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# Geometric predicates as arrangements of hypersurfaces: Application to comparison of algebraic numbers

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## Predicates

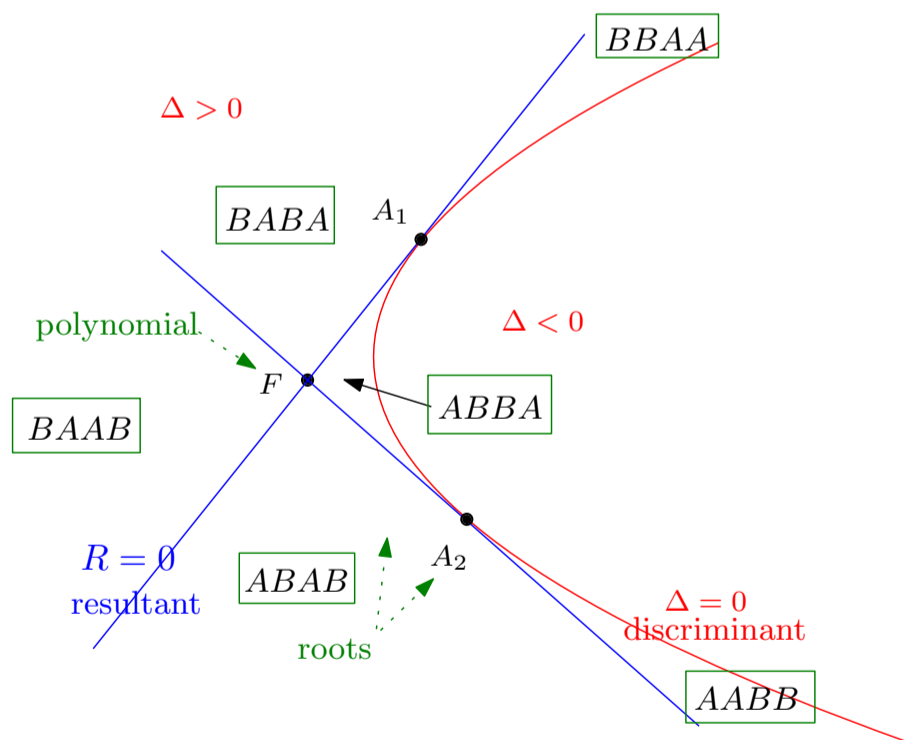
Formulation of a wide range of geometric predicates:

- arrangement of hypersurfaces in  $\mathbb{R}^d$
- cells of dimension  $\leq d$  (i.e., vertices, edges, faces)
- point location queries

## Geometry in the space of coefficients

Given 2 polynomials of deg. 2, determine their root ordering among the 6 non-degenerate cases *AABB*, *ABAB*, *ABBA*, *BAAB*, *BABA*, *BBAA*.

Coefficients	$\longleftrightarrow$	point in affine space
$F(x) = x^2 + Xx + Y$	$\longleftrightarrow$	$(X, Y) \in \mathbb{R}^2$
$A_1^2 + XA_1 + Y = 0$	$\longleftrightarrow$	line for root $A_1$ of $F(x)$
all polynomials with $A_1$ as root	$\longleftrightarrow$	points on the line
quadratic $F(x)$ has 2 roots	$\longleftrightarrow$	2 lines intersect at a point
<b>Resultant</b> of 2 quadrics	$\longleftrightarrow$	union of 2 lines
<b>Discriminant</b> $X^2 - 4Y$	$\longleftrightarrow$	parabola $\Delta$
roots of $F$	$\longleftrightarrow$	lines <i>tangent</i> to $\Delta$
# roots of $F$	$\longleftrightarrow$	rel. pos. of $F$ wrt. $\Delta$



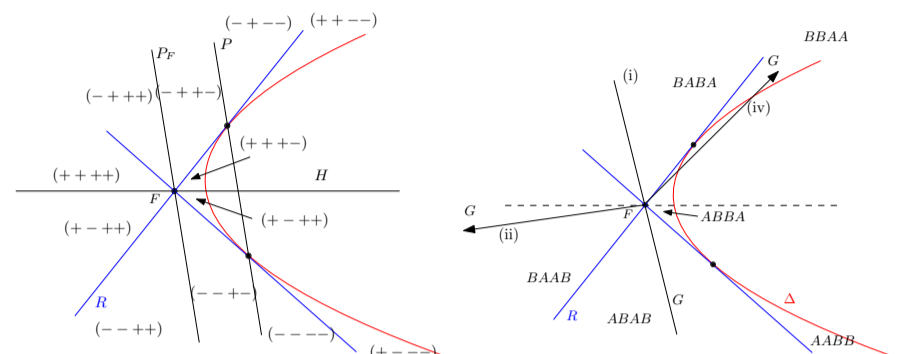
Arrangement consisting of the following hypersurfaces:

- **Resultant** (pair of lines)
- **Discriminant** (parabola)

points in same cell	$\longleftrightarrow$	same root ordering
crossing of <b>resultant</b>	$\longleftrightarrow$	roots of 2 polys equal
crossing of <b>discriminant</b>	$\longleftrightarrow$	roots of same poly equal

## Decomposition

- exploiting conic properties of pole and polar (duality) or intersecting rays from  $F$  with  $\Delta$



- When  $F(x) = a_2x^2 + a_1x + a_0$  and  $G(x) = b_2x^2 + b_1x + b_0$  the tested quantities are:

$$R = b_0^2 a_2^2 - 2 b_0 a_0 b_2 a_2 + a_0^2 b_2^2 - a_1 b_1 b_0 a_2 - a_1 b_1 a_0 b_2 + a_1^2 b_0 b_2 + a_0 b_1^2 a_2$$

(resultant)

$$H = b_1 a_2 - a_1 b_2 \quad (\text{horizontal line})$$

$$P = -2 b_0 a_2 + a_1 b_1 - 2 a_0 b_2 \quad (\text{polar})$$

$$P_F = a_1 b_1 a_2 - a_1^2 b_2 - 2 b_0 a_2^2 + 2 a_0 a_2 b_2 \quad (\text{polar translated})$$

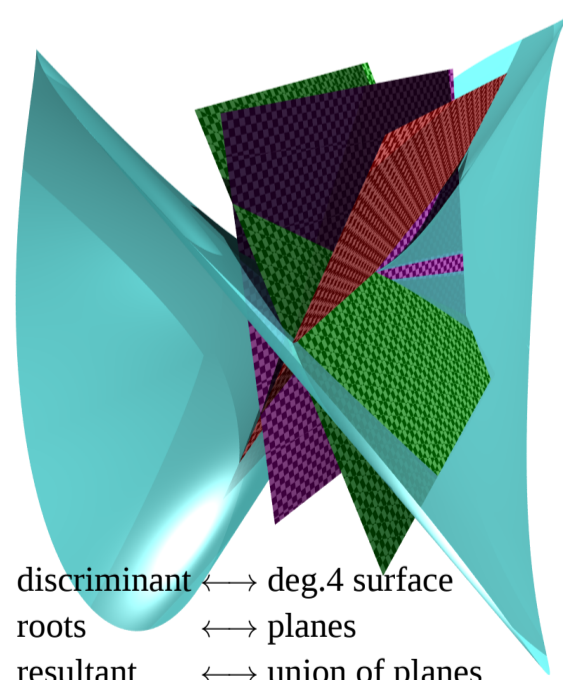
To compare algebraic numbers of degree 2, we have to consider quantities of at most algebraic degree 4 in the worst case. This bound is tight and therefore optimal.

## Proof

- walls of arrangement defined by **resultant** (deg.4) and **discriminant** (deg.2)
- both irreducible, separating full-dimensional cells
- $\Rightarrow$  they are *necessary* to define the arrangement
- all subsidiary equations have alg. deg.  $\leq 4$

## Cubics

Given 2 polynomials of deg. 3, determine their root ordering among the  $\binom{6}{3} + 2\binom{4}{1} + 2 = 30$  non-degenerate cases.



discriminant	$\longleftrightarrow$	deg.4 surface
roots	$\longleftrightarrow$	planes
resultant	$\longleftrightarrow$	union of planes

- quantities of alg. degree at most 6
- optimal, but geometric interpretation not yet known

## Higher degree

- difficult to visualise
- *open problem*: optimal decomposition of the arrangement