



HAL
open science

Geometric predicates as arrangements of hypersurfaces: Application to comparison of algebraic numbers

George Tzoumas

► **To cite this version:**

George Tzoumas. Geometric predicates as arrangements of hypersurfaces: Application to comparison of algebraic numbers. Fall School Shapes, Geometry, and Algebra - SAGA 2010, Oct 2010, Kolympari, Greece. inria-00538839

HAL Id: inria-00538839

<https://inria.hal.science/inria-00538839>

Submitted on 23 Nov 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Geometric predicates as arrangements of hypersurfaces: Application to comparison of algebraic numbers

George M. Tzoumas

INRIA Nancy, France

Predicates

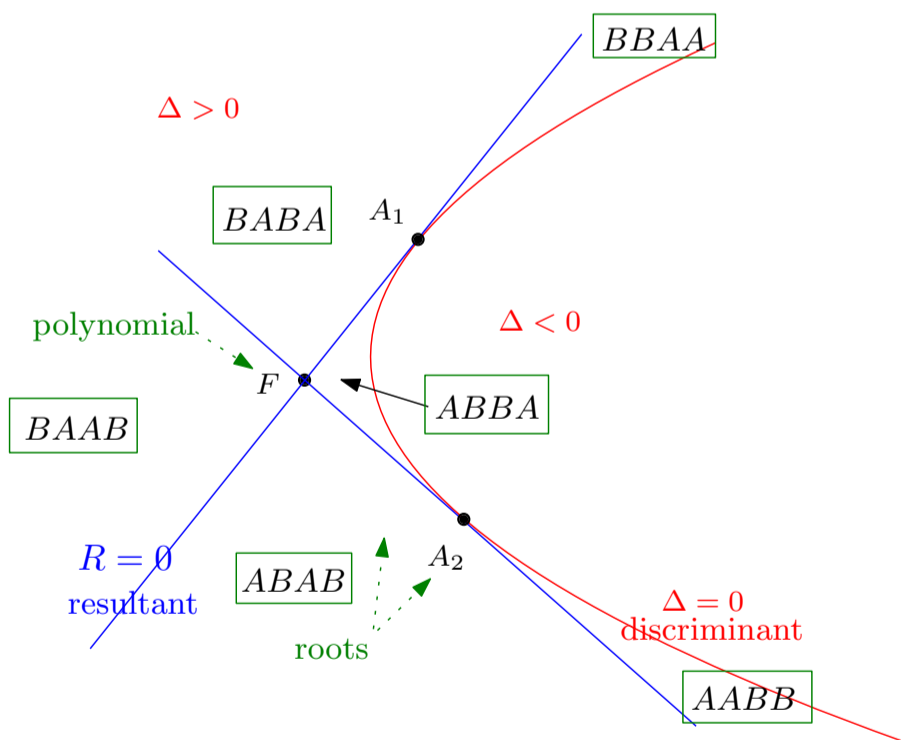
Formulation of a wide range of geometric predicates:

- arrangement of hypersurfaces in \mathbb{R}^d
- cells of dimension $\leq d$ (i.e., vertices, edges, faces)
- point location queries

Geometry in the space of coefficients

Given 2 polynomials of deg. 2, determine their root ordering among the 6 non-degenerate cases *AABB*, *ABAB*, *ABBA*, *BAAB*, *BABA*, *BBAA*.

Coefficients	\longleftrightarrow	point in affine space
$F(x) = x^2 + Xx + Y$	\longleftrightarrow	$(X, Y) \in \mathbb{R}^2$
$A_1^2 + XA_1 + Y = 0$	\longleftrightarrow	line for root A_1 of $F(x)$
all polynomials with A_1 as root	\longleftrightarrow	points on the line
quadratic $F(x)$ has 2 roots	\longleftrightarrow	2 lines intersect at a point
Resultant of 2 quadrics	\longleftrightarrow	union of 2 lines
Discriminant $X^2 - 4Y$	\longleftrightarrow	parabola Δ
roots of F	\longleftrightarrow	lines <i>tangent</i> to Δ
# roots of F	\longleftrightarrow	rel. pos. of F wrt. Δ



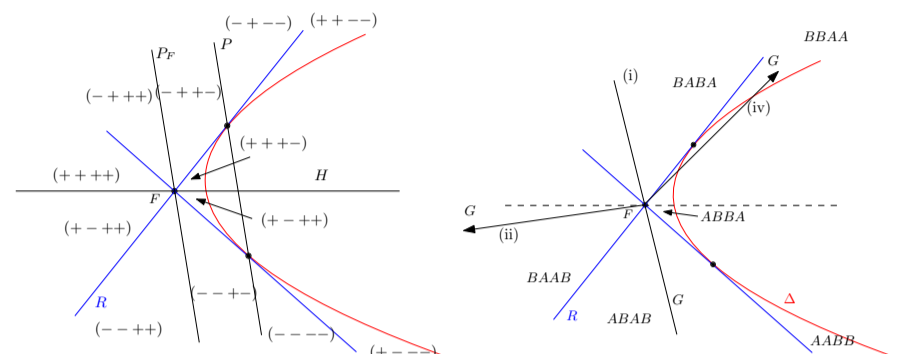
Arrangement consisting of the following hypersurfaces:

- **Resultant** (pair of lines)
- **Discriminant** (parabola)

points in same cell	\longleftrightarrow	same root ordering
crossing of resultant	\longleftrightarrow	roots of 2 polys equal
crossing of discriminant	\longleftrightarrow	roots of same poly equal

Decomposition

- exploiting conic properties of pole and polar (duality) or intersecting rays from F with Δ



- When $F(x) = a_2x^2 + a_1x + a_0$ and $G(x) = b_2x^2 + b_1x + b_0$ the tested quantities are:

$$R = b_0^2 a_2^2 - 2 b_0 a_0 b_2 a_2 + a_0^2 b_2^2 - a_1 b_1 b_0 a_2 - a_1 b_1 a_0 b_2 + a_1^2 b_0 b_2 + a_0 b_1^2 a_2$$

(resultant)

$$H = b_1 a_2 - a_1 b_2 \quad (\text{horizontal line})$$

$$P = -2 b_0 a_2 + a_1 b_1 - 2 a_0 b_2 \quad (\text{polar})$$

$$P_F = a_1 b_1 a_2 - a_1^2 b_2 - 2 b_0 a_2^2 + 2 a_0 a_2 b_2 \quad (\text{polar translated})$$

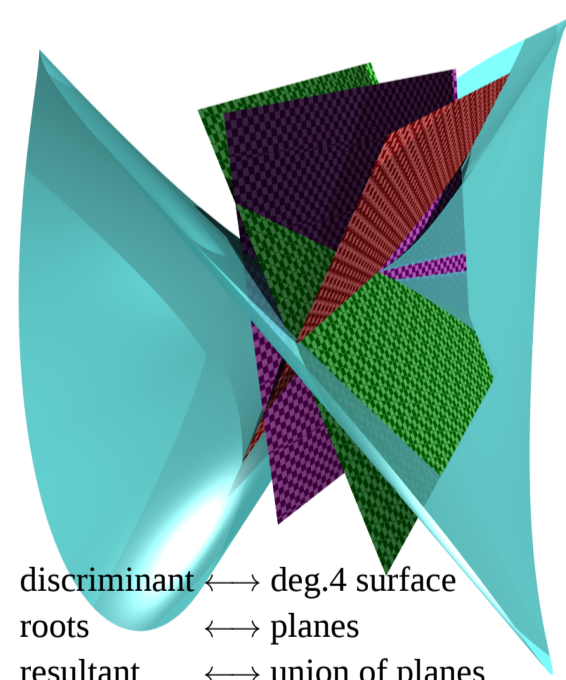
To compare algebraic numbers of degree 2, we have to consider quantities of at most algebraic degree 4 in the worst case. This bound is tight and therefore optimal.

Proof

- walls of arrangement defined by **resultant** (deg.4) and **discriminant** (deg.2)
- both irreducible, separating full-dimensional cells
- \Rightarrow they are *necessary* to define the arrangement
- all subsidiary equations have alg. deg. ≤ 4

Cubics

Given 2 polynomials of deg. 3, determine their root ordering among the $\binom{6}{3} + 2\binom{4}{1} + 2 = 30$ non-degenerate cases.



- quantities of alg. degree at most 6
- optimal, but geometric interpretation not yet known

Higher degree

- difficult to visualise
- *open problem*: optimal decomposition of the arrangement