# Geometric predicates as arrangements of hypersurfaces: Application to comparison of algebraic numbers 

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# Geometric predicates as arrangements of hypersurfaces: Application to comparison of algebraic numbers 

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## Predicates

Formulation of a wide range of geometric predicates:

- arrangement of hypersurfaces in $\mathbb{R}^{d}$
- cells of dimension $\leq d$ (i.e., vertices, edges, faces)
- point location queries


## Geometry in the space of coefficients

Given 2 polynomials of deg. 2, determine their root ordering among the 6 non-degenerate cases $A A B B, A B A B$, $A B B A, B A A B, B A B A, B B A A$.

Coefficients $\longleftrightarrow$ point in affine space
$F(x)=x^{2}+X x+Y \quad \longleftrightarrow \quad(X, Y) \in \mathbb{R}^{2}$
$A_{1}^{2}+X A_{1}+Y=0 \longleftrightarrow$ line for root $A_{1}$ of $F(x)$
all polynomials with $A_{1}$ as root $\longleftrightarrow$ points on the line
quadric $F(x)$ has 2 roots $\longleftrightarrow 2$ lines intersect at a point
Resultant of 2 quadrics $\longleftrightarrow$ union of 2 lines
Discriminant $X^{2}-4 Y \longleftrightarrow$ parabola $\Delta$
roots of $F \longleftrightarrow$ lines tangent to $\Delta$
\# roots of $F \longleftrightarrow$ rel. pos. of $F$ wrt. $\Delta$


Arrangement consisting of the following hypersurfaces:

- Resultant (pair of lines)
- Discriminant (parabola)

$$
\begin{aligned}
& \text { points in same cell } \longleftrightarrow \text { same root ordering } \\
& \text { crossing of resultant } \longleftrightarrow \\
& \text { roots of } 2 \text { polys equal } \\
& \text { crossing of discriminant } \longleftrightarrow \text { roots of same poly equal }
\end{aligned}
$$

## Decomposition

- exploiting conic properties of pole and polar (duality) or intersecting rays from $F$ with $\Delta$

- When $F(x)=a_{2} x^{2}+a_{1} x+a_{0}$ and $G(x)=b_{2} x^{2}+b_{1} x+b_{0}$ the tested quanties are:
$R=b_{0}{ }^{2} a_{2}{ }^{2}-2 b_{0} a_{0} b_{2} a_{2}+a_{0}^{2} b_{2}^{2}-a_{1} b_{1} b_{0} a_{2}-a_{1} b_{1} a_{0} b_{2}+a_{1}{ }^{2} b_{0} b_{2}+a_{0} b_{1}^{2} a_{2}$ (resultant)
$H=b_{1} a_{2}-a_{1} b_{2} \quad$ (horizontal line)
$P=-2 b_{0} a_{2}+a_{1} b_{1}-2 a_{0} b_{2} \quad$ (polar)
$P_{F}=a_{1} b_{1} a_{2}-a_{1}{ }^{2} b_{2}-2 b_{0} a_{2}{ }^{2}+2 a_{0} a_{2} b_{2} \quad$ (polar translated)
To compare algebraic numbers of degree 2, we have to consider quantities of at most algebraic degree 4 in the worst case. This bound is tight and therefore optimal.
Proof
- walls of arrangement defined by resultant (deg.4) and discriminant (deg.2)
- both irreducible, separating full-dimensional cells
- $\Rightarrow$ they are necessary to define the arrangement
- all subsidiary equations have alg. deg. $\leq 4$

Cubics
Given 2 polynomials of deg. 3, determine their root ordering among the $\binom{6}{3}+2\binom{4}{1}+2=30$ non-degenerate cases.


- quantities of alg. degree at most 6
- optimal, but geometric interpretation not yet known


## Higher degree

- difficult to visualise
- open problem: optimal decomposition of the arrangement

