Parallelism Constraints

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Abstract. Parallelism constraints are logical desciptions of trees. They are as expressive as context unification, i.e. second-order linear unification. We present a semi-decision procedure enumerating all "most general unifiers" of a parallelism constraint and prove it sound and complete. In contrast to all known procedures for context unification, the presented procedure terminates for the important fragment of dominance constraints and performs reasonably well in a recent application to underspecified natural language semantics.

1 Introduction

Parallelism constraints [7,17] are logical descriptions of trees. They are equal in expressive power to context unification [4], a variant of linear second-order unification [14,19]. The decidability of context unification is a prominent open problem [21] even though several fragments are known decidable [23,22,4].

Parallelism constraints state relations between the nodes of a tree: mother-of, sibling-of and labeling, dominance (ancestor-of), disjointness, inequality, and parallelism. Parallelism $\pi_1/\pi_2{\sim}\pi_3/\pi_4$, as illustrated in Figure 1, holds in a tree if the structure of the tree between the nodes π_1 and π_2 — i.e., the tree below π_1 minus the tree below π_2 — is isomorphic to that between π_3 and π_4.

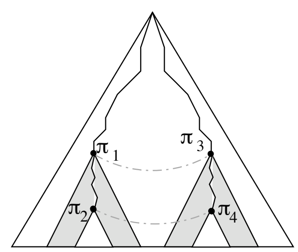

Fig. 1. Parallelism $\pi_1/\pi_2{\sim}\pi_3/\pi_4$

Parallelism constraints differ from context unification in their perspective on trees. They view trees from inside, talking about the *nodes* of a single tree, rather than from the outside, talking about relations between several *trees*. This difference has important consequences. First, it is not only a difference of nodes versus trees but also one of occurrences versus structure. Second, different decidable fragments can be distinguished for parallelism constraints and context unification. Third, different algorithms can be devised. For instance, the language of *dominance constraints* [16,25,1,9] is a decidable fragment of parallelism constraints

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for which powerful solver exist [6,5,17]. But when encoded into context unification, dominance constraints are not subsumed by any of the decidable fragments mentioned above, not even by subtree constraints [24], although they look similar. The difference is again that dominance constraints speak about occurences of subtrees whereas subtree constraints speak about their structure.

Parallelism constraints form the backbone of a recent underspecified analysis of natural language semantics [7,12]. This analysis uses the fragment of *dominance constraints* to describe scope ambiguities in a similar fashion as [20,2], while the full expressivity of parallelism is needed for modeling ellipsis. An earlier treatment of semantic underspecification [18] was based directly on context unification. The implementation used an incomplete procedure [10] which guesses trees top-down by imitation and projection, leaving out flex-flex. This procedure performs well on the parallelism phenomena encountered in ellipsis resolution, but when dealing with scope ambiguities, it consistently runs into combinatoric explosion. To put it differently, this procedure does not perform well enough on the context unification equivalent of dominance constraints.

In this paper, we propose a new semi-decision procedure for parallelism constraints built on top of a powerful, terminating solver for dominance constraints. We prove our procedure sound and complete: We define the notion of a *minimal solved form* for parallelism constraints, which plays the same role as *most general unifiers* in unification theory. We then show that our procedure enumerates all minimal solved forms of a given parallelism constraint.

Plan of the paper. In the following section, we describe the syntax and semantics of dominance and parallelism constraints. Section 3 presents an algorithm for dominance constraints which in section 4 is extended to a semi-decision procedure for parallelism constraints. In sections 5 and 6 we sketch a proof of soundness and completeness. Section 7 concludes. Many proofs are omitted for lack of space but can be found in an extended version [8].

2 Syntax and semantics

Semantics. We assume a signature Σ of function symbols ranged over by $f, g, \ldots$, each of which is equipped with an arity $\mathsf{ar}(f) \geq 0$. Constants are function symbols of arity 0 denoted by a, b. We further assume that Σ contains at least one constant and a symbol of arity at least 2.

A (finite) *tree* τ is a ground term over Σ, for instance $f(g(a, a))$. A *node* of a tree can be identified with its *path* from the root down, expressed by a word over $\mathbb{N}_+$, the set of natural numbers excluding 0. We write ε for the empty path and $\pi_1\pi_2$ for the concatenation of π_1 and π_2. A path π is a prefix of a path π' if there exists some (possibly empty) π'' such that $\pi\pi'' = \pi'$.

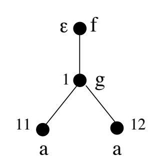

Fig. 2. $f(g(a, a))$

A tree can be characterized uniquely by a tree domain (the set of its paths) and a labeling function. A *tree domain* D is a finite nonempty prefix-closed set of paths. A path $\pi i \in D$ is the i-th child of the node/path $\pi \in D$. A *labeling function* is a function $L : D \to \Sigma$ fulfilling the condition that for every $\pi \in D$ and

$k \geq 1, \pi k \in D$ iff $k \leq \mathsf{ar}(L(\pi))$. We write D_τ for the domain of a tree τ and L_τ for its labeling function. For instance, the tree $\tau = f(g(a,a))$ displayed in Fig. 2 satisfies $D_\tau = \{\epsilon, 1, 11, 12\}, L_\tau(\epsilon) = f, L_\tau(1) = g$, and $L_\tau(11) = a = L_\tau(12)$.

Definition 1. The *tree structure* $\mathcal{M}^\tau$ of a tree τ is a first-order structure with domain D_τ. It provides a labeling relation $:f^\tau \subseteq D_\tau^{\mathsf{ar}(f)+1}$ for each $f \in \Sigma$:

$$:f^\tau = \{(\pi, \pi 1, \ldots, \pi n) \mid L_\tau(\pi) = f, \mathsf{ar}(f) = n\}$$

We write $\mathcal{M}^\tau \models \pi{:}f(\pi_1, \ldots, \pi_n)$ for $(\pi, \pi_1, \ldots, \pi_n) \in :f^\tau$; this relation states that node π of τ is labeled by f and has π_i as its i-th child (for $1 \leq i \leq n$). Every tree structure $\mathcal{M}^\tau$ can be extended conservatively by relations for dominance, disjointness, and parallelism. *Dominance* is the prefix relation between paths $\pi \triangleleft^* \pi'$; restricted to D_τ, it is the ancestor relation of τ; we write $\pi \triangleleft^+ \pi'$ if $\pi \triangleleft^* \pi'$ and $\pi \neq \pi'$. *Disjointness* $\pi \perp \pi'$ holds if neither $\pi \triangleleft^* \pi'$ nor $\pi' \triangleleft^* \pi$. Concerning parallelism, let $\mathsf{betw}_\tau(\pi_1, \pi_2)$ be the set of nodes in the substructure of τ between π_1 and π_2: If $\pi_1 \triangleleft^* \pi_2$ holds in $\mathcal{M}^\tau$, we define

$$\mathsf{betw}_\tau(\pi_1, \pi_2) = \{\pi \in D_\tau \mid \pi_1 \triangleleft^* \pi \text{ but not } \pi_2 \triangleleft^+ \pi\}.$$

The node π_2 plays a special role: it is part of the substructure of τ between π_1 and π_2, but its label is not. This is expressed in Def. 2, which is illustrated in Fig. 1.

Definition 2. *Parallelism* $\mathcal{M}^\tau \models \pi_1/\pi_2 \sim \pi_3/\pi_4$ holds iff $\pi_1 \triangleleft^* \pi_2$ and $\pi_3 \triangleleft^* \pi_4$ are valid in $\mathcal{M}^\tau$ and there exists a *correspondence function* $c : \mathsf{betw}_\tau(\pi_1, \pi_2) \to \mathsf{betw}_\tau(\pi_3, \pi_4)$, a bijective function which satisfies $c(\pi_1) = \pi_3$ and $c(\pi_2) = \pi_4$ and preserves the tree structure of $\mathcal{M}^\tau$, i.e. for all $\pi \in \mathsf{betw}_\tau(\pi_1, \pi_2) - \{\pi_2\}$, $f \in \Sigma$, and $n = \mathsf{ar}(f)$:

$$\mathcal{M}^\tau \models \pi{:}f(\pi 1, \ldots, \pi n) \quad \text{iff} \quad \mathcal{M}^\tau \models c(\pi){:}f(c(\pi 1), \ldots, c(\pi n))$$

Lemma 3. *If* $c : \mathsf{betw}_\tau(\pi_1, \pi_2) \to \mathsf{betw}_\tau(\pi_3, \pi_4)$ *is a correspondence function, then* $c(\pi_1 \pi) = \pi_3 \pi$ *for all* $\pi_1 \pi \in \mathsf{betw}_\tau(\pi_1, \pi_2)$.

Syntax. We assume an infinite set $\mathcal{V}$ of (node) variables ranged over by X, Y, Z, U, V, W. A *(parallelism) constraint* ϕ is a conjunction of *atomic constraints* or *literals* for parallelism, dominance, labeling, disjointness, and inequality. A *dominance constraint* is a constraint without parallelism literals. The abstract syntax of parallelism constraints is defined as follows:

$$\begin{array}{rll} \varphi, \psi ::= & X_1/X_2 \sim Y_1/Y_2 \mid X \triangleleft^* Y \mid X{:}f(X_1, \ldots, X_n) & \quad (\mathsf{ar}(f) = n) \\ \mid & X \perp Y \mid X \neq Y \mid \textbf{false} \mid \varphi \wedge \psi & \end{array}$$

$$\text{Abbreviations: } X{=}Y \text{ for } X \triangleleft^* Y \wedge Y \triangleleft^* X \text{ and } X \triangleleft^+ Y \text{ for } X \triangleleft^* Y \wedge X \neq Y$$

For simplicity, we view parallelism, inequality, and disjointness literals as symmetric. We also write XRY, where $R \in \{\triangleleft^*, \triangleleft^+, \perp, \neq, =\}$. A richer set of relations could be used, as proposed in [6], but this would complicate matters slightly. For a comparision to context unification, we refer to [17]. An example for the simpler case of string unification is given below (see Figure 4).

First order formulas Φ built from constraints and the usual logical connectives are interpreted over the class of tree structures in the usual Tarskian way. We write $\mathcal{V}(\Phi)$ for the set of variables occurring in Φ. If a pair $(\mathcal{M}^\tau, \alpha)$ of a tree structure $\mathcal{M}^\tau$ and a variable assignment $\alpha : \mathcal{G} \to D_\tau$, for some set $\mathcal{G} \supseteq \mathcal{V}(\Phi)$, satisfies Φ, we write this as $(\mathcal{M}^\tau, \alpha) \models \Phi$ and say that $(\mathcal{M}^\tau, \alpha)$ is a *solution* of Φ. We say that Φ is *satisfiable* iff it possesses a solution. Entailment $\Phi \models \Phi'$ means that all solutions of Φ are also solutions of Φ'.

We often draw constraints as graphs with the nodes representing variables; a labeled variable is connected to its children by solid lines, while a dotted line represents dominance. For example, the graph for $X{:}f(X_1, X_2) \wedge X_1 \triangleleft^* Y \wedge X_2 \triangleleft^* Y$ is displayed in Fig. 3. As trees do not branch upwards, this constraint is unsatisfiable.

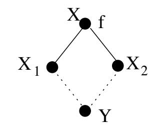

Fig. 3. An unsatisfiable constraint

Parallelism literals are shown graphically as well as textually: the square brackets in Fig. 4 illustrate the parallelism literal written beside the graph. This graph encodes the string unification [15] problem $gx = xg$; the two brackets represent the two occurences of x. Disjointness and inequality literals are not represented graphically.

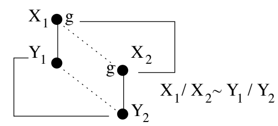

Fig. 4. String unification

3 Solving dominance constraints

Our semi-decision procedure for parallelism constraints consists of two parts: a terminating dominance constraint solver, and a part dealing with parallelism proper. Having our procedure terminate for general dominance constraints and perform well for dominance constraints in linguistic applications was an important design requirement for us.

In this section, we present the first part of our procedure, the solver for dominance constraints. This solver, which is similar to the algorithms in [13,6] and could in principle be replaced by them, terminates in non-deterministic polynomial time. Actually, satisfiability of dominance constraints is NP-complete [13]. Boolean satisfiability is encoded by forcing graph fragments to "overlap" and making the algorithm choose between different possible overlappings. For instance, the constraint to the right entails $X{=}Y \vee X{=}Y_1$. The solver is intended to perform well in cases without overlap, where distinct variables denote distinct values. This can typically be assumed in linguistic applications.

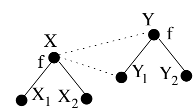

Fig. 5. Overlap

We organize all procedures in this paper as *saturation algorithms*. A saturation algorithm consists of a set of *saturation rules*, each of which has the form $\varphi \to \vee_{i=1}^n \varphi_i$ for some $n \geq 1$. A rule is a *propagation rule* if $n = 1$, and a *distribution rule* otherwise. The only critical rules with respect to termination are those which introduce fresh variables on their right hand side. A rule $\varphi \to \Phi$ is *correct* if $\varphi \models \exists V \Phi$ where $V = \mathcal{V}(\Phi) - \mathcal{V}(\varphi)$.

By a slight abuse of notation, we identify a constraint with the *set* of its literals. This way, subset inclusion defines a partial ordering $\subseteq$ on constraints; we also write $=^{set}$ for

Propagation rules:

(D.Clash.Ineq) $X{=}Y \ \wedge\ X{\neq}Y \rightarrow \mathbf{false}$

(D.Clash.Disj) $X{\perp}X \rightarrow \mathbf{false}$

(D.Dom.Refl) $\varphi \rightarrow X{\lhd^*}X$ where $X \in \mathcal{V}(\varphi)$

(D.Dom.Trans) $X{\lhd^*}Y \ \wedge\ Y{\lhd^*}Z \rightarrow X{\lhd^*}Z$

(D.Eq.Decom) $X{:}f(X_1,\ldots,X_n) \ \wedge\ Y{:}f(Y_1,\ldots,Y_n) \ \wedge\ X{=}Y \rightarrow \wedge_{i=1}^{n} X_i{=}Y_i$

(D.Lab.Ineq) $X{:}f(\ldots) \ \wedge\ Y{:}g(\ldots) \rightarrow X{\neq}Y$ where $f \neq g$

(D.Lab.Disj) $X{:}f(\ldots X_i,\ldots,X_j,\ldots) \rightarrow X_i{\perp}X_j$ for $1 \leq i < j \leq n$

(D.Prop.Disj) $X{\perp}Y \ \wedge\ X{\lhd^*}X' \ \wedge\ Y{\lhd^*}Y' \rightarrow Y'{\perp}X'$

(D.Lab.Dom) $X{:}f(\ldots,Y,\ldots) \rightarrow X{\lhd^+}Y$

Distribution rules:

(D.Distr.NotDisj) $X{\lhd^*}Z \ \wedge\ Y{\lhd^*}Z \rightarrow X{\lhd^*}Y \ \vee\ Y{\lhd^*}X$

(D.Distr.Child) $X{\lhd^*}Y \ \wedge\ X{:}f(X_1,\ldots,X_n) \rightarrow Y{=}X \ \vee\ \bigvee_{i=1}^{n} X_i{\lhd^*}Y$

Fig. 6. Solving dominance constraints: rule set *D*

the corresponding equality $\subseteq \cap \supseteq$, and $\subset$ for the strict variant $\subseteq \cap \neq^{set}$. This way, we can define saturation for a set *S* of saturation rules as follows: We assume that each rule $\rho \in S$ comes with an application condition $C_\rho(\varphi)$ deciding whether ρ can be applied to φ or not. A *saturation step* $\rightarrow_S$ consists of one application of a rule in *S*:

$$\frac{\varphi' \subseteq \varphi \qquad \rho \in S}{\varphi \rightarrow_S \varphi \ \wedge\ \varphi_i} \text{ if } C_\rho(\varphi) \text{ where } \rho \text{ is } \varphi' \rightarrow \vee_{i=1}^{n}\varphi_i$$

For this section, we let $C_{\varphi' \rightarrow \vee_{i=1}^{n}\varphi_i}(\varphi)$ be true iff $\varphi_i \not\subseteq \varphi$ for all $1 \leq i \leq n$. We call a constraint *S-saturated* if it is irreducible with respect to $\rightarrow_S$ and *clash-free* if it does not contain $\mathbf{false}$. We also say that a constraint is in *S-solved form* if it is *S*-saturated and clash-free.

Figure 6 contains schemata for saturation rules that together solve dominance constraints. Let *D* be the (infinite) set of instances of these schemata. Both clash schemata are obvious. Next, there are standard schemata for reflexivity, transitivity, decomposition, and inequality. Schema (D.Lab.Dom) declares that a parent dominates its children.

We illustrate the remaining schemata of propagation rules by an example: We reconsider the unsatisfiable constraint $X{:}f(X_1, X_2) \wedge X_1{\lhd^*}Y \wedge X_2{\lhd^*}Y$ of Fig. 3. By (D.Lab.Disj), we infer $X_1{\perp}X_2$, from which (D.Prop.Disj) yields $Y{\perp}Y$, which then clashes by (D.Clash.Disj).

There are only two situations where distribution is necessary. The situation shown in Fig. 7 is handled by (D.Distr.NotDisj): the tree nodes denoted by X and Y cannot be at disjoint positions because they both dominate Z. The distribution rule (D.Distr.Children) is applicable to the constraint in Fig. 5: As the constraint contains $Y{:}f(Y_1, Y_2) \wedge Y{\lhd^*}X$, we must have either $Y{=}X$ or $Y_1{\lhd^*}X$ or $Y_2{\lhd^*}X$. Propagation proves that the third choice results in a clash, while the others lead to satisfiable constraints.

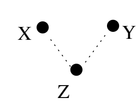

Fig. 7. Nondisjointness

Proposition 4 (Soundness). *Any dominance constraint in D-solved form is satisfiable.*

Along the lines of [13]. On the other hand, the saturation algorithm for *D* is complete in the sense that it computes every *minimal solved form* of a dominance constraint.

Definition 5. Let φ, φ' be constraints, *S* a set of saturation rules and $\preceq$ an partial order on constraints. Then φ' is a $\preceq$*-minimal S-solved form for* φ iff φ' is an *S*-solved form that is $\prec$-minimal satisfying $\varphi \preceq \varphi'$.

For dominance constraints, we can simply use set inclusion. As an example, a $\subseteq$-minimal *D*-solved form for the constraint in Fig. 8 is $X \triangleleft^* Y \wedge X \triangleleft^* Z \wedge X \triangleleft^* X \wedge Y \triangleleft^* Y \wedge Z \triangleleft^* Z$. (Note that X does not need to be labeled.)

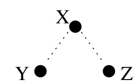

Fig. 8. A solved form

Lemma 6 (Completeness). *Let* φ *be a dominance constraint and* φ' *a* $\subseteq$*-minimal D-solved form for* φ*. Then* $\varphi \to_{\mathrm{D}}^* \varphi'$.

Proof. By well-founded induction on the strict partial order $\supset$ on the set $\{\psi \mid \psi \subseteq \varphi'\}$. If φ is *D*-solved then $\varphi =^{set} \varphi'$ by minimality and we are done. Otherwise, there is a rule $\psi \to \vee_{i=1}^n \psi_i$ in *D* which applies to φ. Since $\varphi \subseteq \varphi'$ and φ' is in *D*-solved form, there exists an i such that $\psi_i \subseteq \varphi'$. By the inductive hypothesis, $\varphi \wedge \psi_i \to_{\mathrm{D}}^* \varphi'$ and thus $\varphi \to_{\mathrm{D}}^* \varphi'$. $\square$

4 Processing parallelism constraints

We extend the dominance constraint solver of the previous section to a semi-decision procedure for parallelism constraints. The main idea is to compute the correspondence functions for all parallelism literals in the input constraint (compare Def. 2). We use a new kind of literals, *path equalities*, to accomplish this with as much propagation and as little case distinction as possible.

We define the set of variables $\mathsf{betw}_\varphi(X_1, X_2)$ *between* X_1 and X_2 as the syntactic counterpart of the set of nodes $\mathsf{betw}_\tau(\pi_1, \pi_2)$: If $X_1 \triangleleft^* X_2 \in \varphi$, then

$$\mathsf{betw}_\varphi(X_1, X_2) = \{X \in \mathcal{V}(\varphi) \mid X_1 \triangleleft^* X \in \varphi \text{ and } (X \triangleleft^* X_2 \in \varphi \text{ or } X \perp X_2 \in \varphi)\}$$

Given a parallelism literal $X_1/X_2 \sim Y_1/Y_2$, we need to establish a syntactic correspondence function $c : \mathsf{betw}_\varphi(X_1, X_2) \to \mathsf{betw}_\varphi(Y_1, Y_2)$. In doing this, we may have to add new local variables to φ. In the following, we always consider a constraint φ together with a set $\mathcal{G} \subseteq \mathcal{V}$ of *global* variables; all other variables are *local*. For an input constraint φ, we assume $\mathcal{V}(\varphi) \subseteq \mathcal{G}$.

We record syntactic correspondences by use of a new, auxiliary kind of constraints: a *path equality* $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & Y \end{smallmatrix}\right)$ states, informally speaking, that X below X_1 corresponds to Y below Y_1. More precisely, a path equality relation $\mathcal{M}^\tau \models \mathrm{p}\left(\begin{smallmatrix} \pi_1 & \pi_3 \\ \pi_2 & \pi_4 \end{smallmatrix}\right)$ is true iff there exists a path π such that $\pi_2 = \pi_1 \pi$ and $\pi_4 = \pi_3 \pi$, and for each $\pi' \triangleleft^+ \pi$, $L_\tau(\pi_1 \pi') = L_\tau(\pi_3 \pi')$.

Figure 9 shows the schemata of the sets *P* and *N* of saturation rules for computing correspondences, and Fig. 14 shows the schemata of the set *T*, which deal with interacting parallelism literals (and thus interacting correspondences). The rule set $D \cup P \cup N \cup T$ forms a sound and complete semi-decision procedure for parallelism constraints, which we abbreviate by *DPNT* (and accordingly for other rule set combinations).

Propagation Rules:

(P.Root) $X_1/X_2{\sim}Y_1/Y_2 \rightarrow \ \mathrm{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ X_1 & Y_1 \end{smallmatrix}) \ \wedge \ \mathrm{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{smallmatrix})$

(P.Copy.Dom) $U_1 R U_2 \ \wedge \ \bigwedge_{i=1}^{2} \mathrm{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ U_i & V_i \end{smallmatrix}) \ \wedge \ X_1/X_2{\sim}Y_1/Y_2 \rightarrow V_1 R V_2$
where $R \in \{\triangleleft^*, \perp, \neq\}$ and $U_1, U_2 \in \mathsf{betw}_\varphi(X_1, X_2)$.

(P.Copy.Lab) $U_0{:}f(U_1, \ldots, U_n) \ \wedge \ \bigwedge_{i=0}^{n} \mathrm{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ U_i & V_i \end{smallmatrix}) \ \wedge \ X_1/X_2{\sim}Y_1/Y_2 \rightarrow$
$V_0{:}f(V_1, \ldots, V_n)$ where $U_0 \perp X_2 \in \varphi$ or $U_0 \triangleleft^+ X_2 \in \varphi$

(P.Path.Sym) $\mathrm{p}\,(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}) \rightarrow \ \mathrm{p}\,(\begin{smallmatrix} Y & X \\ V & U \end{smallmatrix})$

(P.Path.Dom) $\mathrm{p}\,(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}) \rightarrow X \triangleleft^* U \ \wedge \ Y \triangleleft^* V$

(P.Path.Eq.1) $\mathrm{p}\,(\begin{smallmatrix} X_1 & X_3 \\ X_2 & X_4 \end{smallmatrix}) \ \wedge \ \bigwedge_{i=1}^{4} X_i{=}Y_i \rightarrow \ \mathrm{p}\,(\begin{smallmatrix} Y_1 & Y_3 \\ Y_2 & Y_4 \end{smallmatrix})$

(P.Path.Eq.2) $\mathrm{p}\,(\begin{smallmatrix} X & X \\ U & V \end{smallmatrix}) \rightarrow U{=}V$

Distribution Rules:

(P.Distr.Crown) $X_1 \triangleleft^* X \ \wedge \ X_1/X_2{\sim}Y_1/Y_2 \rightarrow X \triangleleft^* X_2 \ \vee \ X \perp X_2 \ \vee \ X_2 \triangleleft^+ X$

(P.Distr.Project) $\varphi \rightarrow X{=}Y \ \vee \ X{\neq}Y$ where $X, Y \in \mathcal{V}(\varphi)$

Introduction of local variables:

(N.New) $\varphi \ \wedge \ X_1/X_2{\sim}Y_1/Y_2 \rightarrow \ \mathrm{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ X & X' \end{smallmatrix})$ where $X \in \mathsf{betw}_\varphi(X_1, X_2)$;
X' new and local

Fig. 9. Schemata of rule sets *P* and *N* for computing correspondence

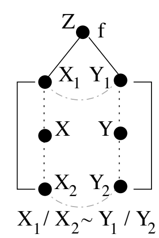

Fig. 10. Correspondence

The main rules. We start out with discussing the most important rules for computing correspondence functions, namely (P.Root), (N.New), (P.Copy.Dom), (P.Copy.Lab). Schema (P.Root) states, with respect to a parallelism literal $X_1/X_2{\sim}Y_1/Y_2$, that X_1 corresponds to Y_1 and X_2 corresponds to Y_2. To see how to go on from there, consider the constraint in Fig. 10. Variable X is between X_1 and X_2, and Y is between Y_1 and Y_2. But they are just *dominated* by X_1 and Y_1, respectively, their position is not fixed. So it would be precipitous to assume that X and Y correspond — there is nothing in the constraint which would force us to do that. Schema (N.New) acts on this idea as follows: Given a literal $X_1/X_2{\sim}Y_1/Y_2$ and a variable $X \in \mathsf{betw}_\varphi(X_1, X_2)$, correspondence $\mathrm{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ X & X' \end{smallmatrix})$ is stated between X and a variable $X' \notin \mathcal{V}(\varphi) \cup \mathcal{G}$. If the structure of the constraint enforces correspondence between X and some other variable $Y \in \mathsf{betw}_\varphi(Y_1, Y_2)$, then this will be inferred by saturation. (N.New) need only be applied if X does not yet possess a correspondent within $X_1/X_2{\sim}Y_1/Y_2$. We adapt the application condition for (N.New) rules accordingly:

$$C_{\varphi' \rightarrow \mathrm{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ X & X' \end{smallmatrix})}(\varphi) \text{ is true iff } X' \notin \mathcal{V}(\varphi) \cup \mathcal{G} \text{ and } \mathrm{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ X & Y \end{smallmatrix}) \notin \varphi \text{ for all variables } Y$$

Recall that $\mathcal{G}$ is the set of global variables with respect to which we saturate our constraint. Given $X_1/X_2{\sim}Y_1/Y_2 \in \varphi$, (P.Copy.Dom) and (P.Copy.Lab) copy dominance,

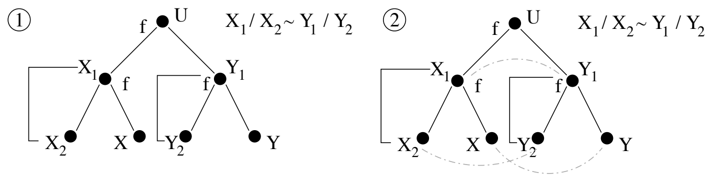

Fig. 11. Resolving an atomic parallelism constraint

disjointness, inequality, and labeling literals from $\mathsf{betw}_\varphi(X_1, X_2)$ to $\mathsf{betw}_\varphi(Y_1, Y_2)$ and vice versa. The condition on the position of U_0 in (P.Copy.Lab) makes sure that the labels of X_2 and Y_2 are not copied.

P contains two additional distribution rule schemata. (P.Distr.Crown) deals with situations like that in Fig. 12: We have to decide whether X is in $\mathsf{betw}_\varphi(X_1, X_2)$ or not. Only then do we know whether we need to apply (N.New) to X. (P.Distr.Project), on the other hand, guesses whether two variables should be identified or not. It is a very powerful schema, so we do not want to use it too often in practice.

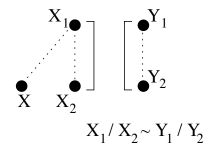

Fig. 12. X "inside" or "outside"?

Examples. Before we turn to the rules in *T*, let us discuss two more examples that can be handled by the rules we have seen up to now. How does syntactic correspondence as established by *DPNT* relate to semantic correspondence functions as defined in Def. 2? (P.Root) implements the first property of correspondence functions, the "preservation of tree structure" property remains to be examined. Consider Fig. 11. Constraint 1 constitutes the input to the procedure, while constraint 2 shows, as grey arcs, the correspondences that must hold by Def. 2. These correspondences are computed by *DPNT*: We infer $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_1 & Y_1 \end{smallmatrix}\right) \wedge \mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{smallmatrix}\right)$ by (P.Root). (N.New) is applicable to X and yields $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & X' \end{smallmatrix}\right)$ for a new local variable X'. We have $X_1 \lhd^+ X_2$ by (D.Lab.Dom), so we may apply (P.Copy.Lab) to $X_1{:}f(X_2, X)$ and get $Y_1{:}f(Y_2, X')$. But since the constraint also contains $Y_1{:}f(Y_2, Y)$, (D.Eq.Decom) gives us $X'{=}Y$, from which (P.Path.Eq.1) infers $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & Y \end{smallmatrix}\right)$. We see that the structure of the constraint has enforced correspondence between X and Y, and saturation has made the correct inferences.

While *DPNT* computes only finitely many solved forms for the constraint in Fig. 11, the constraint in Fig. 13 possesses infinitely many different solved forms. One solved form contains $X_1{=}X_2{=}Y_1{=}Y_2$. Another contains $X_1 \lhd^+ X_2{=}Y_1 \lhd^+ Y_2$. For the case of $X_1 \lhd^+ Y_1 \lhd^+ X_2 \lhd^+ Y_2$, there is one solved form with one local variable, two with two, one with three, two with four, and so on ad infinitum.

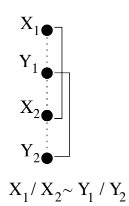

Fig. 13. Self-overlap

Interacting correspondences. We now turn to the set of saturation rules *T*, the schemata of which are shown in Fig. 14. *T* handles the interaction of correspondence functions for "overlapping" parallelism contexts. Schema (T.Trans.H) de-

(T.Trans.H) $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}\right) \wedge \mathrm{p}\left(\begin{smallmatrix} Y & Z \\ V & W \end{smallmatrix}\right) \rightarrow \mathrm{p}\left(\begin{smallmatrix} X & Z \\ U & W \end{smallmatrix}\right)$

(T.Trans.V) $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{smallmatrix}\right) \wedge \mathrm{p}\left(\begin{smallmatrix} X_2 & Y_2 \\ X_3 & Y_3 \end{smallmatrix}\right) \rightarrow \mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_3 & Y_3 \end{smallmatrix}\right)$

(T.Diff.1) $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{smallmatrix}\right) \wedge \mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_3 & Y_3 \end{smallmatrix}\right) \wedge X_2 \triangleleft^* X_3 \wedge Y_2 \triangleleft^* Y_3 \rightarrow \mathrm{p}\left(\begin{smallmatrix} X_2 & Y_2 \\ X_3 & Y_3 \end{smallmatrix}\right)$

(T.Diff.2) $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_3 & Y_3 \end{smallmatrix}\right) \wedge \mathrm{p}\left(\begin{smallmatrix} X_2 & Y_2 \\ X_3 & Y_3 \end{smallmatrix}\right) \wedge X_1 \triangleleft^* X_2 \wedge Y_1 \triangleleft^* Y_2 \rightarrow \mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{smallmatrix}\right)$

Fig. 14. Rule set *T*: interaction of correspondences

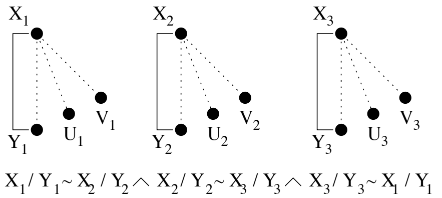

Fig. 15. Using *T*

scribes horizontal transitivity of path equality constraints, while (T.Trans.V), (T.Diff.1) and (T.Diff.2) all deal with vertical transitivity. The correctness of these rules is obvious.

We discuss an example where T is needed to ensure correct interaction of correspondences. Consider the constraint in Fig. 15. We have $X_i \triangleleft^* U_i$ and $X_i \triangleleft^* V_i$ for $1 \leq i \leq 3$, so (P.Distr.Crown) is applicable. Suppose that in each case, we choose $U_i \perp Y_i$ and $V_i \perp Y_i$. Suppose further that using (P.Distr.Project), we choose $U_1 \neq V_1$. (N.New) can be applied to $U_1, V_1 \in \mathsf{betw}_\varphi(X_1, Y_1)$, yielding new local variables U_1' and V_1' with $\mathrm{p}\left(\begin{smallmatrix} X_1 & X_2 \\ U_1 & U_1' \end{smallmatrix}\right)$, $\mathrm{p}\left(\begin{smallmatrix} X_1 & X_2 \\ V_1 & V_1' \end{smallmatrix}\right)$. Suppose that by (P.Distr.Project), we choose $U_1'{=}U_2$ and $V_1'{=}V_2$, hence we get $\mathrm{p}\left(\begin{smallmatrix} X_1 & X_2 \\ U_1 & U_2 \end{smallmatrix}\right)$ and $\mathrm{p}\left(\begin{smallmatrix} X_1 & X_2 \\ V_1 & V_2 \end{smallmatrix}\right)$ by (P.Path.Eq.1). We can use (N.New) on $U_2, V_2 \in \mathsf{betw}_\varphi(X_2, Y_2)$, getting $\mathrm{p}\left(\begin{smallmatrix} X_2 & X_3 \\ U_2 & U_2' \end{smallmatrix}\right)$ and $\mathrm{p}\left(\begin{smallmatrix} X_2 & X_3 \\ V_2 & V_2' \end{smallmatrix}\right)$ for new local variables U_2', V_2'. Suppose that again, we choose $U_2'{=}U_3$ and $V_2'{=}V_3$ by (P.Distr.Project). This yields $\mathrm{p}\left(\begin{smallmatrix} X_2 & X_3 \\ U_2 & U_3 \end{smallmatrix}\right)$ and $\mathrm{p}\left(\begin{smallmatrix} X_2 & X_3 \\ V_2 & V_3 \end{smallmatrix}\right)$ by (P.Path.Eq.1). Now we turn to the third parallelism literal, $X_3/Y_3 \sim X_1/Y_1$. Again by (N.New), we can add $\mathrm{p}\left(\begin{smallmatrix} X_3 & X_1 \\ U_3 & U_3' \end{smallmatrix}\right)$ and $\mathrm{p}\left(\begin{smallmatrix} X_3 & X_1 \\ V_3 & V_3' \end{smallmatrix}\right)$ for new local variables U_3', V_3'.

But now, we choose $U_3'{=}V_1$ and $V_3'{=}U_1$ by (P.Distr.Project), which gives us $\mathrm{p}\left(\begin{smallmatrix} X_3 & X_1 \\ V_3 & U_1 \end{smallmatrix}\right)$ and $\mathrm{p}\left(\begin{smallmatrix} X_3 & X_1 \\ V_3 & U_1 \end{smallmatrix}\right)$. This constraint is unsatisfiable: In a tree structure satisfying this constraint, the path from X_1 to U_1 would have to be the same one as the path from X_1 to V_1, but the constraint contains $U_1 \neq V_1$. However, (T.Trans.H) can detect this: From $\mathrm{p}\left(\begin{smallmatrix} X_1 & X_2 \\ U_1 & U_2 \end{smallmatrix}\right)$ and $\mathrm{p}\left(\begin{smallmatrix} X_2 & X_3 \\ U_2 & U_3 \end{smallmatrix}\right)$, we get $\mathrm{p}\left(\begin{smallmatrix} X_1 & X_3 \\ U_1 & U_3 \end{smallmatrix}\right)$, and combined with $\mathrm{p}\left(\begin{smallmatrix} X_3 & X_1 \\ V_3 & U_1 \end{smallmatrix}\right)$ this gives $\mathrm{p}\left(\begin{smallmatrix} X_3 & X_3 \\ V_3 & U_3 \end{smallmatrix}\right)$, to which we can add $V_3{=}U_3$ by (P.Path.Eq.2). As (P.Copy.Dom) copies $U_1 \neq V_1$ to $U_3 \neq V_3$, this results in a clash by (D.Clash.Ineq).

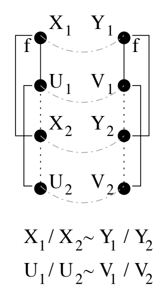

Fig. 16. Vertical transitivity

Each path equality inferred by *DPN* saturation describes a correspondence for some parallelism literal. With *T*, this is different. Consider, for example, Fig. 16 where *DPN* saturation can infer the correspondence $\mathrm{p}\left(\begin{smallmatrix}X_1 & Y_1\\ U_1 & V_1\end{smallmatrix}\right)$. (P.Root) yields $\mathrm{p}\left(\begin{smallmatrix}U_1 & V_1\\ U_2 & V_2\end{smallmatrix}\right)$. Now (T.Trans.V) can add $\mathrm{p}\left(\begin{smallmatrix}X_1 & Y_1\\ U_2 & V_2\end{smallmatrix}\right)$, a path equality that does not describe any syntactic correspondence for any of the two parallelism literals present. In this case, the additional path equality is not vital. But in other cases, e.g. if we extend the example in Fig. 15 by a fourth context and a fourth parallelism literal, the ability to infer path equalities beyond correspondence is necessary to ensure proper interaction of parallelism literals. Actually, the reason why we record correspondence by path equalities, as quadruples of variables, is that they support this.

Implementation. A first prototype implementation of *DPNT* is available as an applet on the Internet [3]. Saturation rules are applied in an order refining the order mentioned above: A distribution rule is only applied to a constraint saturated under the propagation rules from *DPT*. A rule from *N* is only applied to a constraint saturated under *DPT*. This implementation handles ellipses in natural language equally well as the previously mentioned implementation based on context unification [18]. But the two implementations differ with respect to scope ambiguities, i.e. dominance constraint solving: While the context unification based program could handle scope ambiguities with at most 3 quantifiers, the parallelism constraint procedure resolves scope ambiguities of 5 quantifiers in only 6 seconds and can even deal with more quantifiers.

5 Soundness

Clearly, all rules in *DPNT* are correct. For the soundness of *DPNT*-saturation is remains to show that generated *DPNT*-solved forms are satisfiable. First, we show that a special class of *DPNT*-solved forms, called "simple", are satisfiable. Then we lift the result to arbitrary *DPNT*-solved forms.

However, we only regard *generated* constraints, where each path equality either establishes a correspondence for some parallelism literal, or is the result of combining several such correspondence statements by a *T* rule.

Definition 7. Let φ be a constraint.

A path equality $\mathrm{p}\left(\begin{smallmatrix}U_1 & V_1\\ U_2 & V_2\end{smallmatrix}\right) \in \varphi$ is *correspondence-generated in* φ iff there exists some atomic parallelism constraint $X_1/X_2{\sim}Y_1/Y_2 \in \varphi$ such that $U_1{=}X_1 \wedge V_1{=}Y_1$ is in φ, and $U_2 \in \mathsf{betw}_\varphi(X_1, X_2)$ or $V_2 \in \mathsf{betw}_\varphi(Y_1, Y_2)$.

Let $CP(\varphi)$ be the set of correspondence-generated path equalities in φ, and let φ_0 be φ without all its path equalities, then a path equality is *generated in* φ iff it is in the *T*-saturation of $CP(\varphi) \cup \varphi_0$.

φ is called *generated* iff each of its parallelism literals is.

Concerning correspondence-generated path equalities, if $U_2 \in \mathsf{betw}_\varphi(X_1, X_2)$, then it must correspond to V_2 and inference will determine that V_2 must be between Y_1 and Y_2, and vice versa. Every *DPNT*-solved form of a parallelism constraint is generated, so we can safely restrict our attention to generated constraints:

Lemma 8. *Let φ be a constraint without path equalities, and let $\varphi \rightarrow^*_{\mathrm{DPNT}} \varphi'$ with φ' in DPNT-solved form. Then φ' is generated.*

Definition 9. Let φ be a constraint. A variable $X \in \mathcal{V}(\varphi)$ is called *labeled* in φ iff $\exists X' \in \mathcal{V}(\varphi)$ such that $X{=}X'$ and $X'{:}f(X_1, \ldots, X_n)$ are in φ for some term $f(X_1, \ldots, X_n)$. We call φ *simple* if all its variables are labeled and there exists some *root variable* $Z \in \mathcal{V}(\varphi)$ such that $Z \triangleleft^* X$ is in φ for all $X \in \mathcal{V}(\varphi)$.

Proposition 10. *A simple generated constraint in DPNT-solved form is satisfiable.*

Proof. The constraint graph of a simple generated constraint φ in *DPNT*-solved form can be seen as a tree (plus redundant dominance edges, parallelism and path equality literals). So we can transform φ into a tree τ by a standard construction. For every parallelism literal in φ, the corresponding parallelism holds in $\mathcal{M}^\tau$: As suggested by the examples in the previous section, *DPNT* enforces that the computed path equalities encode valid correspondence functions in $\mathcal{M}^\tau$. □

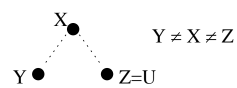

Fig. 17. Extension

Now suppose we have a generated non-simple constraint φ in *DPNT*-solved form. Take for instance the constraint in Fig. 17. We want to show that there is an *extension* $\varphi \wedge \varphi'$ of it that is simple, generated, and in *DPNT*-solved form. We proceed by successively labeling unlabeled variables. Suppose we want to label X first. The main idea is to make all variables minimally dominated by X into X's children, i.e. all variables V with $X \triangleleft^+ V$ such that there is no intervening W with $X \triangleleft^+ W \triangleleft^+ V$.

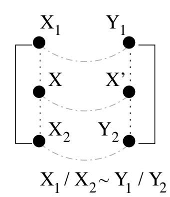

Fig. 18. Extension and parallelism

So in the constraint in Fig. 17, Y, Z, U are minimally dominated. However, we choose only one of Z, U as we have $Z{=}U$. Hence, we would like to label X by some function symbol of arity 2, extending the constraint, for instance, by $X{:}f(Y, Z)$. (If there is no symbol of suitable arity in Σ, we can always simulate it by a constant symbol and a symbol of arity ≥ 2.) However, we have to make sure that we preserve solvedness during extension. For example, when adding $X{:}f(Y, Z)$ to the constraint in Fig. 17, we also add $Y \perp Z$ so as not to make (D.Lab.Disj) applicable. Specifically, we have to be careful when labeling a variable like X_1 in Fig. 18 (where grey arcs stand for path equality literals): X_1 is in $\mathsf{betw}_\varphi(X_1, X_2)$, and when we add $X_1{:}g(X)$ for some unary g, we also have to add $X_2{:}g(X')$, otherwise (P.Copy.Lab) would be applicable.

So, by adding a finite number of atomic constraints and without adding any new local variables, we can label at least one further unlabeled variable in the constraint, while keeping it in *DPNT*-solved form. Thus, if we repeat this process a finite number of times, we can extend our generated constraint in *DPNT*-solved form to a simple generated constraint in *DPNT*-solved form, from which we can then read off a solution right away.

Theorem 11 (Soundness). *A generated constraint in DPNT-solved form is satisfiable.*

6 Completeness

DPNT-saturation is complete in the sense that it computes every *minimal solved form* of a parallelism constraint. For parallelism constraints, the set inclusion order we have

(1) Eliminating/introducing a local variable

$X{=}Z \ \wedge\ \varphi =^{loc}_{\mathcal{G}} \varphi \quad \text{if } X \notin \mathcal{G}, X \notin \mathcal{V}(\varphi), Z \in \mathcal{V}(\varphi)$

(2) Renaming a local variable

$\varphi {=}^{loc}_{\mathcal{G}} \varphi[Y/X] \qquad \text{if } X \notin \mathcal{G}, Y \notin \mathcal{V}(\varphi) \cup \mathcal{G}$

(3) Exchanging representatives of an equivalence class in a constraint

$X{=}Y \ \wedge\ \varphi =^{loc}_{\mathcal{G}} X{=}Y \ \wedge\ \varphi[Y/X]$

(4) Set equivalence (associativity, commutativity, idempotency)

$\varphi {=}^{loc}_{\mathcal{G}} \psi \qquad\qquad \text{if } \varphi =^{set} \psi$

Fig. 20. The equivalence relation $=^{loc}_{\mathcal{G}}$ on constraints handling local variables

used previously is not sufficient; we adapt it such that it takes local variables into account.

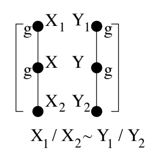

Fig. 19. Local variables?

Consider Fig. 19. If (N.New) is applied to X first, this yields $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & X' \end{smallmatrix}\right)$ for a new local variable X', plus $Y_1{:}g(X')$ and $X'{=}Y$ by (P.Copy.Lab) and (D.Eq.Decom). Accordingly, if (N.New) is applied to Y first, we get $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ Y' & Y \end{smallmatrix}\right) \wedge X_1{:}g(Y') \wedge Y'{=}X$ for a new local variable Y'. The nondeterministic choice in applying (N.New) leads to two *DPNT*-solved forms incomparable by $\subseteq$ which, however, we do not want to distinguish.

To solve this problem, we use an equivalence relation handling local variables: Let $\mathcal{G} \subseteq \mathcal{V}$, then $=^{loc}_{\mathcal{G}}$ is the smallest equivalence relation on constraints satisfying the axioms in Fig. 20. From this equivalence and subset inclusion, we define the new partial order $\leq_{\mathcal{G}}$.

Definition 12. For $\mathcal{G} \subseteq \mathcal{V}$ let $\leq_{\mathcal{G}}$ be the reflexive and transitive closure $(\subseteq \cup =^{loc}_{\mathcal{G}})^*$.

We also write $=_{\mathcal{G}}$ for $\leq_{\mathcal{G}} \cap \geq_{\mathcal{G}}$. We return to our above example concerning Fig. 19. Let $\mathcal{G} = \{X_1, X_2, Y_1, Y_2, X, Y\}$. Then $X_1{:}g(X) \wedge Y_1{:}g(Y) \wedge Y_1{:}g(X') \wedge X'{=}Y =^{loc}_{\mathcal{G}} X_1{:}g(X) \wedge Y_1{:}g(Y) \wedge X'{=}Y$ by axioms (3) and (4). This, in turn, is $=^{loc}_{\mathcal{G}}$ equivalent to $X_1{:}g(X) \wedge Y{:}g(Y)$ by axiom (1). Again by axiom (1), this is $=^{loc}_{\mathcal{G}}$ equivalent to $X_1{:}g(X) \wedge Y_1{:}g(Y) \wedge Y'{=}X$, which equals $X_1{:}g(X) \wedge X_1{:}g(Y') \wedge Y_1{:}g(Y) \wedge Y'{=}X$ by axioms (4) and (3).

Lemma 13. *The partial order $\leq_{\mathcal{G}}$ can be factored out into the relational composition of its components, i.e., $\leq_{\mathcal{G}}$ is $\subseteq \circ =^{loc}_{\mathcal{G}}$.*

Lemma 14. *If $\varphi \leq_{\mathcal{G}} \psi$ and ψ is a DPNT-solved form, then there exists a DPNT-solved form φ' such that $\varphi \subseteq \varphi' =^{loc}_{\mathcal{G}} \psi$.*

Lemma 15. *Let φ be a constraint, $\mathcal{G} \subseteq \mathcal{V}$, and ψ a DPNT-solved form with $\varphi \leq_{\mathcal{G}} \psi$. If a rule $\rho \in$ DPNT is applicable to φ, then there exists a constraint φ' satisfying $\varphi \rightarrow_{\{\rho\}} \varphi'$ and $\varphi' \leq_{\mathcal{G}} \psi$.*

Proof. By Lemma 14 there exists a *DPNT*-solved form ψ' with $\varphi \subseteq \psi' =^{loc}_{\mathcal{G}} \psi$. First, suppose ρ is a rule $\overline{\varphi} \rightarrow \vee^{n}_{i=1} \overline{\varphi}_i$ in *DPT*. Then there exists an i such that $\overline{\varphi}_i \subseteq \psi'$, hence

$\varphi \wedge \overline{\varphi}_i \subseteq \psi'$. Now suppose that $\rho \in N$: Let ρ be $\overline{\varphi} \to \mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & X' \end{smallmatrix}\right)$ with $X' \notin \mathcal{G} \cup \mathcal{V}(\varphi)$. Then $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & Y \end{smallmatrix}\right) \in \psi$ for some variable Y. But then by axiom (2) of Fig. 20, we have $\psi' =^{loc}_{\mathcal{G}} \psi'[Z'/X']$ for some $Z' \notin \mathcal{G} \cup \mathcal{V}(\psi') \cup \mathcal{V}(\varphi)$, which by axiom (1) is $=^{loc}_{\mathcal{G}}$ equivalent to $\psi'[Z'/X'] \wedge Y{=}X'$, which in turn equals $\psi'[Z'/X'] \wedge Y{=}X' \wedge \mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & X' \end{smallmatrix}\right)$ by axiom (3). Call this last constraint ψ'', then $\varphi \wedge \mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & X' \end{smallmatrix}\right) \subseteq \psi'' =^{loc}_{\mathcal{G}} \psi$. □

It remains to show that there exists a *DPNT*-branch of finite length from φ to each of its minimal solved forms. If saturation rules can be applied in any order, *N* can speculatively generate an arbitrary number of local variables. For example, for the constraint in Fig. 21, it could successively postulate $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ Y_1 & Y_1' \end{smallmatrix}\right)$, $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ Y_1' & Y_1'' \end{smallmatrix}\right)$, ... We solve this problem by choosing a special rule application order in our completeness proof: After each $\to_{\mathrm{N}}$ step, we first form a *DPT*-saturation before considering another rule from *N*. We use a *distance measure* between a smaller and a larger constraint to prove completeness for *DPNT* saturation obeying this application order. The two elements of the measure are: the number of distinct variables in the larger constraint not present in the smaller one; and the minimum number of correspondences still to be computed for a constraint.

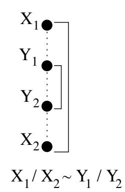

Fig. 21. Termination?

Definition 16. We define the number $\mathsf{lc}(\mathcal{S},\varphi)$ of *lacking correspondents* in φ for a set $\mathcal{S} \subseteq \mathcal{V}(\varphi)$ by

$$\mathsf{lc}(\mathcal{S},\varphi) = \sum \left\{ \mathsf{lc}^{X_1 Y_1}_{X_2 Y_2}(X,\varphi) + \mathsf{lc}^{Y_1 X_1}_{Y_2 X_2}(X,\varphi) \mid X \in S \text{ and } X_1/X_2{\sim}Y_1/Y_2 \in \varphi \right\}$$

where we fix the values of the auxiliary terms be setting for all $W, U, U', V, V' \in \mathcal{V}(\varphi)$:

$$\mathsf{lc}^{UU'}_{VV'}(W,\varphi) = \begin{cases} 1 \text{ if } W \in \mathsf{betw}_{\varphi}(U,V) \text{ and } \mathrm{p}\left(\begin{smallmatrix} U & U' \\ W & W' \end{smallmatrix}\right) \text{ is not in } \varphi \text{ for any } W' \\ 0 \text{ otherwise} \end{cases}$$

Definition 17. For constraints $\varphi_1 {\subseteq} \varphi_2$, let $\mathsf{diff}(\varphi_1,\varphi_2)$ be the size of the set $\{X \in \mathcal{V}(\varphi_2) \mid X{\neq}Y \in \varphi_2 \text{ for all } Y \in \mathcal{V}(\varphi_1)\}$.

We call a set $\mathcal{S} \subseteq \mathcal{V}(\varphi)$ of variables an *inequality set for* φ iff $X{\neq}Y \in \varphi$ for any distinct $X, Y \in \mathcal{S}$.

For constraints φ_2 that are saturated with respect to (P.Distr.Project), $\mathsf{diff}(\varphi_1,\varphi_2)$ is the number of variables X in φ_2 such that $X{=}Y \notin \varphi_2$ for all $Y \in \mathcal{V}(\varphi_1)$.

Definition 18. Let φ, ψ be constraints and $\mathcal{G} \subseteq \mathcal{V}$ with $\varphi \leq_{\mathcal{G}} \psi$. Then the $\mathcal{G}$-*measure* $\mu_{\mathcal{G}}(\varphi,\psi)$ for φ and ψ is the sequence $\left(\mu^1_{\mathcal{G}}(\varphi,\psi), \mu^2(\varphi)\right)$, where:

- $\mu^1_{\mathcal{G}}(\varphi,\psi) = min\{\mathsf{diff}(\varphi,\psi') \mid \varphi \subseteq \psi' =^{loc}_{\mathcal{G}} \psi$ and ψ' is *DPNT*-solved $\}$
- $\mu^2(\varphi) = min\{\mathsf{lc}(\mathcal{S},\varphi) \mid \mathcal{S}$ is a maximal inequality set for $\varphi\}$.

We order $\mathcal{G}$-measures by the lexicographic ordering $<$ on sequences of natural numbers, which is well-founded. The main idea of the following proof is that after each $\to_{\mathrm{N}}$ step and subsequent *DPT*-saturation, the $\mathcal{G}$-measure between a constraint and its solved form has strictly decreased.

Theorem 19 (Completeness). *Let φ be a constraint, $\mathcal{G} \subseteq \mathcal{V}$, and ψ a $\leq_{\mathcal{G}}$-minimal DPNT-solved form for φ. Then there exists a DPNT-solved form $\psi' =_{\mathcal{G}} \psi$ which can be reached from φ, i.e. $\varphi \rightarrow^*_{\mathrm{DPNT}} \psi'$.*

Proof. W.l.o.g. let φ be *DPT*-closed. If no rule from *N* is applicable to φ then $\varphi =_{\mathcal{G}} \psi$ by the minimality of ψ. If a rule $\rho \in N$ is applicable to φ, then by Lemma 15 there exist φ', φ'' such that $\varphi \rightarrow_{\{\rho\}} \varphi'' \rightarrow^*_{\mathrm{DPT}} \varphi' \leq_{\mathcal{G}} \psi$, and φ' is *DPT*-saturated. By induction, it is sufficient to show that $\mu_{\mathcal{G}}(\varphi', \psi) < \mu_{\mathcal{G}}(\varphi, \psi)$. Note that because φ is *DPT*-closed, a maximal inequality set within φ contains exactly one variable from each syntactic variable equivalence class represented in φ; and $\mathsf{lc}(\{X\}, \varphi) = \mathsf{lc}(\{Y\}, \varphi)$ whenever $X{=}Y \in \varphi$ because of saturation under (P.Path.Eq.1). The value of $\mathsf{diff}(\varphi, \psi')$ is minimal, i.e. equal to $\mu^1_{\mathcal{G}}(\varphi, \psi)$, if for any $Y \in \mathcal{V}(\psi')$ with $Y{\neq}X \in \psi'$ for all $X \in \mathcal{V}(\varphi)$ the following holds: Y is local[1] and there is no variable $Z \in \mathcal{V}(\psi')$ distinct from Y with $Y{=}Z \in \psi'$.

Let φ'' be $\varphi \wedge \mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & X' \end{smallmatrix}\right)$. In φ', (P.Distr.Project) has been applied to X' and all variables in $\mathcal{V}(\varphi)$. Let $\psi' =^{loc}_{\mathcal{G}} \psi$ with $\varphi \subseteq \psi'$ and minimal $\mathsf{diff}(\varphi, \psi')$. The constraint ψ' contains $\mathrm{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X & Z \end{smallmatrix}\right)$ for some Z. W.l.o.g. we pick a ψ' that does not contain X'.

- If $X'{=}Y \in \varphi'$ for some $Y \in \mathcal{V}(\varphi)$, then $\mu^2(\varphi') < \mu^2(\varphi)$ and $\mu^1_{\mathcal{G}}(\varphi', \psi) = \mu^1_{\mathcal{G}}(\varphi', \psi)$: $\mathsf{lc}(\{V\}, \varphi') < \mathsf{lc}(\{X\}, \varphi)$ whenever $V{=}X \in \varphi'$, and either X or some other member of its equivalence class must be in each maximal inequality set. At the same time, a maximal inequality set within φ' can contain only one of X' and Y, so X' contributes nothing additional to $\mu^2(\varphi')$.
  Let ψ'' be $\psi' \wedge X'{=}Z \wedge \psi'[X'/Z]$. Then ψ'' is *DPNT*-solved, and $\varphi' \subseteq \psi''$. We have $\mathsf{diff}(\varphi', \psi'') = \mathsf{diff}(\varphi, \psi')$ because for any $V{\neq}Y \in \psi'$, ψ'' contains $V{\neq}Y \wedge V{\neq}X'$. Furthermore, $\mathsf{diff}(\varphi', \psi'')$ is minimal because the only variable in ψ'' not in ψ' is X'.
- If $X'{\neq}Y \in \varphi'$ for all $Y \in \mathcal{V}(\varphi)$, then $\mu^1_{\mathcal{G}}(\varphi', \psi) < \mu^1_{\mathcal{G}}(\varphi, \psi)$: Let ψ'' be $\psi'[X'/Z]$. Thus, $\psi' =^{loc}_{\mathcal{G}} \psi''$ by axiom (2) and because Z must be local, and $Z{=}Z'$ is not in ψ'' for any distinct Z' because of the minimality of $\mathsf{diff}(\varphi, \psi')$, as pointed out above. Obviously ψ'' is a *DPNT*-solved form with $\varphi' \subseteq \psi''$. Furthermore, $\mathsf{diff}(\varphi', \psi'') = \mathsf{diff}(\varphi, \psi) - 1$ because we must have had $Z{\neq}V \in \psi'$ for all $V \in \mathcal{V}(\varphi)$. $\square$

7 Conclusion

We have presented a semi-decision procedure for parallelism constraints which terminates for the important fragment of dominance constraints. It uses path equality constraints to record correspondence, allowing for strong propagation. We have proved the procedure sound and complete. In the process, we have introduced the concept of a minimal solved form for parallelism constraints.

Many things remain to be done. One important problem is to describe the linguistically relevant fragment of parallelism constraints and see whether it is decidable. Then, the prototype implementation we have is not optimized in any way. We would like to

[1] The variable Y is local because $\mathcal{V}(\psi') \cap \mathcal{G} = \mathcal{V}(\psi) \cap \mathcal{G} = \mathcal{V}(\varphi) \cap \mathcal{G}$, otherwise ψ would not be a *minimal* solved form for φ.

replace it by one using constraint technology and to see how that scales up to large examples from linguistics. Also, we would like to apply parallelism constraints to a broader range of linguistic phenomena.

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A Correspondence Functions

In the following appendix sections, we give the proofs omitted earlier for brevity. The first proof we still owe is that of lemma 3: We prove that whenever we have a correspondence function, then corresponding nodes are reached via the same paths from the parallelism roots down.

Lemma 3. *If* $c : \mathsf{betw}_\tau(\pi_1, \pi_2) \rightarrow \mathsf{betw}_\tau(\pi_3, \pi_4)$ *is a correspondence function, then* $c(\pi_1\pi) = \pi_3\pi$ *for all* $\pi_1\pi \in \mathsf{betw}_\tau(\pi_1, \pi_2)$.

Proof. By induction on π. The case of $\pi = \varepsilon$ is obvious. So let $\pi = \pi' i \in D_\tau$ with $\pi_1\pi \in \mathsf{betw}_\tau(\pi_1, \pi_2)$ and let $\overline{\pi} = \pi_1\pi'$. As $\pi' i \in D_\tau$, we have $\overline{\pi} \neq \pi_2$. Suppose $\mathcal{M}^\tau \models \overline{\pi}{:}f(\overline{\pi}1, \ldots, \overline{\pi}n)$, then $\mathcal{M}^\tau \models c(\overline{\pi}){:}f(c(\overline{\pi}1), \ldots, c(\overline{\pi}n))$ as c is a correspondence function. By the inductive hypothesis, $c(\overline{\pi}) = c(\pi_1\pi') = \pi_3\pi'$. Hence $c(\pi_1\pi' j) = \pi_3\pi' j$ for $1 \leq j \leq n$. As $\pi \in D_\tau$, we have $i \in \{1, \ldots, n\}$, so $c(\pi_1\pi) = \pi_3\pi$. □

B Soundness of rule set *D* for dominance constraints

We proceed in two steps, as sketched for the soundness proof of *DPNT* in section 5. First, we identify *simple D-solved forms* and show that they are satisfiable (Proposition 21). Then we show how to extend every *D*-solved form into a simple *D*-solved form by adding further constraints (Lemma 26).

Definition 20. A variable X is *labeled in* φ iff $X{=}Y \in \varphi$ and $Y{:}f(Y_1, \ldots, Y_n) \in \varphi$ for some variable Y and term $f(Y_1, \ldots, Y_n)$. A variable Y is a *root variable* for φ if $Y \triangleleft^* Z \in \varphi$ for all $Z \in \mathcal{V}(\varphi)$. We call a constraint φ *simple* iff all its variables are labeled, and if there is a root variable for φ.

The constraint graph of a simple constraint in *D*-solved form (Def. 9) is tree-shaped.

Lemma 21. *A simple D-solved form is satisfiable.*

Proof. By induction on the number of literals in a simple *D*-solved form φ. Let Z be a root variable in φ. Since all variables in φ are labeled, there is a variable Z' and a term $f(Z_1, \ldots, Z_n)$ such that $Z{=}Z'$ and $Z'{:}f(Z_1, \ldots, Z_n)$ are in φ. Let

$$V = \{X \in \mathcal{V}(\varphi) \mid Z{=}X \in \varphi\} \text{ and } V_i = \{X \in \mathcal{V}(\varphi) \mid Z_i \triangleleft^* X \in \varphi\}.$$

for all $1 \leq i \leq n$. To see that $\mathcal{V}(\varphi) = V \cup V_1 \cup \ldots \cup V_n$, let $X \in \mathcal{V}(\varphi)$ such that $Z_i \triangleleft^* X \notin \varphi$ for all $1 \leq i \leq n$. As Z is a root variable, $Z \triangleleft^* X \in \varphi$, and by saturation with (D.Distr.Child), φ must contain $Z{=}X$.

For a set $W \subseteq \mathcal{V}(\varphi)$ we define $\varphi_{|W}$ as the conjunction of all literals $\psi \in \varphi$ with $\mathcal{V}(\psi) \subseteq W$. We show that

$$\varphi \models\!\!\!\!\!\dashv \varphi' \quad \text{holds where} \quad \varphi' := \varphi_{|V} \wedge Z{:}f(Z_1, \ldots, Z_n) \wedge \bigwedge_{i=1}^{n} \varphi_{|V_i}$$

because φ is in *D*-solved form: Each literal in φ is entailed by φ'.

- Suppose $X{:}g(X_1,\ldots,X_m) \in \varphi$ for some variable X and term $g(X_1,\ldots,X_m)$. If $Z_i \lhd^* X \in \varphi$ for some $1 \leq i \leq n$, then $X{:}g(X_1,\ldots,X_m) \in \varphi_{|V_i}$ since φ is saturated under (D.Lab.Dom) and (D.Dom.Trans). Otherwise, $Z{=}X \in \varphi$, and thus $Z{=}X \in \varphi_{|V}$. In this case, $f = g$ and $n = m$ by saturation with (D.Lab.Ineq) and (D.Clash.Ineq) coupled with the clash-freeness of φ. As φ is saturated under (D.Eq.Decom), it must contain $Z_i{=}X_i$ for $1 \leq i \leq n$, hence $Z_i{=}X_i \in \varphi|_{V_i}$. So, φ' contains $Z{=}X \wedge Z{:}f(Z_1,\ldots,Z_n) \wedge \bigwedge_{i=1}^n Z_i{=}X_i$, which entails $X{:}g(X_1,\ldots,X_m)$ as required.
- Now suppose $XRY \in \varphi$ for some variables X, Y and $R \in \{\lhd^*, \neq, \perp\}$. There are four possible cases:
  - If $X \in V_i, Y \in V_j$ with $1 \leq i \neq j \leq n$, then R cannot be $\lhd^*$ by (D.Dom.Refl), (D.Prop.Disj) and (D.Clash.Disj) combined with the clash-freeness of φ. φ' entails $Z_i \perp Z_j$ and thus $X \perp Y$ as well as $X{\neq}Y$.
  - The cases where X and Y both belong to V or to the same V_i are obvious.
  - If $X \in V$ and $Y \in V_i$ for some i, then $X \lhd^* Y \in \varphi$ by (D.Lab.Dom) and (D.Dom.Trans). R cannot be $\perp$ by saturation under (D.Dom.Refl), (D.Prop.Disj) and (D.Clash.Disj) and the clash-freeness of φ. φ' entails $Z \lhd^+ Z_i$ and thus $X \lhd^* Y$ and $X{\neq}Y$.
  - The case of $X \in V$ and $Y \in V_i$ is symmetric to the previous one.

Next note that all $\varphi_{|V_i}$ are simple *D*-solved forms. By the inductive hypothesis there exist solutions $(\mathcal{M}^{\tau_i}, \alpha_i) \models \varphi_{|V_i}$ for all $1 \leq i \leq n$. Thus, $(\mathcal{M}^{f(\tau_1,\ldots,\tau_n)}, \alpha)$ is a solution of φ if $\alpha_{|V_i} = \alpha_i$ and $\alpha(X) = \alpha(Z)$ is the root node of $f(\tau_1,\ldots,\tau_n)$ for all $X \in V$. □

Now suppose we have a constraint φ in *D*-solved form. We want to show that there is an *extension* $\varphi \wedge \varphi'$ of φ such that $\varphi \wedge \varphi'$ is in *D*-solved form as well as simple. We proceed by successively labeling unlabeled variables $X \in \varphi$, taking as X's children the variables minimally dominated by it, as sketched in Fig. 17. We formalize this as follows: Given a constraint φ we define an ordering $\prec_\varphi$ on its variables such that $X \prec_\varphi Y$ holds iff $X \lhd^* Y \in \varphi$ but not $Y \lhd^* X \in \varphi$.

Definition 22. Let φ be a dominance constraint and $X \in \mathcal{V}(\varphi)$ unlabeled. Then we define the set $\operatorname{con}_\varphi(X)$ of variables *connected to X in φ* as follows:

$$\operatorname{con}_\varphi(X) = \{Y \in \mathcal{V}(\varphi) \mid Y \text{ minimal with } X \prec_\varphi Y\}$$

For the constraint in Fig. 17, $\operatorname{con}_\varphi(X) = \{Y, Z, U\}$. However, when picking variables to serve as children of X, we do not use all of $\operatorname{con}_\varphi(X)$: In the example above, we choose only one of Z, U as we have $Z{=}U$.

Definition 23. We call $V \subseteq \mathcal{V}(\varphi)$ a *φ-disjointness* set if for any two distinct variables $Y_1, Y_2 \in V$, $Y_1{=}Y_2 \notin \varphi$.

The idea is that all variables in a φ-disjointness set can safely be placed at disjoint positions in at least one of the trees solving φ.

Lemma 24. *Let φ be D-saturated and $X \in \mathcal{V}(\varphi)$. If V is a maximal φ-disjointness set within $\mathrm{con}_{\varphi}(X)$, then for all $Y \in \mathrm{con}_{\varphi}(X)$ there exists some $Z \in V$ such that $Y{=}Z \in \varphi$.*

Proof. If $Y{=}Z \notin \varphi$ for all $Z \in V$, then $\{Y\} \cup V$ is a disjointness set; thus $Y \in V$ by the maximality of V. $\square$

Lemma 25 (Extension by Labeling). *Every D-solved form φ with an unlabeled variable X can be extended to a D-solved form in which X is labeled.*

Proof. Let $\{X_1, \ldots, X_n\}$ be a maximal φ-disjointness set in $\mathrm{con}_{\varphi}(X)$. Let us assume for the moment that Σ contains a function symbol f of arity n. We define the following extension $\mathsf{ext}(\varphi)$ of $\varphi \wedge X{:}f(X_1, \ldots, X_n)$:

$$\begin{aligned}\mathsf{ext}(\varphi) := \varphi \ \wedge\ X{:}f(X_1, \ldots, X_n) \ \wedge\ \bigwedge_{i=1}^{n} X{\neq}X_i \ \wedge \\ \bigwedge_{\substack{X_i \triangleleft^* U, X_j \triangleleft^* V \in \varphi,\\ 1 \leq i \neq j \leq n}} U \perp V \ \wedge \bigwedge_{\substack{Z:g(\ldots) \in \varphi,\\ g \neq f \ \vee\ \mathsf{ar}(g) \neq \mathsf{ar}(f)}} Z{\neq}X\end{aligned}$$

Note that X is labeled in $\mathsf{ext}(\varphi)$ since $X{=}X \in \varphi$ by (D.Dom.Refl). We consider each rule of *D* in turn and show that it is not applicable to $\mathsf{ext}(\varphi)$.

(D.Clash.Ineq): No new dominance constraints have been introduced.
Suppose a new inequality $X{\neq}X_i$ has made (D.Clash.Ineq) applicable. Then $X{=}X_i \in \varphi$, but $X_i \in \mathrm{con}_{\varphi}(X)$.
Suppose a new inequality $Z{\neq}X$ has made (D.Clash.Ineq) applicable. Then $Z{:}g(\ldots)$ and $X{=}Z$ are in φ, but X is unlabeled in φ.

(D.Clash.Disj): Suppose a new literal $U \perp V$ has made (D.Clash.Disj) applicable, where $X_i \triangleleft^* U, X_j \triangleleft^* V \in \varphi$ with $i \neq j$. Then $U{=}V \in \varphi$. As φ is saturated under (D.Distr.NotDisj), we must have either $X_i \triangleleft^* X_j$ or $X_j \triangleleft^* X_i$ in φ. But $\{X_i, X_j\}$ is a disjointness set.

(D.Dom.Refl): No new variables have been added.

(D.Dom.Trans): No new dominance constraints have been added.

(D.Eq.Decom): For (D.Eq.Decom) to be applicable to $X{:}f(X_1, \ldots, X_n)$ and some literal $Z{:}f(Z_1, \ldots, Z_n) \in \varphi$, $Z{=}X$ must be in φ already. But X is unlabeled in φ.

(D.Lab.Ineq): The only new labeling constraint is $X{:}f(X_1, \ldots, X_n)$. $Z{\neq}X$ is in $\mathsf{ext}(\varphi)$ for all Z labeled anything but f.

(D.Lab.Disj): The only new labeling constraint is $X{:}f(X_1, \ldots, X_n)$. By saturation under (D.Dom.Refl), $X_i \triangleleft^* X_i \in \varphi$ for $1 \leq i \leq n$, so $X_i \perp X_j$ is in $\mathsf{ext}(\varphi)$ for all $1 \leq i \neq j \leq n$ by definition.

(D.Prop.Disj): The only disjointness constraints new in $\mathsf{ext}(\varphi)$ have the form $U \perp V$, where $X_i \triangleleft^* U, X_j \triangleleft^* V \in \varphi$ for $j \neq i$. If $U \triangleleft^* U'$ and $V \triangleleft^* V'$ are in φ, then by saturation under (D.Dom.Trans) $X_i \triangleleft^* U', X_j \triangleleft^* V' \in \varphi$, so $U' \perp V'$ is in $\mathsf{ext}(\varphi)$.

(D.Lab.Dom): $X{:}f(X_1, \ldots, X_n)$ is the only labeling constraint in $\mathsf{ext}(\varphi) - \varphi$. We have $X \triangleleft^* X_i \in \varphi$ for all $1 \leq i \leq n$ because $\{X_1, \ldots, X_n\} \subseteq \mathrm{con}_{\varphi}(X)$. $X{\neq}X_i$ is in $\mathsf{ext}(\varphi)$ by definition for all $1 \leq i \leq n$.

(D.Distr.Child): Suppose $X \triangleleft^* Z \in \varphi$, but neither $Z \triangleleft^* X$ nor $X_i \triangleleft^* Z$ is in φ for any $i \in \{1, \ldots, n\}$. Then $X \prec_\varphi Z$. If $Z \in \text{con}_\varphi(X)$, we have the following situation: The disjointness set $\{X_1, \ldots, X_n\}$ is maximal within $\text{con}_\varphi(X)$, so $Z{=}X_i$ for some $i \in \{1, \ldots, n\}$ by lemma 24, a contradiction. So suppose Z is not minimal, i.e. there exists some $Y \in \text{con}_\varphi(X)$ such that $Y \triangleleft^* Z \in \varphi$. But then again, $X_i{=}Y$ for some $i \in \{1, \ldots, n\}$, so $X_i \triangleleft^* Z$.

(D.Distr.NotDisj): No new dominance constraints have been added.

We now turn to the case that the signature does not contain a function symbol for the arity we need. We can get around this problem by encoding the symbols with a nullary symbol and one symbol of arity ≥ 2, whose existence we have assumed. This encoding may introduce new variables, but only finitely many. For a detailed description of this construction, see [11], lemma 4.11. If a function symbol of the appropriate arity is present in Σ, then the labeling of X does not introduce new variables. $\square$

Lemma 26. *Every D-solved form can be extended to a simple D-solved form.*

Proof. Let φ be *D*-saturated and without **false**. Without loss of generality, we can assume that φ has a root variable (otherwise, we choose a fresh variable X and consider $\varphi \wedge \bigwedge\{X \triangleleft^* Y | Y \in \mathcal{V}(\varphi)\}$ instead of φ). By Lemma 25, we can successively label all variables in φ. $\square$

Together, lemmas 21 and 26 show the soundness of *D*:

Proposition 4 (Soundness). *Any dominance constraint in D-solved form is satisfiable.*

C Soundness of rule set *DPNT* for parallelism constraints

Generatedness is about where path equality literals may occur. (See Def. 7.) In proving soundness of *DPNT*, we may restrict ourselves to generated constraints, since all solved forms that are computed are generated:

Lemma 8. *Let φ be a constraint without path equalities and let $\varphi \to^*_{\text{DPNT}} \varphi'$ with φ' in DPNT-solved form. Then φ' is generated.*

Proof. Let $\varphi_1, \ldots, \varphi_n$ be a sequence of constraints such that $\varphi_1 =^{set} \varphi$, $\varphi_n =^{set} \varphi'$, and $\varphi_i \to_{\text{DPNT}} \varphi_{i+1}$ for $1 \leq i \leq n-1$. We show by induction on i that (1) each $\text{p}\left(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}\right) \in \varphi_i$ is generated in φ', (2) alongside with $\text{p}\left(\begin{smallmatrix} Y & X \\ V & U \end{smallmatrix}\right)$ and every $\text{p}\left(\begin{smallmatrix} X' & Y' \\ U' & V' \end{smallmatrix}\right)$ with $X'{=}X, U'{=}U, Y'{=}Y, V'{=}V \in \varphi'$.

φ_1 contains no path qualities. So let $\varphi_i \to_{\{\rho\}} \varphi_{i+1}$, where ρ is an instance of (P.Root), (P.Path.Sym), (P.Path.Eq.1) or (N.New), or $\rho \in T$.

If ρ is an instance of (P.Root), then φ_{i+1} has the form $\varphi_i \wedge \text{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_1 & Y_1 \end{smallmatrix}\right) \wedge \text{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{smallmatrix}\right)$ for some X_1, X_2, Y_1, Y_2. Then $X_1/X_2 \sim Y_1/Y_2 \in \varphi$, and we have $X_1, X_2 \in \text{betw}_{\varphi'}(X_1, X_2)$ by closure under (D.Dom.Refl) and (P.Path.Dom). So $\text{p}\left(\begin{smallmatrix} X_1 & Y_1 \\ X_1 & Y_1 \end{smallmatrix}\right)$,

$\mathrm{p}\left(\begin{smallmatrix}X_1 & Y_1\\ X_2 & Y_2\end{smallmatrix}\right)$ are correspondence-generated in φ'. Condition (2) from above holds for $\mathrm{p}\left(\begin{smallmatrix}X_1 & Y_1\\ X_1 & Y_1\end{smallmatrix}\right)$ and $\mathrm{p}\left(\begin{smallmatrix}X_1 & Y_1\\ X_2 & Y_2\end{smallmatrix}\right)$ by closure of φ' under (P.Path.Sym), (P.Path.Eq.1) and (D.Dom.Trans).

If ρ is an instance of (N.New), then φ_{i+1} has the form $\varphi_i \wedge \mathrm{p}\left(\begin{smallmatrix}X_1 & Y_1\\ X & X'\end{smallmatrix}\right)$, and $X_1/X_2{\sim}Y_1/Y_2 \in \varphi_i$ for some X_2, Y_2 such that $X \in \mathsf{betw}_{\varphi'}(X_1, X_2)$. So $\mathrm{p}\left(\begin{smallmatrix}X_1 & Y_1\\ X & X'\end{smallmatrix}\right)$ is correspondence-generated in φ'. Condition (2) holds for $\mathrm{p}\left(\begin{smallmatrix}X_1 & Y_1\\ X & X'\end{smallmatrix}\right)$ by closure under (P.Path.Sym), (P.Path.Eq.1), (D.Dom.Trans) and (D.Prop.Disj).

If $\rho \in T$ and φ_{i+1} has the form $\varphi_i \wedge \mathrm{p}\left(\begin{smallmatrix}X & Y\\ U & V\end{smallmatrix}\right)$, then $\mathrm{p}\left(\begin{smallmatrix}X & Y\\ U & V\end{smallmatrix}\right)$ is generated by definition. Concerning condition (2), we just consider the case of (T.Trans.H), the others are analogous. Suppose ρ has the form $\mathrm{p}\left(\begin{smallmatrix}X & Z\\ U & W\end{smallmatrix}\right) \wedge \mathrm{p}\left(\begin{smallmatrix}Z & Y\\ W & V\end{smallmatrix}\right) \to \mathrm{p}\left(\begin{smallmatrix}X & Y\\ U & V\end{smallmatrix}\right)$. Then $\mathrm{p}\left(\begin{smallmatrix}Z & X\\ W & U\end{smallmatrix}\right)$, $\mathrm{p}\left(\begin{smallmatrix}Y & Z\\ V & W\end{smallmatrix}\right)$ are in φ' by closure under (T.Trans.H) and generated by the inductive hypothesis. So $\mathrm{p}\left(\begin{smallmatrix}Y & X\\ V & U\end{smallmatrix}\right) \in \varphi'$ is generated in φ' as well. The case of a literal $\mathrm{p}\left(\begin{smallmatrix}X' & Y'\\ U' & V'\end{smallmatrix}\right)$ where $X'{=}X, U'{=}U, Y'{=}Y, V'{=}V \in \varphi'$ is analogous.

If ρ is an instance of (P.Path.Sym) or (P.Path.Eq.1) and φ_{i+1} has the form $\varphi_i \wedge \mathrm{p}\left(\begin{smallmatrix}X & Y\\ U & V\end{smallmatrix}\right)$, then $\mathrm{p}\left(\begin{smallmatrix}X & Y\\ U & V\end{smallmatrix}\right)$ is generated in φ' because of inductive hypothesis (2). $\square$

As for the case of dominance constraints, we first prove that simple generated constraints in *DPNT*-solved form are satisfiable.

Proposition 10. *A simple generated constraint in DPNT-solved form is satisfiable.*

Proof. Let φ be a simple generated constraint in *DPNT*-solved form, and let φ_{dom} be the maximal subset of φ that is a dominance constraint. φ_{dom} is in *D*-solved form, so it is satisfiable (Lemma 21). It remains to show that all path equality literals and all parallelism literals of φ are satisfied in a solution $(\mathcal{M}^\tau, \alpha)$ of φ_{dom} as constructed in lemma 21. Note that by this construction, if $\pi \in D_\tau$, then there exists some $X \in \mathcal{V}(\varphi)$ with $\alpha(X) = \pi$.

Path equality literals. Let $\mathrm{p}\left(\begin{smallmatrix}X & Y\\ U & V\end{smallmatrix}\right)$ be a path equality literal in φ. As φ is simple, either $X{=}U \in \varphi$, or there exist $X_0, \ldots, X_n \in \mathcal{V}(\varphi)$ for some n such that $X_0{=}X, X_n{=}U \in \varphi$ and for all $0 \le i \le n-1$, $X_i{:}f_i(X'_{i_1}, \ldots, X'_{i_{m_i}}) \in \varphi$ for some $X'_{i_1}, \ldots, X'_{i_{m_i}} \in \mathcal{V}(\varphi)$ and $f_i \in \Sigma$ of arity m_i, and $X'_{j_i}{=}X_{i+1} \in \varphi$ for some $j_i \in \{i_1, \ldots, i_{m_i}\}$. n and the f_i, $1 \le i \le n$, are unique as φ is clash-free and closed under (D.Distr.NotDisj), (D.Distr.Child) and (D.Lab.Ineq). We show, by induction on the length of a proof of generatedness for $\mathrm{p}\left(\begin{smallmatrix}X & Y\\ U & V\end{smallmatrix}\right)$, that if $X{=}U \in \varphi$ then $Y{=}V \in \varphi$, and that otherwise for all $0 \le i \le n$, $\mathrm{p}\left(\begin{smallmatrix}X & Y\\ X_i & Y_i\end{smallmatrix}\right) \in \varphi$ for some $Y_i \in \mathcal{V}(\varphi)$ in such a way that for $0 \le i \le n-1$, $Y_i{:}f_i(Y'_{i_1}, \ldots, Y'_{i_{m_i}}) \in \varphi$ for some $Y'_{i_1}, \ldots, Y'_{i_{m_i}} \in \mathcal{V}(\varphi)$, and $Y'_{j_i}{=}Y_{i+1} \in \varphi$.

Suppose $\mathrm{p}\left(\begin{smallmatrix}X & Y\\ U & V\end{smallmatrix}\right)$ is correspondence-generated. Then there exists some parallelism literal $W_1/W_2{\sim}W_3/W_3 \in \varphi$ with $W_1{=}X, W_3{=}Y \in \varphi$. W.l.o.g. suppose $U \in \mathsf{betw}_\varphi(W_1, W_2)$, then $V \in \mathsf{betw}_\varphi(W_3, W_4)$ by (P.Copy.Dom). If $X{=}U \in \varphi$, then also $Y{=}V \in \varphi$ by closure under (P.Copy.Dom). Suppose $X{=}U \notin \varphi$. We proceed by induction on n.

Suppose $n = 1$. We have $\mathrm{p}\left(\begin{smallmatrix}X_0 & Y_0\\ X_0 & Y_0\end{smallmatrix}\right) \in \varphi$ by closure under (P.Root) and (P.Path.Eq.1). If $X{:}f(X'_1, \ldots, X'_m) \in \varphi$, then $X'_1, \ldots, X'_m \in \mathsf{betw}_\varphi(W_1, W_2)$ by closure under (D.Lab.Dom), (D.Dom.Trans), (P.Distr.Crown) and the fact that $U \in \mathsf{betw}_\varphi(W_1, W_2)$. φ must contain either $X \triangleleft^* W_2$ or $X \bot W_2$ as $X \in \mathsf{betw}_\varphi(W_1, W_2)$, and if $X \triangleleft^* W_2$ then also $X {\neq} W_2$ by (P.Distr.Project) since $U \in \mathsf{betw}_\varphi(W_1, W_2)$ and $X {\neq} U$ by (P.Lab.Dom); so by closure of φ under (N.New) and (P.Copy.Lab), we must have $Y{:}f(Y'_1, \ldots, Y'_m) \in \varphi$ for some $Y'_1, \ldots, Y'_m \in \mathcal{V}(\varphi)$. Likewise, if $U{=}X'_j$, then $V{=}Y'_j$ by (P.Copy.Lab), (P.Path.Eq.1).

Now suppose $n > 1$. As $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}\right)$ is correspondence-generated, there exists some $W_1/W_2{\sim}W_3/W_3 \in \varphi$ with $W_1{=}X, W_3{=}Y \in \varphi$ and $U \in \mathsf{betw}_\varphi(W_1, W_2)$, $V \in \mathsf{betw}_\varphi(W_3, W_4)$. As $n > 1$, there exists X_{n-1} such that $X \triangleleft^* X_{n-1} \in \varphi$ and $X_{n-1}{:}f(X'_1, \ldots, X'_m) \in \varphi$ for some $f, m, X'_1, \ldots, X'_m$, and $U{=}X'_j \in \varphi$ for some j. As $X, U \in \mathsf{betw}_\varphi(W_1, W_2)$ and $X \triangleleft^* X_{n-1} \triangleleft^* U \in \varphi$ by (D.Lab.Dom), (D.Dom.Trans), $X_{n-1} \in \mathsf{betw}_\varphi(W_1, W_2)$ must have been chosen by (P.Distr.Crown), so by (N.New) there exists Y_{n-1} with $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ X_{n-1} & Y_{n-1} \end{smallmatrix}\right) \in \varphi$. By the inductive hypothesis, $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ X_i & Y_i \end{smallmatrix}\right) \in \varphi$ for all $0 \leq i \leq n-1$. As $X_{n-1}, U \in \mathsf{betw}_\varphi(W_1, W_2)$ and $X_{n-1} {\neq} X'_i$ for $1 \leq i \leq m$ by (D.Lab.Dom), we must have $X'_1, \ldots, X'_m \in \mathsf{betw}_\varphi(W_1, W_2)$ by (P.Distr.Crown). So by (N.New) there are $Y'_1, \ldots, Y'_m$ such that $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ X'_i & Y'_i \end{smallmatrix}\right) \in \varphi$ for $1 \leq i \leq m$. As above, we can argue that either $X_{n-1} \bot W_2$ or $X_{n-1} \triangleleft^+ W_2$ must be in φ, so by (P.Copy.Lab), $Y_{n-1}{:}f(Y'_1, \ldots, Y'_m) \in \varphi$. Furthermore, φ must contain $V{=}Y'_j$ by (P.Path.Eq.1).

Suppose $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}\right)$ is generated but not correspondence-generated, i.e. there exists a rule $\rho \in T$ with rhs $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}\right)$ such that all path equality literals in the lhs of ρ are generated. Suppose ρ is an instance of (T.Trans.H) and the lhs of ρ is $\mathrm{p}\left(\begin{smallmatrix} X & Z \\ U & W \end{smallmatrix}\right) \wedge \mathrm{p}\left(\begin{smallmatrix} Z & Y \\ W & V \end{smallmatrix}\right)$. If $X{=}U \in \varphi$ then $Z{=}W \in \varphi$ and thus also $Y{=}V \in \varphi$ by the inductive hypothesis. So suppose $X{=}U \notin \varphi$, and suppose we have sequences $X{=}X_0, \ldots, X_{n_1}{=}U$ and $Z{=}Z_0, \ldots, Z_{n_2}{=}W$. By the inductive hypothesis, we must have $n_1{=}n_2$.

Now suppose ρ is an instance of (T.Diff.2) and the lhs of ρ is $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ U' & V' \end{smallmatrix}\right) \wedge \mathrm{p}\left(\begin{smallmatrix} U & V \\ U' & V' \end{smallmatrix}\right) \wedge X \triangleleft^* U \wedge Y \triangleleft^* V$. If $X{=}U' \in \varphi$, then $X{=}U \in \varphi$ by (D.Dom.Trans), and by the inductive hypothesis $Y{=}V'$ and thus $Y{=}V$ are in φ by (D.Dom.Trans). If $U{=}U' \in \varphi$, then $V{=}V' \in \varphi$ by the inductive hypothesis, and $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}\right) \in \varphi$ even without application of ρ. Suppose otherwise, and let $X{=}X_0, \ldots, X_{n_1}{=}U'$ and $U{=}U_0, \ldots, U_{n_2}{=}U'$. By closure under (D.Lab.Dom), (D.Dom.Trans) and (D.Distr.NotDisj), there exists a minimal $i \in \{0, \ldots, n_1\}$ with $U_0 \triangleleft^* X_i \in \varphi$. φ is simple, so by (D.Distr.Child), we must have $X_i{=}U_0 \in \varphi$, i.e. we can choose the sequence $X_0, \ldots, X_{n_1}$ such that it equals $X_0, \ldots, X_{i-1}, U_0, \ldots, U_{n_2}$. But then the inductive hypotheses already hold for $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}\right)$ and the sequence $X = X_0, \ldots, X_{i-1}, U_0{=}U$. The cases of ρ being an instance of (T.Trans.V) or (T.Diff.1) are analogous.

Now let

$$\begin{aligned} \varphi' = \varphi_{dom} & \cup \{X_1/X_2{\sim}Y_1/Y_2 \in \varphi\} \cup \\ & \{\mathrm{p}\left(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}\right) \in \varphi \mid \exists f, n, i, X_1, \ldots, X_n.\ X{:}f(X_1, \ldots, X_n), X_i{=}U \in \varphi\}. \end{aligned}$$

Then $\varphi \models\!\mid \varphi'$: $\varphi \models \varphi'$ since $\varphi' \subseteq \varphi$. $\varphi' \models \varphi$ since all path equalities in φ of the form $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ X & Y \end{smallmatrix}\right)$ are entailed anyway, and the remaining path equalities in $\varphi' - \varphi$ are entailed by T and the instances of (P.Path.Eq.1).

Let $(\mathcal{M}^\tau, \alpha)$ be a solution of φ_{dom} constructed as in lemma 21. It remains to show that each path equality in φ' is satisfied by $(\mathcal{M}^\tau, \alpha)$. So let $\mathrm{p}\left(\begin{smallmatrix} X & Y \\ U & V \end{smallmatrix}\right) \in \varphi'$, and let $X{:}f(X_1,\ldots,X_n), X_i{=}U \in \varphi$. Then, as shown, there are $Y_1,\ldots,Y_n$ such that $Y{:}f(Y_1,\ldots,Y_n), Y_i{=}V \in \varphi$. Then by the construction from lemma 21, the subtree τ_X of τ with root $\alpha(X)$ is labeled f, as is the subtree τ_Y of τ with root $\alpha(Y)$, and the path from $\alpha(X)$ to $\alpha(X_i) = \alpha(U)$ in τ_X is i, as is the path from $\alpha(Y)$ to $\alpha(Y_i) = \alpha(V)$ in τ_Y.

Parallelism literals. Let $X_1/X_2{\sim}Y_1/Y_2 \in \varphi$, and let $\alpha(X_1){=}\pi_1, \alpha(X_2){=}\pi_2, \alpha(Y_1){=}\pi_3, \alpha(Y_2){=}\pi_4$. Then $\pi_1{\lhd^*}\pi_2, \pi_3{\lhd^*}\pi_4$ hold in $\mathcal{M}^\tau$ as it is a model of φ_{dom}. We define a function $c : \mathsf{betw}_\tau(\pi_1,\pi_2) \to \mathsf{betw}_\tau(\pi_3,\pi_4)$ by

$$c(\alpha(X)){=}\alpha(Y) \;\text{ iff }\; X \in \mathsf{betw}_\varphi(X_1,X_2) \;\text{ and }\; \mathsf{p}\,\begin{pmatrix} X_1 & Y_1 \\ X & Y \end{pmatrix} \in \varphi.$$

It remains to show that c is the correspondence function for $\pi_1/\pi_2{\sim}\pi_3/\pi_4$.

c is well-defined because if $\mathsf{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ X & Y \end{smallmatrix}), \mathsf{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ X & Z \end{smallmatrix}) \in \varphi$, then by closure under (T.Trans.H), (P.Path.Sym), (P.Path.Eq.1) also $Y{=}Z \in \varphi$.

The domain of c is $\mathsf{betw}_\tau(\pi_1,\pi_2)$: we first show that the domain of c is a subset of $\mathsf{betw}_\tau(\pi_1,\pi_2)$. Let $X \in \mathsf{betw}_\varphi(X_1,X_2)$. As $\mathcal{M}^\tau$ is a model of φ_{dom}, $\pi_1{\lhd^*}\alpha(X)$ and either $\alpha(X){\lhd^*}\pi_2$ or $\alpha(X){\perp}\pi_2$ must hold in $\mathcal{M}^\tau$. So $\alpha(X) \in \mathsf{betw}_\tau(\pi_1,\pi_2)$. We now show that $\mathsf{betw}_\tau(\pi_1,\pi_2)$ is a subset of the domain of c. Let $\pi \in \mathsf{betw}_\tau(\pi_1,\pi_2)$, then, as noted above, there exists an X with $\alpha(X) = \pi$. We need to show that $X \in \mathsf{betw}_\varphi(X_1,X_2)$. φ possesses a root variable, call it X_0, and we have $X_0{\lhd^*}X_1, X_0{\lhd^*}X$ in φ. Let X_0' be a $\lhd^+$-maximal variable such that $X_0'{\lhd^*}X_1, X_0'{\lhd^*}X \in \varphi$. If $X_0'{=}X \in \varphi$, then $X{\lhd^*}X_1$ by closure under (D.Dom.Trans), and φ must contain $X{=}X_1$ by (P.Distr.Project) because $\pi_1{\lhd^*}\pi$. If $X_0'{=}X_0'', X_0''{:}f(Z_1,\ldots,Z_n) \in \varphi$, then we cannot have $Z_i{\lhd^*}X_1, Z_j{\lhd^*}X \in \varphi$ for $1 \le i \ne j \le n$, since then $X{\perp}X_1 \in \varphi$ by closure under (D.Dom.Trans), (D.Prop.Distr). We cannot have $Z_i{\lhd^*}X_1, Z_i{\lhd^*}X \in \varphi$ for some $i \in \{1,\ldots,n\}$ since we have chosen X_0' to be maximal. The only remaining possibility is $X_0'{=}X_1 \in \varphi$ and $Z_i{\lhd^*}X \in \varphi$ for some $i \in \{1,\ldots,n\}$. In any case, $X_1{\lhd^*}X \in \varphi$. By (P.Distr.Crown), we must have chosen either $X{\lhd^*}X_2$ or $X{\perp}X_2$. By an analogous argument, one can see that the range of c is $\mathsf{betw}_\tau(\pi_3,\pi_4)$.

c is one-to-one (injective) because if $\mathsf{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ X & Z \end{smallmatrix}), \mathsf{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ Y & Z \end{smallmatrix}) \in \varphi$ for $X, Y \in \mathsf{betw}_\varphi(X_1,X_2)$, then $X{=}Y \in \varphi$ by closure under (P.Copy.Dom). It is onto (surjective) by closure under (N.New).

$c(\pi_1) = \pi_3$, and $c(\pi_2) = \pi_4$ by closure under (P.Root).

c is structure-preserving: suppose $\psi_0 \in \mathsf{betw}_\tau(\pi_1,\pi_2) - \{\pi_2\}$, and $\mathcal{M}^\tau \models \psi_0{:}f(\psi_1,\ldots,\psi_n)$. Then there exists a $U_0 \in \mathcal{V}(\varphi)$ with $\alpha(U_0) = \psi$ and, as shown above, $U_0 \in \mathsf{betw}_\varphi(X_1,X_2)$. As φ is simple, U_0 must be labeled: φ must contain $U_0{=}U_0', U_0'{:}f(U_1,\ldots,U_n)$ for some $U_0', U_1,\ldots,U_n$. By (P.Distr.Project) we must have $U_0{\neq}X_2 \in \varphi$ since $\psi_0{\neq}\pi_2$. So by (P.Distr.Crown), either $U_0{\lhd^+}X_2$ or $U_0{\perp}X_2 \in \varphi$. Thus $U_1,\ldots,U_n \in \mathsf{betw}_\varphi(X_1,X_2)$. By closure under (N.New), φ contains $\mathsf{p}\,(\begin{smallmatrix} X_1 & Y_1 \\ U_i & V_i \end{smallmatrix}), 0 \le i \le n$, for some $V_0,\ldots,V_n$, and by closure under (P.Path.Eq.1) and (P.Copy.Lab), it contains $V_0{:}f(V_1,\ldots,V_n)$. By the construction of c, we have $c(\psi_i) = c(\alpha(U_i)) = \alpha(V_i)$ for $0 \le i \le n$, and as $(\mathcal{M}^\tau,\alpha) \models \varphi_{dom}$, we must have $\mathcal{M}^\tau \models \alpha(V_0){:}f(\alpha(V_1),\ldots,\alpha(V_n)) = c(\psi_0){:}f(c(\psi_1),\ldots,c(\psi_n))$. The opposite direction, starting from $\mathcal{M}^\tau \models c(\psi_0){:}f(c(\psi_1),\ldots,c(\psi_n))$, is proved by an analogous argument. $\square$

Now we show how to extend a non-simple generated constraint in *DPNT*-solved form to a simple one. As mentioned in Sec. 5, if we label an unlabeled variable X occurring within some parallelism context, we have to label simultaneously the correspondent of X, as well as all *its* correspondents. We formalize this in the notion of the *copy set* of a labeling literal $X{:}f(X_1,\ldots,X_n)$.

Definition 27. Let φ be a constraint with $X, X_1, \ldots, X_n, Y, Y_1, \ldots, Y_n \in \mathcal{V}(\varphi)$ and let f be a function symbol of arity n. Then we define $\hookrightarrow_\varphi$ by

$$X{:}f(X_1,\ldots,X_n) \hookrightarrow_\varphi Y{:}f(Y_1,\ldots,Y_n)$$

iff there exists some $U_1/U_2{\sim}V_1/V_2 \in \varphi$ such that $X, X_1, \ldots, X_n \in \mathsf{betw}_\varphi(U_1,U_2)$ and $X{=}U_2 \notin \varphi$ but $\mathrm{p}\,(\begin{smallmatrix}U_1 & V_1\\ X & Y\end{smallmatrix}) \in \varphi$ and $\mathrm{p}\,(\begin{smallmatrix}U_1 & V_1\\ X_i & Y_i\end{smallmatrix}) \in \varphi$ for $1 \le i \le n$.

Furthermore,

$$\begin{aligned}\mathrm{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big) := \{&Y{:}f(Y_1,\ldots,Y_n) \mid \\ &X{:}f(X_1,\ldots,X_n) \hookrightarrow_\varphi^* Y{:}f(Y_1,\ldots,Y_n)\}\end{aligned}$$

where as usual $\hookrightarrow_\varphi^*$ is the reflexive and transitive closure of $\hookrightarrow_\varphi$.

Lemma 28. *Let φ be a constraint in DPNT-solved form, and let $Y{:}f(Y_1, \ldots, Y_n) \in \mathrm{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)$.*

- *If X is unlabeled in φ, then so is Y.*
- *If $\{X_1,\ldots,X_n\} \subseteq$ con $_\varphi(X)$, then $\{Y_1,\ldots,Y_n\} \subseteq$ con $_\varphi(Y)$.*
- *If $\{X_1,\ldots,X_n\}$ is a maximal φ-disjointness set in con $_\varphi(X)$, then $\{Y_1,\ldots,Y_n\}$ is a maximal φ-disjointness set in con $_\varphi(Y)$.*

Proof. By well-founded induction on the strict partial order $\subset$ on the set $\{\mathcal{S} \mid \{X{:}f(X_1,\ldots,X_n)\} \subseteq \mathcal{S} \subseteq \mathrm{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)\}$.

The case of $\mathcal{S} = \{X{:}f(X_1,\ldots,X_n)\}$ is trivial. Otherwise, $\mathcal{S}$ has the form $\mathcal{S}' \cup \{Y{:}f(Y_1,\ldots,Y_n)\}$ and there exists $Z{:}f(Z_1,\ldots,Z_n) \in \mathcal{S}'$ with $Z{:}f(Z_1,\ldots,Z_n) \hookrightarrow_\varphi Y{:}f(Y_1,\ldots,Y_n)$ (because $X{:}f(X_1,\ldots,X_n) \in \mathcal{S}$, so if there were no such $Z{:}f(Z_1,\ldots,Z_n) \in \mathcal{S}'$, then $\mathcal{S} \not\subseteq \mathrm{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)$). Let $U_1/U_2{\sim}V_1/V_2 \in \varphi$ with $Z, Z_1, \ldots, Z_n \in \mathsf{betw}_\varphi(U_1,U_2)$ and $Z{=}U_2 \notin \varphi$ but $\mathrm{p}\,(\begin{smallmatrix}U_1 & V_1\\ Z & Y\end{smallmatrix}) \in \varphi$ and $\mathrm{p}\,(\begin{smallmatrix}U_1 & V_1\\ Z_i & Y_i\end{smallmatrix}) \in \varphi$ for $1 \le i \le n$. Then $Y, Y_1, \ldots, Y_n \in \mathsf{betw}_\varphi(V_1,V_2)$ by closure under (P.Copy.Dom), and $Y{=}V_2 \notin \varphi$, again by closure under (P.Copy.Dom).

- Suppose Z is unlabeled. Then Y must be unlabeled too, as any labeling literal would have been copied by (P.Copy.Lab).
- Suppose $\{Z_1,\ldots,Z_n\} \subseteq$ con $_\varphi(Z)$. Then by closure under (P.Copy.Dom), $Y{\lhd^*}Y_i \in \varphi$ but $Y_i{\lhd^*}Y \notin \varphi$ for $1 \le i \le n$. Assume that Y_i is not minimal with $Y \prec_\varphi Y_i$, i.e. there exists some W with $Y \prec_\varphi W \prec_\varphi Y_i$. Then $W \in \mathsf{betw}_\varphi(V_1,V_2)$ by closure under (D.Dom.Trans), (D.Prop.Disj), (P.Distr.Crown). So by (N.New), there exists some $W' \in \mathsf{betw}_\varphi(U_1,U_2)$ with $\mathrm{p}\,(\begin{smallmatrix}U_1 & V_1\\ W' & W\end{smallmatrix})$. But then $Z{\lhd^*}W'{\lhd^*}Z_i \in \varphi$ by (P.Copy.Dom), but neither $W'{\lhd^*}Z$ nor $Z_i{\lhd^*}W'$ is in φ, so Z_i is not minimal either, a contradiction.

– Suppose $\{Z_1,\ldots,Z_n\}$ is a maximal φ-disjointness set in $\mathrm{con}_{\varphi}(Z)$. Assume that $\{Y_i,Y_j\}$ is not a disjointness set for some $1\leq i\neq j\leq n$. So $Y_i{=}Y_j\in\varphi$. But then by (P.Copy.Dom), $Z_i{=}Z_j\in\varphi$, a contradiction.
Assume $\{Y_1,\ldots,Y_n\}$ is not maximal, i.e. there exists some $Y'\notin\{Y_1,\ldots,Y_n\}$ such that $\{Y_1,\ldots,Y_n,Y'\}\subseteq\mathrm{con}_{\varphi}(Y)$ is a disjointness set. We must have $V_1\triangleleft^* Y'$ by (D.Dom.Trans) and either $Y'\triangleleft^* V_2$ or $Y'\perp V_2$ or $V_2\triangleleft^+ Y'$ by (P.Distr.Crown). But if $V_2\triangleleft^+ Y'$, then $Y'\notin\mathrm{con}_{\varphi}(Y)$ because $Y{=}V_2\notin\varphi$. So $Y'\in\mathsf{betw}_{\varphi}(V_1,V_2)$. By closure under (N.New) and (P.Copy.Dom), there exists a $Z'\in\mathsf{betw}_{\varphi}(U_1,U_2)$ such that $\mathrm{p}\left(\begin{smallmatrix}U_1 & V_1\\ Z' & Y'\end{smallmatrix}\right)\in\varphi$. By closure under (P.Copy.Dom), we have $Z'\in\mathrm{con}_{\varphi}(Z)$. Z' cannot be in $\{Z_1,\ldots,Z_n\}$: If $Z'{=}Z_i\in\varphi$ for some $i\in\{1,\ldots,n\}$, then $\mathrm{p}\left(\begin{smallmatrix}U_1 & V_1\\ Z_i & Y'\end{smallmatrix}\right)$, $\mathrm{p}\left(\begin{smallmatrix}U_1 & V_1\\ Z_i & Y_i\end{smallmatrix}\right)\in\varphi$ by (P.Path.Eq.1), so $Y'{=}Y_i\in\varphi$ by (P.Path.Eq.2). Hence, $\{Z_1,\ldots,Z_n,Z'\}$ is a φ-disjointness set in $\mathrm{con}_{\varphi}(Z)$ that is bigger than $\{Z_1,\ldots,Z_n\}$, a contradiction.

Proposition 29. *Every DPNT-solved form φ with an unlabeled variable X can be extended to a DPNT-solved form in which X is labeled.*

Proof. Let $\{X_1,\ldots,X_n\}$ be a maximal φ-disjointness set in $\mathrm{con}_{\varphi}(X)$. Let f be a function symbol in Σ of arity n. (If there exists no suitable f, this problem is solved the same way as in Lemma 25). Then we define the extension $\mathsf{ext}(\varphi)$ of $\varphi\wedge X{:}f(X_1,\ldots,X_n)$ as

$$\mathsf{ext}(\varphi) := \varphi\ \wedge \bigwedge_{\substack{Y:f(Y_1,\ldots,Y_n)\in\\ \mathrm{copy}_{\varphi}(X:f(X_1,\ldots,X_n))}} \Big(Y{:}f(Y_1,\ldots,Y_n)\ \wedge\ \textstyle\bigwedge_{i=1}^{n} Y{\neq}Y_i\ \wedge \bigwedge_{\substack{Y_i\triangleleft^* U, Y_j\triangleleft^* V\in\varphi,\\ 1\leq i\neq j\leq n}} U\perp V\ \wedge \bigwedge_{\substack{Z:g(\ldots)\in\varphi,\\ g\neq f\ \vee\ \mathsf{ar}(g)\neq\mathsf{ar}(f)}} Z{\neq}Y\Big)$$

This definition extends the one in Lemma 25 from a single labeling literal $X{:}f(X_1,\ldots,X_n)$ to a set $\mathrm{copy}_{\varphi}\big(X{:}f(X_1,\ldots,X_n)\big)$ of labeling literals.

(D.Clash.Ineq): $\mathsf{ext}(\varphi)$ contains no new dominance literals. If a new inequality literal $Y{\neq}Y_i$ were to make (D.Clash.Ineq) applicable, then φ must contain $Y{=}Y_i$, but $Y{:}f(Y_1,\ldots,Y_n)\in\mathrm{copy}_{\varphi}\big(X{:}f(X_1,\ldots,X_n)\big)$, so $Y_i\in\mathrm{con}_{\varphi}(Y)$ by lemma 28. If a new inequality $Z{\neq}Y$ were to make the clash rule applicable, then $Z{:}g(\ldots)$ and $Y{=}Z$ must be in φ, but by lemma 28, Y is unlabeled because X is.

(D.Clash.Disj): The only new disjointness literals in $\mathsf{ext}(\varphi)$ have the form $U\perp V$ for $Y_i\triangleleft^* U, Y_j\triangleleft^* V$ in φ with $i\neq j$. Assume $U{=}V$ is in φ. Then by (D.Distr.NotDisj), either $Y_i\triangleleft^* Y_j$ or $Y_j\triangleleft^* Y_i$ must be in φ. But $\{X_i,X_j\}$ is a disjointness set, and so, by lemma 28, is $\{Y_i,Y_j\}$.

(D.Dom.Refl): No new variables have been added.

(D.Dom.Trans), (D.Distr.NotDisj): No new dominance literals have been added.

(D.Eq.Decom): Suppose $Y{:}f(Y_1,\ldots,Y_n)\in\mathrm{copy}_{\varphi}\big(X{:}f(X_1,\ldots,X_n)\big)$ and $Y{=}Z$ is in φ. Then Y and Z must be unlabeled by lemma 28, so for (D.Eq.Decom) to be applicable, both $Y{:}f(Y_1,\ldots,Y_n)$ and $Z{:}f(Z_1,\ldots,Z_n)$ must be in $\mathsf{ext}(\varphi)-\varphi$, which means that $Z{:}f(Z_1,\ldots,Z_n)\in\mathrm{copy}_{\varphi}\big(X{:}f(X_1,\ldots,X_n)\big)$, too.

If $\text{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)$ is a singleton, then we must have $X_i{=}Y_i{=}Z_i$ for $1 \leq i \leq n$. So suppose otherwise. Let $U{:}f(U_1,\ldots,U_n) \in \text{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)$. We use induction on the length of a $\hookrightarrow_\varphi$ sequence starting in $X{:}f(X_1,\ldots,X_n)$ and ending in $U{:}f(U_1,\ldots,U_n)$ to show that $\text{p}\,(\begin{smallmatrix} X & U \\ X_i & U_i \end{smallmatrix}) \in \varphi$ for $1 \leq i \leq n$. We start with a sequence of length 0. As $\text{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)$ is not a singleton, there exists some $W_1/W_2{\sim}W_3/W_4 \in \varphi$ with $X, X_1, \ldots, X_n \in \mathsf{betw}_\varphi(W_1,W_2)$. By closure under (N.New), there exist $X', X'_1, \ldots, X'_n$ such that $\text{p}\,(\begin{smallmatrix} W_1 & W_3 \\ X & X' \end{smallmatrix}) \in \varphi$ as well as $\text{p}\,(\begin{smallmatrix} W_1 & W_3 \\ X_i & X'_i \end{smallmatrix}) \in \varphi$ for $1 \leq i \leq n$. By (P.Path.Sym), $\text{p}\,(\begin{smallmatrix} W_3 & W_1 \\ X' & X \end{smallmatrix})$, $\text{p}\,(\begin{smallmatrix} W_3 & W_1 \\ X'_i & X_i \end{smallmatrix}) \in \varphi$, so by (T.Trans.H), $\text{p}\,(\begin{smallmatrix} W_1 & W_1 \\ X & X \end{smallmatrix})$, $\text{p}\,(\begin{smallmatrix} W_1 & W_1 \\ X_i & X_i \end{smallmatrix}) \in \varphi$ for $1 \leq i \leq n$. As $X{\lhd^*}X_i \in \varphi$, closure under (T.Diff.1) yields $\text{p}\,(\begin{smallmatrix} X & X \\ X_i & X_i \end{smallmatrix}) \in \varphi$ for $1 \leq i \leq n$.

Suppose $V{:}f(V_1,\ldots,V_n) \in \text{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)$ with $\text{p}\,(\begin{smallmatrix} X & V \\ X_i & V_i \end{smallmatrix}) \in \varphi$ for $1 \leq i \leq n$, and $V{:}f(V_1,\ldots,V_n) \hookrightarrow_\varphi U{:}f(U_1,\ldots,U_n)$. Then φ contains some $W_1/W_2{\sim}W_3/W_4$ with $V, V_1, \ldots, V_n \in \mathsf{betw}_\varphi(W_1,W_2)$ and $\text{p}\,(\begin{smallmatrix} W_1 & W_3 \\ V & U \end{smallmatrix})$, $\text{p}\,(\begin{smallmatrix} W_1 & W_3 \\ V_i & U_i \end{smallmatrix}) \in \varphi$ for $1 \leq i \leq n$. Then by closure under (T.Diff.1), $\text{p}\,(\begin{smallmatrix} V & U \\ V_i & U_i \end{smallmatrix}) \in \varphi$ for $1 \leq i \leq n$, and so, by (T.Trans.H), are $\text{p}\,(\begin{smallmatrix} X & U \\ X_i & U_i \end{smallmatrix})$.

Hence $\text{p}\,(\begin{smallmatrix} X & Y \\ X_i & Y_i \end{smallmatrix})$, $\text{p}\,(\begin{smallmatrix} X & Z \\ X_i & Z_i \end{smallmatrix}) \in \varphi$ for $1 \leq i \leq n$. By closure under (P.Path.Sym) and (T.Trans.H), φ contains $\text{p}\,(\begin{smallmatrix} Y & Z \\ Y_i & Z_i \end{smallmatrix})$, and as $Y{=}Z \in \varphi$, $\text{p}\,(\begin{smallmatrix} Z & Z \\ Y_i & Z_i \end{smallmatrix}) \in \varphi$ by (P.Path.Eq.1), whence by (P.Path.Eq.2), $Y_i{=}Z_i \in \varphi$ already (all for $1 \leq i \leq n$).

(D.Lab.Ineq): Suppose $Y{:}f(Y_1,\ldots,Y_n) \in \mathsf{ext}(\varphi) - \varphi$. Then $Z{\neq}Y$ is in $\mathsf{ext}(\varphi)$ by definition for all Z labeled anything but f.

(D.Lab.Disj): Suppose $Y{:}f(Y_1,\ldots,Y_n) \in \mathsf{ext}(\varphi) - \varphi$. Since $Y_i{\lhd^*}Y_i, Y_j{\lhd^*}Y_j \in \varphi$ for $1 \leq i \leq n$ by closure under (D.Dom.Refl), $Y_i{\perp}Y_j$ is in $\mathsf{ext}(\varphi)$ by definition.

(D.Prop.Disj): Suppose $Y{:}f(Y_1,\ldots,Y_n) \in \text{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)$ and $U{\perp}V \in \mathsf{ext}(\varphi) - \varphi$ for some $Y_i{\lhd^*}U$, $Y_j{\lhd^*}V$, $j \neq i$. If $U{\lhd^*}U'$ and $V{\lhd^*}V'$ are in φ, then we also have $Y_i{\lhd^*}U', Y_j{\lhd^*}V' \in \varphi$ by closure under (D.Dom.Trans), so $U'{\perp}V'$ is in $\mathsf{ext}(\varphi)$.

(D.Lab.Dom): Suppose $Y{:}f(Y_1,\ldots,Y_n) \in \mathsf{ext}(\varphi) - \varphi$. We have $Y{\lhd^*}Y_i \in \varphi$ by lemma 28. $Y{\neq}Y_i \in \mathsf{ext}(\varphi)$ by definition.

(D.Distr.Child): Suppose $Y{:}f(Y_1,\ldots,Y_n) \in \mathsf{ext}(\varphi) - \varphi$ and $Y{\lhd^*}Z \in \varphi$.
If $Z{\lhd^*}Y \in \varphi$, then (D.Distr.Child) is not applicable in $\mathsf{ext}(\varphi)$. Otherwise $Y \prec_\varphi Z$. If Z is minimal with $Y \prec_\varphi Z$, then $Z \in \text{con}_\varphi(Y)$, and as $\{Y_1,\ldots,Y_n\}$ is a maximal φ-disjointness set in $\text{con}_\varphi(Y)$, we have $Z{=}Y_i \in \varphi$ for some $i \in \{1,\ldots,n\}$. If Z is not minimal, there exists some $Y' \in \text{con}_\varphi(Y)$ such that $Y'{\lhd^*}Z$ is in φ. But then again, $Y_i{=}Y'$ for some $i \in \{1,\ldots,n\}$, so $Y_i{\lhd^*}Z$.

(P.Root), (P.Path.Sym), (P.Path.Dom), (P.Path.Eq.1), (P.Path.Eq.1), (P.Distr.Crown): No new dominance, parallelism, or path equality literals have been added.

(P.Copy.Dom): Any dominance literal in $\mathsf{ext}(\varphi)$ is in φ already, so the case of $R = {\lhd^*}$ does not apply.

- We next consider the case $R = \perp$. Let $U{\perp}V$ be in $\mathsf{ext}(\varphi) - \varphi$, where for some $Y{:}f(Y_1,\ldots,Y_n) \in \text{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)$ and some $1 \leq i \neq j \leq n$, $Y_i{\lhd^*}U, Y_j{\lhd^*}V \in \varphi$. (Thus, $\{Y_1,\ldots,Y_n\} \neq \emptyset$.) Suppose φ contains a parallelism literal $W_1/W_2{\sim}W_3/W_4$ with $U, V \in \mathsf{betw}_\varphi(W_1,W_2)$. By closure under (N.New), there exist U', V' such that $\text{p}\,(\begin{smallmatrix} W_1 & W_3 \\ U & U' \end{smallmatrix})$, $\text{p}\,(\begin{smallmatrix} W_1 & W_3 \\ V & V' \end{smallmatrix}) \in \varphi$. So

$W_1 \lhd^* U, W_1 \lhd^* V \in \varphi$, and by closure under (D.Dom.Trans), $Y \lhd^* U, Y \lhd^* V \in \varphi$. Hence by (D.Distr.NotDisj), φ contains either $Y \lhd^* W_1$ or $W_1 \lhd^* Y$.

If φ contains $Y \lhd^* W_1$ but not $Y{=}W_1$, then $Y \prec_\varphi W_1$. $\{Y_1, \ldots, Y_n\}$ is a maximal φ-disjointness set in $\mathrm{con}_\varphi(Y)$ by lemma 28. So if $W_1 \in \mathrm{con}_\varphi(Y)$, then by lemma 24, $W_1{=}Y_k$ is in φ for some $k \in \{1, \ldots, n\}$. If W_1 is not minimal with $Y \prec_\varphi W_1$, then there exists some $Y' \in \mathrm{con}_\varphi(Y)$ such that $Y' \prec_\varphi W_1$. Again by lemma 24, φ contains $Y'{=}Y_k$ for some $k \in \{1, \ldots, n\}$ and hence, by closure under (D.Dom.Trans), $Y_k \lhd^* W_1 \in \varphi$. But then we cannot have both $W_1 \lhd^* U$ and $W_1 \lhd^* V$ in φ since at least one of $Y_i \bot Y_k$ and $Y_j \bot Y_k$ is in φ, and φ is clash-free. So (D.Distr.NotDisj) must have made the choice $W_1 \lhd^* Y \in \varphi$.

φ is closed under (P.Distr.Crown), but the choice made cannot be $W_2 \lhd^* Y$, since $Y \lhd^+ U, Y \lhd^+ V \in \varphi$ by closure under (D.Dom.Trans), (D.Lab.Dom), (P.Distr.Project) and on the other hand $U, V \in \mathsf{betw}_\varphi(W_1, W_2)$. So either $Y \lhd^+ W_2 \in \varphi$ by (P.Distr.Crown) and (P.Distr.Project), or $Y \bot W_2 \in \varphi$ by (P.Distr.Crown). In the first case, (P.Distr.Crown) must have chosen either $Y_i \bot W_2$ or $Y_i \lhd^* W_2$ for each $1 \leq i \leq n$ because all the Y_i are minimal with $Y \prec_\varphi Y_i$. In the second case, we have $Y_i \bot W_2 \in \varphi$ for $1 \leq i \leq n$ by closure under (D.Prop.Disj). In both cases, $Y, Y_1, \ldots, Y_n \in \mathsf{betw}_\varphi(W_1, W_2)$. By closure under (N.New), there are $Z, Z_1, \ldots, Z_n$ such that $\mathsf{p}\left(\begin{smallmatrix} W_1 & W_3 \\ Y & Z \end{smallmatrix}\right) \in \varphi$ and $\mathsf{p}\left(\begin{smallmatrix} W_1 & W_3 \\ Y_i & Z_i \end{smallmatrix}\right) \in \varphi$ for $1 \leq i \leq n$. Since $Y{=}W_2 \notin \varphi$, $Z{:}f(Z_1, \ldots, Z_n) \in \mathrm{copy}_\varphi\big(X{:}f(X_1, \ldots, X_n)\big)$. By closure under (P.Copy.Dom), $Z_i \lhd^* U', Z_j \lhd^* V' \in \varphi$, so $U' \bot V' \in \mathsf{ext}(\varphi)$ by definition.

– Lastly, we consider the case of $R = \neq$. Let $Y{:}f(Y_1, \ldots, Y_n) \in \mathrm{copy}_\varphi\big(X{:}f(X_1, \ldots, X_n)\big)$.

  Suppose $Y{\neq}Y_i \in \mathsf{ext}(\varphi) - \varphi$ for some $i \in \{1, \ldots, n\}$. (Again, $\{Y_1, \ldots, Y_n\} \neq \emptyset$.) Suppose further that $W_1/W_2{\sim}W_3/W_4 \in \varphi$ with $Y, Y_i \in \mathsf{betw}_\varphi(W_1, W_2)$. By closure under (N.New), there exist Z, Z_i such that $\mathsf{p}\left(\begin{smallmatrix} W_1 & W_3 \\ Y & Z \end{smallmatrix}\right)$, $\mathsf{p}\left(\begin{smallmatrix} W_1 & W_3 \\ Y_i & Z_i \end{smallmatrix}\right) \in \varphi$.

  We must have $Y \lhd^+ W_2 \in \varphi$ by closure under (P.Distr.Crown), (P.Distr.Project) and the fact that $Y_i \in \mathsf{betw}_\varphi(W_1, W_2)$. So $Y_1, \ldots, Y_n \in \mathsf{betw}_\varphi(W_1, W_2)$ by closure under (P.Distr.Crown). $W_2 \lhd^+ Y_j$ cannot have been chosen for any $j \in \{1, \ldots, n\}$ because $Y \lhd^+ W_2$ and each Y_j is minimal with $Y \prec_\varphi Y_j$.

  So there are $Z_1, \ldots, Z_n$ such that $\mathsf{p}\left(\begin{smallmatrix} W_1 & W_3 \\ Y_j & Z_j \end{smallmatrix}\right) \in \varphi$ for $1 \leq j \leq n$. $Y{=}W_2 \notin \varphi$, so $Z{:}f(Z_1, \ldots, Z_n) \in \mathrm{copy}_\varphi\big(X{:}f(X_1, \ldots, X_n)\big)$. Hence, $Z{\neq}Z_i$ is in $\mathsf{ext}(\varphi)$ by definition.

  Now suppose $Z{\neq}Y \in \mathsf{ext}(\varphi){-}\varphi$, where $Z{:}g(\ldots)$ is in φ for some g with either $g \neq f$ or $\mathsf{ar}(g) \neq \mathsf{ar}(f)$. Suppose further that $W_1/W_2{\sim}W_3/W_4 \in \varphi$ with $Y, Z \in \mathsf{betw}_\varphi(W_1, W_2)$. By closure under (P.Distr.Project), we have either $Z{=}Y \in \varphi$ or $Z{\neq}Y \in \varphi$. $Z{=}Y \in \varphi$ is impossible since Y is unlabeled by lemma 28. So $Z{\neq}Y$ must be in φ already.

(P.Copy.Lab): Let $Y{:}f(Y_1, \ldots, Y_n) \in \mathrm{copy}_\varphi\big(X{:}f(X_1, \ldots, X_n)\big)$ with $Y{:}f(Y_1, \ldots, Y_n) \in \mathsf{ext}(\varphi) - \varphi$. Suppose $W_1/W_2{\sim}W_3/W_4 \in \varphi$ with $Y, Y_1, \ldots, Y_n \in \mathsf{betw}_\varphi(W_1, W_2)$. Then there exist $Z, Z_1, \ldots, Z_n$ such that $\mathsf{p}\left(\begin{smallmatrix} W_1 & W_3 \\ Y & Z \end{smallmatrix}\right) \in \varphi$ and $\mathsf{p}\left(\begin{smallmatrix} W_1 & W_3 \\ Y_i & Z_i \end{smallmatrix}\right) \in \varphi$ for $1 \leq i \leq n$.

By closure under (P.Distr.Project), either $Y{\neq}W_2 \in \varphi$ or $Y{=}W_2 \in \varphi$. If $Y{\neq}W_2$ is in φ, then $Z{:}f(Z_1,\ldots,Z_n) \in \text{copy}_\varphi\big(X{:}f(X_1,\ldots,X_n)\big)$, so the labeling literal $Z{:}f(Z_1,\ldots Z_n)$ has been added to $\mathsf{ext}(\varphi)$. If $Y{=}W_2 \in \varphi$, then (P.Copy.Lab) is not applicable since it does not copy the label of the exception.

(P.Distr.Project): No new variables have been added.

(N.New): Suppose $W_1/W_2{\sim}W_3/W_4 \in \varphi$ and $W_1{\lhd}^*Y \in \varphi$ and $Y{\perp}W_2 \in \mathsf{ext}(\varphi)-\varphi$. But then by closure under (P.Distr.Crown), one of $Y{\lhd}^*W_2$, $Y{\perp}W_2$, $W_2{\lhd}^+Y$ must already be in φ.

(T.Trans.H), (T.Trans.V), (T.Diff.1), (T.Diff.2): Now new path equality literals have been added. □

Lemma 30. *Every generated DPNT-solved form can be extended to a simple generated DPNT-solved form.*

Proof. By lemma 29, analogous to lemma 26; generatedness is preserved as no additional path equality literals are added. □

Theorem 11 (Soundness). *A generated constraint in DPNT-solved form is satisfiable.*

Proof. By lemmas 10 and 30. □

D Completeness: handling the order $\leq_{\mathcal{G}}$

Lemma 13. *The partial order $\leq_{\mathcal{G}}$ can be factored out into the relational composition of its components, i.e., $\leq_{\mathcal{G}}$ is $\subseteq \circ =^{loc}_{\mathcal{G}}$.*

Proof. Let φ_1, φ_2 be constraints with $\varphi_1 \leq_{\mathcal{G}} \varphi_2$. There exists a sequence $\psi_0, \ldots, \psi_n$ of constraints such that $\varphi_1 = \psi_0 \preceq_1 \psi_1 \preceq_2 \ldots \preceq_n \psi_n = \varphi_2$ with $\preceq_i \in \{\subseteq, =^{loc}_{\mathcal{G}}\}$ for $1 \leq i \leq n$, and if $\preceq_i$ is $\subseteq$, then $\preceq_{i+1}$ is $=^{loc}_{\mathcal{G}}$ for $1 \leq i \leq n-1$. We use induction on the number of $\subseteq$ relationships that occur to the right of a $=^{loc}_{\mathcal{G}}$ relationship in the sequence.

W.l.o.g. we assume that the sequence starts with $\psi_0 \subseteq \psi_1 =^{loc}_{\mathcal{G}} \psi_2$, and that if $\psi_i =^{loc}_{\mathcal{G}} \psi_{i+1}$, then there exists a single axiom from Fig. 20 by which this holds.

Let k be such that $\psi_0 \subseteq \psi_1 =^{loc}_{\mathcal{G}} \psi_k \subseteq \psi_{k+1}$ holds. (If there is no such k, then $\psi_0 \subseteq \psi_1 =^{loc}_{\mathcal{G}} \psi_n$ and we are done.) We show by induction on k that $\psi_0 \subseteq \psi_1 \subseteq \psi' =^{loc}_{\mathcal{G}} \psi_{k+1}$ holds for some constraint ψ'. We construct a constraint ψ such that $\psi_{k-1} \subseteq \psi =^{loc}_{\mathcal{G}} \psi_{k+1}$. (The basic idea is to move ψ_{k+1} to the left of ψ_k and to use ψ to make the necessary adjustments.)

- Suppose $\psi_{k-1} =^{loc}_{\mathcal{G}} \psi_k$ by axiom (1) of Fig. 20, and ψ_{k-1} has the form $X{=}Z \wedge \psi_k$ where $X \notin \mathcal{G} \cup \mathcal{V}(\psi_k)$ and $Z \in \mathcal{V}(\psi_k)$.
  If X occurs in ψ_{k+1}, it has been introduced by adding constraints. We set $\psi = X{=}Z \wedge \psi_{k+1}[X'/X]$ where $X' \notin \mathcal{G}$ does not occur in $\bigcup_{i=1}^n \psi_i$:

$$\begin{aligned}\psi_{k-1} &=^{set} X{=}Z \wedge \psi_k \subseteq \psi \ =^{loc}_{\mathcal{G}} \psi_{k+1}[X'/X] \\ &=^{loc}_{\mathcal{G}} \ (\psi_{k+1}[X'/X])[X/X'] =^{set} \psi_{k+1}.\end{aligned}$$

– Suppose $\psi_{k-1} =^{loc}_{\mathcal{G}} \psi_k$ by axiom (1) of Fig. 20, and ψ_k has the form $X{=}Z \wedge \psi_{k-1}$ where $X \notin \mathcal{G} \cup \mathcal{V}(\psi_{k-1})$ and $Z \in \mathcal{V}(\psi_{k-1})$. But then we already have $\psi_{k-1} \subseteq \psi_k \subseteq \psi_{k+1}$.
– Suppose $\psi_{k-1} =^{loc}_{\mathcal{G}} \psi_k$ by axiom (2) of Fig. 20. Then ψ_k has the form $\psi_{k-1}[Y/X]$ for $X \notin \mathcal{G}$ and $Y \notin \mathcal{V}(\psi_{k-1}) \cup \mathcal{G}$.
  - If $X \in \mathcal{V}(\psi_{k+1})$, let $\psi'' = \psi_{k+1}[X'/X]$, where $X' \notin \mathcal{G}$ does not occur in $\bigcup_{i=1}^{n} \psi_i$. Otherwise, $\psi'' = \psi_{k+1}$.
  - If $Y \in \mathcal{V}(\psi_{k+1})$, then it has to be replaced by X while ψ_{k+1} is moved to the left of ψ_k. In this case, let $\psi = \psi''[X/Y]$. Otherwise, $\psi = \psi''$.

  We have

$$\psi_{k-1} \subseteq \psi =^{loc}_{\mathcal{G}} \psi[Y/X] =^{loc}_{\mathcal{G}} (\psi[Y/X])[X/X'] =^{set} \psi_{k+1}.$$

– Suppose $\psi_{k-1} =^{loc}_{\mathcal{G}} \psi_k$ by axiom (3) of Fig. 20, and suppose ψ_{k-1} has the form $X{=}Y \wedge \psi'_{k-1}$, ψ_k has the form $X{=}Y \wedge \psi'_{k-1}[Y/X]$, and ψ_{k+1} has the form $X{=}Y \wedge \psi'_{k-1}[Y/X] \wedge \psi'$. We set $\psi = X{=}Y \wedge \psi'_{k-1} \wedge \psi'$.Then

$$\psi_{k-1} \subseteq \psi =^{loc}_{\mathcal{G}} X{=}Y \wedge (\psi'_{k-1} \wedge \psi')[Y/X] =^{loc}_{\mathcal{G}} \psi_{k+1}.$$

– Suppose $\psi_{k-1} =^{loc}_{\mathcal{G}} \psi_k$ by axiom (3) of Fig. 20, and suppose ψ_k has the form $X{=}Y \wedge \psi'_k$, while ψ_{k-1} has the form $X{=}Y \wedge \psi'_k[Y/X]$ and ψ_{k+1} is $X{=}Y \wedge \psi'_k \wedge \psi'$. We set $\psi = X{=}Y \wedge \psi'_k[Y/X] \wedge \psi'$, then

$$\psi_{k-1} \subseteq \psi =^{loc}_{\mathcal{G}} X{=}Y \wedge (\psi'_k[Y/X] \wedge \psi')[Y/X] =^{loc}_{\mathcal{G}} \psi_{k+1}.$$

– The case of axiom (4) is trivial.

Hence, there exists a constraint ψ' such that $\psi_0 \subseteq \psi_1 \subseteq \psi' =^{loc}_{\mathcal{G}} \psi_{k+1}$ holds. This new sequence is longer than $\psi_0 \subseteq \psi_1 =^{loc}_{\mathcal{G}} \ldots =^{loc}_{\mathcal{G}} \psi_k \subseteq \psi_{k+1}$ by a finite number of $=^{loc}_{\mathcal{G}}$ relationships. But we have not introduced any additional $\subseteq$ relationships. So we can still eliminate each $\subseteq$ relationship that is to the right of some $=^{loc}_{\mathcal{G}}$ relationship in finitely many steps.

Lemma 14. *If $\varphi \leq_{\mathcal{G}} \psi$ and ψ is a DPNT-solved form, then there exists a DPNT-solved form ψ' such that $\varphi \subseteq \psi' =^{loc}_{\mathcal{G}} \psi$.*

Proof. For a constraint φ and $X \in \mathcal{V}(\varphi)$, let $\mathrm{Eq}_\varphi(X)$ be the reflexive and transitive closure of $=$ in φ, i.e. $X \in \mathrm{Eq}_\varphi(X)$, and if $Y \in \mathrm{Eq}_\varphi(X)$ and $Y{=}Z \in \varphi$, then $Z \in \mathrm{Eq}_\varphi(X)$. Furthermore, let

$$\begin{aligned}\mathrm{Subs}(\varphi) := \{\varphi'[Y_1/X_1, \ldots, Y_n/X_n] \mid\ & \varphi' \in \varphi, \mathcal{V}(\varphi') = \{X_1, \ldots, X_n\}, \\ & Y_i \in \mathrm{Eq}_\varphi(X_i) \ \text{ for } 1 \leq i \leq n\}\end{aligned}$$

We next show that $\mathrm{Subs}(\varphi) =^{loc}_{\mathcal{G}} \varphi$. Eq_φ forms an equivalence relation on the variables occurring in φ. Let there be n different sets $\mathrm{Eq}_\varphi(X_i)$, and $\mathrm{Eq}_\varphi(X_i) = \{Z^i_1, \ldots Z^i_{m_i}\}$ for $1 \leq i \leq n$. Then

$$\begin{aligned}\varphi =^{loc}_{\mathcal{G}}\ & Z^1_1{=}L_1 \ \wedge\ \ldots\ \wedge\ Z^1_{m_1}{=}L_1 \ \wedge\ \ldots\ \wedge \\ & Z^n_1{=}L_n \ \wedge\ \ldots\ \wedge\ Z^n_{m_n}{=}L_n \ \wedge \\ & \quad \varphi[L_1/Z^1_1, \ldots, L_1/Z^1_{m_1}, \ldots L_n/Z^n_1, \ldots L_n/Z^n_{m_n}]\end{aligned}$$

for $L_1, \ldots, L_n \notin \mathcal{G} \cup \mathcal{V}(\varphi)$: The L_i may be introduced by axiom (1). Axiom (3) lets us replace Z^i_j by L_i for $1 \leq j \leq m_i$, $1 \leq i \leq n$. From there, by duplicating $\varphi[L_1/Z^1_1, \ldots, L_1/Z^1_{m_1}, \ldots L_n/Z^n_1, \ldots L_n/Z^n_{m_n}]$ a suitable number of times, using axiom (4), and replacing L_i by each Z^i_j according to axiom (1), we arrive at $\mathrm{Subs}(\varphi')$.

Now suppose $\varphi \leq_{\mathcal{G}} \psi$, where ψ is in *DPNT*-solved form. By lemma 13, there exists a constraint φ' with $\varphi \subseteq \varphi' =^{loc}_{\mathcal{G}} \psi$. φ' need not be in *DPNT*-solved form, but $\mathrm{Subs}(\varphi')$ is.

Let $\psi = \psi_0 =^{loc}_{\mathcal{G}} \psi_1 =^{loc}_{\mathcal{G}} \ldots =^{loc}_{\mathcal{G}} \psi_n = \varphi'$ where $\psi_i =^{loc}_{\mathcal{G}} \psi_{i+1}$ by a single axiom from Fig. 20 for all $1 \leq i \leq n-1$. We use induction on n to show that $\mathrm{Subs}(\psi_i)$ is *DPNT*-solved for all $i \leq n$. For $\psi_0 = \psi$, this is trivial.

Suppose $\psi_i =^{loc}_{\mathcal{G}} \psi_{i+1}$ by axiom (1) of Fig. 20, and ψ_i has the form $X{=}Z \wedge \psi_{i+1}$, where $X \notin \mathcal{G} \cup \mathcal{V}(\psi_{i+1})$ and $Z \in \mathcal{V}(\psi_{i+1})$. Then X is a superfluous local variable in ψ_i, and $\mathrm{Eq}_{\psi_i}(X) \cap \mathcal{V}(\psi_{i+1}) \neq \emptyset$. So the constraint $\mathrm{Subs}(\psi_i)|_{\mathcal{V}(\psi_i)-\{X\}} =^{set} \mathrm{Subs}(\psi_{i+1})$ must be in solved form, too.

Suppose $\psi_i =^{loc}_{\mathcal{G}} \psi_{i+1}$ by axiom (1), and ψ_{i+1} has the form $X{=}Z \wedge \psi_i$ for variables $X \notin \mathcal{G} \cup \mathcal{V}(\psi_i)$ and $Z \in \mathcal{V}(\psi_i)$. Then $\mathrm{Subs}(\psi_{i+1}) = \mathrm{Subs}(X{=}Z \wedge \psi_i)$. $\mathrm{Subs}(\psi_{i+1})$ is in solved form because for all saturation rules that would become applicable because of the added dominance literals $X{=}Z$, the consequent has already been added by Subs.

Suppose $\psi_i =^{loc}_{\mathcal{G}} \psi_{i+1}$ by axiom (2) of Fig. 20, and ψ_{i+1} has the form $\psi_i[Y/X]$ where $X \notin \mathcal{G}$ and $Y \notin \mathcal{V}(\psi_i) \cup \mathcal{G}$. So all occurrences of a local variables X have been replaced by a new local variable Y, and if $\mathrm{Subs}(\psi_i)$ is in solved form, then so is $\mathrm{Subs}(\psi_i)[Y/X] =^{set} \mathrm{Subs}(\varphi')$.

In both cases where $\psi_i =^{loc}_{\mathcal{G}} \psi_{i+1}$ by axiom (3), we have $\mathrm{Subs}(\psi_i) = \mathrm{Subs}(\psi_{i+1})$. $\square$

The main completeness theorem has already been shown in the main part of the text:

Theorem 19 (Completeness). *Let φ be a constraint, $\mathcal{G} \subseteq \mathcal{V}$, and ψ a $\leq_{\mathcal{G}}$-minimal DPNT-solved form for φ. Then there exists a DPNT-solved form $\psi' =_{\mathcal{G}} \psi$ which can be reached from φ, i.e. $\varphi \rightarrow^*_{\mathrm{DPNT}} \psi'$.*

Lemma 31. *Let φ be a constraint satisfied by $(\mathcal{M}^\tau, \alpha)$. Then there exists a $\leq_{\mathcal{G}}$-minimal DPNT-solved form for φ which is also satisfied by $(\mathcal{M}^\tau, \alpha)$.*

Proof. Let φ be a constraint satisfied by $(\mathcal{M}^\tau, \alpha)$ and let ψ be φ extended by all literals entailed by $(\mathcal{M}^\tau, \alpha)$. ψ is satisfiable – it is satisfied by $(\mathcal{M}^\tau, \alpha)$. It is also in solved form since each saturation rule only adds entailed constraints. It remains to show that there exists a $\leq_{\mathcal{G}}$-minimal *DPNT*-solved form φ' for φ with $\varphi' \subseteq \psi$. There are two possibilities: either no $\psi' \subset \psi'$ is in *DPNT*-solved form; then ψ itself is a $\leq_{\mathcal{G}}$-minimal *DPNT*-solved form for φ. Otherwise, there exists some $\psi' \subset \psi$ such that ψ' is in *DPNT*-solved form but no $\psi'' \subset \psi'$ is. $\square$