Software pipelining is a loop optimization that overlaps the execution of several iterations of a loop to expose more instruction-level parallelism. It can result in first-class performance characteristics, but at the cost of significant obfuscation of the code, making this optimization difficult to test and debug. In this paper, we present a translation validation algorithm that uses symbolic evaluation to detect semantics discrepancies between a loop and its pipelined version. Our algorithm can be implemented simply and efficiently, is provably sound, and appears to be complete with respect to most modulo scheduling algorithms. A conclusion of this case study is that it is possible and effective to use symbolic evaluation to reason about loop transformations.

Categories and Subject Descriptors D.2.4 [Software Engineering]: Software/Program Verification - Correctness proofs; D.3.4 [Programming Languages]: Processors - Optimization

General Terms Languages, Verification, Algorithms

Keywords Software pipelining, translation validation, symbolic evaluation, verified compilers

1. Introduction

There is one last technique in the arsenal of the software optimizer that may be used to make most machines run at tip top speed. It can also lead to severe code bloat and may make for almost unreadable code, so should be considered the last refuge of the truly desperate. However, its performance characteristics are in many cases unmatched by any other approach, so we cover it here. It is called software pipelining [...] Apple Developer Connection

Software pipelining is an advanced instruction scheduling optimization that exposes considerable instruction-level parallelism by overlapping the execution of several iterations of a loop. It produces smaller code and eliminates more pipeline stalls than merely unrolling the loop then performing acyclic scheduling. Software pipelining is implemented in many production compilers and described in several compiler textbooks [1, section 10.5] [2, chapter 20] [16, section 17.4]. In the words of the rather dramatic quote above, “truly desperate" programmers occasionally perform software pipelining by hand, especially on multimedia and signal processing kernels.

Starting in the 1980’s, many clever algorithms were designed to produce efficient software pipelines and implement them either on stock hardware or by taking advantage of special features such as those of the IA64 architecture. In this paper, we are not concerned about the performance characteristics of these algorithms, but rather about their semantic correctness: does the generated, software-pipelined code compute the same results as the original code? As with all advanced compiler optimizations, and perhaps even more so here, mistakes happen in the design and implementation of software pipelining algorithms, causing incorrect code to be generated from correct source programs. The extensive code rearrangement performed by software pipelining makes visual inspection of the generated code ineffective; the additional boundary conditions introduced make exhaustive testing difficult. For instance, after describing a particularly thorny issue, Rau et al. [23] note that...
proof was mechanized using the Coq proof assistant, and informally argued to be complete with respect to a wide class of software pipelining optimizations.

The formal verification of a validator consists in mechanically proving that it returns true, its two input programs do behave identically at run time. This is a worthwhile endeavor for two reasons. First, it brings additional confidence that the results of validation are trustworthy. Second, it provides an attractive alternative to formally verifying the soundness of the optimization algorithm itself; the validator is often smaller and easier to verify than the optimization it validates [26]. The validation algorithm presented in this paper grew out of the desire to add a software pipelining pass to the CompCert high-assurance C compiler [11]. Its formal verification in Coq is not entirely complete at the time of this writing, but appears within reach given the simplicity of the algorithm.

The remainder of this paper is organized as follows. Section 2 recalls the effect of software pipelining on the shape of loops. Section 3 outlines the basic principle of our validator. Section 4 defines the flavor of symbolic evaluation that it uses. Section 5 presents the validation algorithm. Its soundness is proved in section 6; its completion is the new loop body, also called the steady state. In other words, one iteration of S corresponds to δ iterations of the original loop B.

The prolog P is assumed to increment the loop index i by µ; the steady state S, by δ; and the epilogue E, not at all.

The software pipelining algorithms that we have in mind here are combinations of modulo scheduling [22, 7, 15, 23, 4] followed by modulo variable expansion [10]. However, the presentation above seems general enough to accommodate other scheduling algorithms.

Figure 1 illustrates one run of a software pipeliner. The schedule obtained by modulo scheduling performs, at each iteration, the δth store and increment of p, the (i + 1)th multiplication by c, and the (i + 2)th load. To avoid read-after-write hazards, modulo variable expansion was performed, unrolling the schedule by a factor of 2, and replacing variables p and x by two variables each, p1/p2 and x1/x2, which are defined in an alternating manner. These pairs of variables are initialized from p and x in the prolog; the epilogue sets p and x back to their final values. The unrolling factor δ is therefore 2. Likewise, the minimum number of iterations is µ = 2.

Step 3 Finally, the original loop is replaced by the following pseudo-code:

\[
i := 0;\]
\[\text{if } (N \geq \mu) \{\]
\[M := ((N - \mu)/\delta) \times \delta + \mu;\]
\[P\]
\[\text{while } (i < M) \{ S \} \]
\[E\]
\[\text{while } (i < N) \{ B \}\]

In CFG terms, this amounts to replacing the sub-graph shown at the top of figure 2 by the one shown at the bottom.

The effect of the code after pipelining is as follows. If the number of iterations N is less than µ, a copy of the original loop is executed. Otherwise, we execute the prolog P, then iterate the steady state S, then execute the epilogue E, and finally fall through the copy of the original loop. Since P and S morally correspond to µ and δ iterations of the original loop, respectively, S is iterated n times where n is the largest integer such that \(\mu + n\delta \leq N\), namely \(n = (N - \mu)/\delta\). In terms of the loop index i, this corresponds to the upper limit M shown in the pseudo-code above. The copy of the original loop executes the remaining \(N - M\) iterations.
takes as arguments the input \((i, N, B)\) and output \((P, S, E, \mu, \delta)\) of the software pipeliner (step 2). It is responsible for establishing conditions sufficient to prove semantic preservation between the original and transformed codes (top and bottom parts of figure 2).

What are these sufficient conditions and how can a validator establish them? To progress towards an answer, think of complete unrollings of the original and pipelined loops. We see that the run-time behavior of the original loop is equivalent to that of \(B^N\), that is, \(N\) copies of \(B\) executed in sequence. Likewise, the generated code behaves either like \(B^N\) if \(N < \mu\), or like \(P; S^{\infty}(N); E; B^{\infty}(N)\) if \(N \geq \mu\), where

\[
\kappa(N) \overset{\text{def}}{=} (N - \mu)/\delta
\]

\[
\rho(N) \overset{\text{def}}{=} N - \mu - \delta \times \kappa(N)
\]

The verification problem therefore reduces to establishing that, for all \(N\), the two basic blocks

\[
X_N \overset{\text{def}}{=} B^N \quad \text{and} \quad Y_N \overset{\text{def}}{=} P; S^{\infty}(N); E; B^{\infty}(N)
\]

are semantically equivalent: if both blocks are executed from the same initial state (same memory state, same values for variables), they terminate with the same final state (up to the values of temporary variables introduced during scheduling and unused later). For a given value of \(N\), symbolic evaluation provides a simple, effective way to ensure this semantic equivalence between \(X_N\) and \(Y_N\). In a nutshell, symbolic evaluation is a form of denotational semantics for basic blocks where the values of variables \(x, y, \ldots\) at the end of the block are determined as symbolic expressions of their values \(x_0, y_0, \ldots\) at the beginning of the block. For example, the symbolic evaluation of the block \(B\) is implied by (and, in practice, equivalent to) the following two decidable conditions:

\[
\alpha(X_N) = \alpha(Y_N)\]

The fundamental property of symbolic evaluation is that if two blocks have identical symbolic evaluations, whenever they are executed in the same but arbitrary initial state, they terminate in the same final state. Moreover, symbolic evaluation is insensitive to reordering of independent instructions, making it highly suitable to reason over scheduling optimizations [26]. Therefore, for any given \(N\), our validator can check that \(\alpha(X_N) = \alpha(Y_N)\); this suffices to guarantee that \(X_N\) and \(Y_N\) are semantically equivalent.

If \(N\) were a compile-time constant, this would provide us with a validation algorithm. However, \(N\) is in general a run-time quantity, and statically checking \(\alpha(X_N) = \alpha(Y_N)\) for all \(N\) is not decidable. The key discovery of this paper is that this undecidable property \(\forall N, \alpha(X_N) = \alpha(Y_N)\) is implied by (and, in practice, equivalent to) the following two decidable conditions:

\[
\alpha(E; B^\delta) = \alpha(S; E) \quad \text{(1)}
\]

\[
\alpha(B^\delta) = \alpha(P; E) \quad \text{(2)}
\]

The programmer’s intuition behind condition 1 is the following: in a correct software pipeline, if enough iterations remain to be performed, it should always be possible to choose between 1) execute the steady state \(S\) one more time, then leave the pipelined loop and execute \(E\); or 2) leave the pipelined loop immediately by executing \(E\), then run \(\delta\) iterations of the original loop body \(B\). (See figure 3.) Condition 2 is even easier to justify: if the number

![Figure 2. The effect of software pipelining on a control-flow graph](image-url)
Figure 3. A view of an execution of the loop, before (top) and after (bottom) pipelining. The dashed vertical arrows provide intuitions for equations 1 and 2.

of executions is exactly \( \mu \), the original code performs \( B_v \) and the pipelined code performs \( P; E \).

The translation validation algorithm that we define in section 5 is based on checking minor variations of conditions 1 and 2, along with some additional conditions on the evolutions of variable \( i \) (also checked using symbolic evaluation). Before presenting this algorithm, proving that it is sound, and arguing that it is complete in practice, we first need to define symbolic evaluation more precisely and study its algebraic structure.

4. Symbolic evaluation with observables

In this section, we formally define the form of symbolic evaluation used in our validator. Similar forms of symbolic evaluation are at the core of several other translation validators, such as those of Necula [18], Rival [24], and Tristan and Leroy [26].

4.1 The intermediate language

We perform symbolic evaluation over the following simple language of basic blocks:

- Non-branching instructions:
  \[
  I := r := r' \quad \text{variable-variable move}
  \]
  \[
  r := \text{op}(op, r) \quad \text{arithmetic operation}
  \]
  \[
  r := \text{load}(\kappa, t_{addr}) \quad \text{memory load}
  \]
  \[
  \text{store}(\kappa, t_{addr}, t_{val}) \quad \text{memory store}
  \]

- Basic blocks:
  \[
  B := I_1; \ldots; I_n \quad \text{instruction sequences}
  \]

Here, \( r \) ranges over variable names (a.k.a. pseudo-registers or temporaries); \( op \) ranges over arithmetic operators (such as “load integer constant \( n \)” or “add integer immediate \( n \)” or “float multiply”); and \( \kappa \) stands for a memory quantity (such as “8-bit signed integer” or “64-bit float”).

4.2 Symbolic states

The symbolic evaluation of a block produces a pair \((\varphi, \sigma)\) of a mapping \( \varphi \) from resources to terms, capturing the relationship between the initial and final values of resources, and a set of terms \( \sigma \), recording the computations performed by the block.

Resources:
\[
\rho := r \mid \text{Mem}
\]

Terms:
\[
t := t_0 \quad \text{initial value of} \ \rho
\]
\[
\text{Op}(op, t) \quad \text{result of an operation}
\]
\[
\text{Load}(\kappa, t_{addr}, t_m) \quad \text{result of a load}
\]
\[
\text{Store}(\kappa, t_{addr}, t_{val}, t_m) \quad \text{effect of a store}
\]

Resource maps:
\[
\varphi := \rho \mapsto t
\]

Computations performed:
\[
\sigma := \{t_1; \ldots; t_n\} \quad \text{finite set}
\]

Symbolic states:
\[
A := (\varphi, \sigma)
\]

Resource maps symbolically track the values of variables and the memory state, represented like a ghost variable \( \rho \). A resource \( \rho \) that is not in the domain of a map \( \varphi \) is considered mapped to \( \rho_0 \). Memory states are represented by terms \( t_m \) built out of \( \text{Mem}_0 \) and \( \text{Store}(\ldots) \) constructors. For example, the following block
\[
\text{store}(\kappa, x, y); \quad z := \text{load}(\kappa, w)
\]
symbolically evaluates to the following resource map:
\[
\text{Mem} \mapsto \text{Store}(\kappa, x_0, y_0, \text{Mem}_0)
\]
\[
z \mapsto \text{Load}(\kappa, w_0, \text{Store}(\kappa, x_0, y_0, \text{Mem}_0))
\]

Resource maps describe the final state after execution of a basic block, but do not capture all intermediate operations performed. This is insufficient if some intermediate operations can fail at runtime (e.g. integer divisions by zero or loads from a null pointer). Consider for example the two blocks
\[
y := x \quad \text{and} \quad y := \text{load(int32, x)};
\]
\[
y := x
\]

They have the same resource map, namely \( y := x_0 \), yet the rightmost block right can cause a run-time error if the pointer \( x \) is null, while the leftmost block never fails. Therefore, resource maps do not suffice to check that two blocks have the same semantics; they must be complemented by sets \( \sigma \) of symbolic terms, recording all the computations that are performed by the blocks, even if these computations do not contribute directly to the final state. Continuing the example above, the leftmost block has \( \sigma = \emptyset \) (a move is not considered a computation, since it cannot fail) while the rightmost block has \( \sigma' = \{\text{Load(int32, x, Mem}_0)\} \). Since these two sets differ, symbolic evaluation reveals that the two blocks above are not semantically equivalent with respect to run-time errors.

4.3 Composition

A resource map \( \varphi \) can be viewed as a parallel substitution of symbolic terms for resources: the action of a map on a term \( t \) is the term \( \varphi(t) \) obtained by replacing all occurrences of \( \rho_0 \) in \( t \) by the term associated with \( \rho \) in \( \varphi \). Therefore, the reverse composition \( \varphi_1; \varphi_2 \) of two maps is, classically,
\[
\varphi_1; \varphi_2 \overset{\text{def}}{=} \{\rho \mapsto \varphi_1(\varphi_2(\rho)) \mid \rho \in \text{Dom}(\varphi_1) \cup \text{Dom}(\varphi_2)\}
\]

The reverse composition operator extends naturally to abstract states:
\[
(\varphi_1; \sigma_1); (\varphi_2; \sigma_2) \overset{\text{def}}{=} (\varphi_1; \varphi_2; \sigma_1 \cup \{\varphi_1(t) \mid t \in \sigma_2\})
\]

Composition is associative and admits the identity abstract state \( \varepsilon \overset{\text{def}}{=} (\emptyset, \emptyset) \) as neutral element.

4.4 Performing symbolic evaluation

The symbolic evaluation \( \alpha(I) \) of a single instruction \( I \) is the abstract state defined by
\[
\alpha(r := r') = (r \mapsto r', \emptyset)
\]
\[
\alpha(r := \text{op}(op, r)) = (r \mapsto t, \{t\})
\]
where \( t = \text{Op}(op, r_0) \)
\[
\alpha(r := \text{load}(\kappa, r')) = (r \mapsto t, \{t\})
\]
where \( t = \text{Load}(\kappa, r'_0, \text{Mem}_0) \)
\[
\alpha(\text{store}(\kappa, r, r')) = (\text{Mem} \mapsto t_m, \{t_m\})
\]
where \( t_m = \text{Store}(\kappa, r_0, r'_0, \text{Mem}_0) \)
Symbolic evaluation then extends to basic blocks, by composing the evaluations of individual instructions:
\[
\alpha(I_1; \ldots; I_n) \defeq \alpha(I_1) \ldots \alpha(I_n)
\] (5)
By associativity, it follows that symbolic evaluation distributes over concatenation of basic blocks:
\[
\alpha(B_1; B_2) = \alpha(B_1); \alpha(B_2)
\] (6)

4.5 Comparing symbolic states

We use two notions of equivalence between symbolic states. The first one, written \(\sim\), is defined as strict syntactic equality:
\[
(\varphi, \sigma) \sim (\varphi', \sigma') \text{ if and only if } \forall \rho, \varphi(\rho) = \varphi'(\rho) \text{ and } \sigma = \sigma'
\] (7)
The relation \(\sim\) is therefore an equivalence relation, and is compatible with composition:
\[
A_1 \sim A_2 \implies A_1; A \sim A_2; A
\] (8)
\[
A_1 \sim A_2 \implies A; A_1 \sim A; A_2
\] (9)

We need a coarser equivalence between symbolic states to account for the differences in variable usage between the code before and after software pipelining. Recall that pipelining may perform modulo variable expansion to avoid read-after-write hazards. This introduces fresh temporary variables in the pipelined code. These fresh variables show up in symbolic evaluations and prevent strict equivalence \(\sim\) from holding. Consider:
\[
x := \op(op, x) \quad \text{and} \quad x' := \op(op, x)
\]
These two blocks have different symbolic evaluations (in the sense of the \(\sim\) equivalence), yet should be considered as semantically equivalent if the temporary \(x'\) is unused later.

Let \(\theta\) be a finite set of variables: the observable variables. (In our validation algorithm, we take \(\theta\) to be the set of all variable mentioned in the original code before pipelining; the fresh temporaries introduced by modulo variable expansion are therefore not in \(\theta\).) Equivalence between symbolic states up to the observables \(\theta\) is written \(\approx_{\theta}\) and defined as
\[
(\varphi, \sigma) \approx_{\theta} (\varphi', \sigma') \text{ if and only if } \forall \rho \in \theta \cup \{\\text{mem}\}, \varphi(\rho) = \varphi'(\rho) \text{ and } \sigma = \sigma'
\] (10)
The relation \(\approx_{\theta}\) is an equivalence relation, coarser than \(\sim\), and compatible with composition on the right:
\[
A_1 \approx_{\theta} A_2 \implies A_1; A \approx_{\theta} A_2
\] (11)
\[
A_1 \approx_{\theta} A_2 \implies A; A_1 \approx_{\theta} A; A_2
\] (12)
However, it is not, in general, compatible with composition on the left. Consider an abstract state \(A\) that maps a variable \(x \in \theta\) to a term \(t\) containing an occurrence of \(y \notin \theta\). Further assume \(A_1 \approx_{\theta} A_2\). The composition \(A_1; A\) maps \(x\) to \(A_1(t)\); likewise, \(A_2; A\) maps \(x\) to \(A_2(t)\). Since \(y \notin \theta\), we can have \(A_1(y) \neq A_2(y)\) and therefore \(A_1(t) \neq A_2(t)\) without violating the hypothesis \(A_1 \approx_{\theta} A_2\). Hence, \(A_1; A \approx_{\theta} A_2; A\) does not hold in general.

To address this issue, we restrict ourselves to symbolic states \(A\) that are contained in the set \(\theta\) of observables. We say that a symbolic term \(t\) is contained in \(\theta\), and write \(t \subseteq \theta\), if all variables \(x\) appearing in \(t\) belong to \(\theta\). We extend this notion to symbolic states as follows:
\[
(\varphi, \sigma) \subseteq \theta \text{ if and only if } \forall \rho \in \theta \cup \{\\text{mem}\}, \varphi(\rho) \subseteq \theta \text{ and } \forall t \in \sigma, t \subseteq \theta
\] (13)
It is easy to see that \(\approx_{\theta}\) is compatible with composition on the left if the right symbolic state is contained in \(\theta\):
\[
A_1 \approx_{\theta} A_2 \wedge A \subseteq \theta \implies A_1; A \approx_{\theta} A_2; A
\] (14)
The containment relation enjoys additional useful properties:
\[
A_1 \subseteq \theta \wedge A_1 \approx_{\theta} A_2 \implies A_2 \subseteq \theta
\] (15)
\[
A_1 \subseteq \theta \wedge A_2 \subseteq \theta \implies (A_1; A_2) \subseteq \theta
\] (16)

4.6 Semantic soundness

The fundamental property of symbolic evaluation is that if \(\alpha(B_1) \approx_{\theta} \alpha(B_2)\), the two blocks \(B_1\) and \(B_2\) have the same run-time behavior, in a sense that we now make precise.

We equip our language of basic blocks with the straightforward operational semantics shown in figure 4. The behavior of arithmetic and memory operations is axiomatized as functions \(\oplus\), \(\load\) and \(\store\) that return “option” types to model run-time failures: the result \(\emptyset\) (pronounced “none”) denotes failure; the result \([x]\) (pronounced “some \(x\)”) denotes success. In our intended application, these interpretation functions are those of the RTL intermediate language of the CompCert compiler [12, section 6.1] and of its memory model [13].

The semantics for a basic block \(B\) is, then, given as the relation \(B : S \xrightarrow{\cdot} S'\), sometimes also written \(B : S \xrightarrow{\beta} S'\), where \(S\) is the initial state and \(S'\) the final state. States are pairs \((R, M)\) of a variable state \(R\), mapping variables to values, and a memory state \(M\).

We say that two execution states \(S = (R, M)\) and \(S' = (R', M')\) are equivalent up to the observables \(\theta\), and write \(S \approx_{\theta} S'\), if \(M = M'\) and \(R(r) = R'(r)\) for all \(r \in \theta\).
5. The validation algorithm

We can now piece together the intuitions from section 3 and the definitions from section 4 to obtain the following validation algorithm:

\[ \text{validate}(i, N, B)\ (P, S, \mathcal{E}, \mu, \delta) \theta = \]
\[ \alpha(B^i) \approx_\theta \alpha(P; \mathcal{E}) \quad (A) \]
\[ \wedge \alpha(\mathcal{E}; B^i) \approx_\theta \alpha(S; \mathcal{E}) \quad (B) \]
\[ \wedge \alpha(\mathcal{E}; \mu) \nleq \theta \quad (C) \]
\[ \wedge \alpha(B_{(i)}) = 0p(\text{add}i(1), i_0) \quad (D) \]
\[ \wedge \alpha(B)(i) = \alpha(B_{(i)}) \quad (E) \]
\[ \wedge \alpha(\mathcal{E})(i) = i_0 \quad (F) \]
\[ \wedge \alpha(\mathcal{B})(N) = \alpha(\mathcal{P})(N) = \alpha(S)(N) = \alpha(\mathcal{E})(N) = N_0 \quad (H) \]

The inputs of the validator are the components of the loops before \((i, N, B)\) and after \((P, S, \mathcal{E}, \mu, \delta)\) pipelining, as well as the set \(\theta\) of observables. Checks A and B correspond to equations (1) and (2) of section 3, properly reformulated in terms of equivalence up to observables. Check C makes sure that the set \(\theta\) of observables includes at least the variables appearing in the symbolic evaluation of the original loop body \(B\). Checks D to G ensure that the loop index \(i\) is incremented the way we expect it to be: by one in \(B\), by \(\mu\) in \(P\), by \(\delta\) in \(S\), and not at all in \(\mathcal{E}\). (We write \(\alpha(B)(i)\) to refer to the symbolic term associated to \(i\) by the resource map part of the symbolic state \(\alpha(B)\).) Finally, check H makes sure that the loop limit \(N\) is invariant in both loops.

The algorithm above has low computational complexity. Using a hash-consed representation of symbolic terms, the symbolic evaluation \(\alpha(B)\) of a block \(B\) containing \(n\) instructions can be computed in time \(O(n \log n)\): for each of the \(n\) instructions, the algorithm performs one update on the resource map, one insertion in the constraint set, and one hash-consing operation, all of which can be done in logarithmic time. (The resource map and the constraint set have sizes at most \(n\).) Likewise, the equivalence and containment relations (\(\approx_\theta\) and \(\subseteq\)) can be decided in time \(O(n \log n)\). Finally, note that this \(O(n \log n)\) complexity holds even if the algorithm is implemented within the Coq specification language, using persistent, purely functional data structures.

The running time of the validation algorithm is therefore \(O(n \log n)\) where \(n = \max(|B|, \delta|B|, |P|, |S|, |\mathcal{E}|)\) is proportional to the size of the loop after pipelining. Theoretically, software pipelining can increase the size of the loop quadratically. In practice, we expect the external implementation of software pipelining to avoid this blowup and produce code whose size is linear in that of the original loop. In this case, the validator runs in time \(O(n \log n)\) where \(n\) is the size of the original loop.

6. Soundness proof

In this section, we prove the soundness of the validator: if it returns “true”, the software pipelined loop executes exactly like the original loop. The proof proceeds in two steps: first, we show that the basic blocks \(X_N\) and \(Y_N\) obtained by unrolling the two loops have the same symbolic evaluation and therefore execute identically; then, we prove that the concrete execution of the two loops in the CFG match.

6.1 Soundness with respect to the unrolled loops

We first show that the finitary conditions checked by the validator imply the infinitary equivalences between the symbolic evaluations of \(X_N\) and \(Y_N\) sketched in section 3.

Lemma 2. If validate \((i, N, B)\ (P, S, \mathcal{E}, \mu, \delta) \theta\) returns true, then, for all \(n\),
\[
\alpha(B^\mu+n\delta) \approx_\theta \alpha(P; S^\mu; \mathcal{E}) \quad (17) \\
\alpha(P; S^\mu; \mathcal{E}) \nleq \theta \quad (18) \\
\alpha(B^\mu+n\delta)(i) = \alpha(P; S^\mu)(i) = \alpha(P; S^\mu; \mathcal{E})(i) \quad (19)
\]

Proof: By hypothesis, we know that the properties A to H checked by the validator hold. First note that
\[
\forall n, \alpha(B^n) \nleq \theta \quad (20)
\]
as a consequence of check C and property (16). Conclusion (17) is proved by induction on \(n\). The base case \(n = 0\) reduces to \(\alpha(B^0) \approx_\theta \alpha(P; \mathcal{E})\), which is ensured by check A. For the inductive case, assume \(\alpha(B^{\mu+(n-1)\delta}) \approx_\theta \alpha(P; S^\mu; \mathcal{E})\).

Conclusion (18) follows from (17), (20), and property (15). Conclusion (19) follows from checks E, F and G by induction on \(n\). \(\square\)

Theorem 3. Assume that validate \((i, N, B)\ (P, S, \mathcal{E}, \mu, \delta) \theta\) returns true. Consider an execution \(B^N: S \nleq S'\) of the unrolled initial loop, with \(N \geq \mu\). Assume \(S \equiv_\theta T\). Then, there exists a concrete state \(T'\) such that
\[
\begin{align*}
P; S^N; \mathcal{E}; B^N: T & \nleq T' \quad (21) \\
S' & \equiv_\theta T' \quad (22)
\end{align*}
\]
Moreover, execution (21) decomposes as follows:
\[
T \nleq T_0 \nleq \cdots \nleq T_{\kappa(N)} \nleq T_{\rho(N)} \nleq T' \quad (23)
\]
and the values of the loop index \(i\) in the various intermediate states satisfy
\[
\begin{align*}
T_i(i) = S(i) + \mu + \delta & \pmod{2^{32}} \quad (24) \\
T'_i(i) = S(i) + \mu + \delta \times \kappa(N) + j & \pmod{2^{32}} \quad (25)
\end{align*}
\]

Proof: As a corollary of Lemma 2, properties (17) and (18), we obtain
\[
\begin{align*}
\alpha(B^N) & \approx_\theta \alpha(P; S^N; \mathcal{E}; B^N) \quad (26) \\
\alpha(P; S^N; \mathcal{E}; B^N) & \nleq \theta \quad (27)
\end{align*}
\]
since \(N = \mu + \delta \times \kappa(N) + \rho(N)\). The existence of \(T'\) then follows from Theorem 1.

It is obvious that the concrete execution of the pipelined code can be decomposed as shown in (23). Check D in the validator...
The operational semantics (of figure 2. (Note that figure 2 is not a specific example: it describes instruction can be either a non-branching instruction S or a memory state M.

Our final objective is to perform instruction scheduling on the original loop before pipelining, depicted in part (a) of figure 2, and g′ for the loop after pipelining, depicted in part (b) of figure 2. (Note that figure 2 is not a specific example: it describes the general shape of CFGs accepted and produced by all software pipelining optimizations we consider in this paper.) We now use lemma 4 to relate a CFG execution of the original loop g with the basic-block execution of its unrolling.

**Lemma 5.** Assume α(B)(i) = 0p(addi(1), i0) and α(B)(N) = N0. Consider an execution g : (in, S) → (out, S′) from point in to point out in the original CFG, with S(1) = 0. Then, B′′ : S → S′ where N is the value of variable N in state S.

Proof: The CFG execution can be decomposed as follows:

\[(in, S_0) → (x, S_0) ∼ (in, S_1) → ... → (in, S_n) → (out, S_n)\]

with S = S_0 and S′ = S_n and S_n(i) ≥ S_n(N) and S_j(i) < S_j(N) for all j < n. By construction of B, we have g, B : x ∼ in. Therefore, by Lemma 4, B : S_j → S_{j+1} for j = 0, ..., n − 1. It follows that B′′ : S → S′. Moreover, at each loop iteration, variable N is unchanged and variable i is incremented by 1. It follows that the number n of iterations is equal to the value N of variable N in initial state S. This is the expected result.

Symmetrically, we can reconstruct a CFG execution of the pipelined loop g′ from the basic-block execution of its unrolling.

**Lemma 6.** Let T_{init} be an initial state where variable i has value 0 and variable N has value N. Let T be T_{init} where variable M is set to μ + δ × N. Assume that properties (23), (24) and (25) hold, as well as check H. We can, then, construct an execution g′ : (in, T_{init}) → (out, T′) from point in to point out in the CFG g′ after pipelining.

Proof: By construction of g′, we have g′, P : c ∼ d and g′, S : f ∼ d and g′, E : e ∼ g and g′, B : h ∼ g. The result is obvious if N < μ. Otherwise, applying Lemma 4 to the basic-block executions given by property (23), we obtain the following CFG executions:

\[g′ : (in, T_{init}) → (c, T)\]
\[g′ : (c, T) → (d, T_0)\]
\[g′ : (f, T_j) → (d, T_{j+1})\] for j = 0, ..., N − 1
\[g′ : (e, T_{N(N)}) → (g, T_0)\]

\[g′ : (h, T_j) → (g, T_{j+1})\] for j = 0, ..., N − 1

The expected result follows.

Piecing everything together, we obtain the desired semantic preservation result.

**Theorem 7.** Assume that validate (i, N, B) (P, S, E, μ, δ, θ) returns true. Let g and g′ be the CFG before and after pipelining, respectively. Consider an execution g : (in, S) → (out, S′) with S(1) = 0. Then, for all states T such that S ≥_θ T, there exists a state T′ such that g′ : (in, T) → (out, T′) and S′ ≥_θ T′.

Proof: Follows from Theorem 3 and Lemmas 5 and 6.

7. Discussion of completeness

Soundness (rejecting all incorrect code transformations) is the fundamental property of a validator; but relative completeness (not rejecting valid code transformations) is important in practice, since...
a validator that reports many false positives is useless. Semantic equivalence between two arbitrary pieces of code being undecidable, a general-purpose validation algorithm cannot be sound and complete at the same time. However, a specialized validator can still be sound and complete for a specific class of program transformations—in our case, software pipelining optimizations based on modulo scheduling and modulo variable expansion. In this section, we informally discuss the sources of potential incompleteness for our validator.

Mismatch between infinitary and finitary symbolic evaluation A first source of potential incompleteness is the reduction of the infinitary condition \( \forall N, \alpha(X_N) = \alpha(Y_N) \) to the two finitary checks A and B. More precisely, the soundness proof of section 6 hinges on the following implication being true:

\[
(A) \land (B) \implies (17)
\]
or in other words,

\[
\alpha(B^n) \approx_\theta \alpha(P; E) \land \alpha(E; B^i) \approx_\theta \alpha(S; E) \implies \forall n, \alpha(B^{n+i}) \approx_\theta \alpha(P; S^n; E)
\]

However, the reverse implication does not hold in general. Assume that (17) holds. Taking \( n = 0 \), we obtain equation (A). However, we cannot prove (B). Exploiting (17) for \( n \) and \( n + 1 \) iterations, we obtain:

\[
\alpha(B^{n+1}) \approx_\theta \alpha(P; S^n) \land \alpha(E; B^i) \approx_\theta \alpha(S; E)
\]

Combining these two equivalences, it follows that

\[
\alpha(P; S^n) \land \alpha(E; B^i) \approx_\theta \alpha(P; S^n) \land \alpha(S; E) \tag{26}
\]

However, we cannot just “simplify on the left” and conclude \( \alpha(E; B^i) \approx_\theta \alpha(S; E) \). To see why such simplifications are incorrect, consider the two blocks

\[
x := 1; \quad \text{and} \quad x := 1;
\]

\[
y := 1 \quad \text{and} \quad y := x
\]

We have \( \alpha(x := 1); \alpha(y := 1) \approx_\theta \alpha(x := 1); \alpha(y := x) \) with \( \theta = (x, y) \). However, the equivalence \( \alpha(y := 1) \approx_\theta \alpha(y := x) \) does not hold.

Coming back to equation (26), the blocks \( E; B^i \) and \( S; E \) could make use of some property of the state set up by \( P; S^n \) (such as the fact that \( x = 1 \) in the simplified example above). However, (26) holds for any number \( n \) of iteration. Therefore, this hypothetical property of the state that invalidates (B) must be a loop invariant. We are not aware of any software pipelining algorithm that takes advantage of loop invariants such as \( x = 1 \) in the simplified example above.

In conclusion, the reverse implication (17) \( \implies (A) \land (B) \), which is necessary for completeness, is not true in general, but we strongly believe it holds in practice as long as the software pipelining algorithm used is purely syntactic and does not exploit loop invariants.

Mismatch between symbolic evaluation and concrete executions A second, more serious source of incompleteness is the following: equivalence between the symbolic evaluations of two basic blocks is a sufficient, but not necessary condition for those two blocks to execute identically. Here are some counter-examples.

1. The two blocks could execute identically because of some property of the initial state, as in

\[
y := 1 \quad \text{vs.} \quad y := x
\]

assuming \( x = 1 \) initially.

2. Symbolic evaluation fails to account for some algebraic properties of arithmetic operators:

\[
i := i + 1; \quad \text{vs.} \quad i := i + 2
\]

3. Symbolic evaluation fails to account for memory separation properties within the same memory block:

\[
\begin{align*}
\text{store(int32, p + 4, y)}; & \quad \text{vs.} \quad x := \text{load(int32, p)}; \\
\text{store(int32, p + 4, y)} & \quad \text{vs.} \quad x := \text{load(int32, p)}
\end{align*}
\]

4. Symbolic evaluation fails to account for memory separation properties within different memory blocks:

\[
\begin{align*}
\text{store(int32, a, y)}; & \quad \text{vs.} \quad x := \text{load(int32, b)}; \\
\text{store(int32, a, y)} & \quad \text{vs.} \quad x := \text{load(int32, b)}
\end{align*}
\]

assuming that \( a \) and \( b \) point to different arrays.

Mismatches caused by specific properties of the initial state, as in example 1 above, are not a concern for us: as previously argued, it is unlikely that the pipelining algorithm would take advantage of these properties. The other three examples are more problematical: software pipeliners do perform some algebraic simplifications (mostly, on additions occurring within address computations and loop index updates) and do take advantage of nonaliasing properties to hoist loads above stores.

Symbolic evaluation can be enhanced in two ways to address these issues. The first way, pioneered by Necula [18], consists in comparing symbolic terms and states modulo an *equational theory* capturing the algebraic identities and “good variable” properties of interest. For example, the following equational theory addresses the issues illustrated in examples 2 and 3:

\[
\begin{align*}
\text{Op(addi}(n), \text{Op(addi}(m), t)) & \equiv \text{Op(addi}(n + m), t) \\
\text{Load}(\kappa', t_1, \text{Store}(\kappa, t_2, t_3, t_4)) & \equiv \text{Load}(\kappa', t_2, t_3, t_4) \\
\text{if } t_2 - t_1 \geq \text{sizeof}(\kappa)
\end{align*}
\]

The separation condition between the symbolic addresses \( t_1 \) and \( t_2 \) can be statically approximated using a decision procedure such as Pugh’s Omega test [21]. In practice, a coarse approximation, restricted to the case where \( t_1 \) and \( t_2 \) differ by a compile-time constant, should work well for software pipelining.

A different extension of symbolic evaluation is needed to account for nonaliasing properties between separate arrays or memory blocks, as in example 4. Assume that these nonaliasing properties are represented by region annotations on load and store instructions. We can, then, replace the single resource \( \text{Mem} \) that symbolically represent the memory state by a family of resources \( \text{Mem}^a \) indexed by region identifiers \( a \). A load or store in region \( x \) is, then, symbolically evaluated in terms of the resource \( \text{Mem}^a \). Continuing example 4 above and assuming that the store takes place in region \( a \) and the load in region \( b \), both blocks symbolically evaluate to

\[
\begin{align*}
\text{Mem}^a & \mapsto \text{Store(int32, a_0, y_0, \text{Mem}^a_0)} \\
x & \mapsto \text{Load(int32, b_0, \text{Mem}^a_0)}
\end{align*}
\]

8. Implementation and preliminary experiments

The first author implemented the validation algorithm presented here as well as a reference software pipeliner. Both components are implemented in OCaml and were connected with the CompCert C compiler for testing.

The software pipeliner accounts for approximately 2000 lines of OCaml code. It uses a backtracking iterative modulo scheduler [2, chapter 20] to produce a steady state. Modulo variable expansion is then performed following Lam’s approach [10]. The modulo scheduler uses a heuristic function to decide the order in which
to consider the instructions for scheduling. We tested our software pipeliner with two such heuristics. The first one randomly picks the nodes. We use it principally to stress the pipeliner and the validator. The second one picks the node using classical dependencies constraints. We also made a few schedules by hand.

We experimented our pipeliner on the CompCert benchmark suite, which contains, among others, some numerical kernels such as a fast Fourier transform. On these examples, the pipeliner performs significant code rearrangements, although the observed speedups are modest. There are several reasons for this: we do not exploit the results of an alias analysis; our heuristics are not state of the art; and we ran our programs on an out-of-order PowerPC G5 processor, while software pipelining is most effective on in-order processors.

The implementation of the validator follows very closely the algorithm presented in this paper. It accounts for approximately 300 lines of OCaml code. The validator is instrumented to produce a lot of debug information, including in the form of drawings of symbolic states. The validator found many bugs during the development of the software pipeliner, especially in the implementation of modulo variable expansion. The debugging information produced by the validator was an invaluable tool to understand and correct these mistakes. Figure 6 shows the diagrams produced by our validator for one of these bugs. We would have been unable to get the pipeliner right without the help of the validator.

On our experiments, the validator reported no false alarms. The running time of the validator is negligible compared with that of the software pipeliner itself (less than 5%). However, our software pipeliner is not state-of-the-art concerning the determination of the minimal initiation interval, and therefore is relatively costly in terms of compilation time.

Concerning formal verification, the underlying theory of symbolic evaluation (section 4) and the soundness proof with respect to unrolled loops (section 6.1) were mechanized using the Coq proof assistant. The Coq proof is not small (approximately 3500 lines) but is relatively straightforward: the mechanization raised no unexpected difficulties. The only part of the soundness proof that we have not mechanized yet is the equivalence between unrolled loops and their CFG representations (section 6.2): for such an intuitively obvious result, a mechanized proof is infuriatingly difficult. Simply proving that the situation depicted in figure 2(a) holds (we have a proper, counted loop whose body is the basic block B) takes a great deal of bureaucratic formalization. We suspect that the CFG representation is not appropriate for this kind of proofs; see section 10 for a discussion.

9. Related work

This work is not the first attempt to validate software pipelining a posteriori. Leviathan and Pnueli [14] present a validator for a software pipeliner for the IA-64 architecture. The pipeliner makes use of a rotating register file and predicate registers, but from a validation point of view it is almost the same problem as the one studied in this paper. Using symbolic evaluation, the validator generates a set of verification conditions that are discharged by a theorem prover. We chose to go further with symbolic evaluation: instead of using it to generate verification conditions, we use it to directly establish semantic preservation. We believe that the resulting validator is simpler and algorithmically more efficient. However, exploiting nonaliasing information during validation could possibly be easier in Leviathan and Pnueli’s approach than in our approach.

Another attempt at validating software pipelining is the one of Kundu et al. [9]. It proceeds by parametrized translation validation. The software pipelining algorithm they consider is based on code motion, a rarely-used approach very different from the modulo scheduling algorithms we consider. This change of algorithms makes the validation problem rather different. In the case of Kundu et al., the software pipeliner proceeds by repeated rewriting of the original loop, with each rewriting step being validated. In contrast, modulo scheduling builds a pipelined loop in one shot (usually using a backtracking algorithm). It could possibly be viewed as a sequence of rewrites but this would be less efficient than our solution.

Another approach to establishing trust in software pipelining is run-time validation, as proposed by Goldberg et al. [5]. Their approach takes advantages of the IA-64 architecture to instrument the pipelined loop with run-time check that verify properties of the pipelined loop and recover from unexpected problems. This technique was used to verify that aliasing is not violated by instruction switching, but they did not verify full semantic preservation.

10. Conclusions

Software pipelining is one of the pinnacles of compiler optimizations; yet, its semantic correctness boils down to two simple semantic equivalence properties between basic blocks. Building on this novel insight, we presented a validation algorithm that is simple enough to be used as a debugging tool when implementing software pipelining, efficient enough to be integrated in production compilers and executed at every compilation run, and mathematically elegant enough to lend itself to formal verification.

Despite its respectable age, symbolic evaluation—the key technique behind our validator—finds here an unexpected application
to the verification of a loop transformation. Several known extensions of symbolic evaluation can be integrated to enhance the precision of the validator. One, discussed in section 7, is the addition of an equational theory and of region annotations to take nonaliasing information into account. Another extension would be to perform symbolic evaluation over extended basic blocks (acyclic regions of the CFG) instead of basic blocks, like Rival [24] and Tristan and Leroy [26] do. This extension could make it possible to validate pipelined loops that contain predicated instructions or even conditional branches.

From a formal verification standpoint, symbolic evaluation lends itself well to mechanization and gives a pleasant denotational flavor to proofs of semantic equivalence. This stands in sharp contrast with the last step of our proof (section 6.2), where the low-level operational nature of CFG semantics comes back to haunt us. We are led to suspect that control-flow graphs are a poor representation to reason over loop transformations, and to wish for an alternate representation that would abstract the syntactic details of loops just as symbolic evaluation abstracts the syntactic details of (extended) basic blocks. Two promising candidates are the PEG representation of Tate et al. [25] and the representation as sets of mutually recursive HOL functions of Myreen and Gordon [17]. It would be interesting to reformulate software pipelining validation in terms of these representations.

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