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Skin Effect in Electromagnetism and Asymptotic Behaviour of Skin Depth for High Conductivity

VICTOR PÉRON

Projet MC2

INRIA Bordeaux Sud-Ouest

GT HMS, Université Paris 6

25.01.2010

Motivation

Skin effect in a half-space

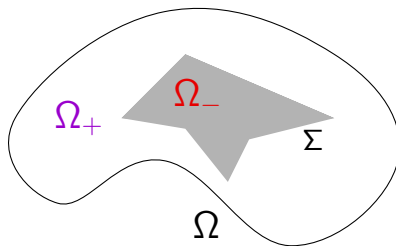
Skin Effect : in $\{z > 0\}$,

$$\|\mathbf{E}_\sigma(z)\| = E_0 e^{-z/\ell(\sigma)}$$

Skin Depth :

$$\ell(\sigma) = \sqrt{2/\omega\mu_0\sigma}$$

The Model Problem



- Ω_- Highly Conducting body $\subset\subset \Omega$: Conductivity $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+ Insulating or Dielectric body: Conductivity $\sigma_+ = 0$

Aim : Understand the Behavior of the Electromagnetic Field as $\sigma \rightarrow \infty$

Tools for the Asymptotic Analysis

- Tensor Calculus & Differential Geometry
- Asymptotic Expansions: *Scaling* Technique. *Boundary Layer* Expansion
- PDE Analysis : *Uniform Estimates*
- Numerical Simulations : the Asymptotic solution Procedure yields a *Fast and Efficient method*

References



E.P. STEPHAN, R.C. McCAMY (83-84-85)

Plane Interface and Eddy Current Problems



H. HADDAR, P. JOLY, H.N. NGUYEN (08)

Smooth Interface and Generalized Impedance Boundary Conditions



V. PÉRON (09)

PhD : Smooth and Polyhedral Interfaces



M. DAUGE, E. FAOU, V. PÉRON (10)

Asymptotic Behavior for High Conductivity of the Skin Depth



G. CALOZ, M. DAUGE, V. PÉRON (09)

Uniform Estimates for Transmission Problems with High Contrast in Heat Conduction and Electromagnetism

Outline

- 1 Uniform Estimates for Interface Problems
- 2 3D Multiscaled Asymptotic Expansion
- 3 Numerical Simulations

Outline

1 Uniform Estimates for Interface Problems

2 3D Multiscaled Asymptotic Expansion

3 Numerical Simulations

Maxwell Problem

Issue and Framework

$$(\mathbf{P}_{\underline{\sigma}}) \quad \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma}) \mathbf{E} = \mathbf{j} \quad \underline{\sigma} = (0, \sigma)$$

Issue: *Uniform* L^2 estimates for solutions (\mathbf{E}, \mathbf{H}) of $(\mathbf{P}_{\underline{\sigma}})$ as $\sigma \rightarrow \infty$?

Maxwell Problem

Issue and Framework

$$(\mathbf{P}_{\underline{\sigma}}) \quad \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma}) \mathbf{E} = \mathbf{j} \quad \underline{\sigma} = (0, \sigma)$$

Issue: *Uniform* L^2 estimates for solutions (\mathbf{E}, \mathbf{H}) of $(\mathbf{P}_{\underline{\sigma}})$ as $\sigma \rightarrow \infty$?

Hypothesis

Σ is a bounded Lipschitz surface in \mathbb{R}^3

Hypothesis (SH)

The angular frequency ω is not an eigenfrequency of the problem

$$\left\{ \begin{array}{ll} \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + i\omega\varepsilon_0 \mathbf{E} = 0 & \text{in } \Omega_+ \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \Sigma \\ \text{B.C.} & \text{on } \partial\Omega \end{array} \right.$$

Maxwell Problem

Uniform $L^2(\Omega)$ Estimates

Boundary Conditions : $\mathbf{E} \times \mathbf{n} = 0$ on $\partial\Omega$

$\mathbf{j} \in \mathbf{H}(\text{div}, \Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \text{div } \mathbf{u} \in L^2(\Omega)\}$

Theorem (CALOZ, DAUGE, P., 09)

Under Hypothesis (SH), there exist σ_0 and $C > 0$, such that for all $\sigma \geq \sigma_0$, $(\mathbf{P}_{\underline{\sigma}})$ with B.C. and $\mathbf{j} \in \mathbf{H}(\text{div}, \Omega)$ has a unique solution (\mathbf{E}, \mathbf{H}) in $L^2(\Omega)^2$, and

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma} \|\mathbf{E}\|_{0,\Omega_-} \leq C \|\mathbf{j}\|_{\mathbf{H}(\text{div},\Omega)}$$

Maxwell Problem

Uniform $L^2(\Omega)$ Estimates

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Application: Convergence of asymptotic expansion for high conductivity

Maxwell Problem

Proof of Uniform Estimates

Lemma

Under Hypothesis (SH), there are σ_0 and $C_0 > 0$ such that if $\sigma \geq \sigma_0$ any solution $(\mathbf{E}, \mathbf{H}) \in L^2(\Omega)^2$ of $(\mathbf{P}_{\underline{\sigma}})$ with B.C. and data $\mathbf{j} \in \mathbf{H}(\text{div}, \Omega)$ satisfies

$$\|\mathbf{E}\|_{0,\Omega} \leq C_0 \|\mathbf{j}\|_{\mathbf{H}(\text{div},\Omega)}$$

Maxwell Problem

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$$\|\mathbf{E}\|_{0,\Omega} \leq C_0 \|\mathbf{j}\|_{\mathbf{H}(\text{div},\Omega)}$$

Proof by contradiction: keys

- 1  C. AMROUCHE, C. BERNARDI, M. DAUGE, V. GIRALT (98)

Vector Potential technique: $\mathbf{E} =: \mathbf{w} + \nabla\varphi$ and $\text{div } \mathbf{w} = 0$

$$\text{curl } \mathbf{H} + \underbrace{(i\omega\varepsilon_0 - \underline{\sigma})}_{=: \underline{a}} (\mathbf{w} + \nabla\varphi) = \mathbf{j} \longrightarrow \text{div } \underline{a} \nabla\varphi = \text{div } \mathbf{j} - \underbrace{\text{div } \underline{a} \mathbf{w}}_{= [\underline{a}]_{\Sigma} \mathbf{w} \cdot \mathbf{n}} \delta_{\Sigma}$$

- 2 Issue: Uniform piecewise regularity for

$$\text{div } \underline{a} \text{grad } \varphi = f + [\underline{a}]_{\Sigma} g \quad \text{as} \quad |a_-/a_+| \rightarrow \infty ?$$

Uniform Estimates

Scalar Problem

$$(\mathbf{P}_{\underline{a}}) : \quad \operatorname{div} \underline{a} \operatorname{grad} \varphi = f + [\underline{a}]_{\Sigma} g \quad \text{as} \quad |\rho| := |\underline{a}_- / \underline{a}_+| \rightarrow \infty$$

Theorem (CALOZ, DAUGE, P., 09)

There exists $\rho_0 > 0$ independent of $\underline{a}_+ \neq 0$ s. t. for all $\underline{a}_- \in \{z \in \mathbb{C} \mid |z| \geq \rho_0 |\underline{a}_+|\}$, and for all $(f, g) \in L^2(\Omega) \times L^2(\Sigma)$ s.t. $\int_{\Sigma} g \, ds = 0$, $(\mathbf{P}_{\underline{a}})$ has a unique solution $\varphi \in H_0^1(\Omega)$, piecewise $H^{3/2}$ and

$$\|\varphi^+\|_{\frac{3}{2}, \Omega_+} + \|\varphi^-\|_{\frac{3}{2}, \Omega_-} \leq C_{\rho_0} (|\underline{a}_+|^{-1} \|f\|_{0, \Omega} + \|g\|_{0, \Sigma})$$

with $C_{\rho_0} > 0$, independent of \underline{a}_+ , \underline{a}_- , f , and g .

Uniform Estimates

Scalar Problem

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with $C_{\rho_0} > 0$, independent of \underline{a}_+ , \underline{a}_- , f , and g .

Key of the Proof:

$$\varphi^+ = \sum_{n=0}^{\infty} \varphi_n^+ \rho^{-n} \quad \Omega_+ \quad \text{and} \quad \varphi^- = \sum_{n=0}^{\infty} \varphi_n^- \rho^{-n} \quad \Omega_-$$

Remarks

- ❶ Similar estimate hold when a_+ and a_- are switched.
More precise estimate when a_+ and a_- play symmetric roles
- ❷ The assumption that Σ is Lipschitz is necessary
- ❸ For *polyhedral* Lipschitz interface Σ

$$\|\varphi^+\|_{s,\Omega_+} + \|\varphi^-\|_{s,\Omega_-} \leq C_{\rho_0} (|a_+|^{-1} \|f\|_{s-2,\Omega} + \|g\|_{s-\frac{3}{2},\Sigma})$$

for $s \leq s_\Sigma$ with some $\frac{3}{2} < s_\Sigma \leq 2$

- ❹ For *smooth* interface Σ , uniform piecewise H^s estimates for any $s \geq 2$

Outline

1 Uniform Estimates for Interface Problems

2 3D Multiscaled Asymptotic Expansion

3 Numerical Simulations

References and Notations

Asymptotic Expansion as $\sigma \rightarrow \infty$ of solutions of (\mathbf{P}_σ) when Σ is smooth :



STEPHAN, MCCAMY (83-84)

Plane Interface



HADDAR, JOLY, NGUYEN (08)



PÉRON (09)

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Hypothesis

- 1 Σ is a Smooth Surface
- 2 ω satisfies the Spectral Hypothesis (SH)
- 3 $\mathbf{j} \in \mathbf{H}(\text{div}, \Omega)$ and $\mathbf{j} = 0$ in Ω_-
- 4 Perfectly Conducting Electric B.C.

Asymptotic Expansion

$$\delta := \sqrt{\frac{\omega \varepsilon_0}{\sigma}} \longrightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty$$

By Theorem there exists δ_0 s.t. for all $\delta \leq \delta_0$, there exists a unique solution $(\mathbf{E}(\delta), \mathbf{H}(\delta))$ to (\mathbf{P}_σ)

$$\mathbf{E}^+_{(\delta)}(\mathbf{x}) \approx \sum_{j \geq 0} \delta^j \mathbf{E}_j^+(\mathbf{x}), \quad \mathbf{E}^-_{(\delta)}(\mathbf{x}) \approx \chi(y_3) \sum_{j \geq 0} \delta^j \mathbf{W}_j(y_\beta, \frac{y_3}{\delta})$$

(y_β, y_3) : “normal coordinates” to Σ in a tubular region \mathcal{U}_- of Σ in Ω_-

$$\mathbf{E}_j^+ \in \mathbf{H}(\text{curl}, \Omega_+) \quad , \quad \mathbf{W}_j \in \mathbf{H}(\text{curl}, \Sigma \times \mathbb{R}_+) \quad \text{profiles}$$

Asymptotic Expansion

$$\delta := \sqrt{\frac{\omega \varepsilon_0}{\sigma}} \longrightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty$$

By Theorem there exists δ_0 s.t. for all $\delta \leq \delta_0$, there exists a unique solution $(\mathbf{E}_{(\delta)}, \mathbf{H}_{(\delta)})$ to $(\mathbf{P}_{\underline{\sigma}})$

$$\mathbf{E}_{(\delta)}^+(\mathbf{x}) \approx \sum_{j \geq 0} \delta^j \mathbf{E}_j^+(\mathbf{x}), \quad \mathbf{E}_{(\delta)}^-(\mathbf{x}) \approx \chi(y_3) \sum_{j \geq 0} \delta^j \mathbf{W}_j(y_\beta, \frac{y_3}{\delta})$$

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Similar Expansion for the Magnetic Field $\mathbf{H}_{(\delta)}$

Validation of the Asymptotic Expansion

Remainders

Aim : proving Estimates for Remainders $\mathbf{R}_{m;\delta}$

$$\mathbf{R}_{m;\delta} := \mathbf{E}(\delta) - \sum_{j=0}^m \delta^j \mathbf{E}_j \quad \text{in } \Omega$$

Evaluation of the right hand side when the Maxwell operator is applied to $\mathbf{R}_{m;\delta}$

$$\left\{ \begin{array}{ll} \text{curl curl } \mathbf{R}_{m;\delta}^+ - \kappa^2 \alpha_+ \mathbf{R}_{m;\delta}^+ & = 0 \quad \text{in } \Omega_+ \\ \text{curl curl } \mathbf{R}_{m;\delta}^- - \kappa^2 \alpha_- \mathbf{R}_{m;\delta}^- & = \mathbf{j}_{m;\delta}^- \quad \text{in } \Omega_- \\ [\mathbf{R}_{m;\delta} \times \mathbf{n}]_{\Sigma} & = 0 \quad \text{on } \Sigma \\ [\text{curl } \mathbf{R}_{m;\delta} \times \mathbf{n}]_{\Sigma} & = \mathbf{g}_{m;\delta} \quad \text{on } \Sigma \\ \text{curl } \mathbf{R}_{m;\delta}^+ \times \mathbf{n} & = 0 \quad \text{on } \partial\Omega \end{array} \right.$$

with $\alpha_+ = 1$ and $\alpha_- = 1 + i \delta^{-2}$, and $\kappa = \omega \sqrt{\varepsilon_0 \mu_0}$

Validation of the asymptotic expansion

Estimates for Remainders

Estimates for the residues $\mathbf{j}_{m;\delta}^-$ and $\mathbf{g}_{m;\delta}$:

$$\|\mathbf{j}_{m;\delta}^-\|_{2,\Omega_-} + \|\mathbf{g}_{m;\delta}\|_{\frac{1}{2},\Sigma} + \|\operatorname{curl}_{\Sigma} \mathbf{g}_{m;\delta}\|_{\frac{3}{2},\Sigma} \leq C_m \delta^{m-1}$$

Validation of the asymptotic expansion

Estimates for Remainders

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Theorem (CALOZ, DAUGE, P., 09)

Under Hypothesis in the framework above, for all $m \in \mathbb{N}$ and $\delta \in (0, \delta_0]$, the remainders $\mathbf{R}_{m;\delta}$ satisfy the optimal estimates

$$\begin{aligned} \|\mathbf{R}_{m;\delta}^+\|_{0,\Omega_+} + \|\operatorname{curl} \mathbf{R}_{m;\delta}^+\|_{0,\Omega_+} \\ + \delta^{-\frac{1}{2}} \|\mathbf{R}_{m;\delta}^-\|_{0,\Omega_-} + \delta^{\frac{1}{2}} \|\operatorname{curl} \mathbf{R}_{m;\delta}^-\|_{0,\Omega_-} \leq C'_m \delta^{m+1} \end{aligned}$$

Profiles of the Magnetic Field

$\mathbf{V}_j =: (\mathcal{V}_j^\alpha ; v_j)$ in coordinates (y_β, Y_3) with $Y_3 = \frac{y_3}{\delta}$

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$$\mathbf{V}_0(y_\beta, Y_3) = \mathbf{h}_0(y_\beta) e^{-\lambda Y_3}$$

$$\mathcal{V}_1^\alpha(y_\beta, Y_3) = \left[h_1^\alpha + Y_3 \left(\mathcal{H} h_0^\alpha + b_\sigma^\alpha h_0^\sigma \right) \right] (y_\beta) e^{-\lambda Y_3}$$

with

\mathcal{H} mean curvature of Σ

$$\mathbf{h}_0(y_\beta) = (\mathbf{n} \times \mathbf{H}_0^+) \times \mathbf{n}(y_\beta, 0) \quad \text{and} \quad h_j^\alpha(y_\beta) := (\mathbf{H}_j^+)^\alpha(y_\beta, 0)$$

Profiles of the Magnetic Field

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Comparison with



HADDAR, JOLY, NGUYEN (08)

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Tool: a Normal Parameterization Technique

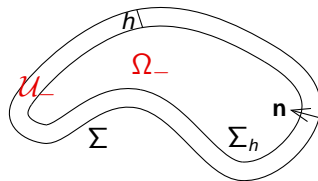


FAOU (02)

Elasticity on a thin shell

Curl Operator and Levi-Civita Tensor

$$h := y_3$$



Curl Operator in Normal Parameterization:

$$(\nabla \times \mathbf{E})^\alpha = \epsilon^{3\beta\alpha} (\partial_3^h E_\beta - \partial_\beta E_3) \quad \text{on } \Sigma_h$$

$$(\nabla \times \mathbf{E})^3 = \epsilon^{3\alpha\beta} D_\alpha^h E_\beta \quad \text{on } \Sigma_h$$

Levi-Civita Tensor ϵ :

$$\epsilon^{ijk} = \epsilon_0(i, j, k) / \sqrt{\det a_{\alpha\beta}(h)}$$

with $a_{\alpha\beta}(h) = \text{metric on } \Sigma_h$ and $\epsilon_0(i, j, k) \in \{0, 1, -1\}$

Covariant Derivative

$$D_\alpha^h E_\beta = \partial_\alpha E_\beta - \Gamma_{\alpha\beta}^\gamma(h) E_\gamma$$

Application 1

A New Definition of the Skin Depth

3D model : $\mathbf{V}_{(\delta)}(y_\alpha, y_3) := \mathbf{H}_{(\delta)}^-(\mathbf{x})$, $y_\alpha \in \Sigma$, $0 \leq y_3 < h_0$

Definition

Let Σ be a smooth surface, and \mathbf{j} s.t. $\mathbf{V}_{(\delta)}(y_\alpha, 0) \neq 0$. The skin depth is the smallest length $\mathcal{L}(\sigma, y_\alpha)$ defined on Σ s.t.

$$\|\mathbf{V}_{(\delta)}(y_\alpha, \mathcal{L}(\sigma, y_\alpha))\| = \|\mathbf{V}_{(\delta)}(y_\alpha, 0)\| e^{-1}$$

Theorem (DAUGE, FAOU, P., 10)

Let Σ be a regular surface with mean curvature \mathcal{H} , and assume $\mathbf{h}_0(y_\alpha) \neq 0$.

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma) \left(1 + \mathcal{H}(y_\alpha) \ell(\sigma) + \mathcal{O}(\sigma^{-1}) \right), \quad \sigma \rightarrow \infty$$

$$\|\mathbf{V}_{(\delta)}\|^2 = \left[\|\mathbf{h}_0\|^2 + 2\delta \operatorname{Re}\langle \mathbf{h}_0, \mathbf{h}_1 \rangle + 2y_3 \mathcal{H} \|\mathbf{h}_0\|^2 + \mathcal{O}((\delta + y_3)^2) \right] e^{-2y_3/\ell(\sigma)}$$

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Framework

- Magnetic Field \mathbf{H}_σ
- Axisymmetric Problem: Ω_- and Ω Axisymmetric Domains
- $\text{curl } \mathbf{j}$ Axisymmetric & Orthoradial $\longrightarrow \mathbf{H}_\sigma$ Axisymmetric & Orthoradial

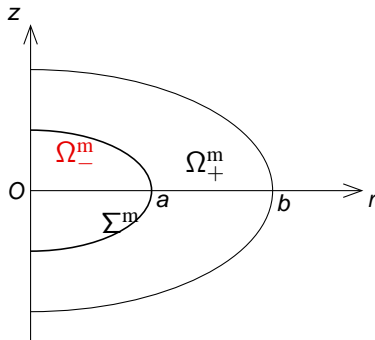
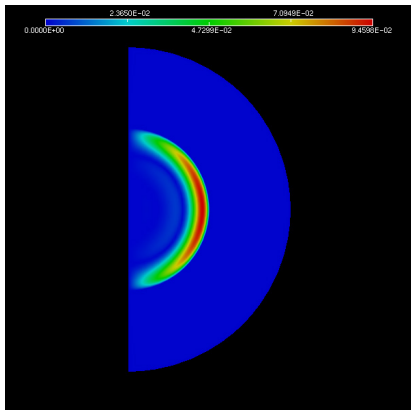


Figure: The meridian domain. Configuration A

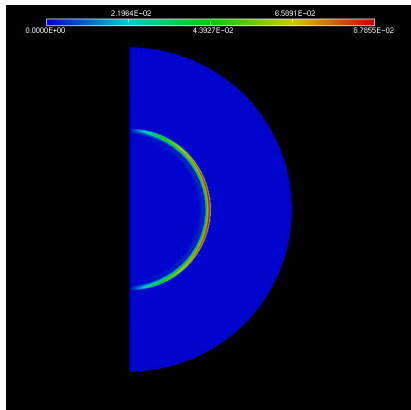
- FEM: Finite Element Library Melina

Spherical Geometry

$|\operatorname{Im} H_\sigma|$. Configuration A



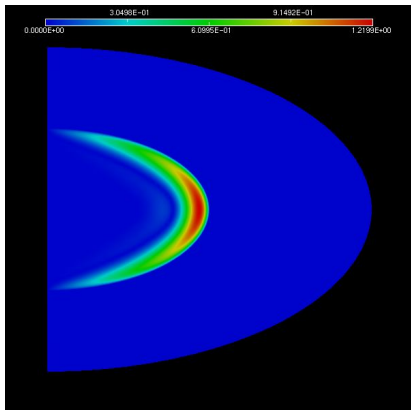
$|\operatorname{Im} H_\sigma|$ as $\sigma = 5 \text{ S.m}^{-1}$



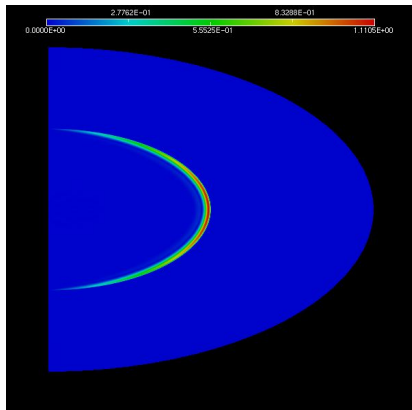
$|\operatorname{Im} H_\sigma|$ as $\sigma = 80 \text{ S.m}^{-1}$

Spheroidal Geometry

$|\operatorname{Im} H_\sigma|$. Configuration A



$|\operatorname{Im} H_\sigma|$ as $\sigma = 5 \text{ S.m}^{-1}$



$|\operatorname{Im} H_\sigma|$ as $\sigma = 80 \text{ S.m}^{-1}$

Influence of the Mean Curvature

Configuration B

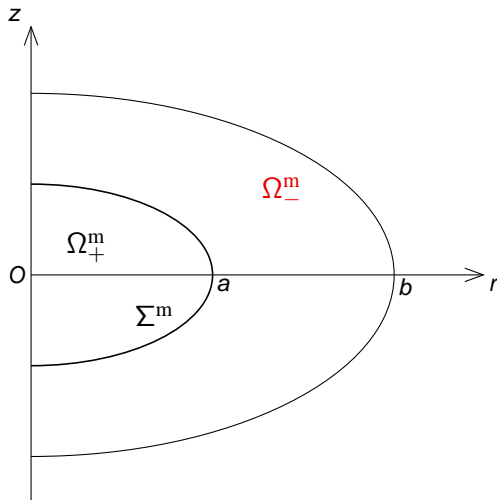
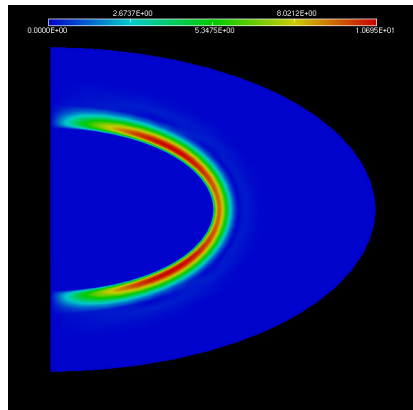
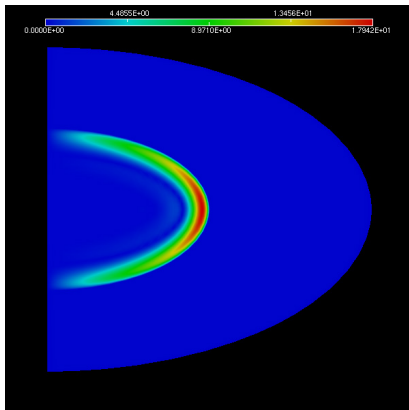


Figure: Configuration B. $\mathcal{H} < 0$

Influence of the Mean Curvature

$|\operatorname{Im} H_\sigma|$ when $\sigma = 5S.m^{-1}$



Numerical Post-Treatment

Linear Regression. Configuration A

Extract $|\mathbf{H}_\sigma(h)|$ along edges of the mesh in Ω_m when $z = 0$: $h = 2 - r$

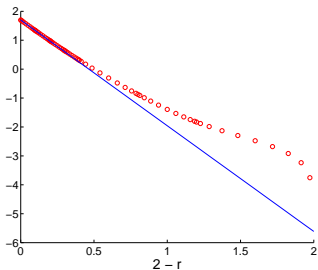
$$\log_{10} |\mathbf{H}_\sigma(h)| = -s(\sigma)h + \alpha$$

Numerical Post-Treatment

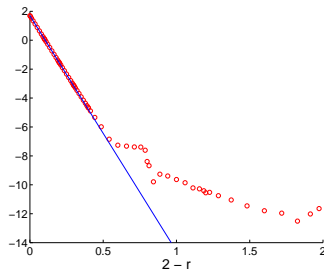
Linear Regression. Configuration A

Extract $|\mathbf{H}_\sigma(h)|$ along edges of the mesh in Ω^m when $z = 0$: $h = 2 - r$

$$\log_{10} |\mathbf{H}_\sigma(h)| = -s(\sigma)h + \alpha$$



○ : $\log_{10} |\mathbf{H}_\sigma(h)|$ when $\sigma = 5$
 $n(\sigma) = 7$, $s(5) = 3,65542$



○ : $\log_{10} |\mathbf{H}_\sigma(h)|$ when $\sigma = 80$
 $n(\sigma) = 3$, $s(80) = 16,279162$

Post-Treatment

Accuracy of Asymptotics

$$\log_{10} \|\mathbf{V}_{(\delta)}(y_\alpha, h)\| = - \underbrace{\frac{1}{\ln 10} \left(\frac{1}{\ell(\sigma)} - \mathcal{H}(y_\alpha) \right)}_{=: S(y_\alpha, \sigma)} h + \beta + \mathcal{O}(\delta) + \mathcal{O}((\delta+h)^2)$$

Relative Error between slopes:

$$\text{error}(\sigma) := \left| \frac{S(\sigma) - s(\sigma)}{S(\sigma)} \right|$$

Post-Treatment

Accuracy of Asymptotics

$$\log_{10} \|\mathbf{V}(\delta)(y_\alpha, h)\| = - \underbrace{\frac{1}{\ln 10} \left(\frac{1}{\ell(\sigma)} - \mathcal{H}(y_\alpha) \right)}_{=: S(y_\alpha, \sigma)} h + \beta + \mathcal{O}(\delta) + \mathcal{O}((\delta+h)^2)$$

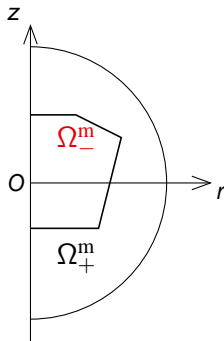
Relative Error between slopes:

$$\text{error}(\sigma) := \left| \frac{S(\sigma) - s(\sigma)}{S(\sigma)} \right|$$

σ	5	20
$\ell(\sigma)$ (cm)	10.3	5.15
$s(\sigma)$	3, 65542	7, 883903
$S(\sigma)$	3, 673319	7, 889506
$\text{error}(\sigma)$ (%)	0, 48	0, 07
$n(\sigma)$	7	5

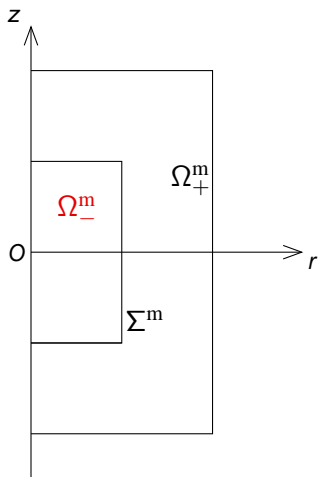
Conclusion

- Efficiency Methods \longrightarrow Skin Effect
- Solution Structure & Optimal Estimates of Remainders
- A Singular Transmission Problem: with C. Poignard and M. Dauge

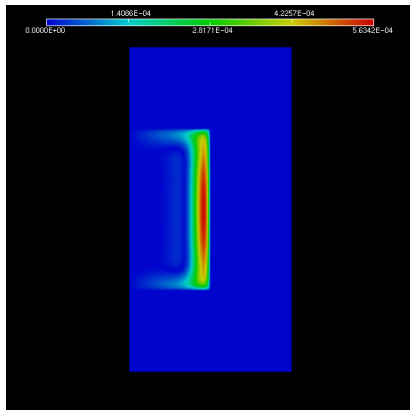
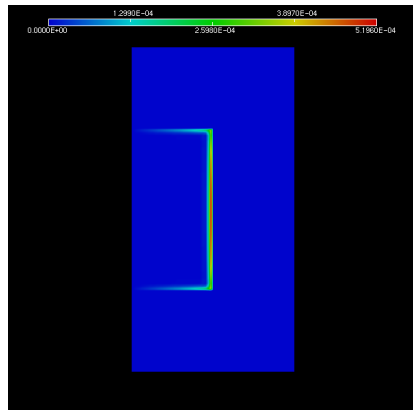


Cylindrical Geometry

Meridian domain Ω^m

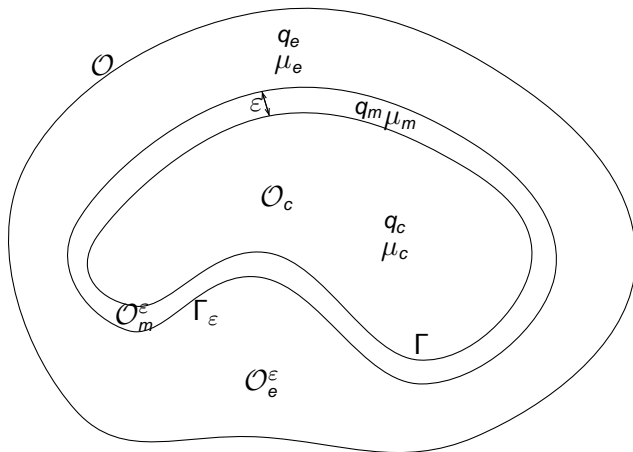


Cylindrical Geometry

 $|\operatorname{Im} H_\sigma|$

 $|\operatorname{Im} H_\sigma|$ as $\sigma = 5\text{S.m}^{-1}$

 $|\operatorname{Im} H_\sigma|$ as $\sigma = 80\text{S.m}^{-1}$

Application 2

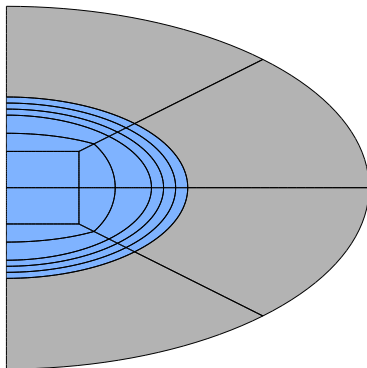
Approximate Transmission Conditions [POIGNARD, P., 08]



Thanks for your attention !

Convergence w.r.t. Interpolation degree p

H_σ^p : the computed solution of the discretize problem with an interpolation degree $p = 1, \dots, 20$ and a fixed mesh



We define

$$A_\sigma^p := \|H_\sigma^p\|_{L_1^2(\Omega_-^m)} \quad (\text{weight function} = r)$$

Interpolation degree p

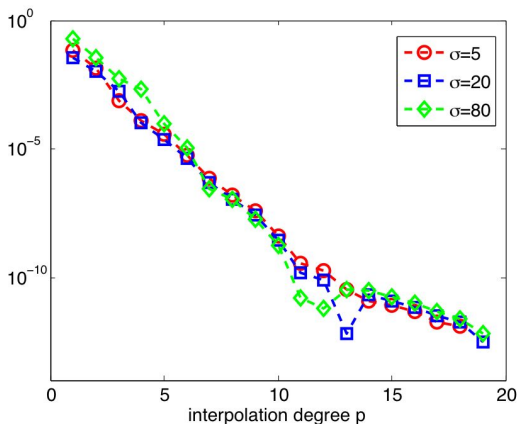
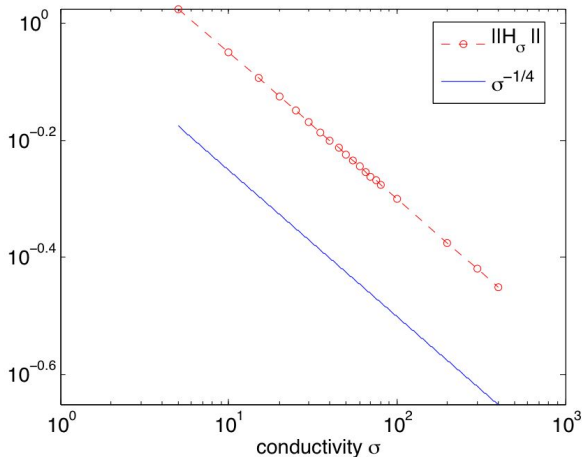


Figure: $|A_\sigma^p - A_\sigma^{20}|$ when $p = 1, \dots, 19$ in semilog scale

Weighted Norm of H_σ^{16}



Behavior consistent with the asymptotic expansion ($\sqrt{\delta} \sim \sigma^{-1/4}$)