

# Skin Effect in Electromagnetism and Asymptotic Behaviour of Skin Depth for High Conductivity

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Projet MC2

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# Motivation

## Skin effect in a half-space

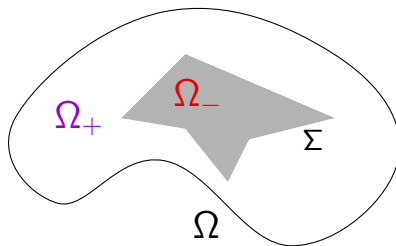
*Skin Effect* : in  $\{z > 0\}$  ,

$$\|\mathbf{E}_\sigma(z)\| = E_0 e^{-z/\ell(\sigma)}$$

*Skin Depth* :

$$\ell(\sigma) = \sqrt{2/\omega\mu_0\sigma}$$

# The Model Problem



- $\Omega_-$  Highly Conducting body  $\subset\subset \Omega$ : Conductivity  $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$ : Interface
- $\Omega_+$  Insulating or Dielectric body: Conductivity  $\sigma_+ = 0$

Aim : Understand the Behavior of the Electromagnetic Field as  $\sigma \rightarrow \infty$

# Tools for the Asymptotic Analysis

- Tensor Calculus & Differential Geometry
- Asymptotic Expansions: *Scaling* Technique. *Boundary Layer* Expansion
- PDE Analysis : *Uniform Estimates*
- Numerical Simulations : the Asymptotic solution Procedure yields a *Fast and Efficient method*

## References



E.P. STEPHAN, R.C. McCAMY (83-84-85)

*Plane Interface and Eddy Current Problems*



H. HADDAR, P. JOLY, H.N. NGUYEN (08)

*Smooth Interface and Generalized Impedance Boundary Conditions*



V. PÉRON (09)

*PhD : Smooth and Polyhedral Interfaces*



M. DAUGE, E. FAOU, V. PÉRON (10)

*Asymptotic Behavior for High Conductivity of the Skin Depth*



G. CALOZ, M. DAUGE, V. PÉRON (09)

*Uniform Estimates for Transmission Problems with High Contrast in Heat Conduction and Electromagnetism*

# Outline

- 1 Uniform Estimates for Interface Problems
- 2 3D Multiscaled Asymptotic Expansion
- 3 Numerical Simulations

# Outline

## 1 Uniform Estimates for Interface Problems

## 2 3D Multiscaled Asymptotic Expansion

## 3 Numerical Simulations

# Maxwell Problem

## Issue and Framework

$$(\mathbf{P}_{\underline{\sigma}}) \quad \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma}) \mathbf{E} = \mathbf{j} \quad \underline{\sigma} = (0, \sigma)$$

Issue: *Uniform*  $L^2$  estimates for solutions  $(\mathbf{E}, \mathbf{H})$  of  $(\mathbf{P}_{\underline{\sigma}})$  as  $\sigma \rightarrow \infty$  ?



# Maxwell Problem

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Issue: *Uniform*  $L^2$  estimates for solutions  $(\mathbf{E}, \mathbf{H})$  of  $(\mathbf{P}_{\underline{\sigma}})$  as  $\sigma \rightarrow \infty$  ?

### Hypothesis

$\Sigma$  is a bounded Lipschitz surface in  $\mathbb{R}^3$

### Hypothesis (SH)

The angular frequency  $\omega$  is not an eigenfrequency of the problem

$$\left\{ \begin{array}{ll} \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + i\omega\varepsilon_0 \mathbf{E} = 0 & \text{in } \Omega_+ \\ \mathbf{E} \times \mathbf{n} = 0 & \text{on } \Sigma \\ \text{B.C.} & \text{on } \partial\Omega \end{array} \right.$$

# Maxwell Problem

## Uniform $L^2(\Omega)$ Estimates

Boundary Conditions :  $\mathbf{E} \times \mathbf{n} = 0$  on  $\partial\Omega$

$\mathbf{j} \in \mathbf{H}(\text{div}, \Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \text{div } \mathbf{u} \in L^2(\Omega)\}$

### Theorem (CALOZ, DAUGE, P., 09)

Under Hypothesis (SH), there exist  $\sigma_0$  and  $C > 0$ , such that for all  $\sigma \geq \sigma_0$ ,  $(\mathbf{P}_{\underline{\sigma}})$  with B.C. and  $\mathbf{j} \in \mathbf{H}(\text{div}, \Omega)$  has a unique solution  $(\mathbf{E}, \mathbf{H})$  in  $L^2(\Omega)^2$ , and

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma} \|\mathbf{E}\|_{0,\Omega_-} \leq C \|\mathbf{j}\|_{\mathbf{H}(\text{div},\Omega)}$$

# Maxwell Problem

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Application: Convergence of asymptotic expansion for high conductivity

# Maxwell Problem

## Proof of Uniform Estimates

### Lemma

Under Hypothesis (SH), there are  $\sigma_0$  and  $C_0 > 0$  such that if  $\sigma \geq \sigma_0$  any solution  $(\mathbf{E}, \mathbf{H}) \in L^2(\Omega)^2$  of  $(\mathbf{P}_{\underline{\sigma}})$  with B.C. and data  $\mathbf{j} \in \mathbf{H}(\text{div}, \Omega)$  satisfies

$$\|\mathbf{E}\|_{0,\Omega} \leq C_0 \|\mathbf{j}\|_{\mathbf{H}(\text{div},\Omega)}$$

# Maxwell Problem

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$$\|\mathbf{E}\|_{0,\Omega} \leq C_0 \|\mathbf{j}\|_{\mathbf{H}(\text{div},\Omega)}$$

Proof by contradiction: keys

- 1  C. AMROUCHE, C. BERNARDI, M. DAUGE, V. GIRALT (98)

Vector Potential technique:  $\mathbf{E} =: \mathbf{w} + \nabla\varphi$  and  $\text{div } \mathbf{w} = 0$

$$\text{curl } \mathbf{H} + \underbrace{(i\omega\varepsilon_0 - \underline{\sigma})}_{=: \underline{a}} (\mathbf{w} + \nabla\varphi) = \mathbf{j} \longrightarrow \text{div } \underline{a} \nabla\varphi = \text{div } \mathbf{j} - \underbrace{\text{div } \underline{a} \mathbf{w}}_{= [\underline{a}]_{\Sigma} \mathbf{w} \cdot \mathbf{n}} \delta_{\Sigma}$$

- 2 Issue: Uniform piecewise regularity for

$$\text{div } \underline{a} \text{grad } \varphi = f + [\underline{a}]_{\Sigma} g \quad \text{as} \quad |a_-/a_+| \rightarrow \infty ?$$

# Uniform Estimates

## Scalar Problem

$$(\mathbf{P}_{\underline{a}}) : \quad \operatorname{div} \underline{a} \operatorname{grad} \varphi = f + [\underline{a}]_{\Sigma} g \quad \text{as} \quad |\rho| := |\underline{a}_- / \underline{a}_+| \rightarrow \infty$$

### Theorem (CALOZ, DAUGE, P., 09)

There exists  $\rho_0 > 0$  independent of  $\underline{a}_+ \neq 0$  s. t. for all  $\underline{a}_- \in \{z \in \mathbb{C} \mid |z| \geq \rho_0 |\underline{a}_+|\}$ , and for all  $(f, g) \in L^2(\Omega) \times L^2(\Sigma)$  s.t.  $\int_{\Sigma} g \, ds = 0$ ,  $(\mathbf{P}_{\underline{a}})$  has a unique solution  $\varphi \in H_0^1(\Omega)$ , piecewise  $H^{3/2}$  and

$$\|\varphi^+\|_{\frac{3}{2}, \Omega_+} + \|\varphi^-\|_{\frac{3}{2}, \Omega_-} \leq C_{\rho_0} (|\underline{a}_+|^{-1} \|f\|_{0, \Omega} + \|g\|_{0, \Sigma})$$

with  $C_{\rho_0} > 0$ , independent of  $\underline{a}_+$ ,  $\underline{a}_-$ ,  $f$ , and  $g$ .

# Uniform Estimates

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with  $C_{\rho_0} > 0$ , independent of  $\underline{a}_+$ ,  $\underline{a}_-$ ,  $f$ , and  $g$ .

Key of the Proof:

$$\varphi^+ = \sum_{n=0}^{\infty} \varphi_n^+ \rho^{-n} \quad \Omega_+ \quad \text{and} \quad \varphi^- = \sum_{n=0}^{\infty} \varphi_n^- \rho^{-n} \quad \Omega_-$$

## Remarks

- ❶ Similar estimate hold when  $a_+$  and  $a_-$  are switched.  
More precise estimate when  $a_+$  and  $a_-$  play symmetric roles
- ❷ The assumption that  $\Sigma$  is Lipschitz is necessary
- ❸ For *polyhedral* Lipschitz interface  $\Sigma$

$$\|\varphi^+\|_{s,\Omega_+} + \|\varphi^-\|_{s,\Omega_-} \leq C_{\rho_0} (|a_+|^{-1} \|f\|_{s-2,\Omega} + \|g\|_{s-\frac{3}{2},\Sigma})$$

for  $s \leq s_\Sigma$  with some  $\frac{3}{2} < s_\Sigma \leq 2$

- ❹ For *smooth* interface  $\Sigma$ , uniform piecewise  $H^s$  estimates for any  $s \geq 2$



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## References and Notations

Asymptotic Expansion as  $\sigma \rightarrow \infty$  of solutions of  $(\mathbf{P}_\sigma)$  when  $\Sigma$  is smooth :



STEPHAN, MCCAMY (83-84)

*Plane Interface*



HADDAR, JOLY, NGUYEN (08)



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## Hypothesis

- 1  $\Sigma$  is a Smooth Surface
- 2  $\omega$  satisfies the Spectral Hypothesis (SH)
- 3  $\mathbf{j} \in \mathbf{H}(\text{div}, \Omega)$  and  $\mathbf{j} = 0$  in  $\Omega_-$
- 4 Perfectly Conducting Electric B.C.

# Asymptotic Expansion

$$\delta := \sqrt{\frac{\omega \varepsilon_0}{\sigma}} \longrightarrow 0 \quad \text{as } \sigma \rightarrow \infty$$

By Theorem there exists  $\delta_0$  s.t. for all  $\delta \leq \delta_0$ , there exists a unique solution  $(\mathbf{E}_{(\delta)}, \mathbf{H}_{(\delta)})$  to  $(\mathbf{P}_{\underline{\sigma}})$

$$\mathbf{E}_{(\delta)}^+(\mathbf{x}) \approx \sum_{j \geq 0} \delta^j \mathbf{E}_j^+(\mathbf{x}), \quad \mathbf{E}_{(\delta)}^-(\mathbf{x}) \approx \chi(y_3) \sum_{j \geq 0} \delta^j \mathbf{W}_j(y_\beta, \frac{y_3}{\delta})$$

$(y_\beta, y_3)$ : “normal coordinates” to  $\Sigma$  in a tubular region  $\mathcal{U}_-$  of  $\Sigma$  in  $\Omega_-$

$$\mathbf{E}_j^+ \in \mathbf{H}(\text{curl}, \Omega_+) \quad , \quad \mathbf{W}_j \in \mathbf{H}(\text{curl}, \Sigma \times \mathbb{R}_+) \quad \text{profiles}$$

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Similar Expansion for the Magnetic Field  $\mathbf{H}_{(\delta)}$

# Validation of the Asymptotic Expansion

## Remainders

Aim : proving Estimates for Remainders  $\mathbf{R}_{m;\delta}$

$$\mathbf{R}_{m;\delta} := \mathbf{E}(\delta) - \sum_{j=0}^m \delta^j \mathbf{E}_j \quad \text{in } \Omega$$

Evaluation of the right hand side when the Maxwell operator is applied to  $\mathbf{R}_{m;\delta}$

$$\left\{ \begin{array}{ll} \text{curl curl } \mathbf{R}_{m;\delta}^+ - \kappa^2 \alpha_+ \mathbf{R}_{m;\delta}^+ & = 0 \quad \text{in } \Omega_+ \\ \text{curl curl } \mathbf{R}_{m;\delta}^- - \kappa^2 \alpha_- \mathbf{R}_{m;\delta}^- & = \mathbf{j}_{m;\delta}^- \quad \text{in } \Omega_- \\ [\mathbf{R}_{m;\delta} \times \mathbf{n}]_{\Sigma} & = 0 \quad \text{on } \Sigma \\ [\text{curl } \mathbf{R}_{m;\delta} \times \mathbf{n}]_{\Sigma} & = \mathbf{g}_{m;\delta} \quad \text{on } \Sigma \\ \text{curl } \mathbf{R}_{m;\delta}^+ \times \mathbf{n} & = 0 \quad \text{on } \partial\Omega \end{array} \right.$$

with  $\alpha_+ = 1$  and  $\alpha_- = 1 + i \delta^{-2}$ , and  $\kappa = \omega \sqrt{\varepsilon_0 \mu_0}$

# Validation of the asymptotic expansion

## Estimates for Remainders

Estimates for the residues  $\mathbf{j}_{m;\delta}^-$  and  $\mathbf{g}_{m;\delta}$  :

$$\|\mathbf{j}_{m;\delta}^-\|_{2,\Omega_-} + \|\mathbf{g}_{m;\delta}\|_{\frac{1}{2},\Sigma} + \|\operatorname{curl}_{\Sigma} \mathbf{g}_{m;\delta}\|_{\frac{3}{2},\Sigma} \leq C_m \delta^{m-1}$$

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### Theorem (CALOZ, DAUGE, P., 09)

*Under Hypothesis in the framework above, for all  $m \in \mathbb{N}$  and  $\delta \in (0, \delta_0]$ , the remainders  $\mathbf{R}_{m;\delta}$  satisfy the optimal estimates*

$$\begin{aligned} \|\mathbf{R}_{m;\delta}^+\|_{0,\Omega_+} + \|\operatorname{curl} \mathbf{R}_{m;\delta}^+\|_{0,\Omega_+} \\ + \delta^{-\frac{1}{2}} \|\mathbf{R}_{m;\delta}^-\|_{0,\Omega_-} + \delta^{\frac{1}{2}} \|\operatorname{curl} \mathbf{R}_{m;\delta}^-\|_{0,\Omega_-} \leq C'_m \delta^{m+1} \end{aligned}$$



## Profiles of the Magnetic Field

$\mathbf{V}_j =: (\mathcal{V}_j^\alpha ; v_j)$  in coordinates  $(y_\beta, Y_3)$  with  $Y_3 = \frac{y_3}{\delta}$

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$$\mathcal{V}_1^\alpha(y_\beta, Y_3) = \left[ h_1^\alpha + Y_3 \left( \mathcal{H} h_0^\alpha + b_\sigma^\alpha h_0^\sigma \right) \right] (y_\beta) e^{-\lambda Y_3}$$

with

$\mathcal{H}$  mean curvature of  $\Sigma$

$$\mathbf{h}_0(y_\beta) = (\mathbf{n} \times \mathbf{H}_0^+) \times \mathbf{n}(y_\beta, 0) \quad \text{and} \quad h_j^\alpha(y_\beta) := (\mathbf{H}_j^+)^\alpha(y_\beta, 0)$$

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Comparison with



HADDAR, JOLY, NGUYEN (08)

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Tool: a Normal Parameterization Technique

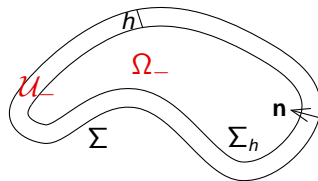


FAOU (02)

*Elasticity on a thin shell*

# Curl Operator and Levi-Civita Tensor

$$h := y_3$$



Curl Operator in Normal Parameterization:

$$(\nabla \times \mathbf{E})^\alpha = \epsilon^{3\beta\alpha} (\partial_3^h E_\beta - \partial_\beta E_3) \quad \text{on } \Sigma_h$$

$$(\nabla \times \mathbf{E})^3 = \epsilon^{3\alpha\beta} D_\alpha^h E_\beta \quad \text{on } \Sigma_h$$

Levi-Civita Tensor  $\epsilon$ :

$$\epsilon^{ijk} = \epsilon_0(i, j, k) / \sqrt{\det a_{\alpha\beta}(h)}$$

with  $a_{\alpha\beta}(h) = \text{metric on } \Sigma_h$  and  $\epsilon_0(i, j, k) \in \{0, 1, -1\}$

Covariant Derivative

$$D_\alpha^h E_\beta = \partial_\alpha E_\beta - \Gamma_{\alpha\beta}^\gamma(h) E_\gamma$$

# Application 1

## A New Definition of the Skin Depth

3D model :  $\mathbf{V}_{(\delta)}(y_\alpha, y_3) := \mathbf{H}_{(\delta)}^-(\mathbf{x})$ ,  $y_\alpha \in \Sigma$ ,  $0 \leq y_3 < h_0$

### Definition

Let  $\Sigma$  be a smooth surface, and  $\mathbf{j}$  s.t.  $\mathbf{V}_{(\delta)}(y_\alpha, 0) \neq 0$ . The skin depth is the smallest length  $\mathcal{L}(\sigma, y_\alpha)$  defined on  $\Sigma$  s.t.

$$\|\mathbf{V}_{(\delta)}(y_\alpha, \mathcal{L}(\sigma, y_\alpha))\| = \|\mathbf{V}_{(\delta)}(y_\alpha, 0)\| e^{-1}$$

### Theorem (DAUGE, FAOU, P., 10)

Let  $\Sigma$  be a regular surface with mean curvature  $\mathcal{H}$ , and assume  $\mathbf{h}_0(y_\alpha) \neq 0$ .

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma) \left( 1 + \mathcal{H}(y_\alpha) \ell(\sigma) + \mathcal{O}(\sigma^{-1}) \right), \quad \sigma \rightarrow \infty$$

$$\|\mathbf{V}_{(\delta)}\|^2 = \left[ \|\mathbf{h}_0\|^2 + 2\delta \operatorname{Re}\langle \mathbf{h}_0, \mathbf{h}_1 \rangle + 2y_3 \mathcal{H} \|\mathbf{h}_0\|^2 + \mathcal{O}((\delta + y_3)^2) \right] e^{-2y_3/\ell(\sigma)}$$

# Outline

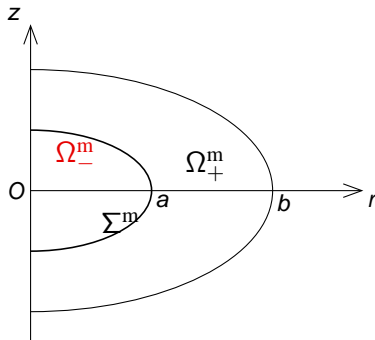
1 Uniform Estimates for Interface Problems

2 3D Multiscaled Asymptotic Expansion

**3 Numerical Simulations**

# Framework

- Magnetic Field  $\mathbf{H}_\sigma$
- Axisymmetric Problem:  $\Omega_-$  and  $\Omega$  Axisymmetric Domains
- $\text{curl } \mathbf{j}$  Axisymmetric & Orthoradial  $\longrightarrow \mathbf{H}_\sigma$  Axisymmetric & Orthoradial



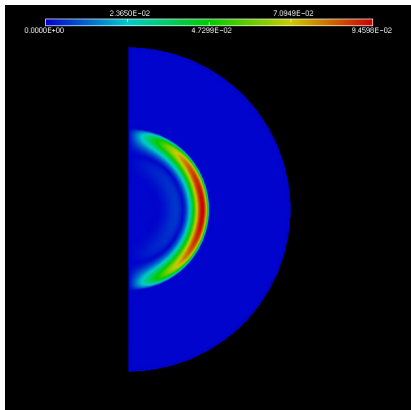
**Figure:** The meridian domain. Configuration A

- FEM: Finite Element Library Melina

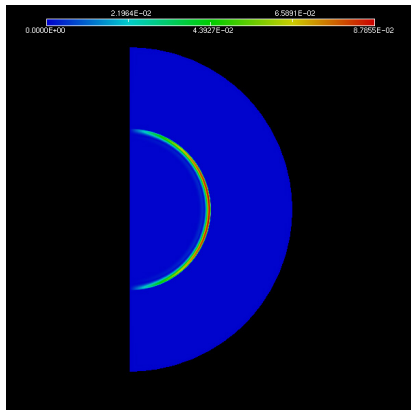


# Spherical Geometry

$|\operatorname{Im} H_\sigma|$ . Configuration A



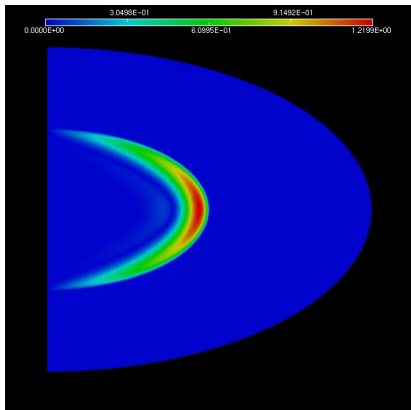
$|\operatorname{Im} H_\sigma|$  as  $\sigma = 5 \text{ S.m}^{-1}$



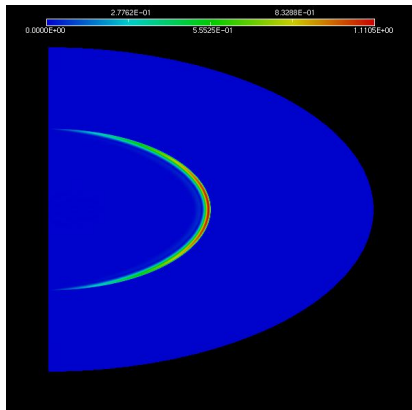
$|\operatorname{Im} H_\sigma|$  as  $\sigma = 80 \text{ S.m}^{-1}$

# Spheroidal Geometry

$|\operatorname{Im} H_\sigma|$ . Configuration A



$|\operatorname{Im} H_\sigma|$  as  $\sigma = 5 \text{ S.m}^{-1}$



$|\operatorname{Im} H_\sigma|$  as  $\sigma = 80 \text{ S.m}^{-1}$

# Influence of the Mean Curvature

## Configuration B

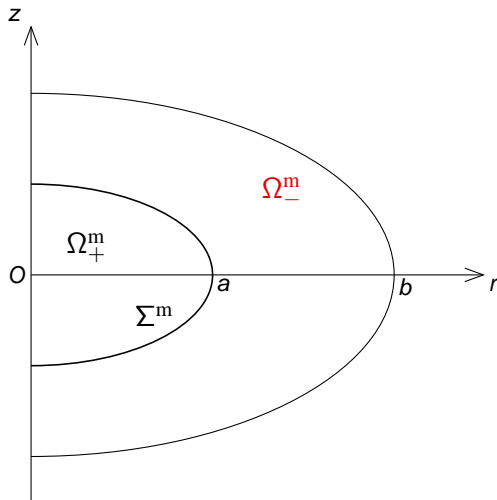
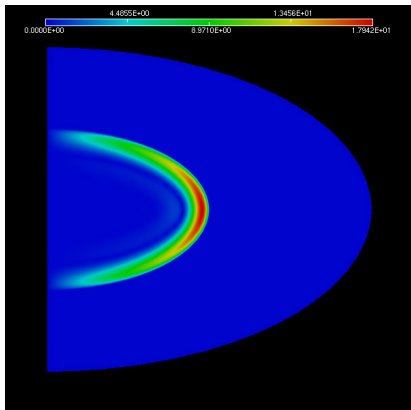


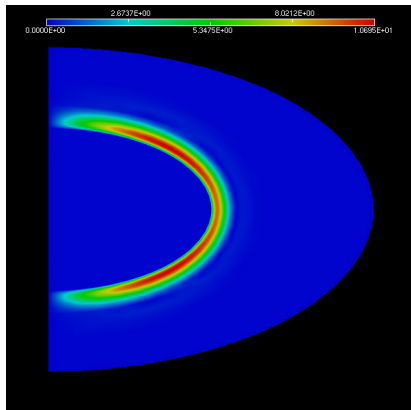
Figure: Configuration B.  $\mathcal{H} < 0$

# Influence of the Mean Curvature

$|\operatorname{Im} H_\sigma|$  when  $\sigma = 5S.m^{-1}$



Configuration A ( $\mathcal{H} > 0$ )



Configuration B ( $\mathcal{H} < 0$ )

# Numerical Post-Treatment

## Linear Regression. Configuration A

Extract  $|\mathbf{H}_\sigma(h)|$  along edges of the mesh in  $\Omega_m$  when  $z = 0$ :  $h = 2 - r$

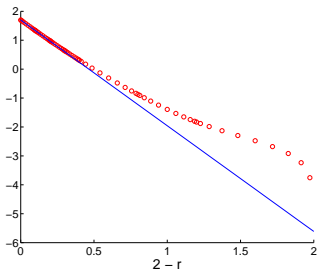
$$\log_{10} |\mathbf{H}_\sigma(h)| = -s(\sigma)h + \alpha$$

# Numerical Post-Treatment

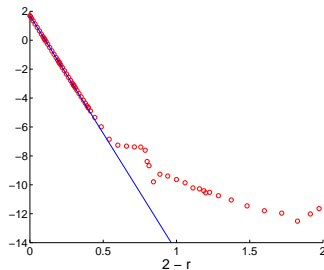
## Linear Regression. Configuration A

Extract  $|\mathbf{H}_\sigma(h)|$  along edges of the mesh in  $\Omega^m$  when  $z = 0$ :  $h = 2 - r$

$$\log_{10} |\mathbf{H}_\sigma(h)| = -s(\sigma)h + \alpha$$



○ :  $\log_{10} |\mathbf{H}_\sigma(h)|$  when  $\sigma = 5$   
 $n(\sigma) = 7$ ,  $s(5) = 3,65542$



○ :  $\log_{10} |\mathbf{H}_\sigma(h)|$  when  $\sigma = 80$   
 $n(\sigma) = 3$ ,  $s(80) = 16,279162$

# Post-Treatment

## Accuracy of Asymptotics

$$\log_{10} \|\mathbf{V}_{(\delta)}(y_\alpha, h)\| = - \underbrace{\frac{1}{\ln 10} \left( \frac{1}{\ell(\sigma)} - \mathcal{H}(y_\alpha) \right)}_{=: S(y_\alpha, \sigma)} h + \beta + \mathcal{O}(\delta) + \mathcal{O}((\delta+h)^2)$$

Relative Error between slopes:

$$\text{error}(\sigma) := \left| \frac{S(\sigma) - s(\sigma)}{S(\sigma)} \right|$$

# Post-Treatment

## Accuracy of Asymptotics

$$\log_{10} \|\mathbf{V}(\delta)(y_\alpha, h)\| = - \underbrace{\frac{1}{\ln 10} \left( \frac{1}{\ell(\sigma)} - \mathcal{H}(y_\alpha) \right)}_{=: S(y_\alpha, \sigma)} h + \beta + \mathcal{O}(\delta) + \mathcal{O}((\delta+h)^2)$$

Relative Error between slopes:

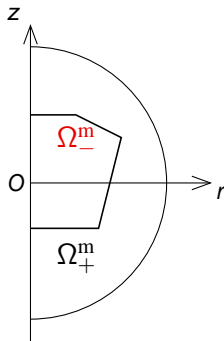
$$\text{error}(\sigma) := \left| \frac{S(\sigma) - s(\sigma)}{S(\sigma)} \right|$$

|                            |           |           |
|----------------------------|-----------|-----------|
| $\sigma$                   | 5         | 20        |
| $\ell(\sigma)$ (cm)        | 10.3      | 5.15      |
| $s(\sigma)$                | 3, 65542  | 7, 883903 |
| $S(\sigma)$                | 3, 673319 | 7, 889506 |
| $\text{error}(\sigma)$ (%) | 0, 48     | 0, 07     |
| $n(\sigma)$                | 7         | 5         |



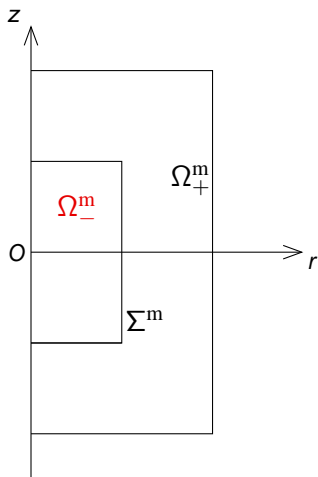
# Conclusion

- Efficiency Methods → Skin Effect
- Solution Structure & Optimal Estimates of Remainders
- A Singular Transmission Problem: with C. Poignard and M. Dauge

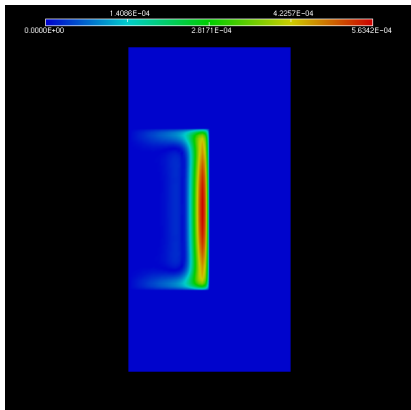
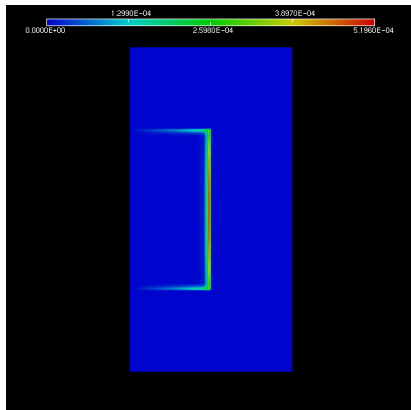


# Cylindrical Geometry

Meridian domain  $\Omega^m$

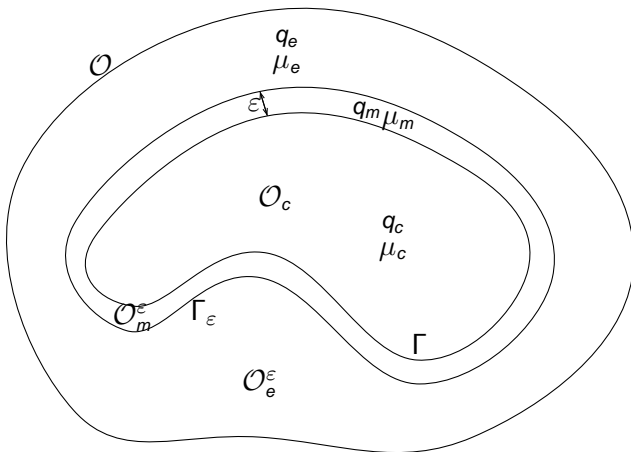


# Cylindrical Geometry

 $|\operatorname{Im} H_\sigma|$ 

 $|\operatorname{Im} H_\sigma|$  as  $\sigma = 5\text{S.m}^{-1}$ 

 $|\operatorname{Im} H_\sigma|$  as  $\sigma = 80\text{S.m}^{-1}$

## Application 2

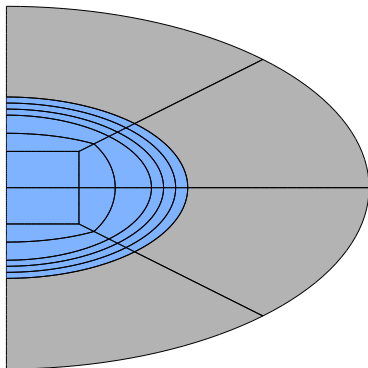
### Approximate Transmission Conditions [POIGNARD, P., 08]



Thanks for your attention !

## Convergence w.r.t. Interpolation degree $p$

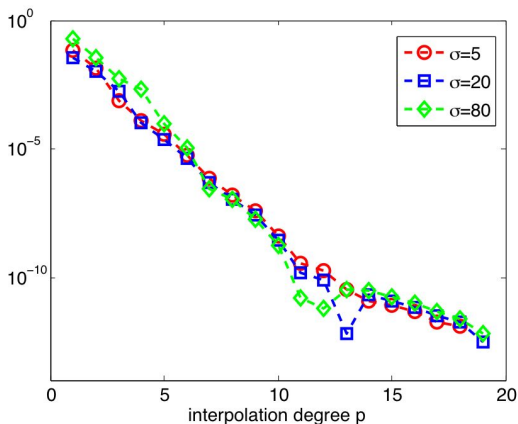
$H_\sigma^p$ : the computed solution of the discretize problem with an interpolation degree  $p = 1, \dots, 20$  and a fixed mesh



We define

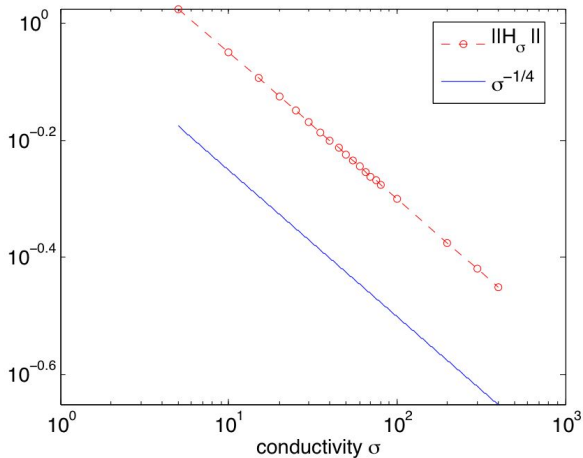
$$A_\sigma^p := \|H_\sigma^p\|_{L_1^2(\Omega_-^m)} \quad (\text{weight function} = r)$$

# Interpolation degree p



**Figure:**  $|A_\sigma^p - A_\sigma^{20}|$  when  $p = 1, \dots, 19$  in semilog scale

# Weighted Norm of $H_\sigma^{16}$



Behavior consistent with the asymptotic expansion ( $\sqrt{\delta} \sim \sigma^{-1/4}$ )