4 rue Jacques Monod - Bat P - 91893 Orsay Cedex, France

jacques.levy-vehel@inria.fr

Abstract

Multistable processes, that is, processes which are, at each "time", tangent to a stable process, but where the index of stability varies along the path, have been recently introduced as models for phenomena where the intensity of jumps is non constant. In this work, we give further results on (multifractional) multistable processes related to their local structure. We show that, under certain conditions, the incremental moments display a scaling behaviour, and that the pointwise Hölder exponent is, as expected, related to the local stability index. We compute the precise value of the almost sure Hölder exponent in the case of the multistable Lévy motion, which turns out to reveal an interesting phenomenon.

Keywords: localisable processes, multistable processes, multifractional processes, pointwise Hölder regularity.

AMS Subject Classification: 60G17; 60G18; 60G22; 60G52.

1   Introduction

Multistable processes are stochastic processes which are "locally stable", but where the index of stability varies with "time". To be more precise, we need to recall the definition

of a *localisable process* [5, 6]: $Y = \{Y(t) : t \in \mathbf{R}\}$ is said to be $h-$localisable at u if there exists an $h \in \mathbf{R}$ and a non-trivial limiting process Y'_u such that

$$\lim_{r \to 0} \frac{Y(u+rt) - Y(u)}{r^h} = Y'_u(t). \tag{1.1}$$

(Note Y'_u may and in general will vary with u.) When the limit exits, $Y'_u = \{Y'_u(t) : t \in \mathbf{R}\}$ is termed the *local form* or tangent process of Y at u. The limit (1.1) may be taken in mainly two ways: convergence in finite dimensional distributions, or in distribution (in which case the process is called *strongly h-localisable*). In the sequel, equality in finite dimensional distributions will be denoted $\stackrel{fdd}{=}$, and equality in distributions $\stackrel{d}{=}$.

A classical example of localisable process is multifractional Brownian motion Y [1, 3, 11, 19] which "looks like" index-$h(u)$ fractional Brownian motion close to time u but where $h(u)$ varies, that is

$$\lim_{r \to 0} \frac{Y(u+rt) - Y(u)}{r^h} = B_{h(u)}(t) \tag{1.2}$$

where B_h is index-h fractional Brownian motion. A generalization of mBm, where the Gaussian measure is replaced by an α-stable one, leads to multifractional stable processes, where the local form is an $h(u)$-self-similar linear α-stable motion [23, 24].

Multifractional multistable processes provide a further step of generalization: they are localisable processes such that the tangent process is again an α-stable random process, but where α now varies with time. Multifractional multistable processes were constructed in [7, 8, 14, 9] using respectively moving averages, sums over Poisson processes, the Ferguson - Klass - LePage series representation, and multistable measures. Section 3.3 below provides several specific examples of such processes.

The aim of this work is twofold:

1. We show that, for a large class of (multifractional) multistable processes, a precise estimate for the incremental moments holds. More precisely, we prove in section 3.1 that there exists a natural scaling relation for $\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^\eta\right]$ and ε small. This class includes (multifractional) multistable processes considered in [8, 14], in particular Lévy multistable motions and linear multistable multifractional motions.

2. We then study the pointwise Hölder regularity of (multifractional) multistable processes. For the same class as above, we obtain an almost sure upper bound for this exponent. In the case of the Lévy multistable motion, we are able to compute its exact value. An interesting phenomenon occurs: when the functional parameter α is smooth, not surprisingly, the Hölder exponent is equal, at each point, almost surely, to the localisability index. However, when α is smaller than one and sufficiently irregular, the regularity of the process is governed by the one of α: their Hölder exponent coincide. Note that a uniform statement, *i.e.* a statement like "almost surely, at each point", cannot hold true in general. Indeed, it already fails for the case of a Lévy stable motion. The right frame in this respect is multifractal analysis, and results in this direction will be presented in a forthcoming work.

The remainder of this work is organized as follows. In the next section, we recall the definition of multistable processes based on the Ferguson - Klass - LePage series

representation used in [14] (this defines processes which are equal in distribution to the ones obtained in [8] through sums over Poisson processes). Our main results on incremental moments and upper bound for the pointwise Hölder exponents are described in subsections 3.1 and 3.2. Subsection 3.3 applies these findings to the linear multistable multifractional motion. In subsection 3.4, we state the result giving the exact value of the pointwise Hölder regularity of the Lévy multistable motion. In section 4, we give intermediate results, some of which being of independent interest, which are used in the proofs of the main statements. Section 5 gathers technical results followed by the proofs of the statements related with the incremental moments and upper bounds on the exponents. Section 6 contains the computation of the exponent for the multistable Lévy motion. Finally, section 7 gives a list of the various technical conditions on multistable processes required by our approach so that their incremental moments and Hölder exponents may be estimated.

2 Multistable processes

Our results will apply to certain processes that are defined as "diagonals" of random fields that we describe in subsections 2.1 and 2.2.

2.1 Finite measure space case

Let $(E, \mathcal{E}, m)$ be a finite measure space, and U be an open interval of $\mathbf{R}$. Let α be a C^1 function defined on U and ranging in $[c, d] \subset (0, 2)$. Let b be a C^1 function defined and bounded on U. Let $f(t, u, .)$ be a family of functions such that, for all $(t, u) \in U^2$, $f(t, u, .) \in \mathcal{F}_{\alpha(u)}(E, \mathcal{E}, m)$. Let $(\Gamma_i)_{i \geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m} = m/m(E)$ on E, and $(\gamma_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i \geq 1}$, $(V_i)_{i \geq 1}$, and $(\gamma_i)_{i \geq 1}$ are independent. As in [14], we will consider the following random field:

$$X(t, u) = b(u)(m(E))^{1/\alpha(u)} C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} f(t, u, V_i), \qquad (2.3)$$

where $C_\eta = \left(\int_0^\infty x^{-\eta} \sin(x) dx \right)^{-1}$.

Note that when the function α is constant, then (2.3) is just the Ferguson - Klass - LePage series representation of a stable random variable (see [4, 10, 15, 16, 21] and [22, Theorem 3.10.1] for the specific properties of this representation that will be needed here).

2.2 σ-finite measure space case

When the space E has infinite measure, one cannot use definition (2.3), since it is no longer possible to renormalize by $m(E)$. However, in the σ-finite case, one may always perform a change of measure that allows to reduce to the finite case, as explained in [22] proposition 3.11.3 (see also section 4 of [14]). In our frame, this simply means adding a term involving the change of measure in the definition of the field.

Let $(E, \mathcal{E}, m)$ be a σ-finite measure space and U be an open interval of $\mathbf{R}$. Let $r : E \to \mathbb{R}_+$ be such that $\hat{m}(dx) = \frac{1}{r(x)} m(dx)$ is a probability measure. Let α be a C^1

function defined on U and ranging in $[c,d] \subset (0,2)$. Let b be a C^1 function defined and bounded on U. Let $f(t,u,.)$ be a family of functions such that, for all $(t,u) \in U^2$, $f(t,u,.) \in \mathcal{F}_{\alpha(u)}(E, \mathcal{E}, m)$. Let $(\Gamma_i)_{i \geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}$ on E, and $(\gamma_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i \geq 1}$, $(V_i)_{i \geq 1}$, and $(\gamma_i)_{i \geq 1}$ are independent. We consider again a random field:

$$X(t,u) = b(u) C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} r(V_i)^{1/\alpha(u)} f(t,u,V_i), \qquad (2.4)$$

with C_α as above.

2.3 The diagonal processes

Multistable processes are obtained by taking diagonals on X, *i.e.* defining $Y(t) = X(t,t)$, both in the finite and σ-finite measure space cases. Indeed, as shown in Theorems 3.3 and 4.5 of [14], provided some conditions are satisfied both by X and by the function f, Y will be a localisable process whose local form is a stable process. In the remaining of this work, we obtain, under certain assumptions (which imply that Y is indeed localisable), estimates on the incremental moments and the pointwise Hölder regularity of Y.

3 Main results

The three following theorems apply to a diagonal process Y defined from the field X given by (2.3) or (2.4). For convenience, the conditions required on X and the function f that appears in (2.3) and (2.4), denoted (C1), ..., (C15), are gathered in section 7.

3.1 Moments of multistable processes

Theorem 3.1 *Let $t \in \mathbf{R}$ and U be an open interval of $\mathbf{R}$ with $t \in U$. Let $\eta \in (0,c)$. Suppose that f satisfies (C1), (C2), (C3) (or (C1), (Cs2), (Cs3), (Cs4) in the σ-finite case), and (C9), and that X verifies (C5) at t. Then, when ε tends to 0,*

$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^\eta\right] \sim \varepsilon^{\eta h(t)} \mathsf{E}\left[|Y_t'(1)|^\eta\right].$$

Proof
See section 5.

Remark: Under the conditions listed in the theorem, Theorems 3.3 and 4.5 of [14] imply that Y is $h(t)-$localisable at t.

3.2 Pointwise Hölder exponent of multistable processes

Let $\mathcal{H}_t = \sup\{\gamma : \lim_{r \to 0} \frac{|Y(t+r) - Y(t)|}{|r|^\gamma} = 0\}$ denote the Hölder exponent of the (non-differentiable) process Y at t.

Theorem 3.2 *(Upper bound) Suppose that there exists a function h defined on U such that (C6), (C7), (C8), (C10), (C11), (C12), (C13), (C14) and (C15) holds for some $t \in U$. Assuming (C1), (C2), (C3), (or (C1), (Cs2), (Cs3), (Cs4) in the σ-finite case), one has:*

$$\mathcal{H}_t \leq h(t).$$

Proof

See section 5.

3.3 Example: the linear multistable multifractional motion

In this section, we apply the results above to the "multistable version" of a classical process known as the linear stable multifractional motion, which is itself a extension of the linear stable fractional motion, defined as follows (in the sequel, M will always denote a symmetric α-stable ($0 < \alpha < 2$) random measure on $\mathbf{R}$ with control measure Lebesgue measure $\mathcal{L}$):

$$L_{\alpha,H,b^+,b^-}(t) = \int_{-\infty}^{\infty} f_{\alpha,H}(b^+, b^-, t, x) M(dx)$$

where $t \in \mathbb{R}$, $H \in (0,1)$, $b^+, b^- \in \mathbb{R}$, and

$$f_{\alpha,H}(b^+, b^-, t, x) = b^+ \left((t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha} \right)$$
$$+ b^- \left((t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha} \right).$$

When $b^+ = b^- = 1$, this process is called well-balanced linear fractional α-stable motion and denoted $L_{\alpha,H}$.

The localisability of the linear fractional α-stable motion simply stems from the fact that it is $1/\alpha$-self-similar with stationary increments [6].

The multistable version of this processes was defined in [7, 8]. Its incremental moments and regularity are described by the following theorems:

Theorem 3.3 *(Linear multistable multifractional motion). Let $\alpha : \mathbf{R} \to [c,d] \subset (0,2)$ and $H : \mathbf{R} \to (0,1)$ be continuously differentiable. Let $(\Gamma_i)_{i \geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}(dx) = \frac{3}{\pi^2} \sum_{j=1}^{+\infty} j^{-2} \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)dx$ on $\mathbf{R}$, and $(\gamma_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i \geq 1}$, $(V_i)_{i \geq 1}$, and $(\gamma_i)_{i \geq 1}$ are independent and define*

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)} \left(\frac{\pi^2 j^2}{3}\right)^{1/\alpha(u)} \sum_{i,j=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} (|t-V_i|^{H(u)-1/\alpha(u)} - |V_i|^{H(u)-1/\alpha(u)}) \mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(V_i)$$

$$(3.5)$$

and the linear multistable multifractional motion

$$Y(t) = X(t,t).$$

Then for all $t \in \mathbf{R}$ and $\eta < c$, when ε tends to 0,

$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^{\eta}\right] \sim \frac{2^{\eta-1}\Gamma(1-\frac{\eta}{\alpha(t)})}{\eta \int_0^{\infty} u^{-\eta-1}\sin^2(u)du} \left(\int_{\mathbf{R}} \left||1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}}\right|^{\alpha(t)} dx\right)^{\frac{\eta}{\alpha(t)}} \varepsilon^{\eta H(t)}.$$

Proof
See section 5.

Theorem 3.4 *Let Y be the linear multistable multifractional motion defined on $\mathbf{R}$ with $H - \frac{1}{\alpha}$ a non-negative function. For all $t \in \mathbf{R}$, almost surely,*

$$\mathcal{H}_t \leq H(t).$$

Proof
See section 5.

3.4 Example: the Lévy multistable motion

In the case of the Lévy multistable motion, we are able to provide a more precise result, to the effect that, at each point, the exact almost sure value of the Hölder exponent is known. Let us first recall some definitions. With M again denoting a symmetric α-stable $(0 < \alpha < 2)$ random measure on $\mathbf{R}$ with control measure Lebesgue measure $\mathcal{L}$, we write

$$L_{\alpha}(t) := \int_0^t M(dz)$$

for α-stable Lévy motion.

The localisability of Lévy motion is a consequence of the fact that it is $1/\alpha$-self-similar with stationary increments [6]. Its multistable version and incremental moments are described in the following theorem:

Theorem 3.5 *(symmetric multistable Lévy motion). Let $\alpha : [0,1] \to [c,d] \subset (1,2)$ be continuously differentiable. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process with unit arrival time, $(V_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $\hat{m}(dx) = dx$ on $[0,1]$, and $(\gamma_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume finally that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$, and $(\gamma_i)_{i\geq 1}$ are independent and define*

$$X(t,u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \mathbf{1}_{[0,t]}(V_i) \tag{3.6}$$

and the symmetric multistable Lévy motion

$$Y(t) = X(t,t).$$

Then for all $t \in (0,1)$ and $\eta < c$, when ε tends to 0,

$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^{\eta}\right] \sim \frac{2^{\eta-1}\Gamma(1-\frac{\eta}{\alpha(t)})}{\eta \int_0^{\infty} u^{-\eta-1}\sin^2(u)du} \varepsilon^{\frac{\eta}{\alpha(t)}}.$$

6

Proof
See section 5.

Theorem 3.6 *Let Y be the symmetric multistable Lévy motion defined on $(0, 1)$ with $\alpha : [0, 1] \to [c, d] \subset (0, 2)$. For all $t \in (0, 1)$, almost surely,*

$$\mathcal{H}_t \leq \frac{1}{\alpha(t)}.$$

Proof
See section 5.

Theorem 3.7 *Let $u \in U \subset (0, 1)$.*

1. *If $0 < \alpha(u) < 1$, almost surely,*

$$\mathcal{H}_u = \min\left(\frac{1}{\alpha(u)}, \mathcal{H}_u^\alpha\right),$$

where $\mathcal{H}_u^\alpha$ denotes the Hölder exponent of α at u, at least when $\frac{1}{\alpha(u)} \neq \mathcal{H}_u^\alpha$.

2. *If $1 \leq \alpha(u) < 2$, and α is $\mathcal{C}^1$, almost surely,*

$$\mathcal{H}_u = \frac{1}{\alpha(u)}.$$

Proof
See section 6.

Thus, in the case $0 < \alpha(u) < 1$, the regularity of the multistable Lévy motion is the smallest number between $\frac{1}{\alpha(u)}$ and the regularity of the function α at u. This is very similar to the case of the multifractional Brownian motion, where the Hölder exponent is the minimum between the functional parameter h and its regularity [11, 12]. We conjecture that the same result holds even when $\alpha \geq 1$.

4 Intermediate results

Let φ_X denote the characteristic function of the random variable X. We first state the following almost obvious fact :

Proposition 4.8 *Assume that for a given $t \in \mathbf{R}$ there exists $\varepsilon_0 > 0$ such that*

$$\sup_{r \in B(0, \varepsilon_0)} \int_0^{+\infty} \left| \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \right| dv < +\infty,$$

where Y is a symmetrical process. Then there exists $K > 0$ which depends only on t and ε_0 such that for all $x > 0$, and all $r \in (0, \varepsilon_0)$,

$$\mathsf{P}\left(|Y(t+r) - Y(t)| < x\right) \leq K \frac{x}{r^{h(t)}}.$$

If furthermore we suppose that $\sup_{t \in U} \sup_{r \in B(0, \varepsilon_0)} \int_0^{+\infty} \left| \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \right| dv < +\infty$, then for all $t \in U$, for all $r \in (0, \varepsilon_0)$, $\mathsf{P}\left(|Y(t+r) - Y(t)| < x\right) \leq K \frac{x}{r^{h(t)}}$.

Proof

This is a straightforward consequence of the inversion formula. Let $x > 0$ and $r < \varepsilon_0$. Since Y is a symmetrical process, $\varphi_{Y(t+r)-Y(t)}$ is an even function and

$$
\begin{aligned}
\mathsf{P}\left(|Y(t+r) - Y(t)| < x\right) &= \frac{1}{\pi} \left| \int_0^{+\infty} \varphi_{Y(t+r)-Y(t)}\left(\frac{v}{r^{h(t)}}\right) \sin\left(\frac{vx}{r^{h(t)}}\right) \frac{dv}{v} \right| \\
&\leq \frac{1}{\pi} \frac{x}{r^{h(t)}} \sup_{r \in B(0,\varepsilon_0)} \int_0^{+\infty} \left| \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \right| dv \\
&\leq K \frac{x}{r^{h(t)}} \blacksquare
\end{aligned}
$$

We now consider multistable processes, first in the finite measure space case, and then in the σ-finite measure space case:

Proposition 4.9 *Assuming (C1), (C2) and (C3), there exists $K_U > 0$ such that for all $u \in U$, $v \in U$ and $x > 0$,*

$$
\mathsf{P}\left(|X(v,v) - X(v,u)| > x\right) \leq K_U \left(\frac{|v-u|^d}{x^d}(1 + |\log \frac{|v-u|}{x}|^d) + \frac{|v-u|^c}{x^c}(1 + |\log \frac{|v-u|}{x}|^c) \right).
$$

Proof

See section 5.

In the σ-finite space case, a similar property holds:

Proposition 4.10 *Assuming (C1), (Cs2), (Cs3) and (Cs4), there exists $K_U > 0$ such that for all $u \in U$, $v \in U$ and $x > 0$,*

$$
\mathsf{P}\left(|X(v,v) - X(v,u)| > x\right) \leq K_U \left(\frac{|v-u|^d}{x^d}(1 + |\log \frac{|v-u|}{x}|^d) + \frac{|v-u|^c}{x^c}(1 + |\log \frac{|v-u|}{x}|^c) \right).
$$

Proof

We shall apply Proposition 4.9 to the function $g(t,w,x) = r(x)^{1/\alpha(w)} f(t,w,x)$ on $(E, \mathcal{E}, \hat{m})$.

- By (C1), the family of functions $v \to f(t,v,x)$ is differentiable for all (v,t) in U^2 and almost all x in E thus $v \to g(t,v,x)$ is differentiable too i.e (C1) holds for g.

- Choose $\delta > \frac{d}{c} - 1$ such that (Cs2) holds.

$$
\sup_{w \in U}(|g(t,w,x)|^{\alpha(w)}) = r(x) \sup_{w \in U}(|f(t,w,x)|^{\alpha(w)})
$$

  One has

$$
\begin{aligned}
\int_{\mathbf{R}} \left[\sup_{w \in U}(|g(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx) &= \int_{\mathbf{R}} r(x)^{1+\delta} \left[\sup_{w \in U}(|f(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx) \\
&= \int_{\mathbf{R}} \left[\sup_{w \in U}(|f(t,w,x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^{\delta} m(dx)
\end{aligned}
$$

  thus (C2) holds.

- Choose $\delta > \frac{d}{c} - 1$ such that (Cs3) and (Cs4) hold.

$$g'_u(t, w, x) = r(x)^{1/\alpha(w)}(f'_u(t, w, x) - \frac{\alpha'(w)}{\alpha^2(w)} \log(r(x)) f(t, w, x))$$

and

$$\int_{\mathbf{R}} \left[\sup_{w \in U}(|g'_u(t, w, x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx)$$

$$\leq \int_{\mathbf{R}} \left[\sup_{w \in U} \left[|f'_u(t, w, x) - \frac{\alpha'(w)}{\alpha^2(w)} \log(r(x)) f(t, w, x)|^{\alpha(w)} \right] \right]^{1+\delta} r(x)^\delta m(dx).$$

The inequality $|a + b|^\delta \leq \max(1, 2^{\delta-1})(|a|^\delta + |b|^\delta)$ shows that (C3) holds.

Proposition 4.9 allows to conclude.

∎

Proposition 4.11 *We suppose that there exists a function h defined on U such that (C8), (C10) and (C14) hold. Assuming (C1), (C6), (C7), (C11), (C12), (C13), (C15), one has:*

$$\sup_{r \in B(0,\varepsilon)} \int_0^{+\infty} \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) dv < +\infty.$$

If in addition we suppose (Cu8), (Cu10), (Cu11), (Cu12), (Cu14) and (Cu15), then

$$\sup_{t \in U} \sup_{r \in B(0,\varepsilon)} \int_0^{+\infty} \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) dv < +\infty.$$

Proof
The expression of the characteristic function $\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}$ is given in [14] :

$$\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) = \exp\left(-2 \int_{\mathbf{R}} \int_0^{+\infty} \sin^2\left(\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r, t+r, x)}{2r^{h(t)} y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)} f(t, t, x)}{2r^{h(t)} y^{1/\alpha(t)}} \right) dy\, m(dx) \right).$$

For $v \leq 1$, $\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \leq 1$. For $v \geq 1$, we fix $\varepsilon < \frac{1}{d}$. Lemma (5.14) entails that there exists $K_U > 0$ such that

$$\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \leq \exp\left(-\int_{\mathbf{R}} \int_{\frac{K_U v^{\frac{d}{1-\varepsilon d}}}{r}} \left| \frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r, t+r, x)}{2r^{h(t)} y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)} f(t, t, x)}{2r^{h(t)} y^{1/\alpha(t)}} \right|^2 dy\, m(dx) \right).$$

Let

$$N(v, t, r) = \int_{\mathbf{R}} \int_{\frac{K_U v^{\frac{d}{1-\varepsilon d}}}{r}} \left| \frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r, t+r, x)}{2r^{h(t)} y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)} f(t, t, x)}{2r^{h(t)} y^{1/\alpha(t)}} \right|^2 dy\, m(dx).$$

Using lemma (5.15), there exist $K_U > 0$ and $\varepsilon_0 > 0$ such that for all $v \geq 1$,

$$N(v, t, r) \geq K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})}.$$

The inequality becomes

$$\varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) \leq \exp\left(-K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})}\right),$$

and

$$\int_0^{+\infty} \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v)dv \;\leq\; 1+\int_1^{\infty} \exp\left(-K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})}\right) dv$$
$$<\;\; +\infty$$

■

5   Proofs and technical results

Proof of proposition 4.9

We proceed as in [14]. Note that condition **(C2)** implies that there exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t\in U} \int_{\mathbf{R}} \left[\sup_{w\in U}\left[|f(t,w,x)\log|f(t,w,x)||^{\alpha(w)}\right]\right]^{1+\delta} \hat{m}(dx) < \infty. \tag{5.7}$$

The function $u \mapsto C_{\alpha(u)}^{1/\alpha(u)}$ is a C^1 function since $\alpha(u)$ ranges in $[c,d] \subset (0,2)$. We shall denote $a(u) = (m(E))^{1/\alpha(u)} C_{\alpha(u)}^{1/\alpha(u)}$. The function a is thus also C^1. Let $(u,v) \in U^2$. We estimate:

$$X(v,v) - X(v,u) = \sum_{i=1}^{\infty} \gamma_i(\Phi_i(v) - \Phi_i(u)) + \sum_{i=1}^{\infty} \gamma_i(\Psi_i(v) - \Psi_i(u)),$$

where

$$\Phi_i(u) = a(u)i^{-1/\alpha(u)}f(v,u,V_i)$$

and

$$\Psi_i(u) = a(u)\left(\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}\right)f(v,u,V_i).$$

Thanks to the assumptions on a and f, Φ_i and Ψ_i are differentiable and one computes :

$$\Phi_i'(u) = a'(u)i^{-1/\alpha(u)}f(v,u,V_i)+a(u)i^{-1/\alpha(u)}f_u'(v,u,V_i)+a(u)\frac{\alpha'(u)}{\alpha(u)^2}\log(i)i^{-1/\alpha(u)}f(v,u,V_i),$$

and

$$\Psi_i'(u) = a'(u)\left(\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}\right)f(v,u,V_i) + a(u)\left(\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}\right)f_u'(v,u,V_i)$$

$$+a(u)\frac{\alpha'(u)}{\alpha(u)^2}\left(\log(\Gamma_i)\Gamma_i^{-1/\alpha(u)} - \log(i)i^{-1/\alpha(u)}\right)f(v,u,V_i).$$

Using the mean value theorem, there exists a sequence of independent random numbers $w_i \in (u,v)$ (or (v,u)) and a sequence of random numbers $x_i \in (u,v)$ (or (v,u)) such that:

$$X(v,u) - X(v,v) = (u-v)\sum_{i=1}^{\infty}(Z_i^1 + Z_i^2 + Z_i^3) + (u-v)\sum_{i=1}^{\infty}(Y_i^1 + Y_i^2 + Y_i^3), \tag{5.8}$$

10

where

$$Z_i^1 = \gamma_i a'(w_i) i^{-1/\alpha(w_i)} f(v, w_i, V_i),$$

$$Z_i^2 = \gamma_i a(w_i) i^{-1/\alpha(w_i)} f'_u(v, w_i, V_i),$$

$$Z_i^3 = \gamma_i a(w_i) \frac{\alpha'(w_i)}{\alpha(w_i)^2} \log(i) i^{-1/\alpha(w_i)} f(v, w_i, V_i),$$

$$Y_i^1 = \gamma_i a'(x_i) \left(\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)} \right) f(v, x_i, V_i),$$

$$Y_i^2 = \gamma_i a(x_i) \left(\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)} \right) f'_u(v, x_i, V_i),$$

$$Y_i^3 = \gamma_i a(x_i) \frac{\alpha'(x_i)}{\alpha(x_i)^2} \left(\log(\Gamma_i) \Gamma_i^{-1/\alpha(x_i)} - \log(i) i^{-1/\alpha(x_i)} \right) f(v, x_i, V_i).$$

Note that each w_i depends on a, f, α, u, v, V_i, and each x_i depends on $a, f, \alpha, u, v, V_i, \Gamma_i$ but not on γ_i. This remark will be useful in the sequel.

In [14], it is proved that each series $\sum_{i=1}^{\infty} Z_i^j$ and $\sum_{i=1}^{\infty} Y_i^j$, $j = 1, 2, 3$, converges almost surely. Let $x > 0$. We consider $\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j| > x \right)$ and $\mathsf{P}\left(|\sum_{i=1}^{\infty} Y_i^j| > x \right)$ for $j = 1, 2, 3$.

Let $\eta \in (0, \min(\frac{2c}{d} - 1, \frac{c}{d}(\delta + 1) - 1))$. Markov inequality yields

$$\begin{aligned}
\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j| > x \right) &\leq \frac{1}{x^d} \mathsf{E}\left[|\sum_{i=1}^{\infty} Z_i^j|^d \right] \\
&\leq \frac{1}{x^d} \left(\mathsf{E}\left[|\sum_{i=1}^{\infty} Z_i^j|^{d(1+\eta)} \right] \right)^{\frac{1}{1+\eta}}.
\end{aligned}$$

The random variables Z_i^j are independent with mean 0 thus, by theorem 2 of [2]:

$$\mathsf{E}\left[|\sum_{i=1}^{+\infty} Z_i^j|^{d(1+\eta)} \right] \leq 2 \sum_{i=1}^{+\infty} \mathsf{E}[|Z_i^j|^{d(1+\eta)}].$$

For $j = 1$,

$$\begin{aligned}
\mathsf{E}\left[|Z_i^1|^{d(1+\eta)} \right] &= \mathsf{E}\left[|a'(w_i)|^{d(1+\eta)} i^{-\frac{d(1+\eta)}{\alpha(w_i)}} |f(v, w_i, V_i)|^{d(1+\eta)} \right] \\
&\leq \frac{K_U}{i^{1+\eta}} \mathsf{E}\left[\left(\sup_{w \in B(u,\varepsilon)} |f(v, w, V_i)|^{\alpha(w)} \right)^{\frac{d(1+\eta)}{\alpha(w_i)}} \right] \\
&\leq \frac{K_U}{i^{1+\eta}} \mathsf{E}\left[(\sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)})^{1+\eta} + (\sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)})^{\frac{d}{c}(1+\eta)} \right] \\
&\leq \frac{K_U}{i^{1+\eta}}.
\end{aligned}$$

For $j = 2$,

$$\mathsf{E}\left[|Z_i^2|^{d(1+\eta)}\right] \;\leq\; \frac{K_U}{i^{1+\eta}}\mathsf{E}\left[(\sup_{w\in B(u,\varepsilon)}|f_u'(v,w,V_1)|^{\alpha(w)})^{1+\eta} + (\sup_{w\in B(u,\varepsilon)}|f_u'(v,w,V_1)|^{\alpha(w)})^{\frac{d}{c}(1+\eta)}\right]$$

$$\leq\; \frac{K_U}{i^{1+\eta}}.$$

For $j = 3$,

$$\mathsf{E}\left[|Z_i^3|^{d(1+\eta)}\right] \;=\; \mathsf{E}\left[|a(w_i)\frac{\alpha'(w_i)}{\alpha(w_i)^2}|^{d(1+\eta)}|f(v,w_i,V_i)|^{d(1+\eta)}\frac{(\log i)^{d(1+\eta)}}{i^{\frac{d(1+\eta)}{\alpha(w_i)}}}\right]$$

$$\leq\; K_U\frac{(\log i)^{d(1+\eta)}}{i^{1+\eta}}.$$

Finally, $\displaystyle\sup_{v\in U}\sum_{i=1}^{+\infty}\mathsf{E}\left[|Z_i^j|^{d(1+\eta)}\right] < +\infty$ thus

$$\mathsf{P}\left(|\sum_{i=1}^{\infty}Z_i^j| > x\right) \leq \frac{K_U}{x^d}.$$

We consider now $\mathsf{P}\left(|\sum_{i=1}^{\infty}Y_i^j| > x\right)$ for $j = 1, 2, 3$.

$$\mathsf{P}\left(|\sum_{i=1}^{\infty}Y_i^j| > x\right) \leq \mathsf{P}\left(|Y_1^j| \geq \frac{x}{2}\right) + \mathsf{P}\left(|\sum_{i=2}^{\infty}Y_i^j| \geq \frac{x}{2}\right).$$

Since $\mathsf{P}\left(|\sum_{i=2}^{\infty}Y_i^j| \geq \frac{x}{2}\right) \leq \frac{2^d}{x^d}\left(\mathsf{E}\left[|\sum_{i=2}^{\infty}Y_i^j|^{d(1+\eta)}\right]\right)^{\frac{1}{1+\eta}}$, we want to apply theorem 2 of [2] again. Let $S_m = \sum_{i=1}^{m}Y_i^j$ and write $Y_i^j = \gamma_i W_i^j$. Note that γ_i is independent of W_i^j and S_{i-1}.

$$\mathsf{E}\left(Y_{m+1}^j|S_m\right) \;=\; \mathsf{E}\left(\mathsf{E}(Y_{m+1}^j|S_m,W_{m+1})|S_m\right)$$

$$=\; \mathsf{E}\left(\mathsf{E}(\gamma_{m+1}W_{m+1}^j|S_m,W_{m+1})|S_m\right)$$

$$=\; \mathsf{E}\left(W_{m+1}^j\mathsf{E}(\gamma_{m+1}|S_m,W_{m+1})|S_m\right)$$

$$=\; \mathsf{E}\left(W_{m+1}^j\mathsf{E}(\gamma_{m+1})|S_m\right)$$

$$=\; 0.$$

We apply theorem 2 of [2] with $(d(1+\eta) < 2)$,

$$\mathsf{E}\left[|\sum_{i=2}^{\infty}Y_i^j|^{d(1+\eta)}\right] \leq 2\sum_{i=2}^{\infty}\mathsf{E}|Y_i^j|^{d(1+\eta)},$$

and

$$\mathsf{P}\left(|\sum_{i=1}^{\infty}Y_i^j| > x\right) \leq \mathsf{P}\left(|Y_1^j| \geq \frac{x}{2}\right) + \frac{2^d}{x^d}\left(2\sum_{i=2}^{\infty}\mathsf{E}|Y_i^j|^{d(1+\eta)}\right)^{\frac{1}{1+\eta}}.$$

12

For $j = 1$,

$$\mathsf{P}\left(|Y_1^1| \geq \frac{x}{2}\right) = \mathsf{P}\left(|a'(x_1)|^{\alpha(x_1)}|\frac{1}{\Gamma_1^{1/\alpha(x_1)}} - 1|^{\alpha(x_1)}|f(v, x_1, V_1)|^{\alpha(x_1)} \geq \frac{x^{\alpha(x_1)}}{2^{\alpha(x_1)}}\right)$$

$$\leq \mathsf{P}\left(|\frac{1}{\Gamma_1^{1/\alpha(x_1)}} - 1|^{\alpha(x_1)} \sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)} \geq K_U x^{\alpha(x_1)}\right).$$

For $x < 1$,

$$\mathsf{P}\left(|Y_1^1| \geq \frac{x}{2}\right) \leq \mathsf{P}\left(|\frac{1}{\Gamma_1^{1/\alpha(x_1)}} - 1|^{\alpha(x_1)} \sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)} \geq K_U x^d\right)$$

$$\leq \mathsf{P}\left(\{\sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)} \geq K_U x^d\} \cap \{\Gamma_1 > 1\}\right)$$

$$+ \mathsf{P}\left(\{|\frac{1}{\Gamma_1}|\Gamma_1^{1/\alpha(x_1)} - 1|^{\alpha(x_1)} \sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)} \geq K_U x^d\} \cap \{\Gamma_1 \leq 1\}\right).$$

$$\mathsf{P}\left(\{\sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)} \geq K_U x^d\} \cap \{\Gamma_1 > 1\}\right) \leq \frac{K_U}{x^d} \mathsf{E}\left(\sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)}\right)$$

$$\leq \frac{K_U}{x^d}.$$

Let $W(v, x) = \sup_{w \in B(u,\varepsilon)} |f(v, w, x)|^{\alpha(w)}$ and F_{v,V_1} be the distribution of $W(v, V_1)$.

$$\mathsf{P}\left(\{|\frac{1}{\Gamma_1}|\Gamma_1^{1/\alpha(x_1)} - 1|^{\alpha(x_1)} W(v, V_1) \geq K_U x^d\} \cap \{\Gamma_1 \leq 1\}\right) \leq \mathsf{P}\left(W(v, V_1) \geq K_U x^d \Gamma_1\right)$$

$$= \int_0^{+\infty} \mathsf{P}\left(z \geq K_U x^d \Gamma_1\right) F_{v,V_1}(dz)$$

$$= \int_0^{+\infty} \left(1 - e^{-\frac{z}{K_U x^d}}\right) F_{v,V_1}(dz)$$

$$\leq \int_0^{+\infty} \frac{z}{K_U x^d} F_{v,V_1}(dz)$$

$$\leq \frac{K_U}{x^d}.$$

For $x \geq 1$,

$$\mathsf{P}\left(|Y_1^1| \geq \frac{x}{2}\right) \leq \mathsf{P}\left(|\frac{1}{\Gamma_1^{1/\alpha(x_1)}} - 1|^{\alpha(x_1)} \sup_{w \in B(u,\varepsilon)} |f(v, w, V_1)|^{\alpha(w)} \geq K_U x^c\right)$$

$$\mathsf{P}\left(|Y_1^1| \geq \frac{x}{2}\right) \leq \frac{K_U}{x^c}.$$

For $i \geq 2$,

$$\mathsf{E}|Y_i^1|^{d(1+\eta)} = \mathsf{E}\left(|a'(x_i)|^{d(1+\eta)}|\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}|^{d(1+\eta)}\left(|f(v,x_i,V_i)|^{\alpha(x_i)}\right)^{\frac{d(1+\eta)}{\alpha(x_i)}}\right)$$

$$\leq K_U\mathsf{E}\left(i^{-\frac{d(1+\eta)}{\alpha(x_i)}}W(v,V_i)^{\frac{d(1+\eta)}{\alpha(x_i)}}|(\frac{i}{\Gamma_i})^{1/\alpha(x_i)} - 1|^{d(1+\eta)}\right)$$

$$\leq \frac{K_U}{i^{1+\eta}}\mathsf{E}\left(\left[W(v,V_i)^{1+\eta} + W(v,V_i)^{\frac{d}{c}(1+\eta)}\right]\left[|(\frac{i}{\Gamma_i})^{1/c} - 1|^{d(1+\eta)} + |(\frac{i}{\Gamma_i})^{1/d} - 1|^{d(1+\eta)}\right]\right)$$

$$\leq \frac{K_U}{i^{1+\eta}}\mathsf{E}\left(W(v,V_i)^{1+\eta} + W(v,V_i)^{\frac{d}{c}(1+\eta)}\right)\mathsf{E}\left(|(\frac{i}{\Gamma_i})^{1/c} - 1|^{d(1+\eta)} + |(\frac{i}{\Gamma_i})^{1/d} - 1|^{d(1+\eta)}\right).$$

Using the fact that $\eta \leq \delta$ and $\frac{d}{c}(1+\eta) \leq 1+\delta$,

$$\mathsf{E}\left(W(v,V_i)^{1+\eta} + W(v,V_i)^{\frac{d}{c}(1+\eta)}\right) = \mathsf{E}\left(W(v,V_1)^{1+\eta} + W(v,V_1)^{\frac{d}{c}(1+\eta)}\right)$$

$$\leq K_U,$$

$$\mathsf{E}|(\frac{i}{\Gamma_i})^{1/c} - 1|^{d(1+\eta)} \leq K_U(1 + \mathsf{E}\left((\frac{i}{\Gamma_i})^{\frac{d}{c}(1+\eta)}\right))$$

$$\leq K_U,$$

and

$$\mathsf{E}|(\frac{i}{\Gamma_i})^{1/d} - 1|^{d(1+\eta)} \leq K_U.$$

As a consequence:

$$\sup_{v\in U}\sum_{i=2}^{+\infty}\mathsf{E}|Y_i^1|^{d(1+\eta)} \leq K_U$$

and

$$\mathsf{P}\left(\left|\sum_{i=1}^{\infty}Y_i^1\right| > x\right) \leq K_U\left(\frac{1}{x^c} + \frac{1}{x^d}\right).$$

For $j = 2$, since the conditions required on (a', f) are also satisfied by $(a, f'u)$, one gets in a similar fashion

$$\mathsf{P}\left(\left|\sum_{i=1}^{\infty}Y_i^2\right| > x\right) \leq K_U\left(\frac{1}{x^c} + \frac{1}{x^d}\right).$$

For $j = 3$,

$$\mathsf{P}\left(|Y_1^3| \geq \frac{x}{2}\right) = \mathsf{P}\left(|a(x_1)\frac{\alpha'(x_1)}{\alpha(x_1)^2}\log(\Gamma_1)\Gamma_1^{-1/\alpha(x_1)}f(v,x_1,V_1)| \geq \frac{x}{2}\right)$$

$$\leq \mathsf{P}\left(K_U\frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \geq \frac{\Gamma_1}{|\log\Gamma_1|^{\alpha(x_1)}}\right).$$

14

Let $g(z) = \frac{z}{|\log z|^{\alpha(x_1)}}$, for $z < 1$.

g is a one-to-one increasing function, and for all $z < 1$ such that $z|\log z|^{\alpha(x_1)} < 1$ and $|1 + \alpha(x_1)\frac{\log|\log z|}{|\log z|}|^{\alpha(x_1)} \leq 2$,

$$g\left(z|\log z|^{\alpha(x_1)}\right) = \frac{z|\log z|^{\alpha(x_1)}}{|\log z + \alpha(x_1)\log|\log z||^{\alpha(x_1)}} \geq \frac{z}{2}$$

thus $g^{-1}(\frac{z}{2}) \leq z|\log z|^{\alpha(x_1)}$.

Fix $A > 0$ such that for all $0 < z < A$, $g^{-1}(z) \leq 2z|\log 2 + \log z|^{\alpha(x_1)}$ *i.e.*

$$g^{-1}(z) \leq K_U z|\log z|^{\alpha(x_1)}.$$

Let $B = \{K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \geq \frac{\Gamma_1}{|\log \Gamma_1|^{\alpha(x_1)}}\}$.

$$\mathsf{P}(B) = \mathsf{P}(B \cap \{\Gamma_1 > 1\}) + \mathsf{P}(B \cap \{\Gamma_1 < 1\} \cap \{0 \leq K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \leq A\})$$

$$+ \mathsf{P}(B \cap \{\Gamma_1 < 1\} \cap \{K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} > A\}).$$

Each of these three terms will be treated separately.

- $$\mathsf{P}\left(B \cap \{\Gamma_1 > 1\}\right) \leq \mathsf{P}\left(K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} |\log \Gamma_1|^{\alpha(x_1)} \geq 1\right)$$

$$\leq \mathsf{P}\left(K_U \sup_{w \in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}(|\log \Gamma_1|^c + |\log \Gamma_1|^d) \geq x^{\alpha(x_1)}\right).$$

For $x \geq 1$,

$$\mathsf{P}\left(B \cap \{\Gamma_1 > 1\}\right) \leq \mathsf{P}\left(K_U \sup_{w \in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}(|\log \Gamma_1|^c + |\log \Gamma_1|^d) \geq x^c\right)$$

$$\leq \frac{K_U}{x^c} \mathsf{E}\left(\sup_{w \in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}\right) \mathsf{E}\left(|\log \Gamma_1|^c + |\log \Gamma_1|^d\right)$$

$$\leq \frac{K_U}{x^c}.$$

For $x < 1$,

$$\mathsf{P}\left(B \cap \{\Gamma_1 > 1\}\right) \leq \mathsf{P}\left(K_U \sup_{w \in B(u,\varepsilon)} |f(v,w,V_1)|^{\alpha(w)}(|\log \Gamma_1|^c + |\log \Gamma_1|^d) \geq x^d\right)$$

$$\leq \frac{K_U}{x^d}.$$

- $$\mathsf{P}\left(B \cap \{\Gamma_1 < 1\} \cap \{K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} > A\}\right) \leq \mathsf{P}\left(K_U |f(v,x_1,V_1)|^{\alpha(x_1)} \geq A x^{\alpha(x_1)}\right)$$

$$\leq \frac{K_U}{x^c} + \frac{K_U}{x^d}.$$

- $\mathsf{P}\left(B \cap \{\Gamma_1 < 1\} \cap \{0 \leq K_U \dfrac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \leq A\}\right)$

$$= \mathsf{P}\left(\{g(\Gamma_1) \leq K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}\} \cap \{\Gamma_1 < 1\} \cap \{0 \leq K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \leq A\}\right)$$

$$\leq \mathsf{P}\left(\Gamma_1 \leq K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} + K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \left|\log \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}}\right|^{\alpha(x_1)}\right)$$

$$\leq \mathsf{P}\left(\Gamma_1 \leq K_U |f(v,x_1,V_1)|^{\alpha(x_1)}(\frac{1 + |\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}}) + K_U \frac{||f(v,x_1,V_1)|\log|f(v,x_1,V_1)||^{\alpha(x_1)}}{x^{\alpha(x_1)}}\right).$$

With $W(v,x) = \sup_{w \in B(u,\varepsilon)} |f(v,w,x)|^{\alpha(w)}$ and $Z(v,x) = \sup_{w \in B(u,\varepsilon)} |f(v,w,x)\log|f(v,w,x)||^{\alpha(w)}$,

$$\mathsf{P}\left(B \cap \{\Gamma_1 < 1\} \cap \{0 \leq K_U \frac{|f(v,x_1,V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \leq A\}\right)$$

$$\leq \mathsf{P}\left(\Gamma_1 \leq K_U W(v,V_1)(\frac{1 + |\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}}) + K_U \frac{Z(v,V_1)}{x^{\alpha(x_1)}}\right)$$

$$\leq \mathsf{P}\left(\Gamma_1 \leq K_U W(v,V_1)(\frac{1 + |\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})\right) + \mathsf{P}\left(\Gamma_1 \leq K_U Z(v,V_1)(\frac{1 + |\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})\right).$$

Since $\frac{1 + |\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \leq K_U(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d})$,

$$\mathsf{P}\left(\Gamma_1 \leq K_U W(v,V_1)(\frac{1 + |\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})\right) \leq \mathsf{P}\left(\Gamma_1 \leq K_U W(v,V_1)(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d})\right)$$

$$\leq K_U(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d}),$$

and

$$\mathsf{P}\left(\Gamma_1 \leq K_U Z(v,V_1)(\frac{1 + |\log x|^{\alpha(x_1)}}{x^{\alpha(x_1)}})\right) \leq \mathsf{P}\left(\Gamma_1 \leq K_U Z(v,V_1)(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d})\right).$$

Denoting G_{v,V_1} the distribution of $Z(v,V_1)$,

$$\mathsf{P}\left(\Gamma_1 \leq K_U Z(v,V_1)(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d})\right)$$

$$= \int_0^{+\infty} (1 - \exp(-K_U(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d})z)) G_{v,V_1}(dz)$$

$$\leq K_U(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d}) \int_0^{+\infty} z G_{v,V_1}(dz)$$

$$\leq K_U(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d}),$$

16

since $\sup_{v \in B(u,\varepsilon)} \mathsf{E}\left(Z(v, V_1)\right) < +\infty$.

Finally,

$$\mathsf{P}\left(B \cap \{\Gamma_1 < 1\} \cap \{0 \le K_U \frac{|f(v, x_1, V_1)|^{\alpha(x_1)}}{x^{\alpha(x_1)}} \le A\}\right) \le K_U(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d})$$

and

$$\mathsf{P}\left(|Y_1^3| \ge \frac{x}{2}\right) \le K_U(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d}).$$

For $i \ge 2$,

$$
\begin{aligned}
\mathsf{E}|Y_i^3|^{d(1+\eta)} &\le& K_U \frac{|\log i|^{d(1+\eta)}}{i^{1+\eta}} \mathsf{E}\left(W(v, V_i)^{1+\eta} + W(v, V_i)^{\frac{d}{c}(1+\eta)}\right) \mathsf{E}\left(|\frac{\log \Gamma_i}{\log i}||(\frac{i}{\Gamma_i})^{1/\alpha(x_i)} - 1|^{d(1+\eta)}\right) \\
&\le& K_U \frac{|\log i|^{d(1+\eta)}}{i^{1+\eta}} \mathsf{E}\left(|\frac{\log \Gamma_i}{\log i}||(\frac{i}{\Gamma_i})^{1/\alpha(x_i)} - 1|^{d(1+\eta)}\right) \\
&\le& K_U \frac{|\log i|^{d(1+\eta)}}{i^{1+\eta}} \mathsf{E}\left(|\frac{\log \Gamma_i}{\log i}||(\frac{i}{\Gamma_i})^{1/c} - 1|^{d(1+\eta)} + |\frac{\log \Gamma_i}{\log i}||(\frac{i}{\Gamma_i})^{1/d} - 1|^{d(1+\eta)}\right) \\
&\le& K_U \frac{|\log i|^{d(1+\eta)}}{i^{1+\eta}},
\end{aligned}
$$

thus

$$\sup_{v \in U} \sum_{i=2}^{+\infty} \mathsf{E}|Y_i^3|^{d(1+\eta)} \le K_U$$

and

$$\mathsf{P}\left(\left|\sum_{i=1}^{\infty} Y_i^3\right| > x\right) \le K_U(\frac{1 + |\log x|^c}{x^c} + \frac{1 + |\log x|^d}{x^d}).$$

Let us go back to $\mathsf{P}\left(|X(v, v) - X(v, u)| > x\right)$.

$$
\begin{aligned}
\mathsf{P}\left(|X(v, v) - X(v, u)| > x\right) &=& \mathsf{P}\left(|u - v||\sum_{i=1}^{\infty}(Z_i^1 + Z_i^2 + Z_i^3 + Y_i^1 + Y_i^2 + Y_i^3)| > x\right) \\
&\le& \sum_{j=1}^{3}\left(\mathsf{P}\left(|\sum_{i=1}^{\infty} Z_i^j| \ge \frac{x}{6|u - v|}\right) + \mathsf{P}\left(|\sum_{i=1}^{\infty} Y_i^j| \ge \frac{x}{6|u - v|}\right)\right) \\
&\le& K_U\left(\frac{|v - u|^d}{x^d}(1 + |\log \frac{|v - u|}{x}|^d) + \frac{|v - u|^c}{x^c}(1 + |\log \frac{|v - u|}{x}|^c)\right)
\end{aligned}
$$

and the proof is complete ∎

Lemma 5.12 *Assume (C11), (C12), (C14), (C15). There exists a function $l \ge 0$ such that*

$$\lim_{r \to 0} |\Delta(r, t) - l(t)| = 0,$$

17

where

$$\Delta(r,t) =: \frac{1}{r^{2h(t)}} \int_{\mathbf{R}} \int_{\frac{K}{r}} \left| \frac{C_{\alpha(t+r)}^{1/\alpha(t+r)}}{y^{1/\alpha(t+r)}} f(t+r,t+r,x) - \frac{C_{\alpha(t)}^{1/\alpha(t)}}{y^{1/\alpha(t)}} f(t,t,x) \right|^2 dy\, m(dx).$$

Assuming in addition (Cu11), (Cu12), (Cu14), (Cu15), the convergence is uniform on U.

Proof

Let $l(t) = \frac{C_{\alpha(t)}^{2/\alpha(t)} K^{1-\frac{2}{\alpha(t)}}}{\frac{2}{\alpha(t)}-1} g(t)$. Note that condition (C14) implies the following:

$$\forall \varepsilon > 0, \exists K_U > 0, \forall r \leq \varepsilon, \frac{1}{|r|^{1+2(h(t)-\frac{1}{\alpha(t)})}} \int_{\mathbf{R}} |f(t+r,t,x) - f(t,t,x)|^2 m(dx) \leq K_U.$$

$$(5.9)$$

The uniform condition (Cu14) implies also that:

$$\exists K_U > 0, \forall v \in U, \forall u \in U, \frac{1}{|v-u|^{1+2(h(u)-\frac{1}{\alpha(u)})}} \int_{\mathbf{R}} |f(v,u,x) - f(u,u,x)|^2 m(dx) \leq K_U.$$

$$(5.10)$$

Expanding the square, we can write $\Delta(r,t) - l(t) = \Delta_1(r,t) + \Delta_2(r,t) + \Delta_3(r,t)$ where

$$\Delta_1(r,t) = \frac{1}{r^{2h(t)}} \int_{\mathbf{R}} \int_{\frac{K}{r}} \left| \frac{C_{\alpha(t+r)}^{1/\alpha(t+r)}}{y^{1/\alpha(t+r)}} f(t+r,t+r,x) - \frac{C_{\alpha(t)}^{1/\alpha(t)}}{y^{1/\alpha(t)}} f(t+r,t,x) \right|^2 dy\, m(dx),$$

$$\Delta_2(r,t) = \frac{2C_{\alpha(t)}^{1/\alpha(t)}}{r^{2h(t)}} \int_{\mathbf{R}} \int_{\frac{K}{r}} \frac{1}{y^{1/\alpha(t)}} g_1(r,t,x,y) g_2(r,t,x) dy\, m(dx),$$

and

$$\Delta_3(r,t) = \frac{1}{r^{2h(t)}} \int_{\mathbf{R}} \int_{\frac{K}{r}} \frac{C_{\alpha(t)}^{2/\alpha(t)}}{y^{2/\alpha(t)}} (f(t+r,t,x) - f(t,t,x))^2 dy\, m(dx) - l(t),$$

with $g_1(r,t,x,y) = \frac{C_{\alpha(t+r)}^{1/\alpha(t+r)}}{y^{1/\alpha(t+r)}} f(t+r,t+r,x) - \frac{C_{\alpha(t)}^{1/\alpha(t)}}{y^{1/\alpha(t)}} f(t+r,t,x)$ and $g_2(r,t,x) = f(t+r,t,x) - f(t,t,x)$. Since α is continuous, there exists a positive constant K_U (that may change from line to line) such that

$$|\Delta_2(r,t)| \leq \frac{K_U}{r^{2h(t)}} \int_{\mathbf{R}} \int_{\frac{K}{r}} \left| \frac{g_1(r,t,x,y)g_2(r,t,x)}{y^{1/\alpha(t)}} \right| dy \, m(dx)$$

$$\leq \frac{K_U}{r^{2h(t)}} \left(\int_{\mathbf{R}} \int_{\frac{K}{r}} |g_1(r,t,x,y)|^2 \, dy \, m(dx) \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}} \int_{\frac{K}{r}} \left| \frac{g_2(r,t,x)}{y^{1/\alpha(t)}} \right|^2 dy \, m(dx) \right)^{\frac{1}{2}}$$

$$\leq \frac{K_U}{r^{2h(t)}} r^{h(t)} \sqrt{\Delta_1(r,t)} \left(\int_{\mathbf{R}} \int_{\frac{K}{r}} \left| \frac{g_2(r,t,x)}{y^{1/\alpha(t)}} \right|^2 dy \, m(dx) \right)^{\frac{1}{2}}$$

$$\leq \frac{K_U}{r^{h(t)}} \sqrt{\Delta_1(r,t)} \left(\int_{\mathbf{R}} |g_2(r,t,x)|^2 \, m(dx) \right)^{\frac{1}{2}} r^{\frac{1}{\alpha(t)} - \frac{1}{2}} K^{\frac{1}{2} - \frac{1}{\alpha(t)}} \sqrt{\frac{\alpha(t)}{2 - \alpha(t)}}$$

$$\leq K_U \sqrt{\Delta_1(r,t)} \left(\frac{1}{r^{1 + 2(h(t) - \frac{1}{\alpha(t)})}} \int_{\mathbf{R}} |f(t+r,t,x) - f(t,t,x)|^2 \, m(dx) \right)^{\frac{1}{2}}$$

$$\leq K_U \sqrt{\Delta_1(r,t)} \text{ with } (\mathbf{5.9}).$$

Let us show that $\lim_{r \to 0} \sqrt{\Delta_1(r,t)} = 0$. The triangle inequality yields $\sqrt{\Delta_1(r,t)}) \leq \delta_1(r,t) + \delta_2(r,t) + \delta_3(r,t)$ where

$$\delta_1(r,t) = \frac{1}{2r^{h(t)}} \left(\int_{\mathbf{R}} \int_{\frac{K}{r}} \left| C_{\alpha(t+r)}^{1/\alpha(t+r)} - C_{\alpha(t)}^{1/\alpha(t)} \right|^2 \frac{|f(t+r,t+r,x)|^2}{y^{2/\alpha(t+r)}} dy \, m(dx) \right)^{\frac{1}{2}},$$

$$\delta_2(r,t) = \frac{1}{2r^{h(t)}} \left(\int_{\mathbf{R}} \int_{\frac{K}{r}} \frac{C_{\alpha(t)}^{2/\alpha(t)}}{y^{2/\alpha(t+r)}} |f(t+r,t+r,x) - f(t+r,t,x)|^2 \, dy \, m(dx) \right)^{\frac{1}{2}},$$

and

$$\delta_3(r,t) = \frac{1}{2r^{h(t)}} \left(\int_{\mathbf{R}} \int_{\frac{K}{r}} C_{\alpha(t)}^{2/\alpha(t)} |f(t+r,t,x)|^2 \left(\frac{1}{y^{1/\alpha(t+r)}} - \frac{1}{y^{1/\alpha(t)}} \right)^2 dy \, m(dx) \right)^{\frac{1}{2}}.$$

Now,

$$\delta_1(r,t) \leq K_U \frac{|C_{\alpha(t+r)}^{1/\alpha(t+r)} - C_{\alpha(t)}^{1/\alpha(t)}|}{r^{h(t)}} \left(\frac{1}{\frac{2}{\alpha(t+r)} - 1} (\frac{K}{r})^{1 - \frac{2}{\alpha(t+r)}} \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}} |f(t+r,t+r,x)|^2 \, m(dx) \right)^{\frac{1}{2}}.$$

Since the function $u \mapsto C_{\alpha(u)}^{1/\alpha(u)}$ is a C^1 function,

$$\delta_1(r,t) \leq K_U r^{1 - h(t) + \frac{1}{\alpha(t+r)} - \frac{1}{2}} \left(\int_{\mathbf{R}} |f(t+r,t+r,x)|^2 \, m(dx) \right)^{\frac{1}{2}}$$

$$\leq K_U r^{1 - h(t) + \frac{1}{\alpha(t+r)} - \frac{1}{2}} \text{ with } (\mathbf{C12})$$

$$\leq K_U r^{\frac{1}{2} + \frac{1}{d} - h_+}.$$

Since $\frac{1}{2} + \frac{1}{d} - h_+ > 0$, $\lim_{r \to 0} \delta_1(r,t) = 0$.

$$
\begin{aligned}
\delta_2(r,t) &\leq \frac{C_{\alpha(t)}^{1/\alpha(t)}}{2r^{h(t)}} \left(\frac{1}{\frac{2}{\alpha(t+r)} - 1} (\frac{K}{r})^{1 - \frac{2}{\alpha(t+r)}} \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}} |f(t+r, t+r, x) - f(t+r, t, x)|^2 \, m(dx) \right)^{\frac{1}{2}} \\
&\leq K_U r^{\frac{1}{\alpha(t+r)} - h(t) - \frac{1}{2}} \left(\int_{\mathbf{R}} |f(t+r, t+r, x) - f(t+r, t, x)|^2 \, m(dx) \right)^{\frac{1}{2}} \\
&\leq K_U r^{\frac{1}{2} + \frac{1}{\alpha(t+r)} - h(t)} \text{ with } (\mathbf{C15}) \\
&\leq K_U r^{\frac{1}{2} + \frac{1}{d} - h_+},
\end{aligned}
$$

thus $\lim_{r \to 0} \delta_2(r,t) = 0$.

$$
\delta_3(r,t) \leq \frac{C_{\alpha(t)}^{1/\alpha(t)}}{2r^{h(t)}} \left(\int_{\mathbf{R}} |f(t+r, t, x)|^2 \, m(dx) \right)^{\frac{1}{2}} \left(\int_{\frac{K}{r}} \left(\frac{1}{y^{1/\alpha(t+r)}} - \frac{1}{y^{1/\alpha(t)}} \right)^2 dy \right)^{\frac{1}{2}}
$$

Since the function $u \mapsto \alpha(u)$ is a C^1 function, $\forall \eta < \frac{1}{d}$,

$$
\begin{aligned}
\delta_3(r,t) &\leq \frac{K_U}{r^{h(t)}} \left(\int_{\mathbf{R}} |f(t+r, t, x)|^2 \, m(dx) \right)^{\frac{1}{2}} K_U r^{\frac{1}{2} + \frac{1}{d} - \eta} \\
&\leq K_U r^{\frac{1}{2} + \frac{1}{d} - \eta - h_+} \text{ with } (\mathbf{C11})
\end{aligned}
$$

thus $\lim_{r \to 0} \delta_3(r,t) = 0$. Finally, $\lim_{r \to 0} \sqrt{\Delta_1(r,t)} = 0$.

Let us now consider the last term $\Delta_3(r,t)$:

$$
\Delta_3(r,t) = \frac{C_{\alpha(t)}^{2/\alpha(t)} K^{1 - \frac{2}{\alpha(t)}}}{\frac{2}{\alpha(t)} - 1} \left(\frac{1}{r^{1 + 2(h(t) - 1/\alpha(t))}} \int_{\mathbf{R}} (f(t+r, t, x) - f(t, t, x))^2 \, m(dx) - g(t) \right)
$$

thus, with $(\mathbf{C14})$, $\lim_{r \to 0} |\Delta_3(r,t)| = 0$ ∎

Lemma 5.13 *Assume (C6), (C10), (C12), (C13), (C15), and let:*

$$
\Delta(r,t) =: \frac{1}{r^{1 + 2(h(t) - 1/\alpha(t))}} \left(\frac{C_{\alpha(t)}^{1/\alpha(t)} K^{\frac{1}{\alpha(t+r)} - \frac{1}{\alpha(t)}} (\frac{2}{\alpha(t+r)} - 1) \int_{\mathbf{R}} f(t+r, t+r, x) f(t, t, x) m(dx)}{C_{\alpha(t+r)}^{1/\alpha(t+r)} r^{\frac{1}{\alpha(t+r)} - \frac{1}{\alpha(t)}} (\frac{1}{\alpha(t+r)} + \frac{1}{\alpha(t)} - 1) \int_{\mathbf{R}} f(t+r, t+r, x)^2 m(dx)} - 1 \right)^2 .
$$

Then:

$$
\lim_{r \to 0} |\Delta(r,t)| = 0.
$$

If in addition we suppose (Cu10), (Cu12), (Cu15), the convergence is uniform on U.

Proof

Since the function $t \mapsto \alpha(t)$ is a C^1 function, there exists $K_U > 0$ such that

$$\left| \frac{C_{\alpha(t)}^{1/\alpha(t)}}{C_{\alpha(t+r)}^{1/\alpha(t+r)}} - 1 \right| \le r K_U, \tag{5.11}$$

$$\left| K^{\frac{1}{\alpha(t+r)} - \frac{1}{\alpha(t)}} - 1 \right| \le r K_U, \tag{5.12}$$

and

$$\left| \frac{\frac{2}{\alpha(t+r)} - 1}{\frac{1}{\alpha(t+r)} + \frac{1}{\alpha(t)} - 1} - 1 \right| \le r K_U. \tag{5.13}$$

Increasing K_U if necessary, we also have, $\forall a > 0$,

$$\left| \frac{1}{r^{\frac{1}{\alpha(t+r)} - \frac{1}{\alpha(t)}}} - 1 \right| \le r^a K_U. \tag{5.14}$$

For the last term, we write

$$\frac{\int_{\mathbf{R}} f(t+r, t+r, x) f(t, t, x) m(dx)}{\int_{\mathbf{R}} f(t+r, t+r, x)^2 m(dx)} - 1 = \Delta_1(r, t) + \Delta_2(r, t)$$

where

$$\Delta_1(r, t) = \frac{1}{\int_{\mathbf{R}} f(t+r, t+r, x)^2 m(dx)} \left(\int_{\mathbf{R}} f(t+r, t+r, x) \left(f(t, t, x) - f(t+r, t, x) \right) m(dx) \right)$$

and

$$\Delta_2(r, t) = \frac{1}{\int_{\mathbf{R}} f(t+r, t+r, x)^2 m(dx)} \left(\int_{\mathbf{R}} f(t+r, t+r, x) \left(f(t+r, t, x) - f(t+r, t+r, x) \right) m(dx) \right).$$

With **(C13)**, we may choose K_U such that

$$|\Delta_1(r, t)| \le K_U \int_{\mathbf{R}} |f(t+r, t+r, x)| \, |f(t, t, x) - f(t+r, t, x)| \, m(dx).$$

Let $p \in (\alpha(t), 2)$, $p \ge 1$ satisfying **(C10)**, and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Hölder inequality entails:

$$
\begin{aligned}
|\Delta_1(r, t)| &\le K_U \left(\int_{\mathbf{R}} |f(t+r, t+r, x)|^q m(dx) \right)^{1/q} \left(\int_{\mathbf{R}} |f(t, t, x) - f(t+r, t, x)|^p m(dx) \right)^{1/p} \\
&\le K_U \left(\int_{\mathbf{R}} |f(t+r, t, x) - f(t, t, x)|^p m(dx) \right)^{1/p} \quad \text{with } \textbf{(C6)} \text{ and } \textbf{(C12)} \\
&\le K_U r^{\frac{1}{p} + h(t) - \frac{1}{\alpha(t)}} \quad \text{with } \textbf{(C10)}.
\end{aligned}
$$

With **(C12)**, **(C13)** and Cauchy-Schwarz inequality, we select K_U such that

$$
\begin{aligned}
|\Delta_2(r, t)| &\le K_U \left(\int_{\mathbf{R}} |f(t+r, t+r, x) - f(t+r, t, x)|^2 m(dx) \right)^{\frac{1}{2}} \\
&\le K_U r \quad \text{with } \textbf{(C15)}.
\end{aligned}
$$

Finally, since $h(t) + \frac{1}{p} - \frac{1}{\alpha(t)} \leq 1$,

$$\left| \frac{\int_{\mathbf{R}} f(t+r,t+r,x) f(t,t,x) m(dx)}{\int_{\mathbf{R}} f(t+r,t+r,x)^2 m(dx)} - 1 \right| \leq K_U r^{h(t) + \frac{1}{p} - \frac{1}{\alpha(t)}}. \tag{5.15}$$

Using the inequalities (5.11), (5.12), (5.13), (5.14) and (5.15), we may find a constant K_U such that for all $a > 0$,

$$|\Delta(r,t)| \leq \frac{1}{r^{1+2(h(t)-1/\alpha(t))}} K_U \left(r^2 + r^{2a} + r^{2(h(t) + \frac{1}{p} - \frac{1}{\alpha(t)})} \right).$$

Choosing $a \in \left(h(t) + \frac{1}{p} - \frac{1}{\alpha(t)}, 1 \right)$, this entails:

$$\begin{aligned}
|\Delta(r,t)| &\leq \frac{3}{r^{1+2(h(t)-1/\alpha(t))}} K_U r^{2(h(t) + \frac{1}{p} - \frac{1}{\alpha(t)})} \\
&\leq 3 K_U r^{\frac{2}{p}-1}.
\end{aligned}$$

Since $\frac{2}{p} - 1 > 0$, $\lim\limits_{r \to 0} |\Delta(r,t)| = 0$ ∎

Lemma 5.14 *Assuming (C1), (C6), (C7), (C8), one has:*

$$\forall \varepsilon < \frac{1}{d}, \exists K_U \leq 1 \text{ such that } \forall v \geq 1, \forall r \leq \varepsilon_0,$$

$$\begin{aligned}
y \geq K_U \frac{v^{\frac{d}{1-\varepsilon d}}}{r} \quad \Rightarrow \quad & \sin^2 \left(\frac{v C_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2 r^{h(t)} y^{1/\alpha(t+r)}} - \frac{v C_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2 r^{h(t)} y^{1/\alpha(t)}} \right) \\
&\geq \frac{1}{2} \left| \frac{v C_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2 r^{h(t)} y^{1/\alpha(t+r)}} - \frac{v C_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2 r^{h(t)} y^{1/\alpha(t)}} \right|^2.
\end{aligned}$$

If in addition we suppose (Cu8),

$$\begin{aligned}
y \geq K_U \frac{v^{\frac{d}{1-\varepsilon d}}}{r} \quad \Rightarrow \quad & \forall t \in U, \sin^2 \left(\frac{v C_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2 r^{h(t)} y^{1/\alpha(t+r)}} - \frac{v C_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2 r^{h(t)} y^{1/\alpha(t)}} \right) \\
&\geq \frac{1}{2} \left| \frac{v C_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2 r^{h(t)} y^{1/\alpha(t+r)}} - \frac{v C_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2 r^{h(t)} y^{1/\alpha(t)}} \right|^2.
\end{aligned}$$

Proof

Let $\varepsilon < \frac{1}{d}$. We write $\frac{v C_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{2 r^{h(t)} y^{1/\alpha(t+r)}} - \frac{v C_{\alpha(t)}^{1/\alpha(t)} f(t,t,x)}{2 r^{h(t)} y^{1/\alpha(t)}} = \kappa_1(r,t,v,x,y) + \kappa_2(r,t,v,x,y)$, with

$$\kappa_1(r,t,v,x,y) = \frac{v}{2 r^{h(t)}} \left(\frac{C_{\alpha(t+r)}^{1/\alpha(t+r)} f(t+r,t+r,x)}{y^{1/\alpha(t+r)}} - \frac{C_{\alpha(t)}^{1/\alpha(t)} f(t+r,t,x)}{y^{1/\alpha(t)}} \right)$$

and
$$\kappa_2(r,t,v,x,y) = \frac{vC_{\alpha(t)}^{1/\alpha(t)}}{2r^{h(t)}y^{1/\alpha(t)}}\left(f(t+r,t,x) - f(t,t,x)\right).$$

Using the finite-increments theorem,

$$|\kappa_1(r,t,v,x,y)| \leq \frac{v}{2r^{h(t)}}r(\sup_{a\in U}\left|\frac{K_U|f(t+r,a,x)|}{y^{1/\alpha(a)}}\right| + \sup_{a\in U}\left|\frac{C_{\alpha(a)}^{1/\alpha(a)}|f_v(t+r,a,x)|}{y^{1/\alpha(a)}}\right|$$
$$+ \sup_{a\in U}\left|\frac{|\alpha'(a)|}{\alpha^2(a)}|\ln y|\frac{C_{\alpha(a)}^{1/\alpha(a)}|f(t+r,a,x)|}{y^{1/\alpha(a)}}\right|).$$

For $y \geq 1$, conditions **(C6)** and **(C7)** imply

$$\frac{K_U|f(t+r,a,x)|}{y^{1/\alpha(a)}} \leq \frac{K_U}{y^{1/d}},$$

$$\frac{K_U|f_v(t+r,a,x)|}{y^{1/\alpha(a)}} \leq \frac{K_U}{y^{1/d}},$$

and

$$\frac{|\alpha'(a)|}{\alpha^2(a)}|\ln y|\frac{C_{\alpha(a)}^{1/\alpha(a)}|f(t+r,a,x)|}{y^{1/\alpha(a)}} \leq \frac{K_U|\ln y|}{y^{1/d}}.$$

Finally,

$$|\kappa_1(r,t,v,x,y)| \leq \frac{K_U v r^{1-h(t)}}{y^{1/d}}(1+|\ln y|)$$
$$\leq \frac{K_U v}{y^{1/d-\varepsilon}}.$$

Condition **(C8)** allows to estimate $\kappa_2(r,t,v,x,y)$ as follows:

$$|\kappa_2(r,t,v,x,y)| \leq \frac{K_U v}{(ry)^{1/\alpha(t)}}.$$

Finally, $\forall K > 0$, $\forall \varepsilon < \frac{1}{d}, \exists K_U \geq 1, \forall v \geq 1, \forall r < \varepsilon_0$, $\forall y \geq K_U\frac{v^{\frac{d}{1-\varepsilon d}}}{r}$,

$$\left|\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)}f(t+r,t+r,x)}{2r^{h(t)}y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)}f(t,t,x)}{2r^{h(t)}y^{1/\alpha(t)}}\right| \leq K$$

∎

Lemma 5.15 *Assuming (C6), (C10), (C11), (C12), (C13), (C14), (C15), there exist ε_0 and $K_U > 0$ such that $\forall r < \varepsilon_0$, $\forall v \geq 1$:*

$$N(v,t,r) \geq K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})},$$

where

$$N(v,t,r) =: \int_{\mathbf{R}}\int_{\frac{K_U v^{\frac{d}{1-\varepsilon d}}}{r}}\left|\frac{vC_{\alpha(t+r)}^{1/\alpha(t+r)}f(t+r,t+r,x)}{2r^{h(t)}y^{1/\alpha(t+r)}} - \frac{vC_{\alpha(t)}^{1/\alpha(t)}f(t,t,x)}{2r^{h(t)}y^{1/\alpha(t)}}\right|^2 dy\, m(dx).$$

If in addition we suppose (Cu10), (Cu11), (Cu12), (Cu14), (Cu15), the constant K_U does not depend on t.

Proof
Expanding the square above, we may write

$$N(v,t,r) = A_1(r,t)v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{\alpha(t+r)})} - A_2(r,t)v^{2+\frac{d}{1-\varepsilon d}(1-\frac{1}{\alpha(t+r)}-\frac{1}{\alpha(t)})} + A_3(r,t)v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{\alpha(t)})},$$

where

$$A_1(r,t) = \frac{C_{\alpha(t+r)}^{2/\alpha(t+r)}(K_U)^{1-\frac{2}{\alpha(t+r)}}}{4\left(\frac{2}{\alpha(t+r)}-1\right)r^{1+2(h(t)-\frac{1}{\alpha(t+r)})}} \int_{\mathbf{R}} |f(t+r,t+r,x)|^2 m(dx),$$

$$A_2(r,t) = \frac{C_{\alpha(t+r)}^{1/\alpha(t+r)}C_{\alpha(t)}^{1/\alpha(t)}(K_U)^{1-\frac{1}{\alpha(t+r)}-\frac{1}{\alpha(t)}}}{2\left(\frac{1}{\alpha(t+r)}+\frac{1}{\alpha(t)}-1\right)r^{1+2h(t)-\frac{1}{\alpha(t+r)}-\frac{1}{\alpha(t)}}} \int_{\mathbf{R}} f(t+r,t+r,x)f(t,t,x)m(dx),$$

and

$$A_3(r,t) = \frac{C_{\alpha(t)}^{2/\alpha(t)}(K_U)^{1-\frac{2}{\alpha(t)}}}{4\left(\frac{2}{\alpha(t)}-1\right)r^{1+2(h(t)-\frac{1}{\alpha(t)})}} \int_{\mathbf{R}} |f(t,t,x)|^2 m(dx).$$

We obtain

$$N(v,t,r) = v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{\alpha(t)})}\left(A_1(r,t)(v^{\frac{2d}{1-\varepsilon d}(\frac{1}{\alpha(t)}-\frac{1}{\alpha(t+r)})})^2 - A_2(r,t)(v^{\frac{2d}{1-\varepsilon d}(\frac{1}{\alpha(t)}-\frac{1}{\alpha(t+r)})}) + A_3(r,t)\right).$$

Let $P(r,t,X) = A_1(r,t)X^2 - A_2(r,t)X + A_3(r,t)$. Write:

$$P(r,t,X) = P(r,t,X) - P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) + P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1) + P(r,t,1).$$

Since $P(\frac{A_2(r,t)}{2A_1(r,t)})$ is the minimum of P,

$$P(r,t,X) \geq P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1) + P(r,t,1).$$

Note that $P(r,t,1) = N(1,t,r)$, thus lemma (5.12) entails that there exists a positive function l such that $\lim_{r\to 0} P(r,t,1) = l(t)$. For $P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1)$, we use lemma (5.13). With the same notations,

$$\begin{aligned}|P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1)| &= |A_1(r,t)|r^{1+2(h(t)-\frac{1}{\alpha(t)})}\Delta(r,t)\\ &\leq K_U\Delta(r,t)\end{aligned}$$

thus $\lim_{r\to 0}|P(r,t,\frac{A_2(r,t)}{2A_1(r,t)}) - P(r,t,1)| = 0$. As a consequence, there exist a positive constant K_U and $\varepsilon_0 > 0$ such that for all $x \in \mathbf{R}$ and $r \in (0,\varepsilon_0)$, $P(r,t,x) \geq K_U$. We obtain $N(v,t,r) \geq v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{\alpha(t)})}K_U$ for all $v \in \mathbf{R}$ and $r \in (0,\varepsilon_0)$. Since $\alpha(t) > c$, $N(v,t,r) \geq K_U v^{2+\frac{d}{1-\varepsilon d}(1-\frac{2}{c})}$ ∎

Proof of theorem 3.1

Consider

$$\mathsf{E}\left[\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^{\eta}\right]=\int_0^\infty \mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^{\eta}>x\right)dx.$$

Thanks to **(C1)**, **(C2)**, **(C3)** and **(C5)**, Y is $h(t)$-localisable at t [14], thus for all $x>0$,

$$\mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^{\eta}>x\right)\to \mathsf{P}\left(|Y_t'(1)|^{\eta}>x\right).$$

We shall make use of Lebesgue dominated convergence theorem.

For $x\le 1$, $\mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^{\eta}>x\right)\le 1$.

For $x>1$,

$$\begin{aligned}
\mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|^{\eta}>x\right) &= \mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h(t)}}\right|>x^{1/\eta}\right) \\
&\le \mathsf{P}\left(\left|\frac{X(t+\varepsilon,t+\varepsilon)-X(t+\varepsilon,t)}{\varepsilon^{h(t)}}\right|>\frac{x^{1/\eta}}{2}\right) \\
&+ \mathsf{P}\left(\left|\frac{X(t+\varepsilon,t)-X(t,t)}{\varepsilon^{h(t)}}\right|>\frac{x^{1/\eta}}{2}\right).
\end{aligned}$$

For the first term, by proposition 4.9 (or 4.10),

$$\mathsf{P}\left(\left|\frac{X(t+\varepsilon,t+\varepsilon)-X(t+\varepsilon,t)}{\varepsilon^{h(t)}}\right|>\frac{x^{1/\eta}}{2}\right) \le \frac{K_U}{x^{d/\eta}}\left(1+|\log x|^d\right)+\frac{K_U}{x^{c/\eta}}\left(1+|\log x|^c\right).$$

For the second term, let $p\in(\eta,\alpha(t))$.

$$\mathsf{P}\left(\left|\frac{X(t+\varepsilon,t)-X(t,t)}{\varepsilon^{h(t)}}\right|>\frac{x^{1/\eta}}{2}\right) = \mathsf{P}\left(\left|\frac{X(t+\varepsilon,t)-X(t,t)}{\varepsilon^{h(t)}}\right|^p>\frac{x^{p/\eta}}{2^p}\right).$$

With Markov inequality and **(C9)**,

$$\begin{aligned}
\mathsf{P}\left(\left|\frac{X(t+\varepsilon,t)-X(t,t)}{\varepsilon^{h(t)}}\right|>\frac{x^{1/\eta}}{2}\right) &\le \frac{2^p}{x^{p/\eta}\varepsilon^{ph(t)}}C_{\alpha(t),0}(p)^p\|f(t+\varepsilon,t,.)-f(t,t,.)\|_{\alpha(t)}^p \\
&\le \frac{2^p C_{\alpha(t),0}(p)^p}{x^{p/\eta}\varepsilon^{ph(t)}}\left(\int_{\mathbf{R}}|f(t+\varepsilon,t,x)-f(t,t,x)|^{\alpha(t)}m(dx)\right)^{p/\alpha(t)} \\
&\le \frac{K_{p,\alpha(t)}}{x^{p/\eta}},
\end{aligned}$$

thus

$$\mathsf{P}\left(\left|\frac{Y(t+\varepsilon)-Y(t)}{\varepsilon^{h}}\right|^{\eta}>x\right)\le K_U\left(\frac{1}{x^{d/\eta}}\left(1+|\log x|^d\right)+\frac{1}{x^{c/\eta}}\left(1+|\log x|^c\right)+\frac{1}{x^{p/\eta}}\right)\mathbf{1}_{x>1}+\mathbf{1}_{x\le 1}\blacksquare$$

25

Proof of theorem 3.2
Let $\gamma > h(t)$ and $x > 0$.

$$\mathsf{P}\left(\frac{r^\gamma}{|Y(t+r) - Y(t)|} > x\right) = \mathsf{P}\left(|Y(t+r) - Y(t)| < \frac{r^\gamma}{x}\right).$$

Applying proposition (4.11), there exists $\varepsilon_0 > 0$ such that

$$\sup_{r \in B(0,\varepsilon_0)} \int_0^{+\infty} \varphi_{\frac{Y(t+r)-Y(t)}{r^{h(t)}}}(v) dv < +\infty.$$

Thus with proposition (4.8), there exists $K_U > 0$ such that

$$\mathsf{P}\left(|Y(t+r) - Y(t)| < \frac{r^\gamma}{x}\right) \leq K_U \frac{r^{\gamma-h(t)}}{x}.$$

Let $r_n = \frac{1}{n^\eta}$ with $\eta(\gamma - h(t)) > 1$. $\forall x > 0$, $\sum_n \mathsf{P}\left(\frac{r_n^\gamma}{|Y(t+r_n)-Y(t)|} > x\right) < +\infty$. Borel Cantelli lemma entails that, almost surely, $\lim_{n \to +\infty} \frac{|Y(t+r_n)-Y(t)|}{r_n^\gamma} = +\infty$. As a consequence, almost surely, $\limsup_{r \to 0} \frac{|Y(t+r)-Y(t)|}{r^\gamma} = +\infty$, and

$$\mathcal{H}_t \leq h(t).$$

∎

Proof of theorem 3.5
We want to apply theorem (3.1) with $f(t, u, x) = \mathbf{1}_{[0,t]}(x)$. Let us show that conditions (C1), (C2), (C3), (C5) and (C9) are satisfied.

- (C1) The family of functions $v \to f(t, v, x)$ is differentiable for all (v, t) in $(0,1)^2$ and almost all x in E. The derivatives of f with respect to u vanish.

- (C2)
$$|f(t, w, x)|^{\alpha(w)} = \mathbf{1}_{[0,t]}(x)$$
  thus, for all $\delta > 0$, all $t \in (0,1)$,
$$\int_{\mathbf{R}} \left[\sup_{w \in (0,1)} (|f(t, w, x)|^{\alpha(w)})\right]^{1+\delta} dx = t$$
  and (C2) holds.

- (C3) $f'_u = 0$ thus (C3) holds.

- (C5) $X(t, u)$ (as a process in t) is localisable at u with exponent $\frac{1}{\alpha(u)} \in (\frac{1}{d}, \frac{1}{c}) \subset (0,1)$, with local form $X_u(t, u)$, and $u \mapsto \frac{1}{\alpha(u)}$ is a C^1 function (see [14]).

- (C9)
$$\frac{1}{r^{h(t)\alpha(t)}} \int_{\mathbf{R}} |f(t+r, t, x) - f(t, t, x)|^{\alpha(t)} m(dx) = \frac{1}{r} \int_t^{t+r} dx$$
$$= 1,$$

  thus (C9) holds.

26

From theorem (3.1), we get that

$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^\eta\right] \sim \varepsilon^{\frac{\eta}{\alpha(t)}}\,\mathsf{E}\left[|Y'_t(1)|^\eta\right].$$

Since $Y'_t(1)$ is an $S_{\alpha(t)}(1,0,0)$ random variable, property 1.2.17 of [22] allows to conclude.
∎

Proof of theorem 3.6

We want to apply Theorem (3.2) with $f(t,u,x) = \mathbf{1}_{[0,t]}(x)$ and $h(t) = \frac{1}{\alpha(t)}$ in order to obtain the inequality. Let us show that the conditions (C6), (C7), (Cu8), (Cu10), (Cu11), (Cu12), (C13), (Cu14) and (Cu15) are satisfied.

- (C6) Obvious.

- (C7) Obvious.

- (Cu8) $\forall v \in U,\ \forall u \in U,\ \forall x \in \mathbf{R}$,

$$
\begin{aligned}
\frac{1}{|v-u|^{h(u)-1/\alpha(u)}}\,|f(v,u,x) - f(u,u,x)| &= \mathbf{1}_{[u,v]}(x) \\
&\leq 1
\end{aligned}
$$

thus (Cu8) holds.

- (Cu10) $\forall v \in U, \forall u \in U$,

$$
\begin{aligned}
\frac{1}{|v-u|^{1+p(h(u)-\frac{1}{\alpha(u)})}}\int_{\mathbf{R}}|f(v,u,x) - f(u,u,x)|^p\,m(dx) &= \frac{1}{|v-u|}\int_{\mathbf{R}}|\mathbf{1}_{[u,v]}(x)| \\
&= 1
\end{aligned}
$$

thus (Cu10) holds.

- (Cu11) $\forall v \in U,\ \forall u \in U$,
$$\int_{\mathbf{R}}|f(v,u,x)|^2\,m(dx) = v$$
thus (Cu11) holds $(U = (0,1))$.

- (Cu12) For the same reason as (Cu11), (Cu12) holds.

- (C13) Since $t \in (0,1)$ (in particular $t \neq 0$), one can choose U such that $\inf_{v\in U} v > 0$ thus (C13) holds.

- (Cu14)

$$
\begin{aligned}
\frac{1}{r^{1+2(h(t)-1/\alpha(t))}}\int_{\mathbf{R}}(f(t+r,t,x) - f(t,t,x))^2\,m(dx) &= \frac{1}{r}\int_{\mathbf{R}}\mathbf{1}_{[t,t+r]}(x)dx \\
&= 1
\end{aligned}
$$

thus (Cu14) holds.

27

- (Cu15) $\forall v \in U$, $\forall u \in U$,

$$\frac{1}{|v-u|^2} \int_{\mathbf{R}} |f(v,v,x) - f(v,u,x)|^2 \, m(dx) = 0$$

thus (Cu15) holds. ∎

Proof of theorem 3.3

We want to apply theorem (3.1) with $f(t,u,x) = |t-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(u)-\frac{1}{\alpha(u)}}$. Let us show that conditions (C1), (Cs2), (Cs3), (Cs4), (C5) and (C9) are satisfied.

- (C1) The family of functions $u \to f(t,u,x)$ is differentiable for all (u,t) in a neighbourhood of t_0 and almost all x in E. The derivatives of f with respect to u read:

$$f_u'(t,w,x) = (h'(w) + \frac{\alpha'(w)}{\alpha^2(w)}) \left[(\log|t-x|)|t-x|^{h(w)-1/\alpha(w)} - (\log|x|)|x|^{h(w)-1/\alpha(w)} \right].$$

- (Cs2) In [8], it is shown that, given $t_0 \in \mathbf{R}$, one may choose $\varepsilon > 0$ small enough and numbers a, b, h_-, h_+ with $0 < a < \alpha(w) < b < 2$, $0 < h_- < h(w) < h_+ < 1$ and $\frac{a}{b}(\frac{1}{a} - \frac{1}{b}) < h_- - (\frac{1}{a} - \frac{1}{b}) < h_- < h_+ < h_+ + (\frac{1}{a} - \frac{1}{b}) < 1 - (\frac{1}{a} - \frac{1}{b})$ such that, for all t and w in $U := (t_0 - \varepsilon, t_0 + \varepsilon)$ and all real x:

$$|f(t,w,x)|, |f_{t_0}'(t,w,x)| \leq k_1(t,x) \tag{5.16}$$

where

$$k_1(t,x) = \begin{cases} c_1 \max\{1, |t-x|^{h_- - 1/a} + |x|^{h_- - 1/a}\} & (|x| \leq 1 + 2\max_{t \in U} |t|) \\ c_2 |x|^{h_+ - 1/b - 1} & (|x| > 1 + 2\max_{t \in U} |t|) \end{cases} \tag{5.17}$$

for appropriately chosen constants c_1 and c_2. One has, for all $\delta > 0$,

$$\int_{\mathbf{R}} \left[\sup_{w \in U} |f(t,w,x)|^{\alpha(w)} \right]^{1+\delta} r(x)^\delta dx \leq \int_{\mathbf{R}} \left(k_1(t,x)^a + k_1(t,x)^b \right)^{1+\delta} r(x)^\delta dx$$

$$\leq K_\delta \int_{\mathbf{R}} k_1(t,x)^{a(1+\delta)} r(x)^\delta dx$$

$$+ K_\delta \int_{\mathbf{R}} k_1(t,x)^{b(1+\delta)} r(x)^\delta dx.$$

Let us study $\int_{\mathbf{R}} k_1(t,x)^{p(1+\delta)} r(x)^\delta dx$, where $p = a$ or $p = b$.

$$\int_{\mathbf{R}} k_1(t,x)^{p(1+\delta)} r(x)^\delta dx = \frac{\pi^{2\delta}}{3^\delta} \sum_{j=0}^{+\infty} (j+1)^{2\delta} \int_j^{j+1} (k_1(t,-x)^{p(1+\delta)} + k_1(t,x)^{p(1+\delta)}) dx$$

$$= \frac{\pi^{2\delta}}{3^\delta} \sum_{j=0}^{+\infty} (j+1)^{2\delta} \int_j^{j+1} (k_1(-t,x)^{p(1+\delta)} + k_1(t,x)^{p(1+\delta)}) dx.$$

28

We consider now $\int_j^{j+1} k_1(\pm t, x)^{p(1+\delta)} dx$. There exists $K_{p,\delta} > 0$ such that, for all real x such that $|x| \leq 1 + 2\max_{t \in U} |t|$:

$$k_1(\pm t, x)^{p(1+\delta)} \leq K_{p,\delta} \left(1 + |\pm t - x|^{p(1+\delta)(h_- - 1/a)} + |x|^{p(1+\delta)(h_- - 1/a)}\right),$$

and for all real x such that $|x| > 1 + 2\max_{t \in U} |t|$,

$$k_1(\pm t, x)^{p(1+\delta)} \leq K_{p,\delta} |x|^{p(1+\delta)(h_+ - 1/b - 1)}.$$

Let $j_0 = [1 + 2\max_{t \in U} |t|)]$. For $j < j_0$,

$$\int_j^{j+1} k_1(\pm t, x)^{p(1+\delta)} dx \leq K_{p,\delta}\left(1 + \int_j^{j+1} |\pm t - x|^{p(1+\delta)(h_- - 1/a)} dx + \int_j^{j+1} |x|^{p(1+\delta)(h_- - 1/a)} dx\right).$$

Choose δ such that $p(1+\delta)(h_- - 1/a) > -1$ (we show below that such a δ exists). Then

$$
\begin{aligned}
\int_j^{j+1} |\pm t - x|^{p(1+\delta)(h_- - 1/a)} dx &= \int_{\pm t - j - 1}^{\pm t - j} |u|^{p(1+\delta)(h_- - 1/a)} du \\
&\leq \frac{|\pm t - j|^{1+p(1+\delta)(h_- - 1/a)} + |\pm t - j - 1|^{1+p(1+\delta)(h_- - 1/a)}}{1 + p(1+\delta)(h_- - 1/a)} \\
&\leq K_U |t|^{1+p(1+\delta)(h_- - 1/a)} |1 + j|^{1+p(1+\delta)(h_- - 1/a)} \\
&\leq K_U (1+j)^{1+p(1+\delta)(h_- - 1/a)}
\end{aligned}
$$

where $K_U > 0$ depends on U and may have changed from line to line. We deduce:

$$\int_j^{j+1} k_1(\pm t, x)^{p(1+\delta)} dx \leq K_U(1 + j^{1+p(1+\delta)(h_- - 1/a)}).$$

For $j = j_0$,

$$
\begin{aligned}
\int_{j_0}^{j_0+1} k_1(\pm t, x)^{p(1+\delta)} dx &\leq K_U |j_0|^{1+p(1+\delta)(h_- - 1/a)} + K_U \int_{j_0}^{j_0+1} |x|^{p(1+\delta)(h_+ - 1/b - 1)} dx \\
&\leq K_U.
\end{aligned}
$$

For $j > j_0$,

$$
\begin{aligned}
\int_j^{j+1} k_1(\pm t, x)^{p(1+\delta)} dx &\leq K_U \int_j^{j+1} |x|^{p(1+\delta)(h_+ - 1/b - 1)} dx \\
&\leq K_U j^{p(1+\delta)(h_+ - 1/b - 1)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sup_{t \in U} \int_{\mathbf{R}} k_1(t, x)^{p(1+\delta)} r(x)^\delta dx &\leq K_U \left(1 + \sum_{j=0}^{j_0-1} j^{2\delta}(1 + j^{1+p(1+\delta)(h_- - 1/a)})\right) \\
&\quad + K_U \sum_{j=j_0+1}^{\infty} j^{2\delta + p(1+\delta)(h_+ - 1/b - 1)}.
\end{aligned}
$$

To conclude, we need to show that we may chose $\delta > \frac{b}{a} - 1$ such that $p(1+\delta)(h_- - 1/a) > -1$ and $2\delta + p(1+\delta)(h_+ - 1/b - 1) < -1$, for $p = a$ and $p = b$. We consider several cases.

<u>First case</u> : $h_- - \frac{1}{a} \geq 0$ and $h_+ - \frac{1}{b} - 1 \leq -\frac{2}{a}$.

Let $\delta > \frac{b}{a} - 1$. One has $p(1+\delta)(h_- - \frac{1}{a}) \geq 0 > -1$. We consider $1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1)$.

$$
\begin{aligned}
1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) &\leq 1 + 2\delta - \frac{2}{a}p(1+\delta) \\
&= 1 - \frac{2p}{a} + 2\delta(1 - \frac{p}{a}).
\end{aligned}
$$

Since $1 - \frac{2p}{a} < 0$ and $1 - \frac{p}{a} \leq 0$, $1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) < 0$.

<u>Second case</u> : $h_- - \frac{1}{a} \geq 0$ and $h_+ - \frac{1}{b} - 1 > -\frac{2}{a}$.

Let $\delta \in \left(\frac{b}{a} - 1, \frac{\frac{1}{b} - \frac{1}{a} + 1 - h_+}{\frac{2}{a} - \frac{1}{b} - 1 + h_+} \right)$. One has $p(1+\delta)(h_- - \frac{1}{a}) \geq 0 > -1$. We consider $1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1)$.

<u>For $p = a$</u> :

$$
\begin{aligned}
1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) &= a\delta(\frac{2}{a} + h_+ - \frac{1}{b} - 1) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&< a(\frac{1}{b} - \frac{1}{a} + 1 - h_+) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&= 0.
\end{aligned}
$$

<u>For $p = b$</u> :

$$
1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) = b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1).
$$

If $\frac{1}{b} + h_+ - 1 \leq 0$, then $b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) < 0$. Else

$$
\begin{aligned}
b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) &< b\frac{\frac{1}{b} - \frac{1}{a} + 1 - h_+}{\frac{2}{a} - \frac{1}{b} - 1 + h_+}(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) \\
&= \frac{b}{\frac{2}{a} - \frac{1}{b} - 1 + h_+}(\frac{1}{a} - \frac{1}{b})(h_+ - 1 - \frac{1}{b}) \\
&< 0.
\end{aligned}
$$

<u>Third case</u> : $h_- - \frac{1}{a} < 0$ and $h_+ - \frac{1}{b} - 1 \leq -\frac{2}{a}$.

Let $\delta \in \left(\frac{b}{a} - 1, \frac{ah_- + \frac{a}{b} - 1}{1 - ah_-} \right)$.

For $p = a$:

$$\begin{aligned}
1 + p(1+\delta)(h_- - \frac{1}{a}) &= ah_- + \delta(ah_- - 1) \\
&> ah_- + (ah_- - 1)\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-} \\
&= ah_- + 1 - \frac{a}{b} - ah_- \\
&> 0,
\end{aligned}$$

and

$$\begin{aligned}
1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) &= a\delta(\frac{2}{a} + h_+ - \frac{1}{b} - 1) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&\leq a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&\leq -1 \\
&< 0.
\end{aligned}$$

For $p = b$:

$$\begin{aligned}
1 + p(1+\delta)(h_- - \frac{1}{a}) &= b(h_- - \frac{1}{a} + \frac{1}{b}) + b\delta(h_- - \frac{1}{a}) \\
&> b(h_- - \frac{1}{a} + \frac{1}{b}) + b(h_- - \frac{1}{a})\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-} \\
&= b(h_- - \frac{1}{a} + \frac{1}{b}) + b(\frac{1}{a} - \frac{1}{b} - h_-) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) &= b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) \\
&\leq b\delta(\frac{2}{b} - \frac{2}{a}) + b(h_+ - 1) \\
&< 0.
\end{aligned}$$

<u>Fourth case</u> : $h_- - \frac{1}{a} < 0$ and $h_+ - \frac{1}{b} - 1 > -\frac{2}{a}$.

Let $\delta \in \left(\frac{b}{a} - 1, \min(\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-}, \frac{\frac{1}{b} - \frac{1}{a} + 1 - h_+}{\frac{2}{a} - \frac{1}{b} - 1 + h_+}) \right)$.

For $p = a$:

$$
\begin{aligned}
1 + p(1+\delta)(h_- - \frac{1}{a}) &= ah_- + \delta(ah_- - 1) \\
&> ah_- + (ah_- - 1)\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-} \\
&= ah_- + 1 - \frac{a}{b} - ah_- \\
&> 0,
\end{aligned}
$$

and

$$
\begin{aligned}
1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) &= a\delta(\frac{2}{a} + h_+ - \frac{1}{b} - 1) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&> a(\frac{1}{b} - \frac{1}{a} + 1 - h_+) + a(h_+ - \frac{1}{b} - 1 + \frac{1}{a}) \\
&= 0.
\end{aligned}
$$

For $p = b$:

$$
\begin{aligned}
1 + p(1+\delta)(h_- - \frac{1}{a}) &= b(h_- - \frac{1}{a} + \frac{1}{b}) + b\delta(h_- - \frac{1}{a}) \\
&> b(h_- - \frac{1}{a} + \frac{1}{b}) + b(h_- - \frac{1}{a})\frac{ah_- + \frac{a}{b} - 1}{1 - ah_-} \\
&= b(h_- - \frac{1}{a} + \frac{1}{b}) + b(\frac{1}{a} - \frac{1}{b} - h_-) \\
&= 0,
\end{aligned}
$$

and

$$
1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) = b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1).
$$

If $\frac{1}{b} + h_+ - 1 \leq 0$, then $1 + 2\delta + p(1+\delta)(h_+ - 1/b - 1) < 0$, else

$$
\begin{aligned}
b\delta(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) &< b(\frac{\frac{1}{b} - \frac{1}{a} + 1 - h_+}{\frac{2}{a} - \frac{1}{b} - 1 + h_+})(\frac{1}{b} + h_+ - 1) + b(h_+ - 1) \\
&= \frac{b}{\frac{2}{a} - \frac{1}{b} - 1 + h_+}(\frac{1}{a} - \frac{1}{b})(h_+ - 1 - \frac{1}{b}) \\
&< 0.
\end{aligned}
$$

- (Cs3) is obtained with (5.16) for the same reason as in (Cs2).

- (Cs4) For j large enough ($j > j^*$),

$$|f(t,w,x)\log(r(x))|^{\alpha(w)} \leq K_1|k_1(t,x)|^{\alpha(w)}$$
$$+ K_2\sum_{j=j^*}^{+\infty}|f(t,w,x)|^{\alpha(w)}(\log(j))^d\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x).$$

$$|f(t,w,x)|^{\alpha(w)}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x) \leq K_2\frac{1}{|x|^{a(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x)$$

(K_2 may have changed from line to line). Thus

$$\left[\sup_{w\in U}\left[|f(t,w,x)\log(r(x))|^{\alpha(w)}\right]\right]^{1+\delta}r(x)^\delta \leq K|k_1(t,x)|^{a(1+\delta)}r(x)^\delta + K|k_1(t,x)|^{b(1+\delta)}r(x)^\delta$$
$$+ K\sum_{j=j^*}^{+\infty}\frac{j^{2\delta}(\log(j))^d}{|x|^{a(1+\delta)(1+1/b-h_+)}}\mathbf{1}_{[-j,-j+1[\cup[j-1,j[}(x).$$

Let $\delta > \frac{b}{a} - 1$ be such that (Cs2) holds. Since $2\delta + a(1+\delta)(h_+ - 1 - \frac{1}{b}) < -1$, (Cs4) holds .

- (C5) $X(t,u)$ (as a process in t) is localisable at u with exponent $H(u) \in (h_-, h_+) \subset (0,1)$, with local form $X_u(t,u)$, and $u \mapsto H(u)$ is a C^1 function (see [14]).

- (C9)

$$\frac{1}{r^{H(t)\alpha(t)}}\int_{\mathbf{R}}|f(t+r,t,x) - f(t,t,x)|^{\alpha(t)}m(dx) = \int_{\mathbf{R}}\left||1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}}\right|^{\alpha(t)}dx$$

so (C9) holds.

From theorem 3.1, we obtain that

$$\mathsf{E}\left[|Y(t+\varepsilon) - Y(t)|^\eta\right] \sim \varepsilon^{\eta H(t)}\mathsf{E}\left[|Y_t'(1)|^\eta\right].$$

Since $Y_t'(1)$ is an $S_{\alpha(t)}(\sigma,0,0)$ random variable with $\sigma = \left(\int_{\mathbf{R}}\left||1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}}\right|^{\alpha(t)}dx\right)^{\frac{1}{\alpha(t)}}$,
property 1.2.17 of [22] allows to conclude. ∎

Proof of theorem 3.4
We want to apply theorems 3.2 with $f(t,u,x) = |t-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(u)-\frac{1}{\alpha(u)}}$ in order to obtain the inequality. Let us show that conditions (C6), (C7), (Cu8), (Cu10), (Cu11), (Cu12), (C13), (Cu14) and (Cu15) are satisfied.

- (C6) Since $H(t) - \frac{1}{\alpha(t)} \geq 0$, (C6) holds.

- (C7) We also use the fact that $H(t) - \frac{1}{\alpha(t)} \geq 0$ in order to prove that (C7) holds.

- (Cu8) $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbf{R}$,

$$\frac{1}{|v-u|^{h(u)-1/\alpha(u)}} |f(v,u,x) - f(u,u,x)| = \frac{1}{|v-u|^{H(u)-1/\alpha(u)}} \left| |v-x|^{H(u)-\frac{1}{\alpha(u)}} - |u-x|^{H(u)-\frac{1}{\alpha(u)}} \right|$$
$$\leq 1$$

  thus (Cu8) holds.

- (Cu10) $\forall v \in U$, $\forall u \in U$,

$$\frac{1}{|v-u|^{1+p(h(u)-\frac{1}{\alpha(u)})}} \int_{\mathbf{R}} |f(v,u,x) - f(u,u,x)|^p \, m(dx) = \int_{\mathbf{R}} \left| |1-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(u)-\frac{1}{\alpha(u)}} \right|^p dx$$

  so (Cu10) holds.

- (Cu11) $\forall v \in U$, $\forall u \in U$,

$$\int_{\mathbf{R}} |f(v,u,x)|^2 \, m(dx) = v^{1+2(H(u)-\frac{1}{\alpha(u)})} \int_{\mathbf{R}} \left| |1-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(u)-\frac{1}{\alpha(u)}} \right|^2 dx$$

  thus (Cu11) holds.

- (Cu12) For the same reason as (Cu11), (Cu12) holds.

- (C13) For $t \neq 0$, one can choose U such that $\inf_{v \in U} v^{1+2(H(v)-\frac{1}{\alpha(v)})} > 0$ thus (C13) holds.

- (Cu14)

$$\frac{1}{r^{1+2(h(t)-1/\alpha(t))}} \int_{\mathbf{R}} (f(t+r,t,x) - f(t,t,x))^2 \, m(dx) = \int_{\mathbf{R}} \left| |1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}} \right|^2 dx$$

  thus, choosing $g(t) = \int_{\mathbf{R}} \left| |1-x|^{H(t)-\frac{1}{\alpha(t)}} - |x|^{H(t)-\frac{1}{\alpha(t)}} \right|^2 dx$, (Cu14) holds.

- (Cu15) $\forall v \in U$, $\forall u \in U$,

$$\frac{1}{|v-u|^2} \int_{\mathbf{R}} |f(v,v,x) - f(v,u,x)|^2 \, m(dx) =$$

$$\frac{1}{|v-u|^2} \int_{\mathbf{R}} \left| |v-x|^{H(v)-\frac{1}{\alpha(v)}} - |v-x|^{H(u)-\frac{1}{\alpha(u)}} - |x|^{H(v)-\frac{1}{\alpha(v)}} + |x|^{H(u)-\frac{1}{\alpha(u)}} \right|^2 dx$$

  thus (Cu15) holds ∎

6 Proof of Theorem 3.7

Recall the definition of the Lévy Multistable field on $[0,1]$:

$$X(v,u) = C_{\alpha(u)}^{1/\alpha(u)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} \mathbf{1}_{[0,v]}(V_i).$$

To prove Theorem 3.7, we need a series of lemma:

Lemma 6.16 *Assume α is $\mathcal{C}^1$. Then, for all $u \in (0,1)$, almost surely,*

$$\sup_{v\in[0,1]} \frac{|X(v,v) - X(v,u)|}{|v-u|} < +\infty.$$

Proof
in the case of the Lévy multistable field, (5.8) reads:

$$X(v,v) - X(v,u) = (v-u) \left(\sum_{i=1}^{+\infty} Z_i^1(v) + \sum_{i=1}^{+\infty} Z_i^3(v) + \sum_{i=1}^{+\infty} Y_i^1(v) + \sum_{i=1}^{+\infty} Y_i^3(v) \right),$$

where $Z_i^1, \ldots$ are defined as above. Let $A > 0$ and $B > 0$ be constants such that $\forall w \in U$, $|a'(w)| \leq A$ and $|a(w)\frac{\alpha'(w)}{\alpha^2(w)}| \leq B$. We write $\sum\limits_{i=1}^{+\infty} Z_i^1(v) = \sum\limits_{j=1}^{+\infty} \left(\sum\limits_{i=2^j}^{2^{j+1}-1} Z_i^1(v) \right) =: \sum\limits_{j=1}^{+\infty} X_j^1(v)$

and $\sum\limits_{i=1}^{+\infty} Z_i^3(v) = \sum\limits_{j=1}^{+\infty} \left(\sum\limits_{i=2^j}^{2^{j+1}-1} Z_i^3(v) \right) =: \sum\limits_{j=1}^{+\infty} X_j^3(v)$. We consider $\liminf_j \{ \sup\limits_{v\in[0,1]} |X_j^1(v)| \leq \frac{Aj\sqrt{2^j}}{2^{j/d}} \}$ and $\liminf_j \{ \sup\limits_{v\in[0,1]} |X_j^3(v)| \leq \frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}} \}$. Let $V^{(1)}, V^{(2)}, ..., V^{(2^j)}$ denote the order statistics of the V_i (*i.e.* $V^{(1)} = \min V_i, \ldots$). Then:

$$\{ \sup_{v\in[0,1]} |X_j^1(v)| > \frac{Aj\sqrt{2^j}}{2^{j/d}} \} \subset \cup_{N\geq 1}^{2^j} \cup_{l_1,...,l_N \in [\![2^j, 2^{j+1}-1]\!]} \left(\{ |\sum_{i=1}^{N} \gamma_{l_i} a'(w_{l_i}) l_i^{-1/\alpha(w_{l_i})}| > \frac{Aj\sqrt{2^j}}{2^{j/d}} \} .. \right.$$

$$\left. .. \cap \{ V^{(1)} = V_{l_1}, V^{(2)} = V_{l_2}, ..., V^{(N)} = V_{l_N} \} \right).$$

$$\begin{aligned}
\mathsf{P}\left(\sup_{v\in[0,1]} |X_j^1(v)| > \frac{Aj\sqrt{2^j}}{2^{j/d}} \right) &\leq \sum_{N=1}^{2^j} \sum_{l_1,...,l_N \in [\![2^j, 2^{j+1}-1]\!]} \frac{(2^j - N)!}{(2^j)!} \mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} a'(w_{l_i}) l_i^{-1/\alpha(w_{l_i})}| > \frac{Aj\sqrt{2^j}}{2^{j/d}} \right) \\
&\leq \sum_{N=1}^{2^j} \sum_{l_1,...,l_N \in [\![2^j, 2^{j+1}-1]\!]} \frac{(2^j - N)!}{(2^j)!} \mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} \frac{a'(w_{l_i})}{A} \frac{2^{j/d}}{l_i^{1/\alpha(w_{l_i})}}| > j\sqrt{N} \right) \\
&\leq \sum_{N=1}^{2^j} \sum_{l_1,...,l_N \in [\![2^j, 2^{j+1}-1]\!]} \frac{(2^j - N)!}{(2^j)!} 2e^{-\frac{j^2}{2}} \\
&\leq 2e^{-\frac{j^2}{2}} \sum_{N=1}^{2^j} \frac{1}{N!} \\
&\leq 2e^{1-\frac{j^2}{2}}
\end{aligned}$$

where we have used the following inequality (lemma 1.5, chapter 1 in [17]):

$$\mathsf{P}\left(|\sum_{i=1}^{n} u_i| \geq \lambda\sqrt{n}\right) \leq 2e^{-\frac{\lambda^2}{2}}$$

for $(u_i)_i$ independent centered random variables verifying $-1 \leq u_i \leq 1$, with $u_i = \gamma_{l_i}\frac{a'(w_{l_i})}{A}\frac{2^{j/d}}{l_i^{1/\alpha(w_{l_i})}}$ and $\lambda = j$.

We deduce that $\mathsf{P}\left(\liminf_j \left\{\sup_{v\in[0,1]} |X_j^1(v)| \leq \frac{Aj\sqrt{2^j}}{2^{j/d}}\right\}\right) = 1.$

Similarly:

$$\mathsf{P}\left(\sup_{v\in[0,1]} |X_j^3(v)| > \frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}}\right) \leq 2e^{1-\frac{j^2}{2}}$$

and $\mathsf{P}\left(\liminf_j \left\{\sup_{v\in[0,1]} |X_j^3(v)| \leq \frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}}\right\}\right) = 1.$ We work on the event

$$\liminf_j \left\{\sup_{v\in[0,1]} |X_j^1(v)| \leq \frac{Aj\sqrt{2^j}}{2^{j/d}}\right\} \cap \liminf_j \left\{\sup_{v\in[0,1]} |X_j^3(v)| \leq \frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}}\right\} \cap \liminf_i \{\Gamma_i > 1\}.$$

There exists $J_0 \in \mathbb{N}$ such that $\forall j \geq J_0$, $\sup_{v\in[0,1]} |X_j^1(v)| \leq \frac{Aj\sqrt{2^j}}{2^{j/d}}$ and $\sup_{v\in[0,1]} |X_j^3(v)| \leq \frac{\log(2)Bj(j+1)\sqrt{2^j}}{2^{j/d}}$.

$$\left|\sum_{i=1}^{+\infty} Z_i^1(v)\right| \leq \sum_{j=0}^{2^{J_0}-1} \frac{A}{i^{1/d}} + \sum_{j=J_0}^{+\infty} A\frac{j}{2^{j(\frac{1}{d}-\frac{1}{2})}}$$

and

$$\left|\sum_{i=1}^{+\infty} Z_i^3(v)\right| \leq \sum_{j=0}^{2^{J_0}-1} \frac{B\log(i)}{i^{1/d}} + \sum_{j=J_0}^{+\infty} B\log(2)\frac{j(j+1)}{2^{j(\frac{1}{d}-\frac{1}{2})}},$$

thus $\sup_{v\in[0,1]} \left|\sum_{i=1}^{+\infty} Z_i^1(v)\right| < +\infty$ and $\sup_{v\in[0,1]} \left|\sum_{i=1}^{+\infty} Z_i^3(v)\right| < +\infty$.

Fix $i_0 \in \mathbb{N}$ such that $\forall i \geq i_0$, $\Gamma_i > 1$.

$$|\sum_{i=1}^{i_0} Y_i^1(v)| \leq A\sum_{i=1}^{i_0}(\frac{1}{\Gamma_i^{1/c}} + \frac{1}{i^{1/d}})$$

and

$$|\sum_{i=1}^{i_0} Y_i^3(v)| \leq B\sum_{i=1}^{i_0}\left(|\frac{\log\Gamma_i}{\Gamma_i^{1/c}}| + \frac{\log(i)}{i^{1/d}}\right).$$

$$|\sum_{i=i_0}^{+\infty} Y_i^1(v)| \leq A\sum_{i=i_0}^{+\infty} |\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}|\mathbf{1}_{\{1 < \Gamma_i \leq \frac{i}{2}\}}$$

$$+A\sum_{i=i_0}^{+\infty} |\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}|\mathbf{1}_{\{\frac{i}{2} < \Gamma_i \leq 2i\}}$$

$$+A\sum_{i=i_0}^{+\infty} |\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}|\mathbf{1}_{\{\Gamma_i > 2i\}},$$

$$|\sum_{i=i_0}^{+\infty} Y_i^1(v)| \leq 2A\sum_{i=i_0}^{+\infty} (\mathbf{1}_{\{1 < \Gamma_i \leq \frac{i}{2}\}} + \mathbf{1}_{\{\Gamma_i > 2i\}}) + A\sum_{i=i_0}^{+\infty} |\Gamma_i^{-1/\alpha(x_i)} - i^{-1/\alpha(x_i)}|\mathbf{1}_{\{\frac{i}{2} < \Gamma_i \leq 2i\}}$$

$$\leq 2A\sum_{i=i_0}^{+\infty} (\mathbf{1}_{\{1 < \Gamma_i \leq \frac{i}{2}\}} + \mathbf{1}_{\{\Gamma_i > 2i\}}) + K_{c,d}\sum_{i=i_0}^{+\infty} \frac{1}{i^{\frac{1}{d}}}|\frac{\Gamma_i}{i} - 1|.$$

$$|\sum_{i=i_0}^{+\infty} Y_i^3(v)| \leq B\sum_{i=i_0}^{+\infty} |\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i)i^{-1/\alpha(x_i)}|\mathbf{1}_{\{1 < \Gamma_i \leq \frac{i}{2}\}}$$

$$+B\sum_{i=i_0}^{+\infty} |\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i)i^{-1/\alpha(x_i)}|\mathbf{1}_{\{\frac{i}{2} < \Gamma_i \leq 2i\}}$$

$$+B\sum_{i=i_0}^{+\infty} |\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i)i^{-1/\alpha(x_i)}|\mathbf{1}_{\{\Gamma_i > 2i\}},$$

$$|\sum_{i=i_0}^{+\infty} Y_i^3(v)| \leq K\sum_{i=i_0}^{+\infty} \log(i)(\mathbf{1}_{\{1 < \Gamma_i \leq \frac{i}{2}\}} + \mathbf{1}_{\{\Gamma_i > 2i\}})$$

$$+B\sum_{i=i_0}^{+\infty} |\log(\Gamma_i)\Gamma_i^{-1/\alpha(x_i)} - \log(i)i^{-1/\alpha(x_i)}|\mathbf{1}_{\{\frac{i}{2} < \Gamma_i \leq 2i\}}$$

$$\leq K\sum_{i=i_0}^{+\infty} \log(i)(\mathbf{1}_{\{1 < \Gamma_i \leq \frac{i}{2}\}} + \mathbf{1}_{\{\Gamma_i > 2i\}}) + K_{c,d}\sum_{i=i_0}^{+\infty} \frac{\log(i)}{i^{\frac{1}{d}}}|\frac{\Gamma_i}{i} - 1|.$$

Finally, $\sup_{v \in [0,1]} \left|\sum_{i=1}^{+\infty} Y_i^1(v)\right| < +\infty$ and $\sup_{v \in [0,1]} \left|\sum_{i=1}^{+\infty} Y_i^3(v)\right| < +\infty.$

As a consequence, $\sup_{v \in [0,1]} \frac{|X(v,v) - X(v,u)|}{|v-u|} < +\infty$ ∎

Lemma 6.17 *For all $u \in (0,1)$ and all $\eta \in (0, \frac{1}{\alpha(u)})$, one has, almost surely,*

$$\sup_{v \in [0,1]} \left|\frac{X(v,u) - X(u,u)}{|v-u|^\eta}\right| < +\infty.$$

Proof

Let $\eta \in (0, \frac{1}{\alpha(u)})$, $m \in \mathbb{N}$, $C_j = \cap_{i=2^j}^{2^{j+1}-1}\{V_i \notin [u - \frac{1}{j^2 2^j}, u + \frac{1}{j^2 2^j}]\}$,

$$D_j^m = \left\{ \sup_{\frac{1}{2^{m+1}} \leq |v-u| \leq \frac{1}{2^m}} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| \leq \frac{1}{j^2} \right\},$$

and $D_j = \cap_{m \geq 0} D_j^m$. D_j may be written:

$$D_j = \left\{ \sup_{v \in [0,1]} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| \leq \frac{1}{j^2} \right\}.$$

Let us evaluate $\liminf C_j$.

$$\mathsf{P}\left(\overline{C_j}\right) \leq \sum_{i=2^j}^{2^{j+1}-1} \frac{1}{j^2 2^j} = \frac{1}{j^2}$$

and thus $\mathsf{P}\left(\liminf_j C_j\right) = 1$. Now,

$$
\begin{aligned}
\mathsf{P}\left(\overline{D_j}\right) &\leq \frac{1}{j^2} + \mathsf{P}(\overline{D_j} \cap C_j) \\
&= \frac{1}{j^2} + \mathsf{P}\left(\cup_{m \geq 0}(\overline{D_j^m} \cap C_j)\right) \\
&\leq \frac{1}{j^2} + \sum_{m=0}^{+\infty} \mathsf{P}\left(\overline{D_j^m} \cap C_j\right).
\end{aligned}
$$

We consider several cases, depending on the respective values of j and m:

- If $m > j + \frac{2}{\log(2)} \log j$,

$$\mathsf{P}\left(\overline{D_j^m} \cap C_j\right) = 0.$$

- If $j + \frac{2}{\log(2)} \log j \geq m \geq j$,

$$\mathsf{P}\left(\overline{D_j^m}\right) \leq \mathsf{P}\left(\sup_{\frac{1}{2^{m+1}} \leq |v-u| \leq \frac{1}{2^m}} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \mathbf{1}_{[u,v]}(V_i) \right| \geq \frac{1}{2^{(m+1)\eta} j^2} \right).$$

Let $J_0 \in \mathbb{N}$ be such that for all $j > J_0$, $2^{j(\frac{1}{\alpha(u)} - \eta)} > 2^\eta j^{3 + \frac{2\eta}{\log(2)}}$. The event:

$$\left\{ \sup_{\frac{1}{2^{m+1}} \leq |v-u| \leq \frac{1}{2^m}} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \mathbf{1}_{[u,v]}(V_i) \right| \geq \frac{1}{2^{(m+1)\eta} j^2} \right\}$$

is included in the event

$$\cup_{N \geq 1}^{2^j} \left(\cup_{l_1,\dots,l_N \in [\![2^j, 2^{j+1}-1]\!]} \{ |\sum_{i=1}^N \gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta} j^2} \} \cap (\cap_{i=1}^N \{|V_{l_i} - u| \in [\frac{1}{2^{m+1}}, \frac{1}{2^m}]\}) \right) ..$$

38

$$.. \cap \left(\cap_{k \neq l_i} \{|V_k - u| \notin [\frac{1}{2^{m+1}}, \frac{1}{2^m}]\} \right).$$

Notice that for $j \geq J_0$ and $N < j$, $\mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta}j^2} \right) = 0$, and thus

$$
\begin{aligned}
\mathsf{P}\left(\overline{D_j^m} \right) \; &\leq \; \sum_{N=j}^{2^j} \sum_{l_1,\ldots,l_N \in [\![2^j, 2^{j+1}-1]\!]} \mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta}j^2} \right) \mathsf{P}\left(\cap_{i=1}^{N} \{|V_{l_i} - u| \in [\frac{1}{2^{m+1}}, \frac{1}{2^m}]\} \right) \\
&\leq \; \sum_{N=j}^{2^j} \frac{1}{2^{(m+1)N}} \sum_{l_1,\ldots,l_N \in [\![2^j, 2^{j+1}-1]\!]} \mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta}j^2} \right) \\
&\leq \; \sum_{N=j}^{2^j} \frac{1}{2^{(m+1)N}} \sum_{l_1,\ldots,l_N \in [\![2^j, 2^{j+1}-1]\!]} j^4 2^{2(m+1)\eta} \sum_{i=2^j}^{2^{j+1}-1} \frac{1}{i^{\frac{2}{\alpha(u)}}} \\
&\leq \; \sum_{N=j}^{2^j} \frac{j^4 2^{2(m+1)\eta}}{2^{(m+1)N}} 2^{j(1-\frac{2}{\alpha(u)})} C_{2^j}^N \\
&\leq \; j^4 2^{2(j+\frac{2}{\log(2)}\log j + 1)\eta - j\frac{2}{\alpha(u)}} \sum_{N=j}^{2^j} \frac{2^j C_{2^j}^N}{2^{(m+1)N}} \\
&\leq \; j^{4+\frac{4\eta}{\log(2)}} 2^{2j(\eta - \frac{1}{\alpha(u)})} \sum_{N=j}^{2^j} \frac{2^{j-N} 2^{(j-m)N}}{N!} \\
&\leq \; 3 j^{4+\frac{4\eta}{\log(2)}} 2^{2j(\eta - \frac{1}{\alpha(u)})}.
\end{aligned}
$$

- When $j \geq m \geq \frac{\log(j)}{\log(2)}$, the same computations lead to:

$$
\begin{aligned}
&\sum_{N=j2^{j-m}}^{2^j} \sum_{l_1,\ldots,l_N \in [\![2^j, 2^{j+1}-1]\!]} \mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta}j^2} \right) \mathsf{P}\left(\cap_{i=1}^{N} \{|V_{l_i} - u| \in [\frac{1}{2^{m+1}}, \frac{1}{2^m}]\} \right) \\
&\leq \; \sum_{N=j2^{j-m}}^{2^j} \frac{j^4 2^{2(m+1)\eta}}{2^{(m+1)N}} 2^{j(1-\frac{2}{\alpha(u)})} C_{2^j}^N \\
&\leq \; j^4 2^{2(m+1)\eta - 2j/\alpha(u)} \sum_{N=j2^{j-m}}^{2^j} \frac{2^{j-N} 2^{(j-m)N}}{N!} \\
&\leq \; j^4 2^{2\eta} 2^{2j(\eta - \frac{1}{\alpha(u)})} \sum_{N=j2^{j-m}}^{+\infty} \frac{2^{(j-m)N}}{N!} \\
&\leq \; K j^4 2^{2j(\eta - \frac{1}{\alpha(u)})} \frac{e^{2^{j-m}} 2^{(j-m)(j2^{j-m}+1)}}{(j2^{j-m}+1)!}
\end{aligned}
$$

where we have used the estimate $\sum_{n \geq N} \frac{x^n}{n!} \leq e^x \frac{x^{N+1}}{(N+1)!}$. We arrive at:

$$
\begin{aligned}
\mathsf{P}\left(\overline{D_j^m}\right) \;\leq\; & K j^4 2^{2j(\eta - \frac{1}{\alpha(u)})} + \\
& \sum_{N=1}^{j 2^{j-m}} \frac{1}{2^{(m+1)N}} (1 - \frac{1}{2^{m+1}})^{2^j - N} \sum_{l_1,\ldots,l_N \in [\![2^j, 2^{j+1}-1]\!]} \mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta} j^2} \right).
\end{aligned}
$$

We need to distinguish two cases depending on the value of η. If $\eta \leq \frac{1}{2}$, fix $J_1 \in \mathbb{N}$ such that for all $j \geq J_1$, $2^{j(\frac{1}{\alpha(u)} - \frac{1}{2})} > 2^{1/\alpha(u)} j^3 \sqrt{j}$. If $\eta > \frac{1}{2}$, fix $J_1 \in \mathbb{N}$ such that for all $j \geq J_1$, $2^{j(\frac{1}{\alpha(u)} - \eta)} > 2^{1/\alpha(u)} j^3 \sqrt{j}$. Then for all η and all $j \geq J_1$, one has $\frac{2^{j/\alpha(u)}}{j^3 \sqrt{j 2^{j-m}} 2^{(m+1)\eta}} \geq 1$ and

$$
\begin{aligned}
\mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta} j^2} \right) \;\leq\; & \mathsf{P}\left(\left| \sum_{i=1}^{N} \gamma_{l_i} \left(\frac{2^j}{l_i} \right)^{1/\alpha(u)} \right| > j\sqrt{N} \right) \\
\leq\; & 2 e^{-j^2/2}.
\end{aligned}
$$

We then get

$$
\begin{aligned}
\mathsf{P}\left(\overline{D_j^m}\right) \;\leq\; & K j^4 2^{2j(\eta - \frac{1}{\alpha(u)})} + \sum_{N=1}^{j 2^{j-m}} \frac{1}{2^{(m+1)N}} (1 - \frac{1}{2^{m+1}})^{2^j - N} C_{2^j}^N 2 e^{-j^2/2} \\
\leq\; & K j^4 2^{2j(\eta - \frac{1}{\alpha(u)})} + 2 e^{-j^2/2} \sum_{N=1}^{2^j} \frac{1}{2^{(m+1)N}} (1 - \frac{1}{2^{m+1}})^{2^j - N} C_{2^j}^N \\
\leq\; & K j^4 2^{2j(\eta - \frac{1}{\alpha(u)})} + 2 e^{-j^2/2}.
\end{aligned}
$$

- Assume finally that $m \leq \frac{\log(j)}{\log(2)}$.

Fix $J_2 \in \mathbb{N}$ such that for all $j \geq J_2$, $2^{j(\frac{1}{\alpha(u)} - \frac{1}{2})} > 2^{1/\alpha(u)} j^{3+\eta}$. Then, for $j \geq J_2$, one has $\frac{2^{j/\alpha(u)}}{j^3 \sqrt{2^j} 2^{(m+1)\eta}} \geq 1$ and computations similar the ones above lead to

$$
\begin{aligned}
\mathsf{P}\left(\overline{D_j^m}\right) \;\leq\; & \sum_{N=1}^{2^j} \frac{1}{2^{(m+1)N}} (1 - \frac{1}{2^{m+1}})^{2^j - N} \sum_{l_1,\ldots,l_N \in [\![2^j, 2^{j+1}-1]\!]} \mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} l_i^{-1/\alpha(u)}| > \frac{1}{2^{(m+1)\eta} j^2} \right) \\
\leq\; & \sum_{N=1}^{2^j} \frac{1}{2^{(m+1)N}} (1 - \frac{1}{2^{m+1}})^{2^j - N} \sum_{l_1,\ldots,l_N \in [\![2^j, 2^{j+1}-1]\!]} \mathsf{P}\left(|\sum_{i=1}^{N} \gamma_{l_i} (\frac{2^{j/\alpha(u)}}{l_i})^{1/\alpha(u)}| > j\sqrt{N} \right) \\
\leq\; & 2 e^{-j^2/2}.
\end{aligned}
$$

We thus get that, for $j \geq \max(J_0, J_1, J_2)$,

$$
\sum_{m=0}^{+\infty} \mathsf{P}\left(\overline{D_j^m} \cap C_j\right) \;\leq\; K \log(j) j^{4 + \frac{4\eta}{\log(2)}} 2^{2j(\eta - \frac{1}{\alpha(u)})},
$$

40

and thus $\mathsf{P}\left(\liminf_j D_j\right) = 1$.

On the event $\liminf_j C_j \cap \liminf_j D_j$, we may fix $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$\sup_{v \in [0,1]} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| \leq \frac{1}{j^2}.$$

Since $\displaystyle\sup_{v \in [0,1]} \left| \sum_{i=1}^{2^{j_0}-1} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| < +\infty$, we obtain

$$\sup_{v \in [0,1]} \left| \sum_{i=1}^{+\infty} \gamma_i i^{-1/\alpha(u)} \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| < +\infty.$$

Let us now deal with

$$E_j = \left\{ \sup_{v \in [0,1]} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i (\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}) \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta} \right| \leq \frac{1}{j^2} \right\}.$$

$$
\begin{aligned}
\mathsf{P}\left(\overline{E_j}\right) &\leq \frac{1}{j^2} + \mathsf{P}(\overline{E_j} \cap C_j) \\
&\leq \frac{1}{j^2} + \mathsf{P}\left(2^{j\eta} j^{2\eta} \sup_{v \in [0,1]} \left| \sum_{i=2^j}^{2^{j+1}-1} \gamma_i (\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}) \mathbf{1}_{[u,v]}(V_i) \right| > \frac{1}{j^2} \right) \\
&\leq \frac{1}{j^2} + \mathsf{P}\left(\sum_{i=2^j}^{2^{j+1}-1} \left| (\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}) \right| > \frac{1}{2^{j\eta} j^{2(1+\eta)}} \right) \\
&\leq \frac{1}{j^2} + 2^{j\eta} j^{2(1+\eta)} \sum_{i=2^j}^{2^{j+1}-1} \mathsf{E}|\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}| \\
&\leq \frac{1}{j^2} + 2^{j\eta} j^{2(1+\eta)} \sum_{i=2^j}^{2^{j+1}-1} 2\left(\mathsf{P}(\Gamma_i < \frac{i}{2}) + \mathsf{P}(\Gamma_i > 2i)\right) \\
&\quad + 2^{j\eta} j^{2(1+\eta)} \sum_{i=2^j}^{2^{j+1}-1} \mathsf{E}|\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}| \mathbf{1}_{\{\frac{i}{2} < \Gamma_i < 2i\}}.
\end{aligned}
$$

However

$$
\begin{aligned}
\mathsf{E}|\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}| \mathbf{1}_{\{\frac{i}{2} < \Gamma_i < 2i\}} &\leq \frac{1}{i^{1/\alpha(u)}} K_u \mathsf{E}\left|\frac{\Gamma_i}{i} - 1\right| \\
&\leq K_u \frac{1}{i^{1 + \frac{1}{\alpha(u)}}}
\end{aligned}
$$

and

$$2^{j\eta} j^{2(1+\eta)} \sum_{i=2^j}^{2^{j+1}-1} \mathsf{E}|\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}| \mathbf{1}_{\{\frac{i}{2} < \Gamma_i < 2i\}} \leq K j^{2(1+\eta)} 2^{j(\eta - \frac{1}{\alpha(u)})}.$$

41

We thus obtain $\mathsf{P}(\liminf_j E_j) = 1$. As a consequence, $\sup\limits_{v\in[0,1]} \left|\sum\limits_{i=1}^{+\infty} \gamma_i(\Gamma_i^{-1/\alpha(u)} - i^{-1/\alpha(u)}) \frac{\mathbf{1}_{[u,v]}(V_i)}{|v-u|^\eta}\right| < +\infty$ and finally

$$\sup_{v\in[0,1]} \left|\frac{X(v,u) - X(u,u)}{|v-u|^\eta}\right| < +\infty$$

■

Lemma 6.18 *For all $u \in (0,1)$, one has almost surely, for all $\eta \in (0, \frac{1}{\alpha(u)})$,*

$$\sup_{v\in[0,1]} \frac{|X(v,u) - X(u,u)|}{|v-u|^\eta} < +\infty.$$

Proof
Fix $u \in (0,1)$. Lemma 6.17 yields that, for all $\eta \in (0, \frac{1}{\alpha(u)})$, we may choose an Ω_η having probability one and such that, on Ω_η, $\sup\limits_{v\in[0,1]} \left|\frac{X(v,u)-X(u,u)}{|v-u|^\eta}\right| < +\infty$. Thus, on $\Omega = \cap_{j\geq 0}\Omega_{\frac{1}{\alpha(u)} - \frac{1}{2^j}}$, which still has probability one, it holds that, for all $\eta \in (0, \frac{1}{\alpha(u)})$, $\sup_{v\in[0,1]} \frac{|X(v,u)-X(u,u)|}{|v-u|^\eta} < +\infty$ ■

Proof of Theorem 3.7
From Theorem 3.6, we already know that $\mathcal{H}_u \leq \frac{1}{\alpha(u)}$. To prove the reverse inequality, we treat separately the situations where $\alpha < 1$ and $\alpha \geq 1$.

- Consider first the case $0 < \alpha(u) < 1$.

Write:

$$Y(v) - Y(u) = X(v,v) - X(v,u) + X(v,u) - X(u,u).$$

By Lemma 6.18, we know that the Hölder regularity of $v \mapsto X(v,u) - X(u,u)$ at u is almost surely not smaller than $\frac{1}{\alpha(u)}$. Now, by applying the finite increments theorem to the functions $t \mapsto C_t^{1/t}\Gamma_i^{-1/t}$, we get

$$
\begin{aligned}
X(v,v) - X(v,u) &= \sum_{i=1}^{\infty} \gamma_i \mathbf{1}_{[0,v]}(V_i) \left(C_{\alpha(v)}^{1/\alpha(v)}\Gamma_i^{-1/\alpha(v)} - C_{\alpha(u)}^{1/\alpha(u)}\Gamma_i^{-1/\alpha(u)}\right) \\
&= (\alpha(v) - \alpha(u)) \sum_{i=1}^{\infty} \gamma_i \mathbf{1}_{[0,v]}(V_i) \left(CP(\alpha(w_i)) - C_{\alpha(w_i)}^{1/\alpha(w_i)} \frac{\log\Gamma_i}{\alpha(w_i)^2}\right) \Gamma_i^{-1/\alpha(w_i)},
\end{aligned}
$$

where, for each i, $w_i \in [u,v]$ (or $[v,u]$), and CP denotes the derivative of the function $t \mapsto C_t^{1/t}$. However,

$$
\begin{aligned}
\left|\sum_{i=1}^{\infty} \gamma_i \mathbf{1}_{[0,v]}(V_i) \left(Cp(\alpha(w_i)) - \frac{\log\Gamma_i}{\alpha(w_i)^2}\right) \Gamma_i^{-1/\alpha(w_i)}\right| &\leq \sum_{i=1}^{\infty} \left|CP(\alpha(w_i)) - C_{\alpha(w_i)}^{1/\alpha(w_i)} \frac{\log\Gamma_i}{\alpha(w_i)^2}\right| \Gamma_i^{-1/\alpha(w_i)} \\
&\leq K \sum_{i=1}^{\infty} (1 + |\log\Gamma_i|) \left(\Gamma_i^{-1/c} + \Gamma_i^{-1/d}\right).
\end{aligned}
$$

Thus the quantity $T(u,v) = \sum_{i=1}^{\infty} \gamma_i \mathbf{1}_{[0,v]}(V_i) \left(CP(\alpha(w_i)) - C_{\alpha(w_i)}^{1/\alpha(w_i)} \frac{\log\Gamma_i}{\alpha(w_i)^2}\right) \Gamma_i^{-1/\alpha(w_i)}$ is, uniformly in v, almost surely finite and not 0. As a consequence, the function $v \mapsto$

$X(v,v) - X(v,u) = (\alpha(u) - \alpha(v))T(u,v)$ has almost surely the same Hölder exponent at u as the function $v \mapsto \alpha(v)$ at u. If $\mathcal{H}_u^\alpha < \frac{1}{\alpha(u)}$, this entails that Y has exponent $\mathcal{H}_u^\alpha$ at u. If $\mathcal{H}_u^\alpha > \frac{1}{\alpha(u)}$, then the exponent of Y at u is at least $\frac{1}{\alpha(u)}$ and thus exactly $\frac{1}{\alpha(u)}$ by Theorem 3.6.

- Assume now that $1 \leq \alpha(u) < 2$.

Let $\eta < \frac{1}{\alpha(u)}$ and $\delta \in (\eta, \frac{1}{\alpha(u)})$. Then:

$$\frac{|Y(v) - Y(u)|}{|v - u|^\eta} \leq \frac{|X(v,v) - X(v,u)|}{|v - u|^\eta} + \frac{|X(v,u) - X(u,u)|}{|v - u|^\eta}.$$

By lemma 6.18, there exists $K > 0$ such that $\frac{|X(v,u) - X(u,u)|}{|v-u|^\eta} \leq K|v - u|^{\delta - \eta}$, and, by Lemma 6.16, there exists $K > 0$ such that $\frac{|X(v,v) - X(v,u)|}{|v-u|^\eta} \leq K|v - u|^{1-\eta}$. This entails $\lim_{v \to u} \frac{|Y(v) - Y(u)|}{|v-u|^\eta} = 0$ and

$$\mathcal{H}_u \geq \frac{1}{\alpha(u)}$$

∎

7 Assumptions

This section gathers the various conditions required on the considered processes so that our results hold.

- (C1) The family of functions $v \to f(t, v, x)$ is differentiable for all (v, t) in U^2 and almost all x in E. The derivatives of f with respect to v are denoted by f'_v.

- (C2) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[\sup_{w \in U} (|f(t, w, x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx) < \infty.$$

- (Cs2) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[\sup_{w \in U} (|f(t, w, x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^\delta \, m(dx) < \infty.$$

- (C3) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[\sup_{w \in U} (|f'_v(t, w, x)|^{\alpha(w)}) \right]^{1+\delta} \hat{m}(dx) < \infty.$$

- (Cs3) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[\sup_{w \in U} (|f'_v(t, w, x)|^{\alpha(w)}) \right]^{1+\delta} r(x)^\delta \, m(dx) < \infty.$$

- (Cs4) There exists $\delta > \frac{d}{c} - 1$ such that :

$$\sup_{t \in U} \int_{\mathbf{R}} \left[\sup_{w \in U} \left[|f(t, w, x) \log(r(x))|^{\alpha(w)} \right] \right]^{1+\delta} r(x)^{\delta} \, m(dx) < \infty.$$

- (C5) $X(t, u)$ (as a process in t) is localisable at u with exponent $h(u) \in (h_-, h_+) \subset (0, 1)$, with local form $X'_u(t, u)$, and $u \mapsto h(u)$ is a C^1 function .

- (C6) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbf{R}$,

$$|f(v, u, x)| \leq K_U.$$

- (C7) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbf{R}$,

$$|f'_v(v, u, x)| \leq K_U.$$

- (C8) There exists a function h defined on U, $\varepsilon_0 \in (0, 1)$ and $K_U > 0$ such that $\forall r < \varepsilon_0$, $\forall x \in \mathbf{R}$,

$$\frac{1}{r^{h(t)-1/\alpha(t)}} \, |f(t + r, t, x) - f(t, t, x)| \leq K_U.$$

- (Cu8) There exists a function h defined on U and $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$, $\forall x \in \mathbf{R}$,

$$\frac{1}{|v - u|^{h(u)-1/\alpha(u)}} \, |f(v, u, x) - f(u, u, x)| \leq K_U.$$

- (C9) There exists a function h defined on U, $\varepsilon_0 > 0$ and $K_U > 0$ such that $\forall r < \varepsilon_0$,

$$\frac{1}{r^{h(t)\alpha(t)}} \int_{\mathbf{R}} |f(t + r, t, x) - f(t, t, x)|^{\alpha(t)} m(dx) \leq K_U.$$

- (C10) There exists a function h defined on U and $p \in (\alpha(t), 2)$, $p \geq 1$, such that for all $\varepsilon > 0$, there exists $K_U > 0$ such that, $\forall r \leq \varepsilon$,

$$\frac{1}{r^{1+p(h(t)-\frac{1}{\alpha(t)})}} \int_{\mathbf{R}} |f(t + r, t, x) - f(t, t, x)|^p \, m(dx) \leq K_U.$$

- (Cu10) There exists a function h defined on U, $p \in (d, 2)$, $p \geq 1$ and $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$,

$$\frac{1}{|v - u|^{1+p(h(u)-\frac{1}{\alpha(u)})}} \int_{\mathbf{R}} |f(v, u, x) - f(u, u, x)|^p \, m(dx) \leq K_U.$$

- (C11) $\forall \varepsilon > 0$, $\exists K_U > 0$ such that, $\forall r \leq \varepsilon$,

$$\int_{\mathbf{R}} |f(t + r, t, x)|^2 \, m(dx) \leq K_U.$$

- (Cu11) There exists $K_U > 0$ such that $\forall v \in U$, $\forall u \in U$,

$$\int_{\mathbf{R}} |f(v, u, x)|^2 \, m(dx) \leq K_U.$$

- (C12) $\forall \varepsilon > 0$, $\exists K_U > 0$ such that $\forall r \leq \varepsilon$,

$$\int_{\mathbf{R}} |f(t + r, t + r, x)|^2 \, m(dx) \leq K_U.$$

- (Cu12) There exists $K_U > 0$ such that $\forall v \in U$,

$$\int_{\mathbf{R}} |f(v, v, x)|^2 \, m(dx) \leq K_U.$$

- (C13)

$$\inf_{v \in U} \int_{\mathbf{R}} f(v, v, x)^2 m(dx) > 0.$$

- (C14) There exists a function h and a positive function g defined on U such that

$$\lim_{r \to 0} \frac{1}{r^{1+2(h(t)-1/\alpha(t))}} \int_{\mathbf{R}} (f(t + r, t, x) - f(t, t, x))^2 \, m(dx) = g(t).$$

- (Cu14) There exists a function h and a positive function g defined on U such that

$$\lim_{r \to 0} \sup_{t \in U} \left| \frac{1}{r^{1+2(h(t)-1/\alpha(t))}} \int_{\mathbf{R}} (f(t + r, t, x) - f(t, t, x))^2 \, m(dx) - g(t) \right| = 0.$$

- (C15) $\forall \varepsilon > 0$, $\exists K_U > 0$ such that $\forall r \leq \varepsilon$,

$$\frac{1}{|r|^2} \int_{\mathbf{R}} |f(t + r, t + r, x) - f(t + r, t, x)|^2 \, m(dx) \leq K_U.$$

- (Cu15) $\exists K_U > 0$ such that, $\forall v \in U$, $\forall u \in U$,

$$\frac{1}{|v - u|^2} \int_{\mathbf{R}} |f(v, v, x) - f(v, u, x)|^2 \, m(dx) \leq K_U.$$

References

[1] AYACHE, A. AND LÉVY VÉHEL, J. (2000). The generalized multifractional Brownian motion. *Stat. Inference Stoch. Process.* **3,** 7–18.

[2] VON BAHR, B. AND ESSEN, C.G. (1965). Inequalities for the rth Absolute Moment of a Sum of Random Variables, $1 <= r <= 2$ *The Annals of Mathematical Statistics* **36,** (1), 299–303.

[3] BENASSI, A., JAFFARD, S. AND ROUX, D. (1997). Gaussian processes and pseudodifferential elliptic operators. *Rev. Mat. Iberoamericana* **13,** 19–89.

[4] BENTKUS, V., JUOZULYNAS, A. AND PAULAUSKAS, V. (2001). Lévy-LePage series representation of stable vectors: convergence in variation. *J. Theo. Prob.*(2) **14,** (4) 949–978.

[5] FALCONER, K.J. (2002). Tangent fields and the local structure of random fields. *J. Theoret. Probab.* **15,** 731–750.

[6] FALCONER, K.J. (2003). The local structure of random processes. *J. London Math. Soc.*(2) **67,** 657–672.

[7] FALCONER, K.J., LE GUÉVEL, R. AND LÉVY VÉHEL, J. (2009). Localisable moving average stable and multistable processes, *Stochastic Models*, 2009, no.4, 648–672.

[8] FALCONER, K.J. AND LÉVY VÉHEL, J. (2008). *Multifractional, multistable, and other processes with prescribed local form, J. Theoret. Probab.*, DOI 10.1007/s10959-008-0147-9.

[9] FALCONER, K.J., AND LINING, L. (2009). Multistable random measures and multistable processes, preprint.

[10] FERGUSON, T.S. AND KLASS, M.J. (1972). A representation of independent increment processes without Gaussian components. *Ann. Math. Stat.* **43,** 1634–1643.

[11] HERBIN, E. (2006). From N-parameter fractional Brownian motions to N-parameter multifractional Brownian motion. *Rocky Mountain J. Math.* **36,** 1249–1284.

[12] HERBIN, E. AND LÉVY VÉHEL, J. (2009). Stochastic 2 micro-local analysis. *Stoch. Proc. Applic.* **119**(7), 2277–2311.

[13] KOLMOGOROV, A.N. (1940). Wiensersche Spiralen und einige andere interessante Kurven in Hilbertchen Raume, *Doklady*, **26**, 115–118.

[14] LE GUÉVEL, R. AND LÉVY VÉHEL, J. (2009). *A Ferguson - Klass - LePage series representation of multistable multifractional motions and related processes*, preprint. Available at http://arxiv.org/abs/0906.5042.

[15] LE PAGE, R. (1980). Multidimensional infinitely divisible variables and processes. I. Stable case *Tech. Rep. 292, Dept. Stat, Stanford Univ.*

[16] LE PAGE, R. (1980). Multidimensional infinitely divisible variables and processes. II *Probability in Banach Spaces III Lecture notes in Math.* **860** 279–284, Springer, New York.

[17] LEDOUX, M. AND TALAGRAND, M. (1996). Probability in Banach spaces. *Springer-Verlag.*

[18] MANDELBROT, B.B. AND VAN NESS, J. (1968). Fractional Brownian motion, fractional noises and applications. *SIAM Rev.* **10,** 422–437.

[19] PELTIER, R.F. AND LÉVY VÉHEL, J. (1995). Multifractional Brownian motion: definition and preliminary results. *Rapport de recherche de l'INRIA*, No. 2645. Available at: http://www-rocq1.inria.fr/fractales/index.php?page=publications

[20] PETROV, V. (1995). *Limit Theorems of Probability Theory*, Oxford Science Publication.

[21] ROSINSKI, J. (1990). On Series Representations of Infinitely Divisible Random Vectors *Ann. Probab.*, **18,** (1) 405–430.

[22] SAMORODNITSKY, G. AND TAQQU, M.S. (1994). *Stable Non-Gaussian Random Processes*, Chapman and Hall.

[23] STOEV, S. AND TAQQU, M.S. (2004). Stochastic properties of the linear multifractional stable motion. *Adv. Appl. Probab.*, **36,** 1085–1115

[24] STOEV, S. AND TAQQU, M.S. (2005). Path properties of the linear multifractional stable motion. *Fractals*, **13,** 157–178.