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Skin Effect Description in Electromagnetism with a Multiscaled Asymptotic Expansion

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Skin-Effect Description in Electromagnetism with a Scaled Asymptotic Expansion

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Macadam Workshop, RENNES

31.08.2009

References



GABRIEL CALOZ, MONIQUE DAUGE, VICTOR PÉRON (2009)

Uniform Estimates for Transmission Problems with High Contrast in Heat Conduction and Electromagnetism

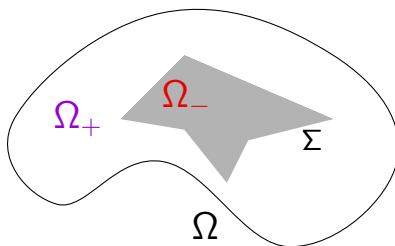


MONIQUE DAUGE, ERWAN FAOU, VICTOR PÉRON (2009)

Asymptotic Behavior at High Conductivity of Skin Depth in Electromagnetism

Transmission Problems with High Contrast Media

ELECTROMAGNETISM

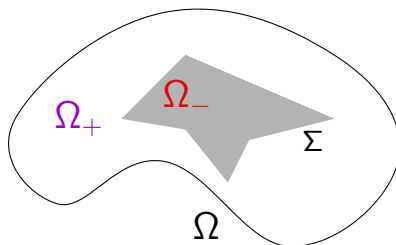


- Ω_- Highly conducting material $\subset\subset \Omega$: Conductivity $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+ Insulating or dielectric body: Conductivity $\sigma_+ = 0$

Aim: Understand the electromagnetic phenomenon as $\sigma \rightarrow \infty$

Transmission Problems with High Contrast Media

HEAT CONDUCTION



- $\Omega_- \subset\subset \Omega$: heat conductivity a_-
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+ : heat conductivity a_+

Aim: Describe the transmission phenomenon as $|a_-/a_+| \gg 1$

Outline

- 1 Uniform Estimates for Transmission Problems
- 2 3D Multiscaled Asymptotic Expansion
- 3 Numerical Simulations

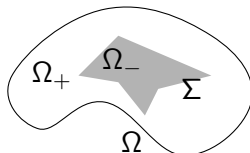
Outline

1 Uniform Estimates for Transmission Problems

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Transmission Problems in High Contrast Media



- Heat transfer equation ($\mathbf{P}_{\underline{a}}$): $\operatorname{div} \underline{a} \operatorname{grad} \varphi = f$

$$\underline{a} = (a_+, a_-) \quad \text{s.t.} \quad |a_-|/|a_+| \rightarrow \infty$$

- Maxwell equations ($\mathbf{P}_{\underline{\sigma}}$):

$$\operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma}) \mathbf{E} = \mathbf{j}$$

$$\underline{\sigma} = (\sigma_+, \sigma_-) \quad \text{s.t.} \quad \sigma_+ = 0 \quad \text{and} \quad \sigma_- \equiv \sigma \rightarrow \infty$$

Issues

- ❶ *Uniform piecewise estimates and regularity* in Sobolev norms for solutions φ of $(\mathbf{P}_{\underline{a}})$
- ❷ *Uniform L^2 estimates* for solutions (\mathbf{E}, \mathbf{H}) of $(\mathbf{P}_{\underline{\sigma}})$

Hypothesis

Σ is a bounded Lipschitz surface in \mathbb{R}^3

Scalar case:



H.P. HUY, E. SANCHEZ-PALENCIA (1974)

Limit problem and Strong convergence results as $a_- / a_+ \rightarrow \infty$



S. HASSANI, S. NICAISE, A. MAGHNOUJI (2009)

Maxwell case:



H. HADDAR, P. JOLY, H.N. NGUYEN (2008)

Uniform estimates under a stronger regularity assumption on Σ

Scalar Problem

Variational Problem

$$V_D = H_0^1(\Omega) \quad \text{or} \quad V_N = \left\{ \varphi \in H^1(\Omega) \mid \int_{\Omega} \varphi \, dx = 0 \right\}$$

(**VP**_a): Find $\varphi \in V = V_D$ or $V = V_N$ such that

$$\begin{aligned} \forall \psi \in V, \quad & \int_{\Omega_+} a_+ \nabla \varphi^+ \cdot \nabla \bar{\psi}^+ \, dx + \int_{\Omega_-} a_- \nabla \varphi^- \cdot \nabla \bar{\psi}^- \, dx = \\ & - \int_{\Omega} f \bar{\psi} \, dx + (a_+ - a_-) \int_{\Sigma} g \bar{\psi} \, ds \end{aligned}$$

Hypothesis (H_a)

- 1 $f \in L^2(\Omega)$ and $g \in L^2(\Sigma)$
- 2 $\int_{\Omega} f \, dx = \int_{\Sigma} g \, ds = 0$ if $V = V_N$ and $\int_{\Sigma} g \, ds = 0$ if $V = V_D$

Uniform Estimates

Theorem

Let us assume that $a_+ \neq 0$. There exist $\rho_0 > 0$ independent of a_+ such that for all $a_- \in \{z \in \mathbb{C} \mid |z| \geq \rho_0 |a_+|\}$, and for all data (f, g) satisfying $(\mathbf{H}_{\underline{a}})$, $(\mathbf{VP}_{\underline{a}})$ has a unique solution $\varphi \in V$, which is piecewise $\mathbf{H}^{3/2}$ and

$$\|\varphi^+\|_{\frac{3}{2}, \Omega_+} + \|\varphi^-\|_{\frac{3}{2}, \Omega_-} \leq C_{\rho_0} (|a_+|^{-1} \|f\|_{0, \Omega} + \|g\|_{0, \Sigma})$$

with $C_{\rho_0} > 0$, independent of a_+ , a_- , f , and g .



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Key of the proof: asymptotic expansion for φ with respect to the powers of $\rho^{-1} := a_+(a_-)^{-1}$ and convergence of these series in the piecewise $\mathbf{H}^{3/2}$ -norm

Remarks

- ❶ Similar estimate when the roles of a_+ and a_- are exchanged. More precise estimate where a_+ and a_- play symmetric roles
- ❷ The assumption that Σ is Lipschitz is necessary
- ❸ For *polyhedral* Lipschitz interface Σ , uniform piecewise H^s estimates:

$$\|\varphi^+\|_{s,\Omega_+} + \|\varphi^-\|_{s,\Omega_-} \leq C_{\rho_0} (|a_+|^{-1} \|f\|_{s-2,\Omega} + \|g\|_{s-\frac{3}{2},\Sigma})$$

for $s \leq s_\Sigma$ with some $\frac{3}{2} < s_\Sigma \leq 2$

- ❹ For *smooth* interface Σ , uniform piecewise H^s estimates for any $s \geq 2$

Maxwell Problem

Framework and Boundary Conditions

$$(\mathbf{P}_{\underline{\sigma}}) \quad \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma}) \mathbf{E} = \mathbf{j} \quad \underline{\sigma} = (0, \sigma)$$

$$\textcircled{1} \quad \mathbf{E} \times \mathbf{n} = 0 \quad \text{and} \quad \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega$$

$$\textcircled{2} \quad \mathbf{E} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{H} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega$$

$$\textcircled{1} \quad \text{B. C. 1:} \quad \mathbf{j} \in \mathbf{H}(\operatorname{div}, \Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{div} \mathbf{u} \in L^2(\Omega)\}$$

$$\textcircled{2} \quad \text{B. C. 2:}$$

$$\mathbf{j} \in \mathbf{H}_0(\operatorname{div}, \Omega) = \{\mathbf{u} \in \mathbf{H}(\operatorname{div}, \Omega) \mid \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\}$$

Hypothesis (SH)

The angular frequency ω is not an eigenfrequency of the problem

$$\left\{ \begin{array}{ll} \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + i\omega\varepsilon_0 \mathbf{E} = 0 & \text{in } \Omega_+ \\ \mathbf{E} \times \mathbf{n} = 0 \quad \text{and} \quad \mathbf{H} \cdot \mathbf{n} = 0 & \text{on } \Sigma \\ \text{B.C.1 or B.C.2} & \text{on } \partial\Omega \end{array} \right.$$

Maxwell Problem

Uniform $L^2(\Omega)$ Estimate

Theorem

Under Hypothesis (SH), there are σ_0 and $C > 0$, such that for all $\sigma \geq \sigma_0$, the Maxwell problem $(\mathbf{P}_{\underline{\sigma}})$ with B.C.2 and $\mathbf{j} \in \mathbf{H}_0(\operatorname{div}, \Omega)$ has a unique solution (\mathbf{E}, \mathbf{H}) in $L^2(\Omega)^2$, which satisfies:

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma} \|\mathbf{E}\|_{0,\Omega_-} \leq C \|\mathbf{j}\|_{\mathbf{H}(\operatorname{div}, \Omega)}$$

Similar result for B.C.1 and $\mathbf{j} \in \mathbf{H}(\operatorname{div}, \Omega)$



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Application: Convergence of asymptotic expansion at high conductivity

Maxwell Problem

Proof of Maxwell Uniform Estimate


Lemma

Under Hypothesis (SH), there are σ_0 and $C_0 > 0$ such that if $\sigma \geq \sigma_0$ any solution $(\mathbf{E}, \mathbf{H}) \in \mathbf{L}^2(\Omega)^2$ of problem $(\mathbf{P}_{\underline{\sigma}})$ with B.C.2 and data $\mathbf{j} \in \mathbf{H}_0(\text{div}, \Omega)$ satisfies the estimate

$$\|\mathbf{E}\|_{0,\Omega} \leq C_0 \|\mathbf{j}\|_{\mathbf{H}(\text{div},\Omega)}$$

Similar statement for B.C.1 and $\mathbf{j} \in \mathbf{H}(\text{div}, \Omega)$.

Keys of the proof:

- Electrical Field Decomposition into a regular field and a gradient field
 -  C. AMROUCHE, C. BERNARDI, M. DAUGE, V. GIRAULT (1998)
- Scalar Estimates used for the gradient field $\nabla\varphi$

Outline

- 1 Uniform Estimates for Transmission Problems
- 2 3D Multiscaled Asymptotic Expansion**
- 3 Numerical Simulations

References and Notations

Asymptotic expansion as $\sigma \rightarrow \infty$ of solutions of (\mathbf{P}_σ) when Σ is smooth:



E. STEPHAN, R.C. MCCAMY (1983-84-85)

Plane Interface and Eddy Current Approximation



H. HADDAR, P. JOLY, H.N. NGUYEN (2008)

Smooth Interface and Impedance boundary conditions



V. PÉRON (2009)

Smooth Interface and Perfectly Conducting Electric or Magnetic Wall B.C.

Hypothesis

- 1 Σ is an oriented smooth compact surface
- 2 Perfectly Conducting Magnetic Wall b.c.
- 3 $\mathbf{j} \in \mathbf{H}_0(\text{div}, \Omega)$ and $\mathbf{j} = 0$ in Ω_-
- 4 Hypothesis (SH) on ω

Asymptotic Expansion

$$\delta := \sqrt{\frac{\omega \varepsilon_0}{\sigma}} \rightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty$$

By Theorem there exists δ_0 s.t. for all $\delta \leq \delta_0$, there exists a unique solution $(\mathbf{E}_{(\delta)}, \mathbf{H}_{(\delta)})$ to $(\mathbf{P}_{\underline{\sigma}})$

Then it is possible to construct series expansions in powers of δ :

$$\mathbf{E}_{(\delta)}^+(\mathbf{x}) \approx \sum_{j \geq 0} \delta^j \mathbf{E}_j^+(\mathbf{x}), \quad \mathbf{E}_{(\delta)}^-(\mathbf{x}) \approx \chi(y_3) \sum_{j \geq 0} \delta^j \mathbf{W}_j(y_\beta, \frac{y_3}{\delta})$$

$$\mathbf{E}_j^+ \in \mathbf{H}(\text{curl}, \Omega_+) \quad , \quad \mathbf{W}_j \in \mathbf{H}(\text{curl}, \Sigma \times \mathbb{R}_+) \quad \text{profiles}$$

(y_β, y_3) : “normal coordinates” to Σ in a tubular neighborhood \mathcal{U}_- of Σ in Ω_-
 Similar result for the magnetic field.

Validation of the asymptotic expansion

Remainders

Aim: proving estimates for remainders $\mathbf{R}_{m;\delta}$

$$\mathbf{R}_{m;\delta} := \mathbf{E}(\delta) - \sum_{j=0}^m \delta^j \mathbf{E}_j \quad \text{in } \Omega$$

Evaluation of the right hand side when the Maxwell operator is applied to $\mathbf{R}_{m;\delta}$:

$$\left\{ \begin{array}{ll} \operatorname{curl} \operatorname{curl} \mathbf{R}_{m;\delta}^+ - \kappa^2 \alpha_+ \mathbf{R}_{m;\delta}^+ & = 0 \quad \text{in } \Omega_+ \\ \operatorname{curl} \operatorname{curl} \mathbf{R}_{m;\delta}^- - \kappa^2 \alpha_- \mathbf{R}_{m;\delta}^- & = \mathbf{j}_{m;\delta}^- \quad \text{in } \Omega_- \\ [\mathbf{R}_{m;\delta} \times \mathbf{n}]_{\Sigma} & = 0 \quad \text{on } \Sigma \\ [\operatorname{curl} \mathbf{R}_{m;\delta} \times \mathbf{n}]_{\Sigma} & = \mathbf{g}_{m;\delta} \quad \text{on } \Sigma \\ \operatorname{curl} \mathbf{R}_{m;\delta}^+ \times \mathbf{n} & = 0 \quad \text{on } \partial\Omega \end{array} \right.$$

with $\alpha_+ = 1$ and $\alpha_- = 1 + i/\delta^2$

Validation of the asymptotic expansion

Estimates for Remainders

$$\|\mathbf{j}_{m;\delta}^-\|_{2,\Omega_-} + \|\mathbf{g}_{m;\delta}\|_{\frac{1}{2},\Sigma} + \|\operatorname{curl}_{\Sigma} \mathbf{g}_{m;\delta}\|_{\frac{3}{2},\Sigma} \leq C_m \delta^{m-1}$$

with $C_m > 0$ independent of δ .

Theorem

Under Hypothesis in the framework above, for all $m \in \mathbb{N}$ and $\delta \in (0, \delta_0]$, the remainders $\mathbf{R}_{m;\delta}$ satisfy the optimal estimates

$$\begin{aligned} \|\mathbf{R}_{m;\delta}^+\|_{0,\Omega_+} + \|\operatorname{curl} \mathbf{R}_{m;\delta}^+\|_{0,\Omega_+} \\ + \delta^{-\frac{1}{2}} \|\mathbf{R}_{m;\delta}^-\|_{0,\Omega_-} + \delta^{\frac{1}{2}} \|\operatorname{curl} \mathbf{R}_{m;\delta}^-\|_{0,\Omega_-} \leq C'_m \delta^{m+1} \end{aligned}$$

Electrical Profiles

$\mathbf{W}_j =: (\mathcal{W}_{\alpha,j} ; w_j)$ in coordinates (y_β, Y_3) with $Y_3 = \frac{Y_3}{\delta}$

$$\mathbf{W}_0 = 0$$

$$\mathcal{W}_{\alpha,1} = -j_{\alpha,0}(y_\beta) e^{-\lambda Y_3} \quad \text{and} \quad w_1 = 0$$

$$\mathcal{W}_{\alpha,2} = \left[\lambda^{-1} \{ b_\alpha^\sigma j_{\sigma,0} - \mathcal{H} j_{\alpha,0} \} - j_{\alpha,1} + Y_3 \{ b_\alpha^\sigma j_{\sigma,0} - \mathcal{H} j_{\alpha,0} \} \right] (y_\beta) e^{-\lambda Y_3}$$

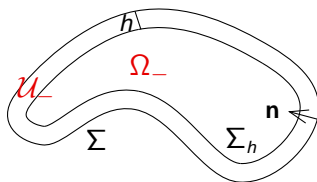
$$w_2 = -\lambda^{-1} D_\alpha j_0^\alpha(y_\beta) e^{-\lambda Y_3}$$

with

\mathcal{H} mean curvature of Σ

$$j_{\alpha,k}(y_\beta) := \lambda^{-1} (\text{curl } \mathbf{E}_k^+ \times \mathbf{n})_\alpha(y_\beta, 0) \quad \text{and} \quad \lambda = \kappa e^{-i\pi/4}$$

Curl Operator and Levi-Civita Tensor



Levi-Civita Tensor ϵ :

$$\epsilon^{ijk} = \epsilon_0(i, j, k) / \sqrt{\det(a_{\alpha\beta}(h))}$$

$a_{\alpha\beta}(h)$: metric on Σ_h and $\epsilon_0(i, j, k) = \text{sign of } (i, j, k)$

Curl Operator in Normal Parameterization:

$$\begin{cases} (\nabla \times \mathbf{E})^\alpha = \epsilon^{3\beta\alpha} (\partial_3^h E_\beta - \partial_\beta E_3) & \text{on } \Sigma_h \\ (\nabla \times \mathbf{E})^3 = \epsilon^{3\alpha\beta} D_\alpha^h E_\beta & \text{on } \Sigma_h \end{cases}$$

D_α^h : Covariant Derivative on Σ_h

Magnetical Profiles

$\mathbf{v}_j := (\mathcal{V}_j^\alpha ; v_j)$ in (y_β, Y_3) coordinates

$$\mathbf{v}_0(y_\beta, Y_3) = H_0(y_\beta) e^{-\lambda Y_3},$$

$$\mathcal{V}_1^\alpha(y_\beta, Y_3) = \left[H_1^\alpha(y_\beta) + Y_3 \left(\mathcal{H} H_0^\alpha + b_\sigma^\alpha H_0^\sigma \right) (y_\beta) \right] e^{-\lambda Y_3},$$

$$v_1(y_\beta, Y_3) = \lambda^{-1} \mathbf{D}_\alpha H_0^\alpha(y_\beta) e^{-\lambda Y_3}$$

with $H_j^\alpha(y_\beta) := (\mathbf{n} \times \mathbf{H}_j^+) \times \mathbf{n}^\alpha(y_\beta, 0)$

Comparison with



H. HADDAR, P. JOLY, H.N. NGUYEN (2008)

Skin Effect and Skin Depth

1D model of *Skin Effect* : $\{z > 0\}$ = half-space conductor body

$$\|\mathbf{E}_\sigma(z)\| = E_0 e^{-\frac{z}{\ell(\sigma)}}, \quad \ell(\sigma) := \sqrt{\frac{2}{\omega\mu_0\sigma}} \quad \text{Skin Depth}$$

$$\ell(\sigma) = \delta / \operatorname{Re} \lambda.$$

3D model of *Skin Effect*, using profiles $\mathbf{V}_0, \mathbf{V}_1$,

$$\mathbf{V}(y_\alpha, y_3) := \mathbf{V}_0(y_\alpha, \frac{y_3}{\delta}) + \delta \mathbf{V}_1(y_\alpha, \frac{y_3}{\delta})$$

Definition

Suppose that for all y_α , $\|\mathbf{V}(y_\alpha, 0)\| \neq 0$. The *Skin Depth* in a conductor body at conductivity σ and y_α on a Regular Surface Σ is the positive real $\mathcal{L}(\sigma, y_\alpha)$ defined on Σ taking the smallest value s.t.

$$\|\mathbf{V}(y_\alpha, \mathcal{L}(\sigma, y_\alpha))\| = \frac{1}{e} \|\mathbf{V}(y_\alpha, 0)\|$$

Asymptotic Behavior for Skin Depth

Theorem

Let Σ be a Regular Surface with mean curvature \mathcal{H} . Suppose that $H_0(y_\alpha) \neq 0$. The Skin Depth $\mathcal{L}(\sigma, y_\alpha)$ has the asymptotic expansion

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma) \left(1 + \mathcal{H}(y_\alpha) \ell(\sigma) + \mathcal{O}(\sigma^{-1}) \right)$$



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Key of the proof:

$$\|\mathbf{v}(y_\alpha, y_3)\|^2 = \left[\|H_0\|^2 + 2\delta \operatorname{Re}\langle H_0, H_1 \rangle + 2y_3 \mathcal{H} \|H_0\|^2 + \mathcal{O}(\delta, y_3) \right] e^{-2y_3/\ell(\sigma)}$$

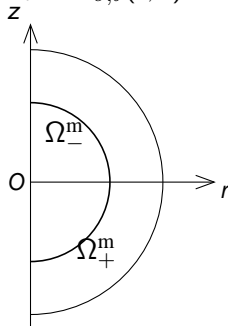
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Numerical Simulations

FEM Discretization

- Magnetic Field
- Framework: Ω_- and Ω Axisymmetric Domains
- \mathbf{F} Axisymmetric & Orthoradial $\rightarrow \mathbf{H}_\sigma$ Axisymmetric & Orthoradial
2D Problem \rightarrow accuracy
- Variational Form : Unknown $H_\sigma = H_{\sigma,\theta}(r, z) \in V$

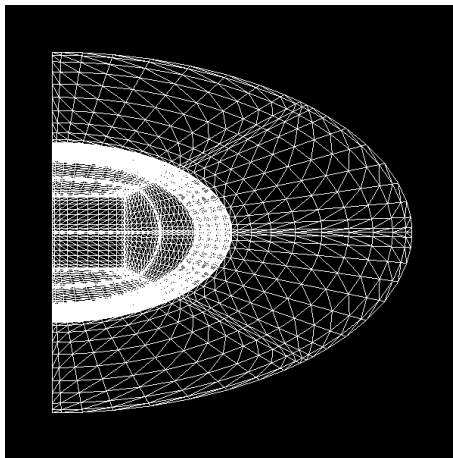


- Finite Element Library Melina

Numerical simulations

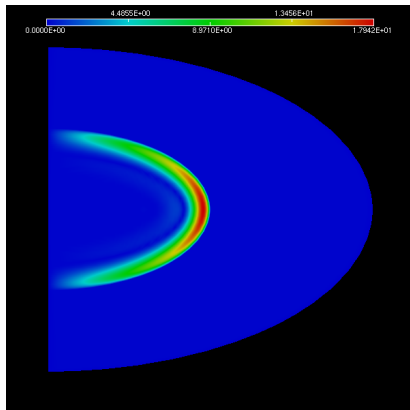
Mesh Refinement

Ω_- & Ω coaxial ellipsoidal domains \rightarrow Ω_-^m & Ω^m coaxial half-ellipsis

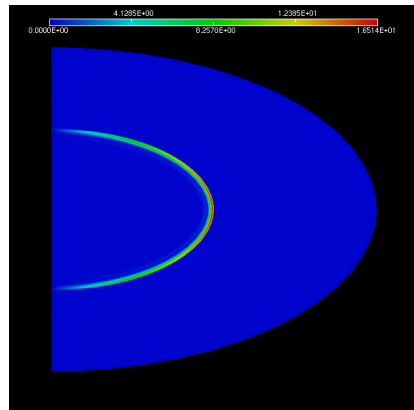


Numerical Simulations

Skin Effect in ellipsoidal geometry



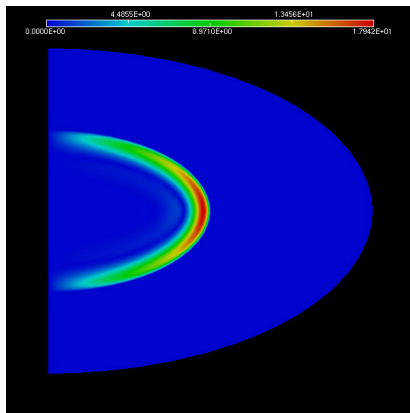
$|\operatorname{Im} H_\sigma|$ for $\sigma = 5\text{S.m}^{-1}$



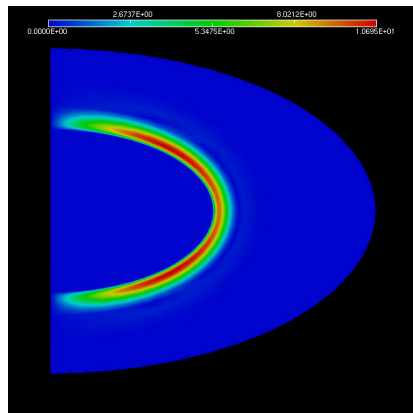
$|\operatorname{Im} H_\sigma|$ for $\sigma = 80\text{S.m}^{-1}$

Asymptotic Behavior for Skin Depth in Ellipsoidal Geometry

$$|\operatorname{Im} H_\sigma| \text{ for } \sigma = 5\text{S.m}^{-1}$$



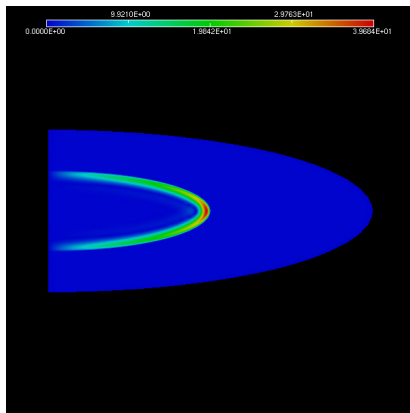
$$\text{Conductor} = \Omega_-^m (\mathcal{H} > 0)$$



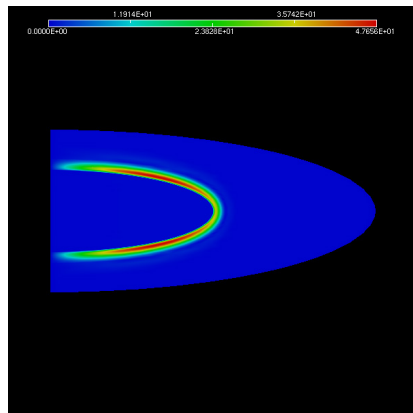
$$\text{Conductor} = \Omega_+^m (\mathcal{H} < 0)$$

Asymptotic Behavior for Skin Depth in Ellipsoidal Geometry

$$|\operatorname{Im} H_\sigma| \text{ for } \sigma = 5\text{S.m}^{-1}$$



$$\text{Conductor} = \Omega_-^m$$



$$\text{Conductor} = \Omega_+^m$$

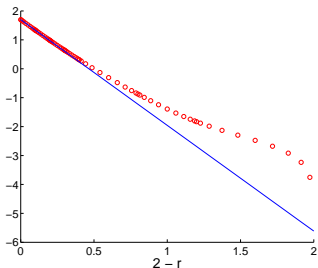
Numerical Post-Treatment

Linear Regression

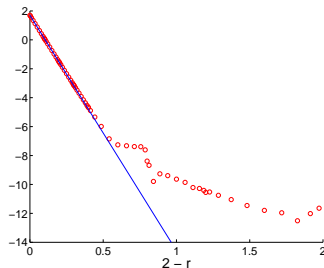
Extract $|H_\sigma(h)|$ along edges of the mesh of Ω_-^m when $z = 0$: $h = 2 - r$.

Linear Regression in $skin(\sigma) \rightarrow n(\sigma)$ points \rightarrow "numerical" slope $s(\sigma)$

$$\log_{10} |H_\sigma(h)| = s(\sigma)h + b$$



○ : $\log_{10} |H_\sigma(2 - r)|$ when $\sigma = 5$
 $n(\sigma) = 7$, $s(\sigma) = -3,65542$



○ : $\log_{10} |H_\sigma(2 - r)|$ when $\sigma = 80$
 $n(\sigma) = 3$, $s(\sigma) = -16,279162$

Post-Treatment

Accuracy of Asymptotics

- Truncated Asymptotic Expansion : $\mathbf{H}_{\theta,1}^- := \mathcal{H}_{\theta,0}^- + \delta \mathcal{H}_{\theta,1}^-$
- $\log_{10} |\mathbf{H}_{\theta,1}^-(\cdot, 2-r)| = c + \beta(\sigma, \Sigma)(2-r) + \mathcal{O}(\frac{1}{\sqrt{\sigma}}; 2-r)$
- "Theoretical" slope $\beta(\sigma, \Sigma) := \frac{1}{\ln 10} (\mathcal{H} - \frac{1}{\ell(\sigma)})$
- Relative Error between slopes: $error(\sigma) := \left| \frac{\beta(\sigma, \Sigma) - s(\sigma)}{\beta(\sigma, \Sigma)} \right|$

σ	5	20	80
$skin(\sigma)$ (cm)	10.3	5.15	2.58
$n(\sigma)$	7	5	3
$s(\sigma)$	-3, 65542	-7, 883903	-16, 279162
$\beta(\sigma, \Sigma)$	-3, 673319	-7, 889506	-16, 32188
$error(\sigma)$ (%)	0, 48	0, 07	0, 26