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STRONG AND WEAK SEMICLASSICAL LIMITS FOR SOME ROUGH HAMILTONIANS

AGISSILAOS ATHANASSOULIS AND THIERRY PAUL

Abstract. We present several results concerning the semiclassical limit of the time dependent Schrödinger equation with potentials whose regularity doesn't guarantee the uniqueness of the underlying classical flow. Different topologies for the limit are considered and the situation where two bicharateristics can be obtained out of the same initial point is emphasized.

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1. Introduction

Consider a wavefunction $u^\varepsilon(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ satisfying the Schrödinger equation,

$$\begin{aligned} i\varepsilon \tfrac{\partial}{\partial t} u_t^\varepsilon(x) &= \left(-\tfrac{\varepsilon^2}{2}\Delta + V(x)\right) u_t^\varepsilon(x), \\ u_{t=0}^\varepsilon &= u_0^\varepsilon(x), \end{aligned} \tag{1.1}$$

or equivalently a density matrix D_t^ε satisfying the Heisenberg-von Neumann equation:

$$\begin{aligned} i\varepsilon \tfrac{\partial}{\partial t} D_t^\varepsilon &= \left[-\tfrac{\varepsilon^2}{2}\Delta + V, D_t^\varepsilon\right], \\ D_{t=0}^\varepsilon &= D_0^\varepsilon, \end{aligned} \tag{1.2}$$

The small parameter ε is called Planck's constant in the context of quantum mechanics, and its physical meaning is that of the lengthscale of oscillation of the wavefunction in space.

Several techniques have been developed for the study of the semiclassical limit $\varepsilon \to 0$ of (1.1). One is based on the Wigner transform (WT). The WT $W_t^\varepsilon(x,k)$ of the wavefunction u_t is, defined in the standard semiclassical scaling,

$$W_t^\varepsilon(x,k) = W^\varepsilon[u_t](x,k) = \int\limits_{y\in\mathbb{R}^n} e^{-2\pi iyk} u_t^\varepsilon(x+\varepsilon\frac{y}{2})\bar{u}_t^\varepsilon(x-\varepsilon\frac{y}{2})dy. \tag{1.3}$$

In that case W_t^ε corresponds to a *pure state*. A Wigner function W^ε need not be the Wigner transform of a wavefunction u; when working with *mixed states*, i.e. with equation (1.2), the Wigner function can be

$$W_t^\varepsilon(x,k) = W^\varepsilon[D_t](x,k) = \int\limits_{y\in\mathbb{R}^n} e^{-2\pi iyk} \rho_t^\varepsilon(x+\varepsilon\frac{y}{2}, x-\varepsilon\frac{y}{2})dy \tag{1.4}$$

where $\rho_t^\varepsilon(x,y)$ is the integral kernel of the density matrix D_t^ε. It evolves in time under the well-known Wigner equation,

$$\begin{gathered}\partial_t W_t^\varepsilon(x,k) + 2\pi k\cdot\partial_x W_t^\varepsilon(x,k) + \tfrac{2}{\varepsilon} Re\left[i\int e^{2\pi iSx}\widehat{V}(S)W_t^\varepsilon(x,k-\tfrac{\varepsilon S}{2})dS\right] = 0,\\ W_{t=0}^\varepsilon = W_0^\varepsilon.\end{gathered} \tag{1.5}$$

An important feature of the Wigner transform is that (under appropriate conditions) it allows for a very natural and compact description of the semiclassical limit: the WT has a physically meaningful limit as ε tends to zero, while in general u^ε does not. The limit (in the weak-$*$ topology of an appropriate algebra of test functions), called the Wigner measure,

$$W_t^\varepsilon \rightharpoonup W_t^0,$$

satisfies the Liouville equation of classical mechanics

$$\begin{gathered}\partial_t W^0 + 2\pi k\partial_x W^0 - \tfrac{1}{2\pi}\partial_x V\partial_k W^0 = 0,\\ W_0^0 = \lim_{\varepsilon\to 0} W_0^\varepsilon.\end{gathered} \tag{1.6}$$

In general the Wigner measure W^0 is a non-negative measure on phase-space, associated with the family of initial data $\{W_0^\varepsilon\}$. For the precise formulation of the result, see e.g. [8]. The limit problem (1.6) is well-posed in general for $V(x) \in C^{1,1}$.

It was already observed in [8] that there are certain cases where semiclassical convergence to the Liouville equation could be shown, but the Liouville equation itself is not well posed – namely for $V(x) \in C^1 \setminus C^{1,1}$. The result then becomes as follows: the Wigner measure is a weak solution of the Liouville equation – but there is not enough information to select the correct

weak solution. Families of initial data whose evolution oscillate between the several solutions of the Liouville equation were constructed in [8].

The case study considered in [8] was essentially equivalent to the following, in dimension one:

One considers the potential

$$V(x) = -|x|^{1+\theta} \cdot \beta(x) \tag{1.7}$$

with $\theta \in (0,1)$, $C^\infty \ni \beta = 1$ on $[-1,+1]$, and $\beta(x) = 0$ for $|x| \geq 2$. Then one can easily check that there exist, for t small enough, two families of bicharateristics starting near the origin at $t = 0$, namely

$$(X^\pm(t), P^\pm(t)) = (\pm c_0 t^\nu, \pm \frac{c_0 \nu}{2\pi} t^{\nu-1}), \tag{1.8}$$

with $\nu = \frac{2}{1-\theta}$ and $c_0 = \left(\frac{(1-\theta)^2}{2}\right)^{1-\theta}$.

In this case it is possible to find semiclassical limits where one wavepacket (a δ-function in classical phase-space) splits in two wavepackets. (In this example it happens immediately, i.e. $\forall t > 0$; different examples can be constructed where it happens after an ε-independent time t_*). The mass of each wavepacket depends on information which can be easily extracted from the quantum problem, but lost if we take the limit as usual in a straightforward way.

Weak semiclassical limits with rough potentials were also considered recently in [2] where the hypothesis on the potential is essentially that its gradient is in BV_{loc} (allowing for a Coulomb part to be added). It was proved there, roughly speaking, that the Wigner measure of the solutions of (1.1) at time t, after averaging over initial data, tends weakly to the push-forward of its initial value by the Ambrosio-DiPerna-Lions flow [1, 6] associated to the underlying classical Hamiltonian system of ODEs (see also [7] for the same construction with density matrices). In particular this averaging condition forbids concentrating initial data, a fact consistent with the weak limit strategy and the fact that the flow is not defined everywhere.

Our results concern less general potentials than those treated in [2, 7] (still giving rise to ill-posed classical dynamics of course), but can deal with initial data concentrating to singular measures.

The first of our results deals with the case study mentionned earlier. Our second Theorem gives a general result concerning the approximation, in strong topology, of the Wigner function of the solution of (1.4) with the solution of a "smoothed" Liouville equation. Our third results goes back to the weak approximation, but with explicit remainder estimates.

2. Main results

We will use the space of test-functions $\mathcal{A}$, from [8], namely

$$\mathcal{A} = \{\phi \in C(\mathbb{R}^{2n}) \mid \int \sup_x |\mathcal{F}_{k\to K}[\phi(x,k)]| dK < \infty\}, \tag{2.1}$$

equipped with the norm

$$||\phi||_{\mathcal{A}} = \int \sup_x |\mathcal{F}_{k\to K}[\phi(x,k)]|dK. \tag{2.2}$$

Our first result will illustrate the case study presented in the introduction; a family of initial data which concentrates at the origin at time $t = 0$ and splits into two separated wave packets at time $t > 0$. This is only an example among several others presented in section 5 and the more general results of the paper are given by the Theorems 2.2 and 2.3 below.

Theorem 2.1. *Let V given by* (1.7) *and*

$$W_0^\varepsilon(x,k) = \lambda^{\frac{7+3\theta}{30}} w(\lambda^{\frac{1+\theta}{6}} x, \lambda^{\frac{1-\theta}{15}} k) \tag{2.3}$$

with $\lambda = log(\frac{1}{\varepsilon})$, $w \in H^2 \cap L^\infty \cap L^1$, $supp\, w \subseteq \{|x|^2 + |k|^2 < 1\}$

Then $\exists T > 0$ such that for all $t \in [0,T]$, the solution W_t^ε of (1.5) *converges in weak-$*$ sense in $\mathcal{A}'$ to*

$$W_t^0 = c_+\delta_{(X^+(t),P^+(t))} + c_-\delta_{(X^-(t),P^-(t))}, \tag{2.4}$$

with $(X^\pm(t), P^p m(t))$ given by (1.8) *and*

$$c_\pm = \int\limits_{\pm x>0} w(x,k)dxdk.$$

Let us compare this result with the results in [8] concerning the potential (1.7). (There are minor differences in the setup, but until the characteristics leave $\{|(x,k)| < 1\}$ the two results are directly comparable). In Remarque IV.3 of [8], one can find a construction of a family of normalized wave functions u_ε^+ concentrating at the origin, and whose Wigner functions, after extraction of a subsequence, converge to a Dirac mass pined-up at $(X^+(t), P^+(t))$. The same construction is obviously possible in order to get a family u_ε^- whose Wigner functions, after extraction of a subsequence, converge to a Dirac mass pined-up at $(X^-(t), P^-(t))$.

Defining $D_0^\varepsilon = \frac{1}{\sqrt{2}}(|u_\varepsilon^+\rangle\langle u_\varepsilon^+| + |u_\varepsilon^-\rangle\langle u_\varepsilon^-|)$ it is easy to check that the Wigner function of D_0^ε will follow the conclusion of Theorem 2.1. However our construction is different and is not obtained through compactness arguments. Therefore there is no need of extraction of subsequence, the scaling property is explicit (it is given by an implicit diagonal argument in a two scale sequence in [8]) and the choice of the intial datum is somewhat more general. Moreover, as shown in section 5 the present construction extends to more general examples.

Let us also remark the similarity with the situation of a regular separatrix extensively studied in [9].

We turn now to the more general results of this paper.

In the sequel we will consider potentials V satisfies the following assumption:

Assumption 1.

$$\int |\widehat{V}(S)| \, \frac{S^2}{1+S^2} \, dS < \infty, \tag{2.5}$$

and moreover there are constants $C > 0, \theta \in (0,1)$ *such that for* $m \in \{0,1,2\}$

$$\forall 1 \leqslant a \leqslant b \leqslant +\infty : \int\limits_{|S| \in (a,b)} |\widehat{V}(S)| \, |S|^m dS \leqslant \tfrac{C}{m-1-\theta} \left(b^{m-1-\theta} - a^{m-1-\theta} \right)$$

This is closely modeled after $V(x) = C|x|^{1+\theta}$; indeed it is easy to check that the aforementioned potential (smoothly cut off at infinity) satisfies this condition. In section 5, we will also see some other relevant types of singularities (also generated from $|x|^{1+\theta}$ in some sense).

Denote

$$\tilde{V}^{\gamma}(x) = \left(\frac{2}{\varepsilon^{\gamma}} \right)^{\frac{n}{2}} \int e^{-\frac{2\pi}{\varepsilon^{\gamma}}|x-x'|^2} V(x') dx'. \tag{2.6}$$

Theorem 2.2. *Let us suppose that Assumption 1 holds, and there exist* $T > 0$, $\delta \in (0, \frac{\theta}{2+\theta})$, $\gamma > \frac{2}{1+\theta}$ *and* $[W_0^{\varepsilon}] \in H^2(\mathbb{R}^{2n})$ *such that*

$$||[W_0^{\varepsilon}] - W_0^{\varepsilon}||_{L^2} = o(1)||W_0^{\varepsilon}||_{L^2} \tag{2.7}$$

and the solution of

$$\partial_t \rho + 2\pi k \partial_x \rho - \frac{1}{2\pi} \partial_x \tilde{V}^{\gamma} \cdot \partial_x \rho = 0, \tag{2.8}$$

with initial condition $\rho(t=0) = [W_0^{\varepsilon}]$ *satisfies*

$$||\rho(t)||_{H^2} = O(\varepsilon^{-\delta} ||W_0^{\varepsilon}||_{L^2}) \tag{2.9}$$

uniformly on $[0,T]$.

Then the Wigner function W_t^{ε} *of the solution of (1.4) satisfies, uniformly on* $[0,T]$,

$$||W_t^{\varepsilon} - \rho_1^{\varepsilon}(t)||_{L^2} = O(\varepsilon^{\kappa} ||W_0^{\varepsilon}||_{L^2} \; + \; ||W_0^{\varepsilon} - [W_0^{\varepsilon}]||_{L^2}) = o(||W_0^{\varepsilon}||_{L^2}). \tag{2.10}$$

where ρ_1^{ε} *is the solution of* (2.8) *with initial datum* $\rho_1^{\varepsilon}(t=0) = W_0^{\varepsilon}$ *and* $\kappa = min\{ \; \gamma \frac{1+\theta}{2} - 1, \; \frac{\theta}{2+\theta} - \delta \; \}$.

Our next result deals with the weak-$*$ limit in ε (which in general is not in L^2). It must be noted that the assumptions are strictly stronger than those of theorem 2.2, therefore the latter also holds.

Theorem 2.3. *Assume that the hypotheses of Theorem 2.2 hold, let us suppose moreover that*

$$||[W_0^{\varepsilon}] - W_0^{\varepsilon}||_{L^2} = o(1), \tag{2.11}$$

and

- *W_0^ε is compactly supported (uniformly in ε),*
- *$W_0^\varepsilon \in L^1 \cap L^2$,*
- *W_0^ε is the Wigner function of a density matrix (positive operator of trace 1),*
- *$||W_0^\varepsilon||_{L^2} = o(\varepsilon^{-\kappa})$ where κ is the number defined in Theorem 2.2.*
- *the total energy is uniformly bounded in ε,*

$$\int (2\pi^2 k^2 + V(x)) W_0^\varepsilon(x,k) dx dk < C_0,$$

- *the potential V is in H^1, and bounded below.*

Then, $\forall \phi \in \mathcal{A}$ and $\forall t \in [0,T]$, the Wigner function W_t^ε of the solution of (1.4) *satisfies*

$$\langle W_t^\varepsilon, \phi \rangle = \langle \rho_1^\varepsilon(t), \phi \rangle + E^\varepsilon, \tag{2.12}$$

where again ρ_1^ε satisfies (2.8) *with initial datum $\rho_1^\varepsilon(t=0) = W_0^\varepsilon$.*

The error E^ε can be estimated explicitly as follows: if $\mu = (\varepsilon^\kappa ||W_0^\varepsilon||_{L^2} + ||W_0^\varepsilon - [W_0^\varepsilon]||_{L^2} + \sqrt{\varepsilon}||W_0^\varepsilon||_{H^1})^{\frac{1}{n}}$,

$$E^\varepsilon = O\left(\mu ||\phi||_{\mathcal{A}} + |\langle W^\varepsilon, \left(\mathbb{1} - e^{-\varepsilon\Delta}\right)\phi \rangle| \right). \tag{2.13}$$

Corollary 2.4. *If the assumptions of Theorem 2.3 are satisfied, the Wigner function W^ε has a weak-$*$ limit in $\mathcal{A}'$ if and only if ρ_1^ε has a weak-$*$ limit, and the two are equal, i.e.*

$$\exists W_t^0 \ : \ \lim_{\varepsilon \to 0} \langle W_t^\varepsilon, \phi \rangle = \langle W_t^0, \phi \rangle \ \ \forall \phi \in \mathcal{A}, \ t \in [0,T], \tag{2.14}$$

if and only if

$$\lim_{\varepsilon \to 0} \langle \rho_1^\varepsilon(t), \phi \rangle = \langle W_t^0, \phi \rangle \ \ \forall \phi \in \mathcal{A}, \ t \in [0,T]. \tag{2.15}$$

Organization of the paper: Proofs of the main Theorems are in section 4. Checking the assumptions of the Theorems for various concrete problems is done in section 5.

3. Definitions and Notations

The Fourier transform is defined as

$$\widehat{f}(k) = \mathcal{F}_{x \to k}[f(x)] = \int_{x \in \mathbb{R}^n} e^{-2\pi i k x} f(x) dx. \tag{3.1}$$

For compactness, we will use the following notations:

$$T_\varepsilon^V W = \tfrac{2}{\varepsilon} Re\left[i \int e^{2\pi i S x} \widehat{V}(S) W(x, k - \tfrac{\varepsilon S}{2}) dS \right] =$$

$$= \mathcal{F}^{-1}_{X,K \to x,k}\left[\int \widehat{V}(S) \widehat{W}(X - S, K) \tfrac{sin(\pi \varepsilon S K)}{\varepsilon} dS \right],$$

$$T_0^V W = -\tfrac{1}{2\pi}\partial_x \overset{\sim\gamma}{V}\cdot\partial_x W =$$

$$= \pi\mathcal{F}^{-1}_{X,K\to x,k}\left[\int \widehat{V}(S)\widehat{W}(X-S,K)S\cdot K dS\right].$$

The Sobolev norm of order m on phase-space will be defined as follows:

$$||f||_{H^m(\mathbb{R}^{2n})} = \sum_{|a|+|b|\leqslant m} ||\partial_x^a\partial_k^b f||_{L^2(\mathbb{R}^{2n})},$$

where of course a and b are multi-indices of length n each.

Denote by Φ the ε-dependent smoothing operator

$$\Phi : f(x) \mapsto \mathcal{F}^{-1}_{a\to x}[e^{-\frac{\pi}{2}(\eta_1\sigma_x^2 a^2)}\mathcal{F}_{z\to a}[f(z)]]. \tag{3.2}$$

As a matter of notation we will use $\widetilde{f} = \Phi f$. The parameter $\eta_1(\varepsilon)$ is calibrated in the proof of lemma 4.1.

Denote $H(x,k,t)$ the Husimi function,

$$\begin{aligned} H^\varepsilon(x,k) &= \tfrac{\sqrt{2}^n}{\varepsilon^n}\int e^{-2\pi\frac{(x-x')^2}{\varepsilon}-2\pi\frac{(k-k')^2}{\varepsilon}} W^\varepsilon(x',k',t)dx'dk' = \\ &= (2\varepsilon)^{-\frac{n}{2}}\left\langle D^\varepsilon e^{-\frac{2\pi i}{\varepsilon}k'(x-x')}\, e^{-\pi\frac{(x-x')^2}{2\varepsilon}}, e^{\frac{2\pi i}{\varepsilon}k'(x-x')}\, e^{-\pi\frac{(x-x')^2}{2\varepsilon}}\right\rangle_{L^2_{x'}}. \end{aligned} \tag{3.3}$$

where of course W^ε is the corresponding Wigner function, and D^ε the corresponding operator,

$$D^\varepsilon u(x) = \varepsilon^n \int e^{2\pi i k(x-y)} W^\varepsilon(\frac{x+y}{2},\varepsilon k)u(y)dydk. \tag{3.4}$$

It is of crucial importance that $D^\varepsilon > 0 \;\Rightarrow\; H(x,k) > 0$.

4. Proofs

4.1. **Proof of Theorem 2.1. Remark on notation:** Here and in the sequel we use the notation $R = \lambda^{-\frac{1}{3}}$. λ as stated is an effective upper bound; if we replace $\lambda(\varepsilon)$ by a function $1 \ll \lambda'(\varepsilon) \leqslant \lambda(\varepsilon)$ theorem 2.1 still holds.

In lemma 5.3 it is shown essentially that Theorem 2.3 applies if $n = 1$, $V(x) = -|x|^{1+\theta}$ in $\{|x| < 1\}$ (with a smooth cutoff outside of that, the specifics of which are irrelevant) and $W_0^\varepsilon = \delta_x^{-1}\delta_k^{-1}w(\frac{x}{\delta_x},\frac{k}{\delta_k})$, where $\delta_k = R^{\frac{1+\theta}{2}}$, $\delta_x = R^{\frac{1-\theta}{5}}$. (It is completely straightforward to see that the assumptions not checked explicitly there also hold). Therefore corollary 2.4 applies, i.e. it suffices to find the limit in ε of ρ_1^ε. (We avoid duplicating parts of the proof of lemma 5.3 here).

With $[W_0^\varepsilon]$ defined as in equation (5.15), we have

$$\begin{aligned} ||W_0^\varepsilon - [W_0^\varepsilon]||_{L^1} &\leqslant C\delta_x^{-1}\delta_k^{-1} \int\limits_{|x|<CR} |w(\tfrac{x}{\delta_x}, \tfrac{k}{\delta_k})| dxdk = \\ &= \int\limits_{|x|<C\frac{R}{\delta_x}} |w(x,k)| dxdk \leqslant C\delta_x^{-1} R = o(1). \end{aligned} \tag{4.1}$$

This means that ρ_2^ε and ρ_1^ε (the evolution under equation (2.8) of $[W_0^\varepsilon]$, W_0^ε respectively) are interchangeable for our purposes, since

$$\langle \rho_1^\varepsilon, \phi\rangle = \langle \rho_2^\varepsilon, \phi\rangle + \langle \rho_1^\varepsilon - \rho_2^\varepsilon, \phi\rangle$$

and

$$|\langle \rho_1^\varepsilon - \rho_2^\varepsilon, \phi\rangle| \leqslant ||\rho_1^\varepsilon - \rho_2^\varepsilon||_{L^1} ||\phi||_{L^\infty} \leqslant ||W_0^\varepsilon - [W_0^\varepsilon]||_{L^1} ||\phi||_{\mathcal{A}}.$$

(We used the obvious bound $||\phi||_{L^\infty} \leqslant ||\phi||_{\mathcal{A}} = \int \sup\limits_x |\mathcal{F}_{k\to K}[\phi(x,k)]| dK$).

In particular, this means we can always work with $|x| > CR$, since ρ_2^ε stays outside of $\{|x| < CR\}$ by construction. This allows certain ODEs we will use to be well-posed. Without loss of generality we will work for $x > 0, k > 0$ (working with $k < 0$ makes no difference other than the opposite sign in the explicit expression for $K(0)$ in equation (4.2), the case $x < 0$ follows by symmetry.). Denote by $X(t), K(t)$ the unique solution of

$$\begin{aligned} \dot{X}(t) = 2\pi K(t), \quad \dot{K}(t) = \tfrac{1+\theta}{2\pi} (X(t))^\theta, \\ X(0) = CR, \quad K(0) = |X(0)|^{\frac{1+\theta}{2}}, \end{aligned} \tag{4.2}$$

i.e. the only branch of the level set of $\{2\pi^2 k^2 + V(x)\} = 0$ in $\{x > R \ \wedge \ k > 0\}$.

The interesting property of this trajectory is that, unlike what happens with regular potentials, it leaves zero in finite time (more generally, i.e. without restricting to $\{x > R \ \wedge \ k > 0\}$, it reaches *and* leaves zero in finite time).

Claim: *We will show that any characteristic of the Liouville equation (2.8) starting in $\{x > R \ \wedge \ k > 0\} \cap supp[W_0^\varepsilon]$ converges to $X(t), K(t)$ defined in equation (4.2).*

Then it follows that

$$\chi_{x>0}(x,k)[W_0^\varepsilon] \rightharpoonup \int \chi_{x>0}(x,k)[W_0^\varepsilon] dxdk \ \delta(x - X(t), k - K(t)).$$

Repeating the argument for $x < 0$ gives a limit of two delta functions leaving zero along different trajectories.

Proof of the claim: Denote by $X_1(t), K_1(t)$ the solution of

$$\begin{aligned} \dot{X}_1(t) = 2\pi K_1(t), \quad \dot{K}_1(t) = \tfrac{-1}{2\pi} \partial_x \overset{\sim\gamma}{V}(X_1(t)), \\ X_1(0) = x_0, \quad K_1(0) = k_0, \end{aligned} \tag{4.3}$$

where of course $(x_0, k_0) \in \{x > R \wedge k > 0\} \cap supp[W_0^\varepsilon]$ are as in the statement of the claim). Then equation (4.3) can be recast as

(4.4)
$$\dot{X}_1(t) = 2\pi K_1(t), \qquad \dot{K}_1(t) = \tfrac{-1}{2\pi} V_x(X_1(t)) + \tfrac{-1}{2\pi}(\partial_x \overset{\sim\gamma}{V}(X_1(t)) - V_x(X_1(t))),$$
$$X_1(0) = x_0, \quad K_1(0) = k_0.$$

Here we use the estimate $|\partial_x \overset{\sim\gamma}{V}(X_1(t)) - V_x(X_1(t))| \leqslant CR^{\theta-1}\varepsilon^{\frac{\gamma}{2}} = o(1)$ (see lemma 6.2 with $V'(x)$ in the place of f). It is clear now that any such characteristic converges to

$$\begin{aligned} \dot{X}_2(t) = 2\pi K_2(t), \qquad \dot{K}_2(t) = \tfrac{-1}{2\pi} V_x(X_2(t)), \\ X_2(0) = x_0, \quad K_2(0) = k_0. \end{aligned} \tag{4.5}$$

To conclude observe that trajectory $X_2(t), K_2(t)$ is squeezed between two level sets of $2\pi^2 k^2 + V(x)$ converging to each other.

Essentially the same analysis applies to the example of lemma 5.4 as well.

4.2. **Proof of Theorem 2.2.** It will be helpful to recall the main objects we are going to use here. The WT for this problem, $W = W^\varepsilon(x, k, t)$, satisfies the well-known Wigner equation (1.5) with initial data W_0^ε. Moreover, W_1^ε and ρ_2^ε are defined as follows

$$\begin{cases} \partial_t W_1^\varepsilon + 2\pi k \cdot \partial_x W_1^\varepsilon + T_\varepsilon^{\overset{\sim\gamma}{V}} W_1^\varepsilon = 0, \\ \qquad W_1^\varepsilon(t=0) = W_0^\varepsilon, \end{cases} \tag{4.6}$$

and ρ_2^ε,

$$\begin{cases} \partial_t \rho_2^\varepsilon + 2\pi k \partial_x \rho_2^\varepsilon + T_0^{\overset{\sim\gamma}{V}} \rho_2^\varepsilon = 0, \\ \qquad \rho_2^\varepsilon(t=0) = [W_0^\varepsilon], \end{cases} \tag{4.7}$$

Of course recall that ρ_1^ε was defined as the solution of equation (2.8) with initial data $\rho_1 \varepsilon(t=0) = W_0^\varepsilon$.

We partition the proof as follows:

(4.8)
$$||W^\varepsilon - \rho_1^\varepsilon||_{L^2} \leqslant ||W^\varepsilon - W_1^\varepsilon||_{L^2} + ||W_1^\varepsilon - \rho_2^\varepsilon||_{L^2} + ||\rho_1^\varepsilon - \rho_2^\varepsilon||_{L^2} = o(1)||W_0^\varepsilon||_{L^2}.$$

We will use without further comments the elementary observation

$$|| \int f(x-s,k) g(s) ds||_{L^2(\mathbb{R}^{2n})} \leqslant ||g||_{L^1(\mathbb{R}^n)} \, ||f||_{L^2(\mathbb{R}^{2n})} \tag{4.9}$$

Lemma 4.1 ($W \approx W_1$)**.** $\forall t \in [0,T]$

$$||W_1^\varepsilon(t) - W^\varepsilon(t)||_{L^2} = O(\ \varepsilon^{\gamma\frac{1+\theta}{2}-1}||W_0^\varepsilon||_{L^2}\ ).$$

Proof: Denote

$$h^\varepsilon = W^\varepsilon - W_1^\varepsilon. \tag{4.10}$$

Obviously,

$$\begin{aligned} &\partial_t h + 2\pi k \partial_x h + T^{\tilde{V}^\gamma} h = -T^{V-\tilde{V}^\gamma} W_1^\varepsilon, \\ &h(x,k,0) = 0. \end{aligned} \tag{4.11}$$

Since the Wigner equation has a bounded L^2 propagator, it suffices to bound in L^2 the rhs. Indeed, for any $a < 0$ we have

(4.12)

$$||T^{V-\tilde{V}^\gamma} W_1^\varepsilon||_{L^2} = C|| \int \widehat{V}(S)(1 - e^{-\frac{\pi}{2}\eta_1 S^2})\widehat{W_1^\varepsilon}(X-S,K)\tfrac{sin(\pi\varepsilon SK)}{\varepsilon} dS||_{L^2} \leqslant$$

$$\leqslant \varepsilon^{-1}||\widehat{V}(S)(1 - e^{-\frac{\pi}{2}\eta_1 S^2})||_{L^1}||W_1^\varepsilon||_{L^2} \leqslant$$

$$\leqslant \varepsilon^{-1}||W_1^\varepsilon||_{L^2}\left[\eta_1 \int\limits_{|S|\leqslant \eta_1^a} |\widehat{V}(S)|\,|S|^2 dS + \int\limits_{|S|\geqslant \eta_1^a} |\widehat{V}(S)| dS\right] =$$

$$= C\varepsilon^{-1}||W_1^\varepsilon||_{L^2}\left[\eta_1 + \eta_1 \int\limits_{\rho\in(1,\eta_1^a)} \rho^{-\theta} d\rho + \int\limits_{\rho\geqslant\eta_1^a} \rho^{-2-\theta} d\rho\right] =$$

$$= \varepsilon^{-1}||W_0^\varepsilon||_{L^2}\left[\eta_1 + \eta_1^{a(1-\theta)+1} + \eta_1^{(-1-\theta)a}\right].$$

Recall that $\eta_1 = \varepsilon^\gamma$. Asking that the remainder is small gives

$$\begin{aligned} &\tfrac{\eta_1}{\varepsilon} = o(1) \ \Leftrightarrow\ \gamma > 1, \\ &\tfrac{\eta_1^{a(1-\theta)+1}}{\varepsilon} = o(1) \ \Leftrightarrow\ \gamma[a(1-\theta)+1] > 1, \\ &\tfrac{\eta_1^{(-1-\theta)a}}{\varepsilon} \ \Leftrightarrow\ \gamma(-1-\theta)a > 1. \end{aligned} \tag{4.13}$$

It is easy to check that, for any $\gamma > 2$, and if we set $a = -\frac{1}{2}$ the above constraints are satisfied. So finally with this calibration the proof of lemma 4.1 is complete.

Moreover:

Lemma 4.2 ($W_1 \approx \rho_2^\varepsilon$)**.** $\forall t \in [0,T]$

$$||W_1^\varepsilon(t) - \rho_2^\varepsilon(t)||_{L^2} = O(\varepsilon^{\frac{\theta}{2+\theta}-\delta}||W_0^\varepsilon||_{L^2}). \tag{4.14}$$

Proof: It is straightforward to check that

$$\begin{aligned} &\partial_t \widehat{W}_1^\varepsilon - 2\pi X \cdot \partial_K \widehat{W}_1^\varepsilon + \int \widehat{\widetilde{V}^\gamma}(S) \widehat{W}_1^\varepsilon (X-S,K) \tfrac{sin(\pi\varepsilon S\cdot K)}{\varepsilon} dS = 0, \\ &\partial_t \widehat{\rho}_2^\varepsilon - 2\pi X \cdot \partial_K \widehat{\rho}_2^\varepsilon + \pi \int \widehat{\widetilde{V}^\gamma}(S) \widehat{\rho}_2^\varepsilon (X-S,K) S\cdot K dS = 0, \end{aligned} \tag{4.15}$$

and therefore, if $f = \widehat{W^\varepsilon}_1 - \widehat{\rho}_2^\varepsilon$,

$$\begin{aligned} &\partial_t f - 2\pi X \cdot \partial_K f + \int \widehat{\widetilde{V}^\gamma}(S) f(X-S,K) \tfrac{sin(\pi\varepsilon S\cdot K)}{\varepsilon} dS = \\ &= \int \widehat{\widetilde{V}^\gamma}(S) \widehat{\rho}_2^\varepsilon (X-S,K) \left(1 - \tfrac{sin(\pi\varepsilon S\cdot K)}{\pi\varepsilon S\cdot K}\right) \pi S\cdot K dS, \\ &f(x,k,0) = W_0^\varepsilon - [W_0^\varepsilon]. \end{aligned} \tag{4.16}$$

Using the Duhamel formula,

$$\begin{aligned} &||f(t)||_{L^2} \leqslant ||W_0^\varepsilon - [W_0^\varepsilon]||_{L^2} + \\ &+T \sup_{t\in[0,T]} || \int \widehat{\widetilde{V}^\gamma}(S) \widehat{\rho}_2^\varepsilon (X-S,K,t) \left(1 - \tfrac{sin(\pi\varepsilon S\cdot K)}{\pi\varepsilon S\cdot K}\right) \pi S\cdot K dS||_{L^2}. \end{aligned} \tag{4.17}$$

Recall that the first term above is (relatively) small by assumption; we will work out the other term:

To bound that, first of all observe that

$$\left| \frac{sin(\pi\varepsilon S\cdot K)}{\pi\varepsilon S\cdot K} - 1 \right| \leqslant C min\{1, |\pi\varepsilon S\cdot K|\} \tag{4.18}$$

and therefore for any $b<0$

(4.19)

$$||\int \widehat{\widetilde{V}^{\gamma}}(S)\widehat{\rho}_2^{\varepsilon}(X-S,K)\left(1-\frac{sin(\pi\varepsilon S\cdot K)}{\pi\varepsilon S\cdot K}\right)\pi S\cdot K dS||_{L^2}\leqslant$$

$$\leqslant C\varepsilon||\int\limits_{0\leqslant|S|\leqslant 1}|\widehat{\widetilde{V}^{\gamma}}(S)|\cdot|S|^2|\widehat{\rho}_2^{\varepsilon}(X-S,K)|\cdot|K|^2dS||_{L^2}+$$

$$+\varepsilon^{1+2b}||\int\limits_{1<|S|<\varepsilon^b}\widehat{\widetilde{V}^{\gamma}}(S)\widehat{\rho}_2^{\varepsilon}(X-S,K)|K|^2dS||_{L^2}+$$

$$+||\int\limits_{|S|>\varepsilon^b}\widehat{\widetilde{V}^{\gamma}}(S)\widehat{\rho}_2^{\varepsilon}(X-S,K)|SK|dS||_{L^2}\leqslant$$

$$\leqslant C\left(\varepsilon^{1+2b}||\widehat{V}(S)min\{|S|^2,1\}||_{L^1}||\rho_2^{\varepsilon}||_{H^2}+\int\limits_{|S|>\varepsilon^b}|\widehat{\widetilde{V}^{\gamma}}(S)|\,|S|dS||\rho_2^{\varepsilon}||_{H^1}\right)\leqslant$$

$$\leqslant C\left(\varepsilon^{1+2b}||\widehat{V}\tfrac{S^2}{1+S^2}||_{L^1}||\rho_2^{\varepsilon}||_{H^2}+\int\limits_{\rho=\varepsilon^b}^{\infty}\rho^{-1-\theta}d\rho||\rho_2^{\varepsilon}||_{H^1}\right)=$$

$$=C\left(\ \varepsilon^{1+2b}||\rho_2^{\varepsilon}||_{H^2}+\varepsilon^{-\theta b}||\rho_2^{\varepsilon}||_{H^1}\leqslant C(\varepsilon^{1+2b-\delta}+\varepsilon^{-\theta b-\delta})||W_0^{\varepsilon}||_{L^2}\ \right)$$

Calibrating the parameters is easy; we are given $\theta\in(0,1)$, δ as in the statement of the theorem and we need to find $b\in(-\frac{1}{2},0)$ so that

$$\begin{gathered}1+2b-\delta>0,\quad -\theta b-\delta>0\quad\Leftrightarrow\\ \Leftrightarrow\ \tfrac{\delta-1}{2}<b<-\tfrac{\delta}{\theta}.\end{gathered}\tag{4.20}$$

It should now be clear that the constraint on δ in statement of the theorem comes from the self-consistency check $\frac{-1}{2}<\frac{\delta-1}{2}<-\frac{\delta}{\theta}<0$. The proof of lemma 4.2 is complete.

Now the last step is to show that $\rho_2^{\varepsilon}\approx\rho_1^{\varepsilon}$; but this follows immediately from assumption 1, since they satisfy the same equation (which has an L^2-continuous propagator): $||\rho_2^{\varepsilon}(t)-\rho_1^{\varepsilon}(t)||_{L^2}=||W_0^{\varepsilon}-[W_0^{\varepsilon}]||_{L^2}$.

The proof of Theorem 2.2 is complete.

4.3. **Proof of Theorem 2.3.** First of all observe that equation (2.10), under the assumptions of theorem 2.3 $||W_0^{\varepsilon}-[W_0^{\varepsilon}]||_{L^2}=o(1)$ and $||W_0^{\varepsilon}||_{L^2}=o(\varepsilon^{-\kappa})$, implies

$$||W^{\varepsilon}(t)-\rho_1^{\varepsilon}(t)||_{L^2}=o(1).\tag{4.21}$$

Therefore the result formally follows by

$$\langle\rho_1^{\varepsilon}(t),\phi\rangle=\langle W^{\varepsilon}(t),\phi\rangle+\langle\rho_1^{\varepsilon}(t)-W^{\varepsilon},\phi\rangle=\langle W^{\varepsilon}(t),\phi\rangle+o(1).$$

However for that computation to be correct we would need to have $||\phi||_{L^2}\leqslant||\phi||_{\mathcal{A}}$, which is not true. This section is mainly devoted to working around

that. An L^1 semiclassical approximation for the Husimi function is a bonus of the method used.

Theorem 4.3. *Suppose that the assumptions of Theorem 2.3 hold. Denote by $W^\varepsilon(x,k,t)$ the solution of the Wigner equation (1.5), and by $H^\varepsilon(x,k,t)$ the corresponding Husimi function.*

Moreover, assume that there is a function $\rho^\varepsilon(x,k,t)$ such that

$$\exists C_0 > 0 \ : \ \bigcup_{t\in[0,T]} supp\rho^\varepsilon(x,k,t) \subseteq \{|(x,k)| < C_0\} \tag{4.22}$$

and

$$\sup_{t\in[0,T]} ||H^\varepsilon(t) - \rho^\varepsilon(t)||_{L^2} = o(1). \tag{4.23}$$

Then

$$\sup_{t\in[0,T]} ||H^\varepsilon(t) - \rho^\varepsilon(t)||_{L^1} = o(1). \tag{4.24}$$

Remark: The proof is essentially an adaptation of an argument from [3].

Proof: Before the main part of the proof observe that our assumptions on the potential imply that $W^\varepsilon(x,k,t) \in L^1_{x,k} \forall t$, and

$$tr(A^\varepsilon(t)) = \int W^\varepsilon(x,k,t)dxdk = 1.$$

The first statement is elementary to prove, and the second well-known; see [3] for details. It is also straightforward that $||W^\varepsilon(t)||_{L^2} = ||W^\varepsilon_0||_{L^2}$ (it follows by the essential self-adjointness of the corresponding Schrödinger operator).

Denote $h^\varepsilon = H^\varepsilon - \rho^\varepsilon$. Then, for any $M > C_0$,

$$\begin{aligned} ||h^\varepsilon||_{L^1} &= \int\limits_{|(x,k)|\leqslant M} |H^\varepsilon - \rho^\varepsilon| dxdk + \int\limits_{|(x,k)|>M} |H^\varepsilon - \rho^\varepsilon| dxdk \leqslant \\ &\leqslant \langle \chi_{[0,M]}\left(\,|(x,k)|\,\right), h^\varepsilon\rangle + \int\limits_{|(x,k)|>M} |H^\varepsilon| dxdk = \\ &= CM^n ||h^\varepsilon||_{L^2} + \int\limits_{|(x,k)|>M} |H^\varepsilon| dxdk. \end{aligned} \tag{4.25}$$

$M = M(\varepsilon)$ will be calibrated so that $M \gg 1$ and

$$\sup_{t\in[0,T]} M^n(\varepsilon) ||h^\varepsilon||_{L^2} = o(1). \tag{4.26}$$

At this point we will need that $V(x) > 0$. However, since V is assumed to be bounded below, non-negativity follows without loss of generality by substituting $V(x) \mapsto V_2(x) = V(x) + ||V_-||_{L^\infty}$, where of course V_- is the negative part, $V_-(x) = V(x)\chi_{\{V(x)<0\}}(x)$.

Now, recalling that $H(x,k) \geqslant 0$, we have

(4.27)
$$\int\limits_{|(x,k)|>M} |H(x,k)|dxdk \leqslant M^{-2} \int\limits_{|(x,k)|>M} k^2 H(x,k)dxdk \leqslant$$

$$\leqslant M^{-2} \int (k^2 + V(x))H(x,k)dxdk =$$

$$= M^{-2} \int (k^2 + V(x))W^\varepsilon(x,k,t)dxdk + M^{-2}C\varepsilon \int W^\varepsilon(x,k)dxdk +$$

$$+M^{-2} \int (V(x) - (\tfrac{2}{\varepsilon})^{\frac{n}{2}} \int e^{-\frac{2\pi|x-x'|^2}{\varepsilon}} V(x')dx')W^\varepsilon(x,k)dxdk \leqslant$$

$$\leqslant O(\varepsilon M^{-2}) +$$
$$+M^{-2}||(V(x) - (\tfrac{2}{\varepsilon})^{\frac{n}{2}} \int e^{-\frac{2\pi|x-x'|^2}{\varepsilon}} V(x')dx')||_{L^2_x} ||W^\varepsilon||_{L^2} \leqslant$$

$$\leqslant O(M^{-2})\,(\varepsilon + \sqrt{\varepsilon}||V||_{H^1}||W^\varepsilon||_{L^2}) = o(1).$$

In this computation we used the energy conservation,

$$\int (2\pi^2 k^2 + V(x))W^\varepsilon(x,k,t)dxdk = tr\left(\ (-\tfrac{\varepsilon}{2}\Delta + V(x))D^\varepsilon(t)\ \right) =$$

$$= tr\left(\ (-\tfrac{\varepsilon}{2}\Delta + V(x))D^\varepsilon(0)\ \right) = \int (2\pi^2 k^2 + V(x))W^\varepsilon(x,k,0)dxdk,$$

and the obvious observation

$$||V(x) - (\frac{2}{\varepsilon})^{\frac{n}{2}} \int e^{-\frac{2\pi|x-x'|^2}{\varepsilon}} V(x')dx'||_{L^2} = O(\sqrt{\varepsilon}||V||_{H^1})$$

The proof is complete.

Proof of Theorem 2.3: Now observe that the conditions of theorem 4.3 are satisfied with $\rho^\varepsilon = \rho_1^\varepsilon$.

Indeed, since W_0^ε is of compact support, $\rho_1^\varepsilon(t)$ is of compact support uniformly in ε (see lemma 6.5) for $t \in [0,T]$. Moreover, using again that $||H^\varepsilon(t) - W^\varepsilon(t)||_{L^2} \leqslant C\sqrt{\varepsilon}||W^\varepsilon(t)||_{H^1}$ (see e.g. lemma A.1 of [4] for a proof), it follows that

$$||H^\varepsilon - \rho_1^\varepsilon||_{L^2} \leqslant C||W^\varepsilon - \rho_1^\varepsilon||_{L^2} + \sqrt{\varepsilon}||W^\varepsilon(t)||_{H^1}.$$

With that in mind we can calibrate more precisely the parameter $M = M(\varepsilon)$ which appeared earlier:

$$M = (\varepsilon^\kappa ||W_0^\varepsilon||_{L^2} \ + \ ||W_0^\varepsilon - [W_0^\varepsilon]||_{L^2} + \sqrt{\varepsilon}||W_0^\varepsilon||_{H^1})^{-\frac{1}{2n}}. \tag{4.28}$$

(The fact that $\sqrt{\varepsilon}||W_0^\varepsilon||_{H^1} = o(1)$ follows from equation (2.9)).

The result follows, making use of theorem 4.3, by virtue of

$$|\langle \rho_1^\varepsilon - W^\varepsilon, \phi\rangle| \leqslant |\langle \rho_1^\varepsilon - H^\varepsilon, \phi\rangle| + |\langle W^\varepsilon - H^\varepsilon, \phi\rangle| \leqslant$$

$$\leqslant ||\rho_1^\varepsilon - H^\varepsilon||_{L^1}||\phi||_{L^\infty} + |\langle W^\varepsilon, \phi - \phi * \frac{\sqrt{2}^n}{\varepsilon^n} e^{-2\pi\frac{|(x,k)|^2}{\varepsilon}}\rangle| = \tag{4.29}$$

$$= O(M^{-2})\,(1+\sqrt{\varepsilon}||W^\varepsilon||_{L^2})\,||\phi||_{\mathcal{A}} + |\langle W^\varepsilon, \left(\mathbb{1} - e^{-\varepsilon\frac{2}{\pi}\Delta}\right)\phi\rangle|.$$

It was shown in [8] that W^ε and H^ε have the same weak-$*$ limit in $\mathcal{A}'$; $\langle W^\varepsilon, \left(\mathbb{1} - e^{-\varepsilon\frac{2}{\pi}\Delta}\right)\phi\rangle \to 0$ follows. One can estimate the rate of convergence if there is more smoothness information for the test-function ϕ.

The proof is complete.

5. Examples

5.1. **Examples for Theorem 2.2.** Here we are concerned with Cauchy problems for the Liouville equation,

$$\begin{cases} \partial_t \rho_1^\varepsilon + 2\pi k \partial_x \rho_1^\varepsilon - \frac{1}{2\pi}\partial_x \tilde{V}^\gamma \cdot \partial_x \rho_1^\varepsilon = 0, \\ \rho_1^\varepsilon(t=0) = W_0^\varepsilon, \end{cases}$$

which have the following

Property 1. *There exist $T > 0$, $\delta \in (0, \frac{\theta}{2+\theta})$ and $[W_0^\varepsilon] \in H^2(\mathbb{R}^{2n})$ such that*

$$||[W_0^\varepsilon] - W_0^\varepsilon||_{L^2} = o(1)||W_0^\varepsilon||_{L^2} \tag{5.1}$$

and the solution of

$$\begin{cases} \partial_t \rho_2^\varepsilon + 2\pi k \partial_x \rho_2^\varepsilon - \frac{1}{2\pi}\partial_x \tilde{V}^\gamma \cdot \partial_x \rho_2^\varepsilon = 0, \\ \rho_2^\varepsilon(t=0) = [W_0^\varepsilon], \end{cases} \tag{5.2}$$

satisfies

$$||\rho_2^\varepsilon(t)||_{H^2} = O(\varepsilon^{-\delta}||W_0^\varepsilon||_{L^2}), \quad \textit{uniformly for } t \in [0,T]. \tag{5.3}$$

Lemma 5.1. *Let $\theta \in (0,1)$, $\delta \in (0, \frac{\theta}{2+\theta})$, $n = 2$, $V(x) = -|x|^{1+\theta}$ with an appropriate smooth cutoff outside $\{|x| < 2\}$. Assume moreover that there exists a function of compact support $w \in H^2(\mathbb{R}^4) \cap L^\infty$, such that*

$$W_0^\varepsilon(x,k) = \delta_x^{-2}\delta_k^{-2} w\left(\frac{x-(z_1,z_2)}{\delta_x}, \frac{k-(z_3,z_4)}{\delta_k}\right), \tag{5.4}$$

with $(z_1,z_2) = -L(z_3,z_4)$ (i.e. the wavepacket is "shot towards zero"). $||w||_{H^2}$, $||w||_{L^\infty}$, $|z|$ have to be bounded uniformly in ε, $||w||_{H^2}, ||w||_{L^\infty}, |z| = O(1)$; other than that w and z can be allowed to depend on ε.

The small parameters involved $\delta_x, \delta_k, R \ll 1$ are calibrated as follows:

$$R = \left(log\left(\frac{1}{\varepsilon}\right)\right)^{\frac{-1}{3}},$$

as in equation (6.26), and

$$\delta_x^{-1}(R+\delta_k) \ll 1,$$
$$(\delta_x+\delta_k)^{-2}(\delta_k+R)^{-2} \leqslant C\varepsilon^{-\frac{\delta}{2}}. \tag{5.5}$$

Then Property 1 is satisfied.

Remarks:

- The difficulties arise from the non-smootheness at zero, and the details of a smooth behaviour away from zero are irrelevant here. So we will work for a small enough time, before any trajectory starting in a neighbourhood of zero reaches the support of the cutoff function. Therefore, to keep the presentation simple, we will not introduce any explicit treatment of the cutoff function. This approach is followed in the sequel as well.
- It is obviously much easier to workout the example if the wavepacket is not "shot towards zero". However in that case it's basically covered by the standard theory; the interesting cases are when in the limit a delta function passes over a set where the flow associated with the original potential $V(x)$ doesn't exist.
- An example of a scaling satisfying the above constraints is $\delta_x = \sqrt{\delta_k} = \sqrt{R}$.

Proof: Assume without loss of generality that $z = (0, -L, 0, 1)$. Consider $\phi : \mathbb{R} \to [0,1]$ to be a C^∞ function such that

$$\begin{aligned} \phi(x) &= 0, \quad |x| < \tfrac{1}{2}, \\ \phi(x) &= 1, \quad |x| > 1. \end{aligned} \tag{5.6}$$

The modified initial data $[W_0^\varepsilon]$ is selected as

$$[W_0^\varepsilon](x,k) = W_0^\varepsilon(x,k)\phi\left(\frac{|x_1|}{R+2L\delta_k}\right). \tag{5.7}$$

The key claim is that ρ_2^ε never touches $\{|x| < R\}$. Indeed $[W_0^\varepsilon]$ has two disjointly supported components. The calibration of the parameters and the cut-off is such that, if $[W_0^\varepsilon]$ was propagated by a Liouville equation corresponding to $V(x) = 0$, each of the components would stay on either side of the strip $\{|x_1| < R\}$ (see Figure 1). Since in fact it is propagated with a potential driving the flow *away* from $x = 0$, the claim follows.

Therefore, in this problem we can take $\tilde{V}{}^{\gamma}_{eff}(x) = \tilde{V}{}^{\gamma}(x)\phi\left(\frac{|x|}{R+2L\delta_k}\right)$. Now for condition (5.3), making use of observation 6.6 (enlarging the cut-off area in fact strengthens observation 6.6, i.e. substituting $R + 2L\delta_k$ for R is

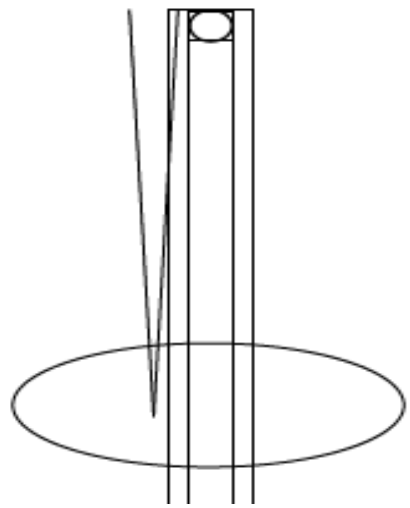

FIGURE 1. Any trajectory leaving the support of $[W_0^\varepsilon]$ has an initial velocity in a small cone around $k = (0, 1)$. If we were in free space, there wouldn't be enough space for sufficient movement in x_1 to reach $\{|x_1| < R\}$. Therefore the solution is the sum of two components, supported on either side of $\{|x_1| < R\} = 0$. It clear that the presence of a repulsive potential will only make this more pronounced.

painless) and lemma 6.4 it follows that

$$\begin{aligned} ||\rho_2^\varepsilon(t)||_{H^2} &\leqslant ||[W_0^\varepsilon]||_{H^2}\varepsilon^{-\frac{\delta}{2}} \leqslant C||W_0^\varepsilon||_{H^2}\varepsilon^{-\frac{\delta}{2}}(\delta_k + R)^{-2} = \\ &= C||W_0^\varepsilon||_{L^2}(\delta_x + \delta_k)^{-2}(\delta_k + R)^{-2}\varepsilon^{-\frac{\delta}{2}}, \end{aligned} \tag{5.8}$$

and therefore – using the scaling of equation (5.5) – $||\rho_2^\varepsilon(t)||_{H^2} \leqslant C\varepsilon^{-\delta}||W_0^\varepsilon||_{L^2}$.

Now for $W_0^\varepsilon - [W_0^\varepsilon]$:

$$\begin{aligned} &||W_0^\varepsilon - [W_0^\varepsilon]||_{L^2}^2 \leqslant \\ &\leqslant \delta_x^{-2n}\delta_k^{-2n} \int\limits_{|x_1|<C(R+\delta_k)} |w(\tfrac{(x_1,x_2)}{\delta_x}, \tfrac{(k_1-1,k_2)}{\delta_k})|^2 dxdk = \\ &= \delta_x^{-n}\delta_k^{-n} \int\limits_{|x_1|<C\frac{R+\delta_k}{\delta_x}} |w(x_1, x_2, k_1 - 1, k_2)|^2 dxdk \leqslant \\ &\leqslant C||W_0^\varepsilon||_{L^2}^2||w||_{L^\infty}^2\delta_x^{-1}(R + \delta_k) \end{aligned} \tag{5.9}$$

The same idea can be applied to different configurations:

Lemma 5.2. *Consider the setup of lemma 5.1 with* $z = (0, -L, 0, 1)$ *and the only difference that*

$$V(x) = -|x_1|^{1+\theta}\psi(x_1)\psi(x_2). \tag{5.10}$$

where ψ *is a smooth cutoff function,* $\psi = 1 - \phi \in \mathcal{S}(\mathbb{R})$ *(*ϕ *was defined in equation (5.6)).*

Then Property 1 holds.

Proof: The geometry is essentially the same as before, and it is clear that the obvious adaptation of observation 6.6, i.e. the one with

$$\widetilde{V}^{\gamma}_{eff}(x) = \widetilde{V}^{\gamma}(x)\phi(\frac{x_1}{R + L\delta_k})$$

holds.

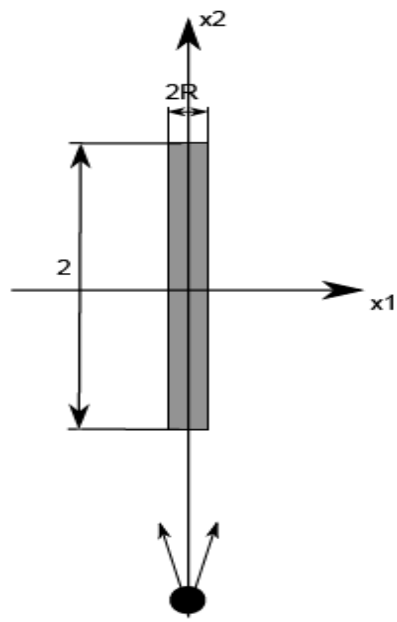

FIGURE 2. The construction of lemma 5.1 holds, with obvious adjustments, to the example of lemma 5.2.

5.2. **Examples for Theorem 2.3.** In relation to Theorem 2.3, we introduce

Property 2. *Given* $V(x)$, W_0^e *with* $||W_0^\varepsilon||_{L^2} = o(\varepsilon^{-\kappa})$, *there exist* $T > 0$, $\delta \in (0, \frac{\theta}{2+\theta})$ *and* $[W_0^\varepsilon] \in H^2(\mathbb{R}^{2n})$ *such that*

$$||[W_0^\varepsilon] - W_0^\varepsilon||_{L^2} = o(1) \tag{5.11}$$

and the solution of

$$\begin{cases} \partial_t \rho_2^\varepsilon + 2\pi k \partial_x \rho_2^\varepsilon - \frac{1}{2\pi}\partial_x \tilde{V}^\gamma \cdot \partial_x \rho_2^\varepsilon = 0, \\ \rho_2^\varepsilon(t=0) = [W_0^\varepsilon], \end{cases} \tag{5.12}$$

satisfies

$$||\rho_2^\varepsilon(t)||_{H^2} = O(\varepsilon^{-\delta}||W_0^\varepsilon||_{L^2}), \quad \textit{uniformly for } t \in [0,T]. \tag{5.13}$$

We treat κ, γ as given parameters; of course when checking the property in the context of Theorem 2.3 they are controlled by the statement of Theorem 2.2.

Lemma 5.3. *Let $n = 1$, $\theta \in (0,1)$, $V(x) = -|x|^{1+\theta}$ with an appropriate smooth cutoff outside $\{|x| < 2\}$. Assume moreover that $w(x,k) \in H^2 \cap L^\infty \cap L^1$, $\operatorname{supp} w \subseteq \{|x|, |k| < 1\}$, and*

$$W_0^\varepsilon = \delta_x^{-1}\delta_k^{-1} w(\frac{x}{\delta_x}, \frac{k}{\delta_k}). \tag{5.14}$$

Finally, set $R = \left(log\left(\frac{1}{\varepsilon}\right)\right)^{\frac{-1}{3}}$.

Then Property 2 is satisfied if $\delta_k = C' R^{\frac{1+\theta}{2}}$, $\delta_x = C'' R^{\frac{1-\theta}{5}}$.

Proof: We will cutoff a strip of the form $\{|x| \leqslant CR\}$, and show that in fact this suffices.

As is illustrated in Figure 5.2, the preimage under the flow of $\{|x| < R \wedge |k| < \delta_k\}$ is contained between the level sets $\{2\pi^2 k^2 - \tilde{V}^\gamma(x) = \tilde{V}^\gamma(0)\}$ and $\{2\pi^2 k^2 + \tilde{V}^\gamma(x) = \tilde{V}^\gamma(R)\}$ (in red).

A big help with the algebra will be to approximate $\tilde{V}^\gamma(x) = -|x|^{1+\theta} + O(\varepsilon^{\frac{\gamma}{2}})$. ($\gamma$ is as in the statement of Theorem 2.2). Since all the other small parameters are (negative) powers of $log(\frac{1}{\varepsilon})$, $O(\varepsilon^{\frac{\gamma}{2}})$ is negligible everywhere. (The justification is that without loss of generality we can localize the problem on a compact set, and then $V(x) \in W^{1,\infty}$ uniformly in ε. The conclusion follows by a standard observation on mollifiers, see lemma 6.2).

It is easily seen that the needed length is x_*, defined by

$$2\pi^2\delta_k^2 + \tilde{V}^\gamma(x_*) = \tilde{V}^\gamma(R)$$

or

$$|x_*|^{1+\theta} = 2\pi^2\delta_k^2 + R^{1+\theta} + O(\varepsilon^{\frac{\gamma}{2}}).$$

Here is where $\delta_k = C' R^{\frac{1+\theta}{2}}$ comes from; making that scaling we get $x_* = ((1+2\pi^2)R^{1+\theta} + O(\varepsilon^{\frac{\gamma}{2}}))^{\frac{1}{1+\theta}} = C'R$.

So far we have ensured by construction that, if $\delta_k = R^{\frac{1+\theta}{2}}$, (there is an $O(1)$ constant C so that) if

$$[W_0^\varepsilon](x,k) = W_0^\varepsilon(x,k)\phi(\frac{x}{CR}), \tag{5.15}$$

its propagation under equation (5.12) will never enter $\{|x| < R\}$.

Now for the approximation errors:

$$||W_0^\varepsilon - [W_0^\varepsilon]||_{L^2}^2 \leqslant C\delta_x^{-2}\delta_k^{-2} \int\limits_{|x|<CR} |w(\tfrac{x}{\delta_x}, \tfrac{k}{\delta_k})|^2 dxdk =$$

$$= C\delta_x^{-1}\delta_k^{-1} \int\limits_{|x|<C\frac{R}{\delta_x}} |w(x,k)|^2 dxdk \leqslant C\delta_x^{-2} R^{\frac{1-\theta}{2}}. \tag{5.16}$$

This gives the constraint $\delta_x \gg R^{\frac{1-\theta}{4}}$ for the error to be small, which is satisfied e.g. by our earlier selection $\delta_x = CR^{\frac{1-\theta}{5}}$.

Again, since all the small parameters are powers of $log(\frac{1}{\varepsilon})$, $||[W_0^\varepsilon]||_{H^2} = o(\varepsilon^{-\frac{\delta}{2}})$ follows automatically for any $\delta \in (0, \frac{\theta}{2+\theta})$.

The proof is complete.

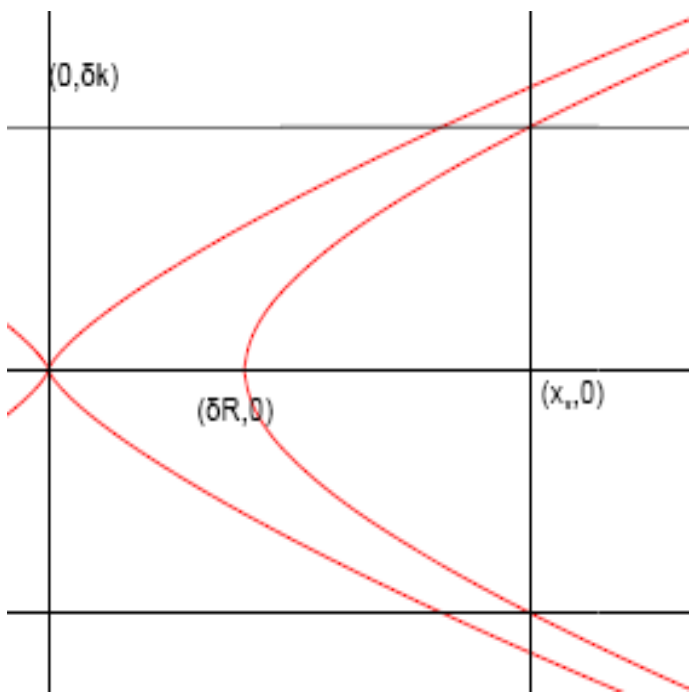

FIGURE 3. Schematic for lemma 5.3; cutting off a strip in x, $\{|x| < x_*\}$, suffices to cut off the preimage of $\{|x| < R\}$

The following is a somewhat artificial example, but it highlights an interesting behaviour.

Lemma 5.4. *Assume $n = 2$, V as in equation (5.10). Moreover, there is a function $f \in \mathcal{S}(\mathbb{R}^4)$, $\int f(x,k)dxdk = 1$, such that*

$$supp\ f \subseteq \{|x| < 1\} \times \{|k| < 1 \ \wedge \ k_1 > 0\}. \tag{5.17}$$

Denote by $\bar{f}$

$$\bar{f}(x_1, x_2, k_1, k_2) = f(-x_1, x_2, -k_1, k_2) \tag{5.18}$$

Now take

$$\begin{aligned} W_0^\varepsilon(x,k) = \tfrac{1}{\delta_x^2\delta_k^2}\Big(c_1 f(\tfrac{x_1-2R, x_2+2}{\delta_x}, \tfrac{k_1, k_2+1}{\delta_k}) + \\ +c_2\bar{f}(\tfrac{x_1-2R, x_2+2}{\delta_x}, \tfrac{k_1, k_2+1}{\delta_k})\Big). \end{aligned} \tag{5.19}$$

If $\delta_k = C'\delta_x = C''\sqrt{R}$*, this problem has Property 2.*

Proof: Two things should be obvious by construction: that $W_0^\varepsilon \rightharpoonup \delta(x + (0,2), k - (0,1))$, and that the propagation of W_0^ε under equation (2.8), i.e. ρ_1^ε, is never supported inside $\{|x_1| < R\}$. In other words, for this specifically constructed data, $W_0^\varepsilon = [W_0^\varepsilon]$.

The result follows.

6. Auxiliary results

Lemma 6.1 (2nd order derivatives equations for the Liouville equation)**.** *Consider the Cauchy problem for the Liouville equation with potential* $V(x)$ *on* $\mathbb{R}^n$*,*

$$\begin{aligned} f_t + 2\pi k\nabla_x f - \tfrac{1}{2\pi}\nabla_x V(x)\nabla_k f = 0, \\ f(t=0) = f_0. \end{aligned} \tag{6.1}$$

There are constants $C_1, C_2 > 0$ *depending only on* n *such that*

$$||f(t)||_{H^2} \leqslant C_1 e^{tC_2 \sup\limits_{|a|\leqslant 3} ||\partial_x^a V(x)||_{L^\infty}} ||f_0||_{H^2}.$$

Proof: The proof follows readily with the method of characteristics.

It suffices to observe that if

$$\dot{x}_i(t) = 2\pi k_i(t), \qquad \dot{k}_i(t) = -\frac{1}{2\pi}\partial_{x_i}V(x(t)), \tag{6.2}$$

$$\begin{gathered} z(t) = f(x(t), k(t)), \\ z_{x_i}(t) = \partial_{x_i} f(x(t), k(t)), z_{x_i x_j}(t) = \partial_{x_i x_j} f(x(t), k(t)), \\ z_{k_i}(t) = \partial_{k_i} f(x(t), k(t)), z_{k_i k_j}(t) = \partial_{k_i k_j} f(x(t), k(t)), \\ z_{x_i k_j}(t) = \partial_{x_i k_j} f(x(t), k(t)), \end{gathered} \tag{6.3}$$

it follows that

$$\dot{z} = 0 \tag{6.4}$$

$$\dot{z}_{x_i}(t) = \frac{1}{2\pi}\sum_m \partial_{x_i x_m} V(x(t))\ z_{x_i k_m}(t), \tag{6.5}$$

$$\dot{z}_{k_i}(t) = -2\pi z_{x_i}(t), \tag{6.6}$$

$$\dot{z}_{x_i x_j}(t) = \tag{6.7}$$

$$= \frac{1}{2\pi}\sum_m \left[\partial_{x_j x_m} V z_{x_i k_m}(t) + \partial_{x_i x_m} V z_{x_j k_m}(t) + \partial_{x_i x_j x_m} V z_{k_m}(t)\right]$$

$$\dot{z}_{k_i k_j}(t) = -2\pi(z_{x_j k_i} + z_{x_i k_j}), \tag{6.8}$$

$$\dot{z}_{k_i x_j}(t) = -2\pi z_{x_i x_j}(t) + \frac{1}{2\pi}\sum_m \partial_{x_m x_j} V(x(t)) z_{k_m k_i}(t). \tag{6.9}$$

The result now follows by the Gronwall inequality.

Lemma 6.2. *Consider a function $f \in W^{1,\infty}$, i.e. $\sup\limits_{|a|\leqslant 1} ||\partial_x^a f||_{L^\infty} < \infty$. Then, if $\widetilde{f} = \left(\frac{2}{\eta}\right)^{\frac{n}{2}} \int e^{-2\pi\frac{|x-x'|^2}{\eta}} f(x')dx'$, we have*

$$||f - \widetilde{f}||_{L^\infty} = O(||f||_{W^{1,\infty}}\sqrt{\eta}). \tag{6.10}$$

Proof: Take any $\zeta \in (0, \frac{1}{2})$. Now we have

$$\begin{aligned}
|f(x) - \widetilde{f}(x)| &= \left|\left(\frac{2}{\eta}\right)^{\frac{n}{2}} \int e^{-2\pi\frac{|x'|^2}{\eta}}[f(x-x') - f(x)]dx'\right| = \\
&= \left(\frac{2}{\eta}\right)^{\frac{n}{2}} \left|\int\limits_{||x'|<\eta^{\frac{1}{2}-\zeta}} e^{-2\pi\frac{|x'|^2}{\eta}}[f(x-x)' - f(x)]dx'\right| + O(\eta^\infty) \leqslant \\
&\leqslant C||f||_{W^{1,\infty}}\left(\frac{2}{\eta}\right)^{\frac{n}{2}} \int\limits_{|x'|<\eta^{\frac{1}{2}-\zeta}} e^{-2\pi\frac{|x'|^2}{\eta}}|x'|dx' + O(||f||_{L^\infty}\eta^\infty) = \\
&= O(\sqrt{\eta}2^{\frac{n}{2}})||f||_{W^{1,\infty}} \int\limits_{|y|<\eta^{-\zeta}} e^{-2\pi|y|^2}|y|dy + O(||f||_{L^\infty}\eta^\infty) = \\
&= O(||f||_{W^{1,\infty}}\sqrt{\eta}).
\end{aligned} \tag{6.11}$$

The proof is complete.

Remark: A sharper version is used in subsection 4.1, namely

$$|f(x)-\widetilde{f}(x)| \leqslant C \left(\sup_{\substack{|a|=1 \\ |x-x'|<\eta^{\frac{1}{2}-\zeta}}} |\partial_x^a f| \right) \left(\frac{2}{\eta}\right)^{\frac{n}{2}} \int_{|x'|<\eta^{\frac{1}{2}-\zeta}} e^{-2\pi\frac{|x'|^2}{\eta}}|x'|dx' +$$

$$+O(||f||_{L^\infty}\eta^\infty).$$

The following observations are used in Section 5:

Observation 6.3 (Locality of the Liouville equation)**.** *For a Liouville equation with initial data of compact support ρ_0 and a $C^{1,1}$ potential, interchanging the potential with any one that coincides with it on the "path" of the solution,*

$$\mathbb{S} = \bigcup_{t\in[0,T]} \phi_t(supp\rho_0), \tag{6.12}$$

does not change the solution.

Also,

Lemma 6.4 (An H^2 estimate)**.** *Consider equation (2.8) with initial data $\rho_0 = [W_0^\varepsilon] \in H^2$. Assume that there is a function $\overset{\sim\gamma}{V}_{eff}$ such that*

$$\overset{\sim\gamma}{V} = \overset{\sim\gamma}{V}_{eff} \quad on\ \mathbb{S}, \tag{6.13}$$

where $\mathbb{S}$ is defined as in (6.12), and

$$\sup_{|A|\leqslant 3} ||\partial_x^A \overset{\sim\gamma}{V}_{eff}(x)||_{L^\infty} \ = O\left(\ log\left(\varepsilon^{-\frac{\delta}{2}}\right) \ \right), \tag{6.14}$$

Then,

$$||\rho_2^\varepsilon||_{H^2} = O(\ \varepsilon^{-\frac{\delta}{2}}||W_0^\varepsilon||_{H^2}\ ). \tag{6.15}$$

Proof: The proof consists of using $\overset{\sim\gamma}{V}_{eff}$ in place of $\overset{\sim\gamma}{V}$, making use of observation 6.3, and then applying directly theorem 6.1.

We will also use the following

Lemma 6.5. *It is easy to observe that, if $\nabla V \in L^\infty$ (which in particular follows from Assumption 1) and $suppW_0^\varepsilon$ is compact, it follows that*

$$\exists M>0 : \forall t\in[0,T],\ \varepsilon>0\ :\ \bigcup_{t\in[0,T]} supp\,\rho_1^\varepsilon(t) \subseteq \{|(x,k)|<M\}. \tag{6.16}$$

Observation 6.6 ($\widetilde{V}^{\gamma}_{eff}$ and R)**.** *Let $V(x) = -|x|^{1+\theta}$. $\widetilde{V}^{\gamma}$ is a mollified version, according to equation (2.6).*

Consider $\phi : \mathbb{R}^n \to [0,1]$ to be a C^∞ function such that

$$\begin{aligned} \phi(x) &= 0, \quad |x| < \tfrac{1}{2}, \\ \phi(x) &= 1, \quad |x| > 1, \end{aligned} \tag{6.17}$$

and $||\phi||_{W^{3,\infty}} \leqslant 10$. It is clear that such a function exists; this is of course an arbitrary requirement, but one that allows us not to carry the cutoff function ϕ to other results.

Then, setting $1 \gg R \geqslant \left(log\left(\varepsilon^{-1}\right)\right)^{\frac{-1}{3}}$, it follows that

$$\widetilde{V}^{\gamma}_{eff}(x) := \phi(\frac{x}{R})\, \widetilde{V}^{\gamma}(x), \tag{6.18}$$

satisfies

$$\sup_{|A|\leqslant 3} ||\partial_x^A \widetilde{V}^{\gamma}_{eff}(x)||_{L^\infty} \ = O(\ log\left(\varepsilon^{-\frac{\delta}{2}}\right)\ ), \tag{6.19}$$

while, of course,

$$\widetilde{V}^{\gamma}_{eff} = \widetilde{V}^{\gamma} \ on \ \mathbb{R}^n \setminus \{|x| < R\}. \tag{6.20}$$

Proof: Observe that $1 - \phi(x) = \psi(x) \in \mathcal{S}(\mathbb{R}^n)$. Moreover, making use of observations 6.5 and 6.3, we can restrict $\widetilde{V}^{\gamma}$ to $\{|x| < M\}$ without loss of generality,

$$\widetilde{V}^{\gamma}(x) \ \mapsto \ \widetilde{V}^{\gamma}(x)\left(1 - \phi\left(\frac{x}{2M}\right)\right). \tag{6.21}$$

Denote

$$\begin{aligned} \widetilde{V}^{\gamma}_{eff}(x) &= \widetilde{V}^{\gamma}(x)\left(1 - \phi\left(\tfrac{x}{2M}\right)\right)\phi\left(\tfrac{x}{R}\right) = \\ &= \widetilde{V}^{\gamma}(x)\left(\psi\left(\tfrac{x}{2M}\right) - \psi\left(\tfrac{x}{R}\right)\right). \end{aligned} \tag{6.22}$$

Since

$$||\partial_x^A \widetilde{f}||_{L^\infty} \leqslant ||\partial_x^A f||_{L^\infty}, \tag{6.23}$$

it suffices to work with

$$\partial_x^A |x|^{1+\theta}\left(\psi\left(\frac{x}{2M}\right) - \psi\left(\frac{x}{R}\right)\right). \tag{6.24}$$

Set $x' = \frac{x}{R}$; then $\partial_x = R^{-1}\partial_{x'}$. Now we have

(6.25)
$$||\partial_x^A |x|^{1+\theta} \left(\psi\left(\tfrac{x}{2M}\right) - \psi\left(\tfrac{x}{R}\right)\right)||_{L^\infty} =$$
$$= R^{\theta-2}||\partial_x^A |x|^{1+\theta} \left(\psi\left(\tfrac{R}{2M}x\right) - \psi(x)\right)||_{L^\infty} \leqslant$$
$$\leqslant R^{\theta-2} \sup_{|A|\leqslant 3} ||\partial_x^A x^{1+\theta}||_{L^\infty([\frac{1}{2},\frac{2M}{R}])} \sup_{|A|\leqslant 3} ||\partial_x^A \left(\psi\left(\tfrac{R}{2M}x\right) - \psi(x)\right)||_{L^\infty(\mathbb{R}^n)} \leqslant$$
$$\leqslant (2M)^{1+\theta}||\phi||_{W^{3,\infty}} R^{-3}.$$

Recall that the constant M is chosen so that any trajectory leaving the support of the initial data doesn't not escape $\{|x| < M\}$ for $t \in [0, T]$. That is, to properly quantify it, one needs to consider initial data of compact support associated with the Liouville equation, and a time-scale T. Assuming that the initial data is supported in $\{|(x, k)| < R_0\}$, it is easy to check that one can set $M = R_0 + 1 + R_0T + ||\nabla V(x)||_{L^\infty}T^2$. So now for (6.19) to hold it suffices that

$$\frac{2^{1+\theta}\left(1 + R_0 + R_0T + ||\nabla V(x)||_{L^\infty}T^2\right)^{1+\theta}10}{C\frac{\delta}{2}} R^{-3} \leqslant log(\varepsilon^{-1})$$

The constant C comes from the $O(\cdot)$ of equation (6.19); it is clear that it can be chosen so that the constraint finally becomes $R \geqslant (log(\varepsilon^{-1}))^{-\frac{1}{3}}$, which is satisfied by choosing

$$R = (log(\varepsilon^{-1}))^{-\frac{1}{3}}. \tag{6.26}$$

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