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## A Transmission Problem in Electromagnetism with a Singular Interface

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# A Transmission Problem in Electromagnetism with a Singular Interface

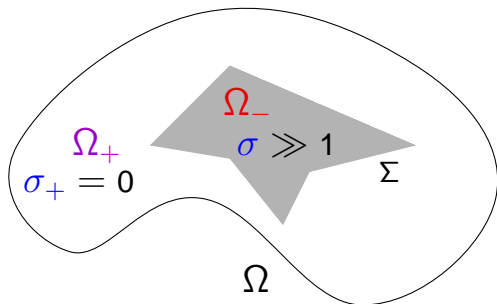
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6th Singular Days on Asymptotic Methods for PDEs  
WIAS Berlin, April 29 – May 1, 2010

## The Skin Effect : A Model Problem



- $\Omega_-$  Highly Conducting body  $\subset\subset \Omega$ : Conductivity  $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$ : Interface
- $\Omega_+$  Insulating or Dielectric body: Conductivity  $\sigma_+ = 0$

The **Skin Effect**: rapid decay of electromagnetic fields with depth inside the conductor.

The **Skin Depth**:  $\ell(\sigma) = \sqrt{2/\omega\mu_0\sigma}$

## References



V. PÉRON (PhD 09)



G. CALOZ, M. DAUGE, V. PÉRON (JMAA 10)

*Uniform Estimates for Transmission Problems with High Contrast in Heat Conduction and Electromagnetism*



M. DAUGE, E. FAOU, V. PÉRON (CRAS 10)

*Asymptotic Behavior for High Conductivity of the Skin Depth Electromagnetism*

- Aim : Understanding the influence of the geometry of a conducting body on the skin effect in electromagnetism.

# Outline

- 1 Framework
- 2 3D Multiscaled Asymptotic Expansion
- 3 Axisymmetric Problems
- 4 Finite Element Computations
- 5 Numerical simulations of skin effect
- 6 Postprocessing

# Outline

- 1 **Framework**
- 2 3D Multiscaled Asymptotic Expansion
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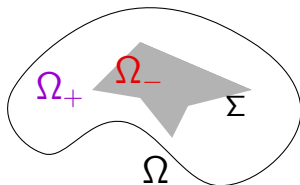
# Framework

## Maxwell Problem

$$(\mathbf{P}_{\underline{\sigma}}) \quad \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma}) \mathbf{E} = \mathbf{j}$$

$$\underline{\sigma} = (0, \sigma \gg 1)$$

$$\mathbf{j} \in \mathbf{H}_0(\operatorname{div}, \Omega) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{div} \mathbf{u} \in L^2(\Omega), \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$



Perfectly Conducting Magnetic Wall B. C.:

$$\mathbf{E} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{H} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega$$

# Existence of solutions

## Hypothesis (SH)

The angular frequency  $\omega$  is not an eigenfrequency of the problem

$$\left\{ \begin{array}{ll} \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 & \text{and } \operatorname{curl} \mathbf{H} + i\omega\varepsilon_0 \mathbf{E} = 0 & \text{in } \Omega_+ \\ \mathbf{E} \times \mathbf{n} = 0 & & \text{on } \Sigma \\ \text{B.C.} & & \text{on } \partial\Omega \end{array} \right.$$

## Theorem (CALOZ, DAUGE, P., 09)

If the surface  $\Sigma$  is Lipschitz, under Hypothesis (SH), there exist  $\sigma_0$  and  $C > 0$ , such that for all  $\sigma \geq \sigma_0$ ,  $(\mathbf{P}_\sigma)$  with B.C. and  $\mathbf{j} \in \mathbf{H}(\operatorname{div}, \Omega)$  has a unique solution  $(\mathbf{E}, \mathbf{H})$  in  $L^2(\Omega)^2$ , and

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma} \|\mathbf{E}\|_{0,\Omega_-} \leq C \|\mathbf{j}\|_{\mathbf{H}(\operatorname{div}, \Omega)}$$

Application: Convergence of asymptotic expansion for large conductivity



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# References and Notations

Asymptotic Expansion as  $\sigma \rightarrow \infty$  of solutions of  $(\mathbf{P}_\sigma)$  when  $\Sigma$  is smooth :



E.P. STEPHAN, R.C. MCCAMY (83-84-85)

*Plane Interface and Eddy Current Problems*



H. HADDAR, P. JOLY, H.N. NGUYEN (08)

*Generalized Impedance Boundary Conditions*



G. CALOZ, M. DAUGE, V. PÉRON (JMAA 10)



M. DAUGE, E. FAOU, V. PÉRON (CRAS 10)



PÉRON (09)

## Hypothesis

- 1  $\Sigma$  is a Smooth Surface
- 2  $\omega$  satisfies the Spectral Hypothesis (SH)
- 3  $\mathbf{j}$  is smooth and  $\mathbf{j} = 0$  in  $\Omega_-$

# Asymptotic Expansion

$$\delta := \sqrt{\omega \varepsilon_0 / \sigma} \longrightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty$$

By Theorem there exists  $\delta_0$  s.t. for all  $\delta \leq \delta_0$ , the solution  $\mathbf{H}(\delta)$  to  $(\mathbf{P}_\sigma)$  :

$$\mathbf{H}_{(\delta)}^+(\mathbf{x}) \approx \mathbf{H}_0^+(\mathbf{x}) + \delta \mathbf{H}_1^+(\mathbf{x}) + \mathcal{O}(\delta^2)$$

$$\mathbf{H}_{(\delta)}^-(\mathbf{x}) \approx \mathbf{H}_0^-(\mathbf{x}; \delta) + \delta \mathbf{H}_1^-(\mathbf{x}; \delta) + \mathcal{O}(\delta^2)$$

$$\text{with } \mathbf{H}_j^-(\mathbf{x}; \delta) = \chi(y_3) \mathbf{V}_j(y_\beta, \frac{y_3}{\delta}).$$

$(y_\beta, y_3)$  : “*normal coordinates*” to  $\Sigma$  in a tubular region  $\mathcal{U}_-$  of  $\Sigma$  in  $\Omega_-$

$\mathbf{H}_j^+ \in \mathbf{H}(\text{curl}, \Omega_+)$  and  $\mathbf{V}_j \in \mathbf{H}(\text{curl}, \Sigma \times \mathbb{R}_+)$  *profiles*.

$$\|\mathbf{H}_j^-(\mathbf{x}; \delta)\|_{0, \Omega_-} \leq C_j \sqrt{\delta} \quad \text{for all } j \in \mathbb{N}$$

## Profiles of the Magnetic Field

$\mathbf{v}_j =: (\mathcal{V}_j^\alpha; v_j)$  in coordinates  $(y_\beta, Y_3)$  with  $Y_3 = \frac{y_3}{\delta}$

$$\mathbf{v}_0(y_\beta, Y_3) = \mathbf{h}_0(y_\beta) e^{-\lambda Y_3}$$

$$\mathcal{V}_1^\alpha(y_\beta, Y_3) = \left[ h_1^\alpha + Y_3 \left( \mathcal{H} h_0^\alpha + b_\sigma^\alpha h_0^\sigma \right) \right] (y_\beta) e^{-\lambda Y_3}$$

Here,

$\mathcal{H}$  mean curvature of  $\Sigma$

$$\mathbf{h}_0(y_\beta) = (\mathbf{n} \times \mathbf{H}_0^+) \times \mathbf{n}(y_\beta, 0) \quad \text{and} \quad h_j^\alpha(y_\beta) := (\mathbf{H}_j^+)^\alpha(y_\beta, 0)$$

$$\lambda = \omega \sqrt{\varepsilon_0 \mu_0} e^{-i\pi/4}$$

# Application

## A New Definition of the Skin Depth

$$\mathbf{V}_{(\delta)}(y_\alpha, y_3) := \mathbf{H}_{(\delta)}^-(\mathbf{x}), \quad y_\alpha \in \Sigma, \quad 0 \leq y_3 < h_0$$

### Definition

Let  $\Sigma$  be a smooth surface, and  $\mathbf{j}$  s.t.  $\mathbf{V}_{(\delta)}(y_\alpha, 0) \neq 0$ . The skin depth is the smallest length  $\mathcal{L}(\sigma, y_\alpha)$  defined on  $\Sigma$  s.t.

$$\|\mathbf{V}_{(\delta)}(y_\alpha, \mathcal{L}(\sigma, y_\alpha))\| = \|\mathbf{V}_{(\delta)}(y_\alpha, 0)\| e^{-1}$$

### Theorem (DAUGE, FAOU, P., 10)

Let  $\Sigma$  be a regular surface with mean curvature  $\mathcal{H}$ , and assume  $\mathbf{h}_0(y_\alpha) \neq 0$ .

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma) \left( 1 + \mathcal{H}(y_\alpha) \ell(\sigma) + \mathcal{O}(\sigma^{-1}) \right), \quad \sigma \rightarrow \infty$$

Key of the proof:

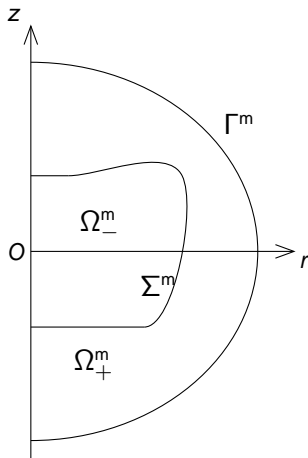
$$\|\mathbf{V}_{(\delta)}\|^2 = \left[ \|\mathbf{h}_0\|^2 + 2y_3 \mathcal{H} \|\mathbf{h}_0\|^2 + 2\delta \operatorname{Re}\langle \mathbf{h}_0, \mathbf{h}_1 \rangle + \mathcal{O}((\delta + y_3)^2) \right] e^{-2y_3/\ell(\sigma)}$$

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# Axisymmetric domains

## The meridian domain



**Figure:** The meridian domain  $\Omega^m = \Omega_-^m \cup \Omega_+^m \cup \Sigma^m$

# Reduction problem

$$\begin{cases} (\operatorname{curl} \mathbf{H})_r = \frac{1}{r} \partial_\theta H_z - \partial_z H_\theta, \\ (\operatorname{curl} \mathbf{H})_\theta = \partial_z H_r - \partial_r H_z, \\ (\operatorname{curl} \mathbf{H})_z = \frac{1}{r} (\partial_r (r H_\theta) - \partial_\theta H_r). \end{cases}$$

The Maxwell problem is axisymmetric : the coefficients do not depend on the angular variable  $\theta$ .



# Reduction problem

$$\begin{cases} (\operatorname{curl} \mathbf{H})_r = \frac{1}{r} \partial_\theta H_z - \partial_z H_\theta, \\ (\operatorname{curl} \mathbf{H})_\theta = \partial_z H_r - \partial_r H_z, \\ (\operatorname{curl} \mathbf{H})_z = \frac{1}{r} (\partial_r (r H_\theta) - \partial_\theta H_r). \end{cases}$$

The Maxwell problem is axisymmetric : the coefficients do not depend on the angular variable  $\theta$ .

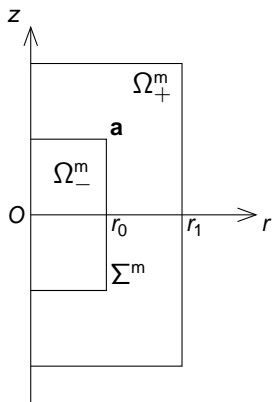
$\mathbf{H}$  is axisymmetric iff  $\check{\mathbf{H}} := (H_r, H_\theta, H_z)$  does not depend on  $\theta$ .

Assume that the right-hand side is axisymmetric and orthoradial. Then,  $\mathbf{H}_{(\delta)}$  is axisymmetric and orthoradial :

$$\check{\mathbf{H}}_{(\delta)}(r, \theta, z) = (0, h_{(\delta)}(r, z), 0).$$

# Configurations chosen for computations

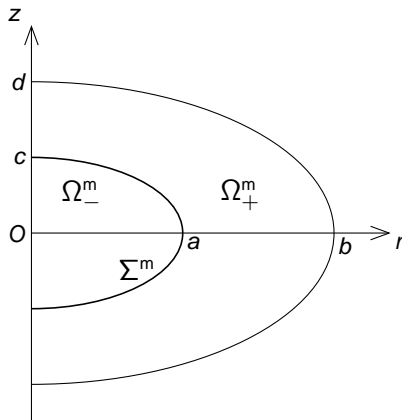
## Configuration A



**Figure:** The meridian domain  $\Omega^m$  in configuration A

# Configurations chosen for computations

## Configuration B



**Figure:** The meridian domain  $\Omega^m$  in configuration B

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# Finite Element Method

- High order elements available in the finite element library MéliNa
- $h_{(\delta)}^{\rho, \mathfrak{M}}$ : the computed solution of the discretized problem with an interpolation degree  $\rho$  and a mesh  $\mathfrak{M}$

$$A_{\sigma}^{\rho, \mathfrak{M}} := \|h_{(\delta)}^{\rho, \mathfrak{M}}\|_{L_1^2(\Omega_-^m)} \quad \text{with} \quad \sigma = \omega \varepsilon_0 \delta^{-2}$$

$$\|v\|_{L_1^2(\Omega_-^m)}^2 = \int_{\Omega_-^m} |v|^2 r dr dz .$$

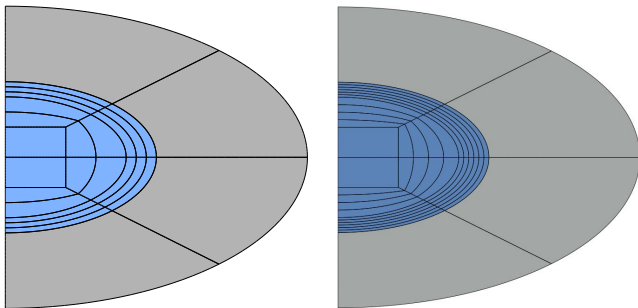
- In the computations, the angular frequency  $\omega = 3.10^7$ .

# Interpolation degree

## Configuration B

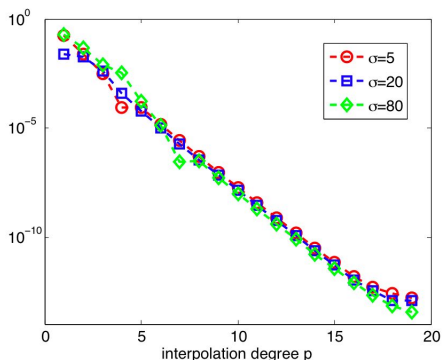
We first check the convergence when the interpolation degree  $p$  of the finite elements increases.

We consider the discretized problem with different degrees:  $Q_p$ ,  $p = 1, \dots, 20$ , and with 2 different meshes:



**Figure:** The meshes  $\mathfrak{M}_3$  and  $\mathfrak{M}_6$

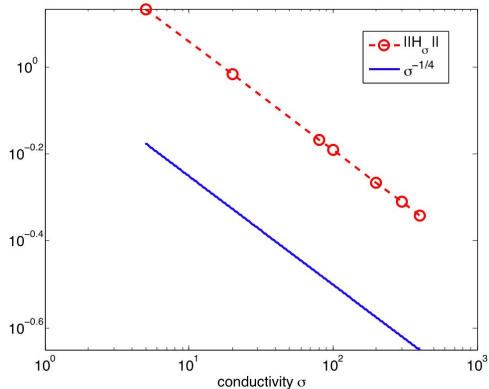
We represent the absolute value of the difference between the weighted norms  $A_{\sigma}^{\rho, \mathcal{M}_3}$  and  $A_{\sigma}^{20, \mathcal{M}_6}$ , versus  $p$  in semilogarithmic coordinates



SCHWAB, SURI (96)

*theoretical results of convergence for the p-version of problems with boundary layers*

We plot in log-log coordinates the weighted norm  $A_{\sigma}^{16, \mathcal{M}_3}$  with respect to  $\sigma$  with red circles.

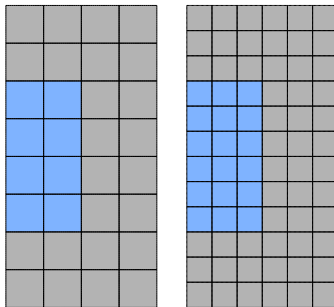


The figure shows that  $A_{\sigma}^{16, \mathcal{M}_3}$  behaves like  $\sigma^{-1/4}$  (solid line) when  $\sigma \rightarrow \infty$ . This behavior is consistent with the asymptotic expansion.



# Configuration A

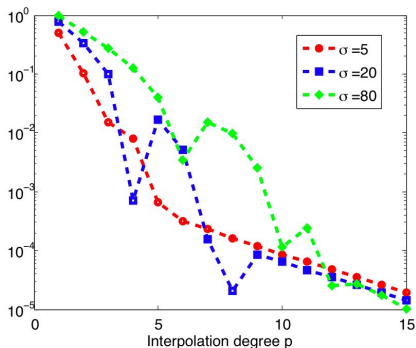
We consider a family of meshes with square elements  $\mathfrak{M}_k$ , with size  $h = 1/k$



**Figure:** Meshes  $\mathfrak{M}_2$ , and  $\mathfrak{M}_3$  in configuration A

## Configuration A

We represent the absolute value of the difference between  $A_\sigma^{\rho, \mathfrak{M}_2}$  and  $A_\sigma^{16, \mathfrak{M}_3}$ , versus  $\rho$  in semilogarithmic coordinates



The figure shows that  $A_\sigma^{\rho, \mathfrak{M}_2}$  approximates  $A_\sigma^{16, \mathfrak{M}_3}$  better than  $10^{-4}$  when  $\rho \geq 12$ .

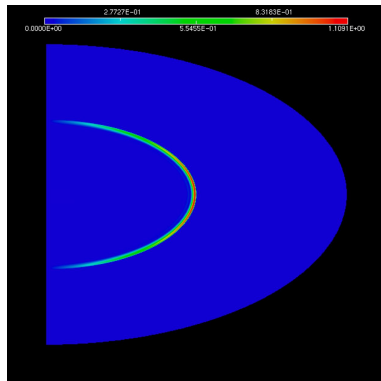
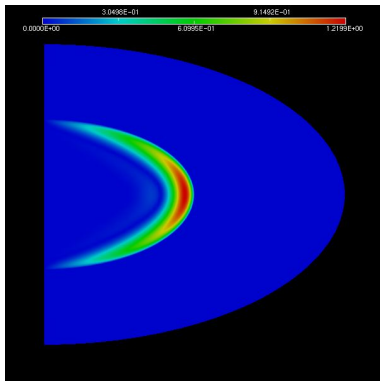
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# Skin effect in configuration B

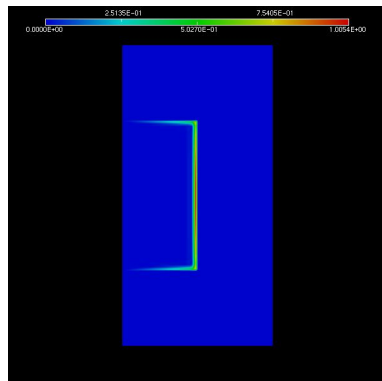
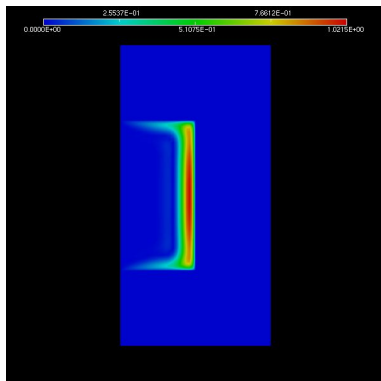
$$|\operatorname{Im} h_{(\delta)}^+| = \mathcal{O}(\delta).$$

Thus, the imaginary part of the computed field is located in the conductor.



**Figure:** Configuration B.  $|\operatorname{Im} H_\sigma|$  when  $\sigma = 5$  and  $\sigma = 80$

# Skin effect in configuration A



**Figure:** Configuration A.  $|\text{Im } \mathbf{H}_\sigma|$  when  $\sigma = 5$  and  $\sigma = 80$

# Influence of the Geometry on the Skin effect

## Configuration B

$\mathcal{H} > 0$  on the left, and  $\mathcal{H} < 0$  on the right

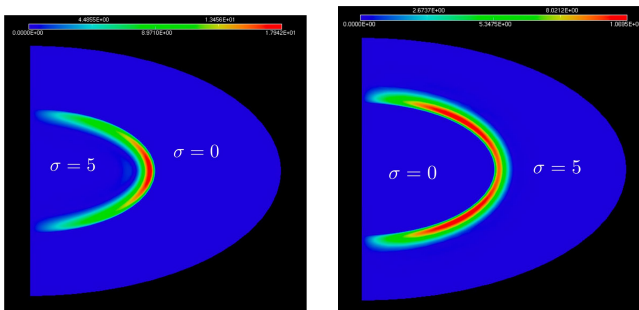


Figure:  $|\operatorname{Im} H_\sigma|, \sigma = 5$

The skin depth is larger when the mean curvature of the conducting body surface is larger.

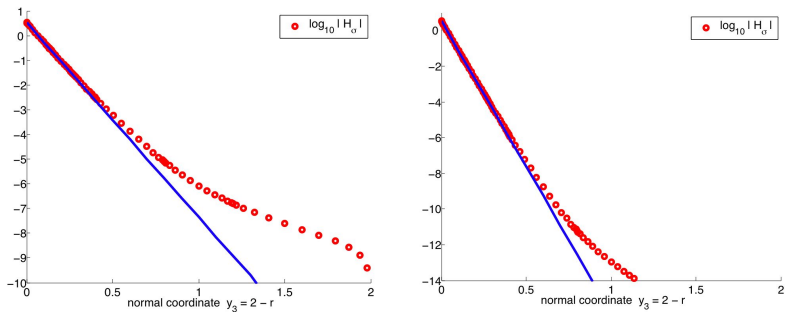
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## Configuration B

We perform numerical treatments from computations in configuration B.

We extract values of  $\log_{10} |\mathbf{H}_\sigma|$  in  $\Omega_-^m$  along the axis  $z = 0 : y_3 = 2 - r$ .



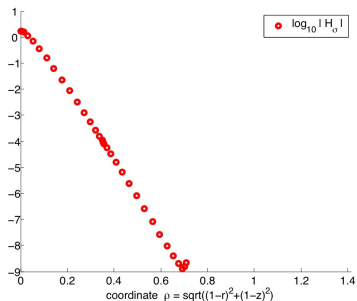
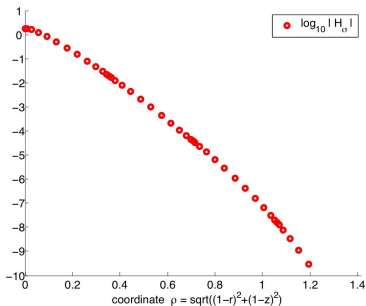
**Figure:** On the left  $\sigma = 20$ . On the right,  $\sigma = 80$ .

The curves exactly behave like lines: the exponential decay is obvious.



# Configuration A

We extract values of  $\log_{10} |H_\sigma|$  in  $\Omega_-^m$  along the diagonal axis  $r = z$



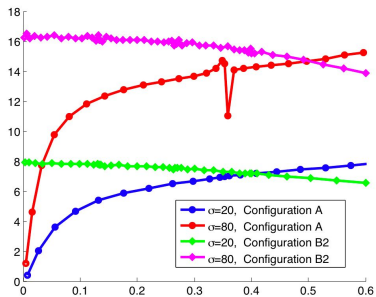
Here,  $\rho = \sqrt{(1-r)^2 + (1-z)^2}$  is the distance to the corner point  $\mathbf{a}(r = 1, z = 1)$ .

The exponential decay is not obvious in configuration A.

To measure a possible exponential decay, we define the slopes

$$\tilde{s}_i(\sigma) := \frac{\log_{10} |\mathbf{H}_\sigma(r_i, z_i)| - \log_{10} |\mathbf{H}_\sigma(z_{i+1}, r_{i+1})|}{\rho_{i+1} - \rho_i}.$$

Here,  $\rho_i := \sqrt{(1 - r_i)^2 + (1 - z_i)^2}$  is the distance from the extraction points  $(r_i, z_i)$  to  $\mathbf{a}$ .



**Figure:** The graphs of the slopes  $\tilde{s}_i(\sigma)$

## Asymptotics in the conducting part

In configuration A, the slopes tend to 0, which means that there is no exponential convergence near the corner.

Nevertheless, a sort of exponential convergence is restored in a region further away from the corner.

The principal asymptotic contribution inside the conductor is a profile globally defined on a sector  $\mathcal{S}$  (of opening  $\frac{\pi}{2}$ ) solving the model Dirichlet problem

$$\begin{cases} -i(\partial_X^2 + \partial_Y^2)v_0 + \kappa^2 v_0 = 0 & \text{in } \mathcal{S}, \\ v_0 = h_0^+(\mathbf{a}) & \text{on } \partial\mathcal{S}, \end{cases}$$

instead the 1D problem in configuration B

$$\begin{cases} -i\partial_Y^2 v_0 + \kappa^2 v_0 = 0 & \text{for } 0 < Y < +\infty, \\ v_0 = h_0^+ & \text{for } Y = 0. \end{cases}$$