

Geometric predicates as arrangements of hypersurfaces: Application to comparison of algebraic numbers

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Predicates

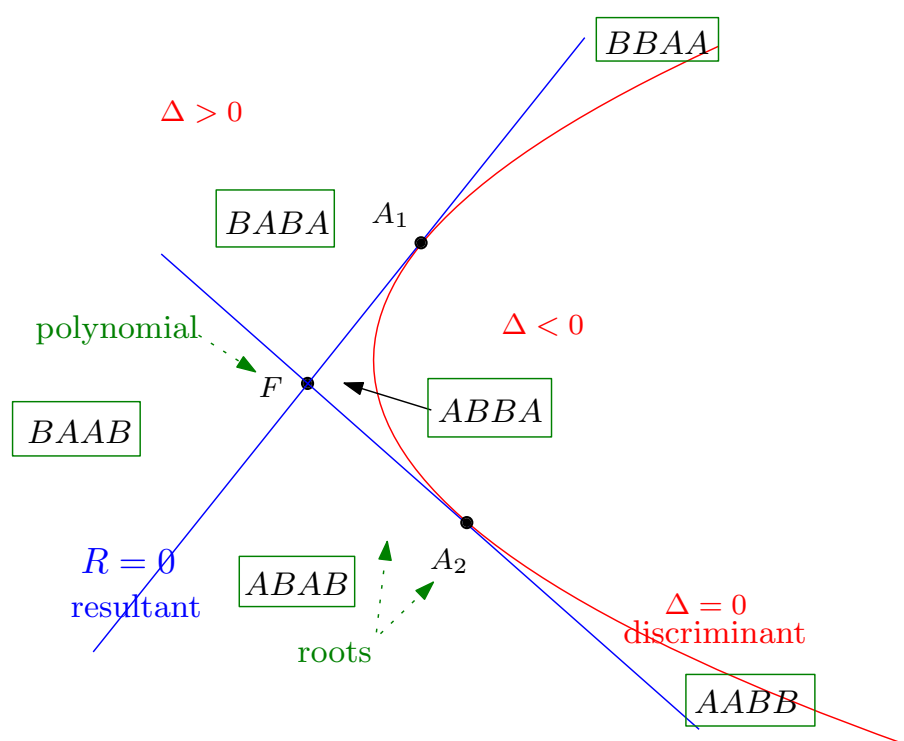
Formulation of a wide range of geometric predicates:

- arrangement of hypersurfaces in \mathbb{R}^d
- cells of dimension $\leq d$ (i.e., vertices, edges, faces)
- point location queries

Geometry in the space of coefficients

Given 2 polynomials of deg. 2, determine their root ordering among the 6 non-degenerate cases *AABB*, *ABAB*, *ABBA*, *BAAB*, *BABA*, *BBAA*.

Coefficients	\longleftrightarrow	point in affine space
$F(x) = x^2 + Xx + Y$	\longleftrightarrow	$(X, Y) \in \mathbb{R}^2$
$A_1^2 + XA_1 + Y = 0$	\longleftrightarrow	line for root A_1 of $F(x)$
all polynomials with A_1 as root	\longleftrightarrow	points on the line
quadratic $F(x)$ has 2 roots	\longleftrightarrow	2 lines intersect at a point
Resultant of 2 quadrics	\longleftrightarrow	union of 2 lines
Discriminant $X^2 - 4Y$	\longleftrightarrow	parabola Δ
roots of F	\longleftrightarrow	lines <i>tangent</i> to Δ
# roots of F	\longleftrightarrow	rel. pos. of F wrt. Δ



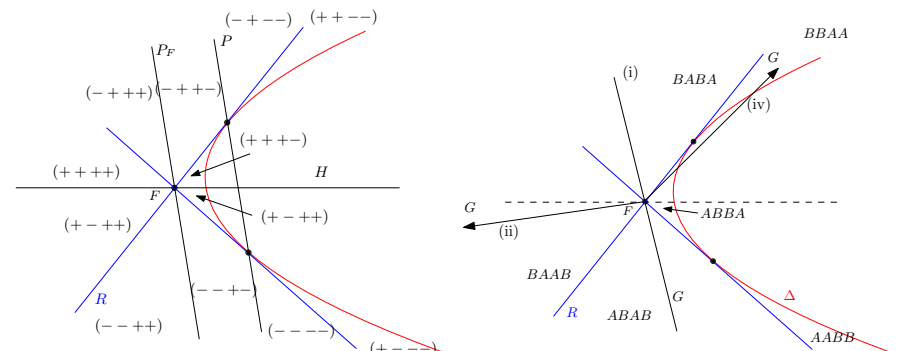
Arrangement consisting of the following hypersurfaces:

- **Resultant** (pair of lines)
- **Discriminant** (parabola)

points in same cell	\longleftrightarrow	same root ordering
crossing of resultant	\longleftrightarrow	roots of 2 polys equal
crossing of discriminant	\longleftrightarrow	roots of same poly equal

Decomposition

- exploiting conic properties of pole and polar (duality) or intersecting rays from F with Δ



- When $F(x) = a_2x^2 + a_1x + a_0$ and $G(x) = b_2x^2 + b_1x + b_0$ the tested quantities are:

$$R = b_0^2 a_2^2 - 2 b_0 a_0 b_2 a_2 + a_0^2 b_2^2 - a_1 b_1 b_0 a_2 - a_1 b_1 a_0 b_2 + a_1^2 b_0 b_2 + a_0 b_1^2 a_2$$

(resultant)

$$H = b_1 a_2 - a_1 b_2 \quad (\text{horizontal line})$$

$$P = -2 b_0 a_2 + a_1 b_1 - 2 a_0 b_2 \quad (\text{polar})$$

$$P_F = a_1 b_1 a_2 - a_1^2 b_2 - 2 b_0 a_2^2 + 2 a_0 a_2 b_2 \quad (\text{polar translated})$$

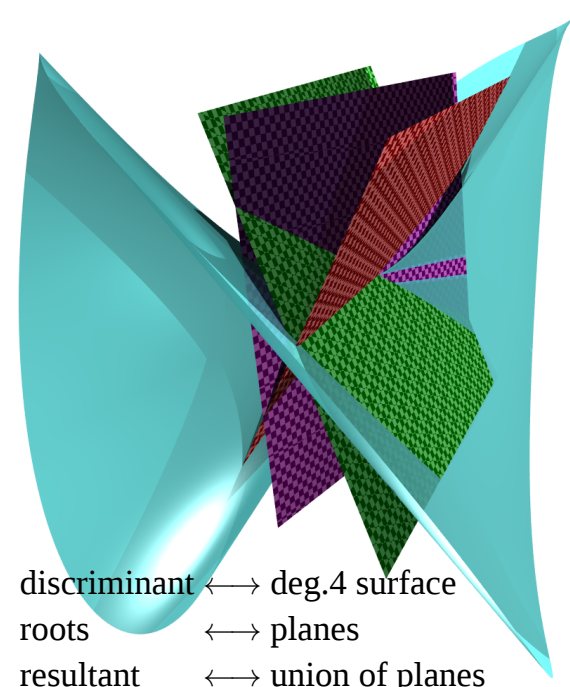
To compare algebraic numbers of degree 2, we have to consider quantities of at most algebraic degree 4 in the worst case. This bound is tight and therefore optimal.

Proof

- walls of arrangement defined by **resultant** (deg.4) and **discriminant** (deg.2)
- both irreducible, separating full-dimensional cells
- \Rightarrow they are *necessary* to define the arrangement
- all subsidiary equations have alg. deg. ≤ 4

Cubics

Given 2 polynomials of deg. 3, determine their root ordering among the $\binom{6}{3} + 2\binom{4}{1} + 2 = 30$ non-degenerate cases.



- quantities of alg. degree at most 6
- optimal, but geometric interpretation not yet known

Higher degree

- difficult to visualise
- *open problem*: optimal decomposition of the arrangement