k-L(2,1)-Labelling for Planar Graphs is NP-Complete for $k \geq 4$.

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Abstract

A mapping from the vertex set of a graph $G = (V,E)$ into an interval of integers $\{0,\ldots,k\}$ is an $L(2,1)$-labelling of G of span k if any two adjacent vertices are mapped onto integers that are at least 2 apart, and every two vertices with a common neighbor are mapped onto distinct integers. It is known that for any fixed $k \geq 4$, deciding the existence of such a labelling is an NP-complete problem while it is polynomial for $k \leq 3$. For $k \geq 8$, it is also remains NP-complete when restricted to planar graphs. planar graphs for $k \geq 8$. In this paper, we show that it remains NP-complete for $4 \leq k \leq 7$.

The Frequency Assignment Problem asks for assigning frequencies to transmitters in a broadcasting network with the aim of avoiding undesired interference. One of the graph theoretical models of this problem which is well elaborated is the notion of distance constrained labelling of graphs. An $L(2,1)$-*labelling* of a graph G is a mapping from the vertex set of G into nonnegative integers such that the labels assigned to adjacent vertices differ by at least 2, and labels assigned to vertices of distance 2 are different. The *span* of such a labeling is the maximum label used. In this model, the vertices of G represent the transmitters and the edges of G express which pairs of transmitters are too close to each other so that an undesired interference may occur, even if the frequencies assigned to them differ by 1. This model was introduced by Roberts [7] and since then the concept has been intensively studied (See the survey of Yeh [8]).

In their seminal paper, Griggs and Yeh [5] proved that determining the minimum span of a graph G, denoted $\lambda_{2,1}(G)$, is an NP-hard problem. Fiala et al. [4] proved that deciding $\lambda_{2,1}(G) \leq k$ is NP-complete for every fixed $k \geq 4$ and later Havet and Thomassé [6] proved that for any $k \geq 4$, it remains NP-complete when restricted to bipartite graphs (and even a restricted family of bipartite graphs, i.e *incidence graphs* or *first division of graphs*). When the span k is part of the input, the problem is nontrivial for trees but a polynomial time algorithm based on bipartite matching was presented in [2]. Moreover, somewhat surprisingly, the problem becomes NP-complete for series-parallel graphs [3], and thus the $L(2,1)$-labelling problem belongs to a handful of problems known to separate graphs of tree-width 1 and 2 by P/NP-completeness dichotomy. Regarding planar graphs, Bodlaender et al. [1] showed that deciding if the span of a planar graph is at most k is NP-complete for any fixed $k \geq 8$. In this paper, we show that it is also NP-complete for $4 \leq k \leq 7$.

1 Preliminaries

Proposition 1 *Let G be a graph admitting a k-$L(2,1)$-labelling. Then the following hold:*

(i) G has no vertex of degree at least k.

(ii) every vertex of degree $k-1$ is labelled 0 or k.

2 $\lambda_{2,1}(G) \leq 5$

Theorem 2 *The following theorem is NP-complete:*
Instance: *A graph G.*
Question: $\lambda_{2,1}(G) \leq 5$*?*

Proof. We give a reduction to BW-colouring.

Let G be a cubic graph. Let $H(G)$ be the graph obtained from G by doing the following.

- For every vertex $v \in V(G)$, we add a vertex w_v adjacent to it; and
- we replace every edge uv by the gadget graph G_{uv} depicted in Figure 1.

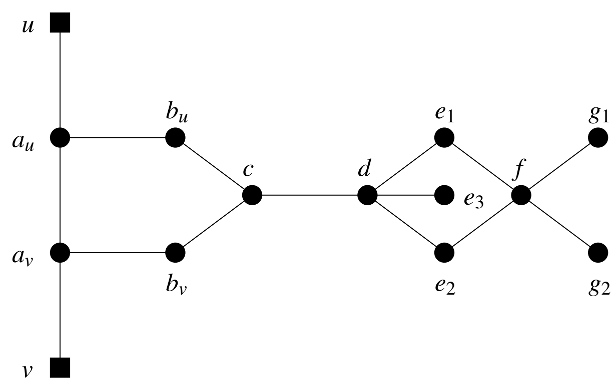

Figure 1: The edge gadget G_{uv}

We will now show that G has a BW-colouring if and only if $H(G)$ has an $L(2,1)$-labelling of span 5.

Suppose first that $H(G)$ has an $L(2,1)$-labelling L of span 5. Then by Proposition 1-(ii), every vertex of $V(G)$ is labelled 0 or 5 because it has degree 4 in $H(G)$. Let B (resp. W) be the set of vertices v of G such that $L(v) = 0$ (resp. 5). We shall prove that in G every $v \in B$ has exactly two neighbours in B and every vertex in W has exactly two neighbours in W. The sets B and W will then be the colour classes of the desired colouring.

In order to prove this, we need the following lemma.

Lemma 3 *(i) If $L(u) = L(v) = 0$ then $(L(a_u), L(a_v)) \in \{(2,5),(5,2),(3,5),(5,3)\}$.*

(ii) If $L(u) = 0$ and $L(v) = 5$ then $(L(a_u), L(a_v)) = (4,1)$.

Proof. (i) By definition of $L(2,1)$-labelling, $L(a_u)$ and $L(a_v)$ are both in $\{2,3,4,5\}$. As $|L(a_u) - L(a_v)| \geq 2$ then $(L(a_u), L(a_v)) \in \{(2,4),(4,2),(2,5),(5,2),(3,5)(5,3)\}$. Suppose for a contradiction that $(L(a_u), L(a_v)) \in \{(2,4),(4,2)\}$. By symmetry, we may assume that $(L(a_u), L(a_v)) = (2,4)$. Then $L(b_u) = 5$ and $L(b_v) = 1$. The vertices d and f have degree four and thus, by Proposition 1-(ii), are labelled in $\{0,5\}$ and since $dist(d,f) = 2$, $\{L(d), L(f)\} = \{0,5\}$. This implies that $\{L(e_1), L(e_2)\} = \{2,3\}$. But then the vertex c cannot be labelled, a contradiction.

(ii) By definition of $L(2,1)$-labelling, $L(a_u) \in \{2,3,4\}$ and $L(a_v) \in \{0,1,2\}$. As $|L(a_u) - L(a_v)| \geq 2$ then $(L(a_u), L(a_v)) \in \{(4,1),(4,2),(3,1)\}$. Suppose for a contradiction that $(L(a_u), L(a_v)) \neq (4,1)$.

By the label symmetry $x \rightarrow 5-x$, we may assume that $(L(a_u), L(a_v)) = (4,2)$. Hence $L(b_u) = 1$ and $L(b_v) = 0$. Now the vertices d and f have degree four and thus, by Proposition 1-(ii), are labelled in $\{0,5\}$. As $dist(b_v, d) = 2$, $L(d) = 5$ and as $dist(d, f) = 2$, $L(f) = 0$. Now $\{L(e_1), L(e_2)\} = \{2,3\}$. But then the vertex c cannot be labelled, a contradiction.

$\square$

By Lemma 3 (i), for any edge $uv \in G[B]$, we have $L(a_u) = 5$ or $L(a_v) = 5$. Let us orient uv from u to v if $L(a_v) = 5$ and from v to u otherwise. We call D the obtained digraph. Let $v \in B$. It has at most one neighbour labelled 5, for every $v \in B$, $d^-(v) \leq 1$. In addition, it has at most one neighbour labelled 4 and thus by Lemma 3-(ii), $d^+(v) + d^-(v) \geq 2$. So $d^+(v) \geq 1$. For $\sum_{v\in B} d^+(v) = \sum_{v\in B} d^-(v)$, it follows that $d^+(v) = d^-(v) = 1$ for every vertex $v \in B$. Hence every $v \in B$ has exactly two neighbours in B.

Analogously, every vertex in $W = \{v \in V(G) \mid L(v) = 5\}$ has exactly two neighbours in W. The sets B and W form the colour classes of the desired colouring.

Suppose now that G has a BW-colouring. Let us now show an $L(2,1)$-labelling L of $H(G)$. The graphs $G[B]$ and $G[W]$ are union of cycles. Orienting each cycle directly we obtain digraphs $D[B]$ and $D[W]$ in which every vertex has indgree and outdegree one. Let us now define the labelling defined. We first label the vertices of G and the w_v. If $v \in B$ then $L(v) = 0$ and $L(w_v) = 2$ and if $v \in W$ then $L(v) = 5$ and $L(w_v) = 3$. Now for each edge uv of G such that $u \in B$ and $v \in W$, we label the vertices of G_{uv} as follows: $L(a_u) = 5$, $L(a_v) = 1$, $L(b_u) = 2$, $L(b_v) = 3$, $L(c) = 0$, $L(d) = 5$, $L(e_1) = 2$, $L(e_2) = 3$, $L(e_3) = 1$, $L(f) = 0$, $L(g_1) = 4$ and $L(g_2) = 5$. For each arc uv of $D[B]$, we label the vertices of G_{uv} as follows: $L(a_u) = 3$, $L(a_v) = 5$, $L(b_u) = 1$, $L(b_v) = 2$, $L(c) = 4$, $L(d) = 0$, $L(e_1) = 2$, $L(e_2) = 3$, $L(e_3) = 5$, $L(f) = 5$, $L(g_1) = 1$ and $L(g_2) = 0$. Analogously, for each arc uv of $D[W]$, we label the vertices of G_{uv} as follows: $L(a_u) = 2$, $L(a_v) = 0$, $L(b_u) = 4$, $L(b_v) = 3$, $L(c) = 1$, $L(d) = 5$, $L(e_1) = 3$, $L(e_2) = 2$, $L(e_3) = 0$, $L(f) = 0$, $L(g_1) = 4$ and $L(g_2) = 5$. It is simple matter to check that L is an $L(2,1)$-labelling of $H(G)$.

$\square$

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