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Matching of asymptotic expansions for the wave propagation in media with thin slot

Sébastien Tordeux, Patrick Joly

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Matching of asymptotic expansions for the wave propagation in media with thin slot

Sébastien Tordeux and Patrick Joly

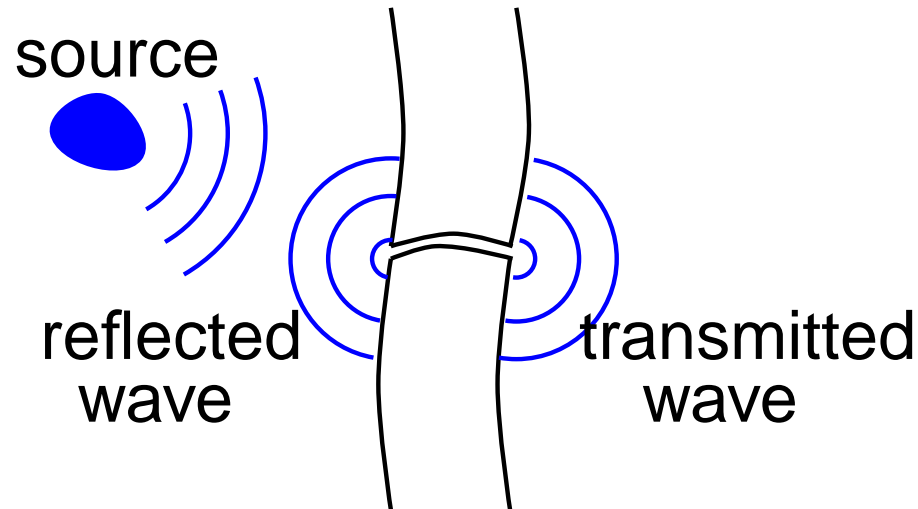
TiSCoPDE workshop, Berlin, September 2005

INRIA-Rocquencourt-Projet POEMS

ETH-SAM

A typical application

How can we study the scattering in media with **thin slot** ?

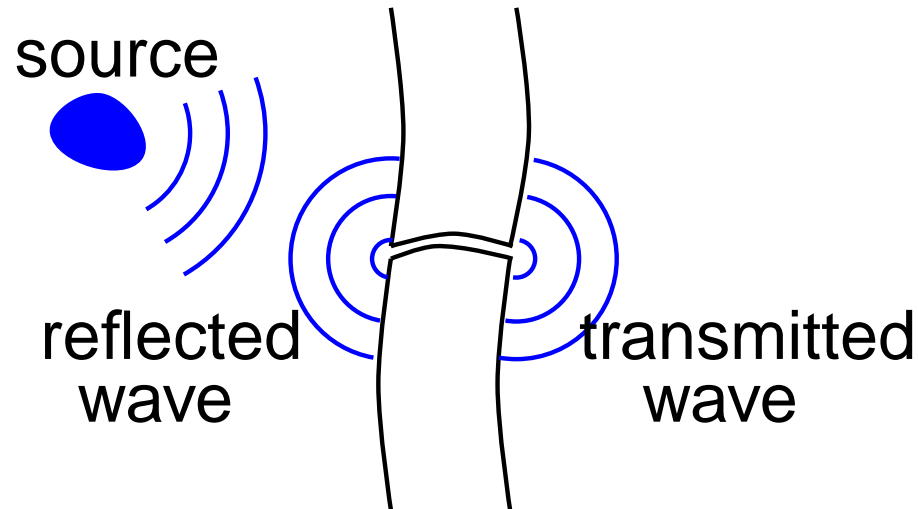


A physical problem with two **characteristical** lengthes

- The **wavelength** λ
- The **width** of the slot ε

A typical application

How can we study the scattering in media with **thin slot** ?

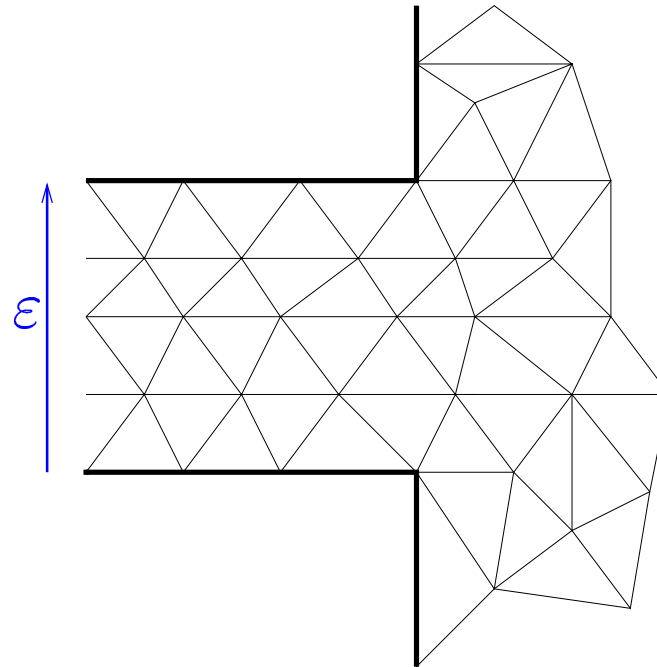


An **asymptotic** case:

$$\varepsilon \ll \lambda$$

The numerical difficulty

A **mesh step** smaller than ε



This leads to **costly** computations

Some references

- Thin slot:
[Harrington, Auckland \(1980\)](#), [Tatout \(1996\)](#).
- Finite differences:
[Taflove \(1995\)](#).
- Thin plates and junction theory,...
[Ciarlet, Le Dret, Dauge-Costabel](#).
- Matching of asymptotic expansions:
[McIver, Rawlins \(1993\)](#), [Il'in \(1992\)](#).
- multiscale analysis
[Maz'ya, Nazarov, Plamenevskii \(1991\)](#).
[Oleinik, Shamaev, Yosifian \(1992\)](#).

A simple problem

Scalar wave equation:

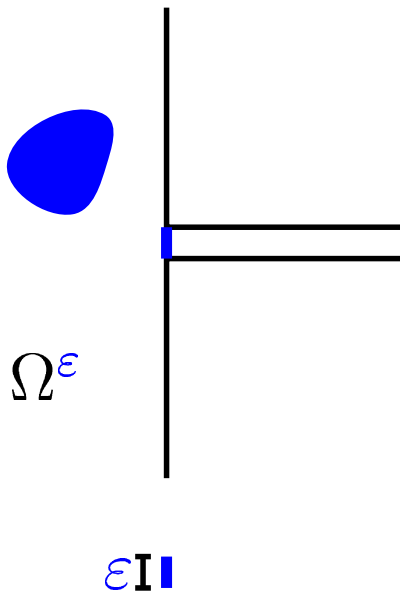
$$\frac{\partial^2 p^\varepsilon}{\partial t^2} - \Delta p^\varepsilon = f$$

Harmonic solution:

$$p^\varepsilon(x, y, t) = \exp(-i\omega t) u^\varepsilon(x, y)$$

Helmholtz Equation:

$$\Delta u^\varepsilon + \omega^2 u^\varepsilon = -f \quad \text{in } \Omega^\varepsilon$$



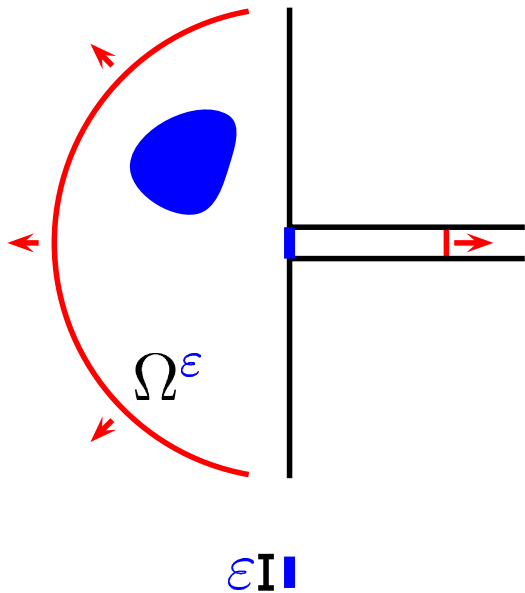
A simple problem

Outgoing solution at infinity:

$$\frac{\partial u^\varepsilon}{\partial n} - i\omega u^\varepsilon \leq \frac{C}{r^2}, \quad \text{for } r \text{ large,}$$

Neumann limit condition
(rigid wall)

$$\frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega^\varepsilon$$



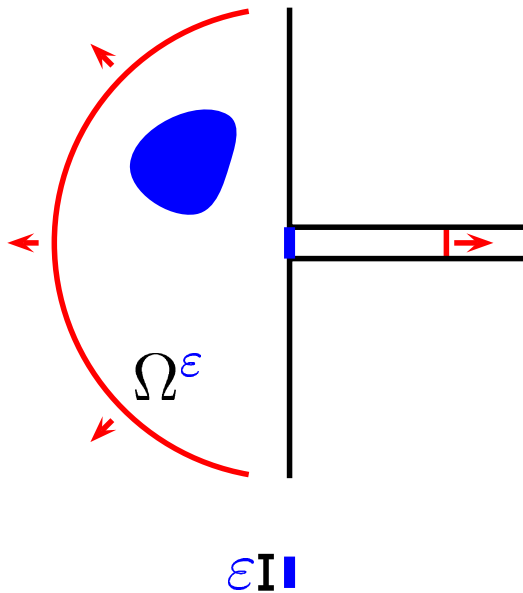
A simple problem

Outgoing solution at infinity:

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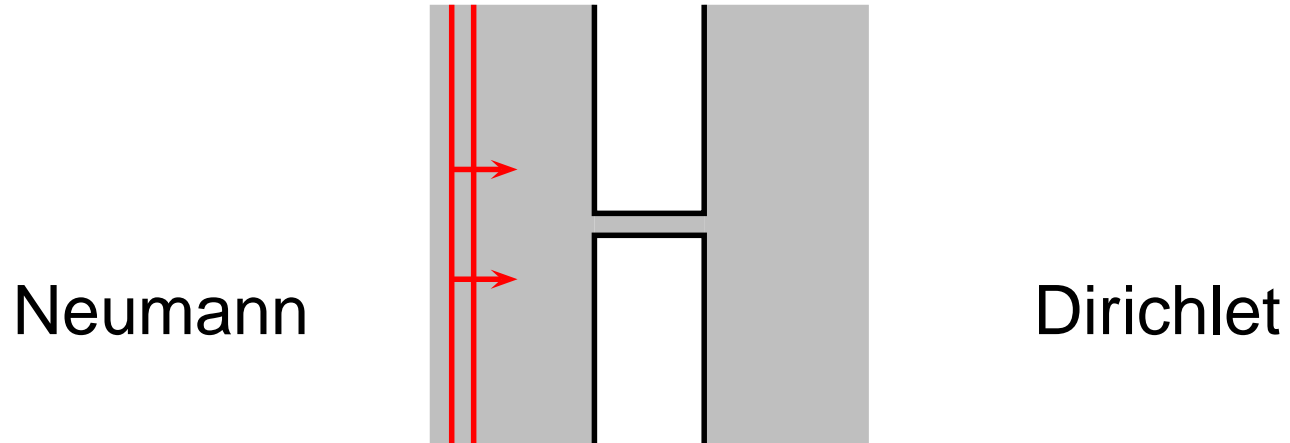
Neumann limit condition
(rigid wall)

$$\frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega^\varepsilon$$



With the **Dirichlet** limit condition, the transmission inside the slot is **negligible** ($o(\varepsilon^\infty)$).

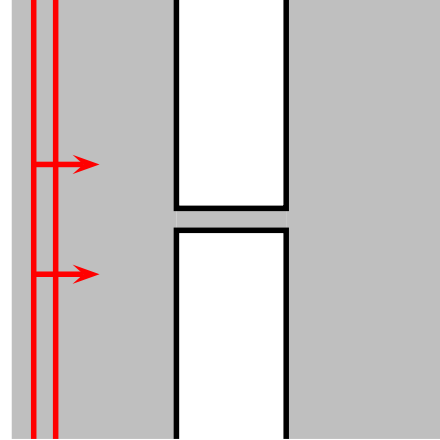
A numerical computation



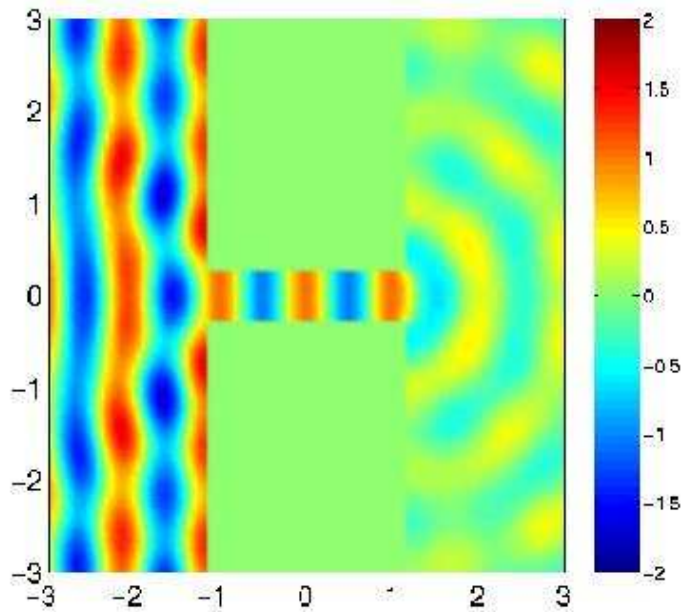
Numerical computation done with the **high order finite elements code** of (M. Duruflé, INRIA)

A numerical computation

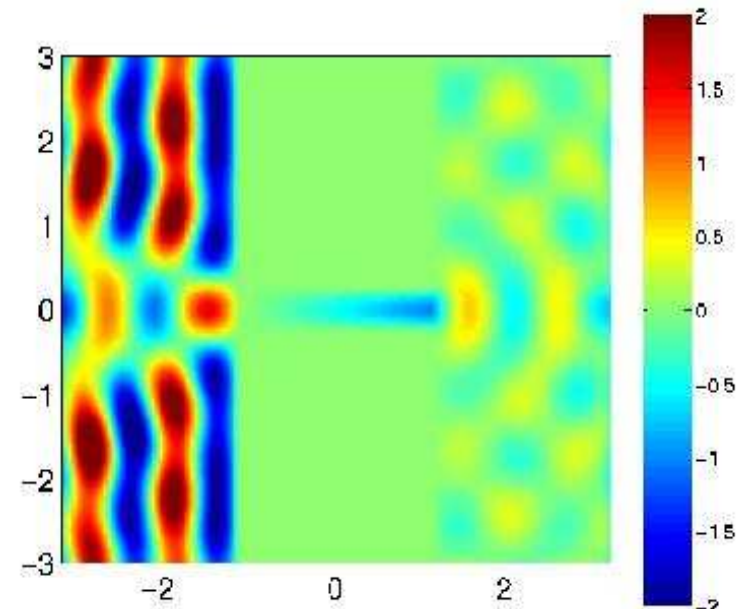
Neumann



Dirichlet

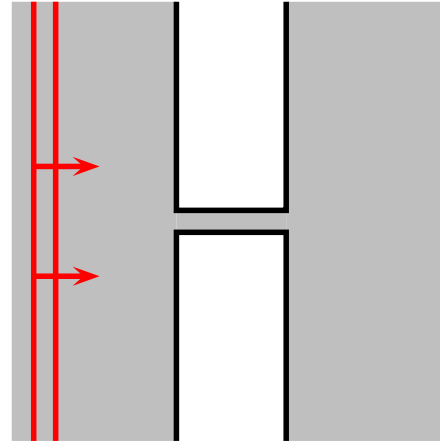


$$\frac{\varepsilon}{\lambda} = 0.5$$

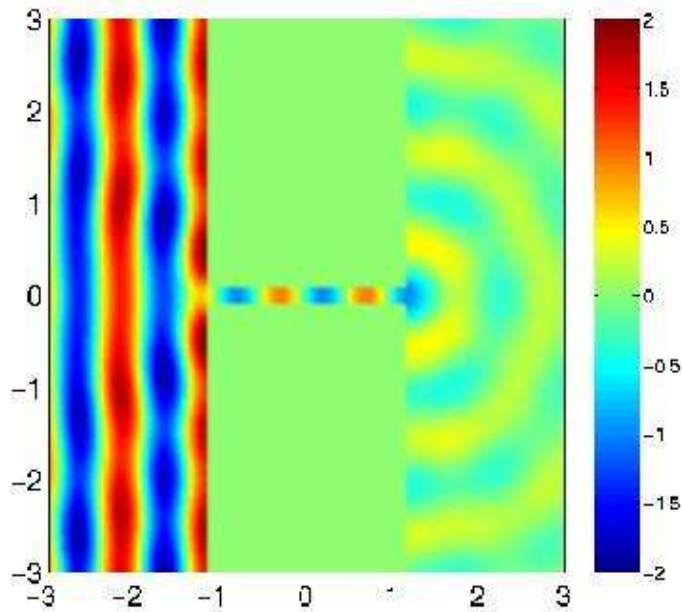


A numerical computation

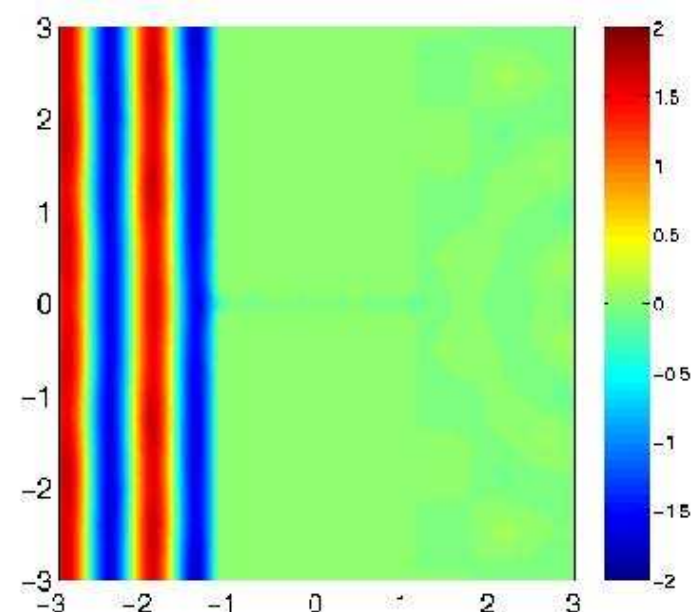
Neumann



Dirichlet

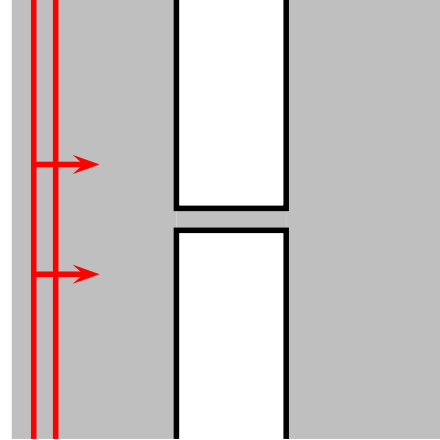


$$\frac{\varepsilon}{\lambda} = 0.2$$

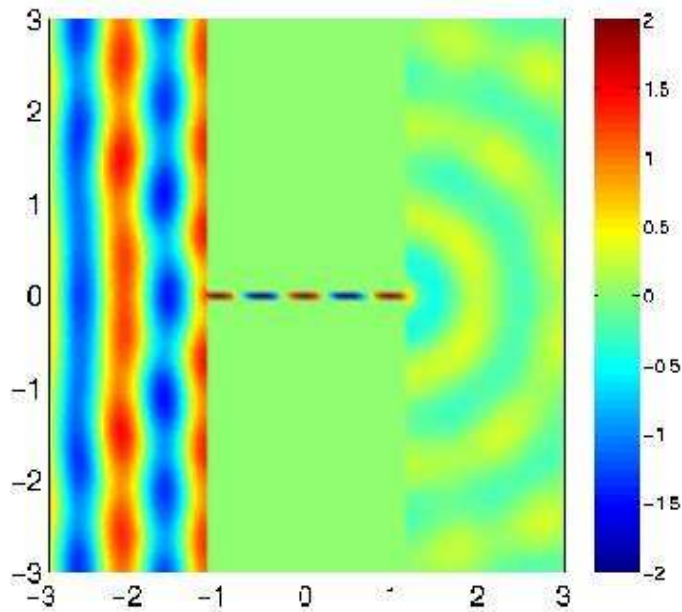


A numerical computation

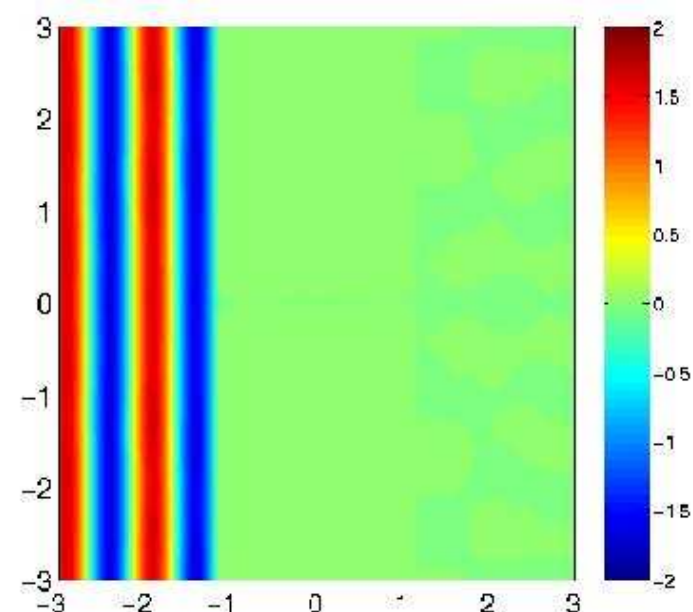
Neumann



Dirichlet



$$\frac{\varepsilon}{\lambda} = 0.1$$

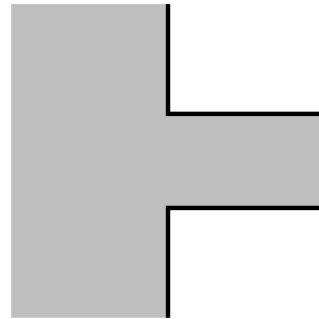


Objectives

- Introduce **accurate** numerical methods

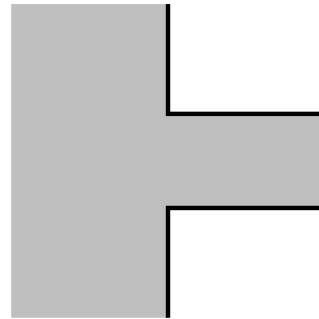
Objectives

- Introduce **accurate** numerical methods
- We need an **intermediate zone**



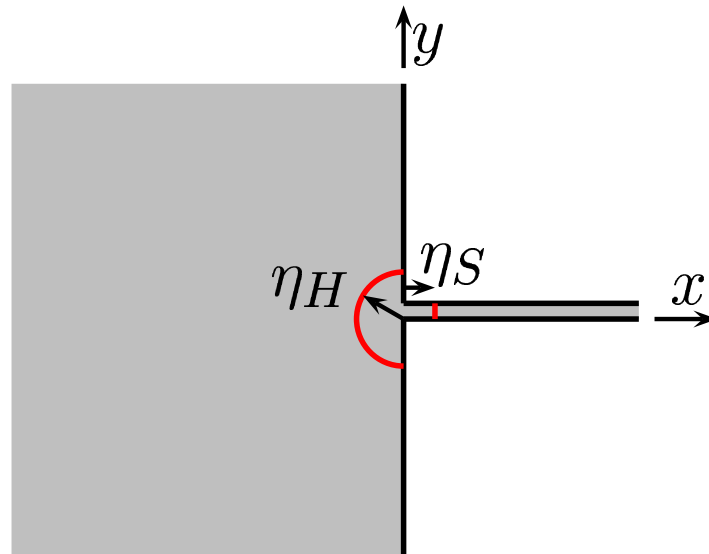
Objectives

- Introduce **accurate** numerical methods
- We need an **intermediate zone**



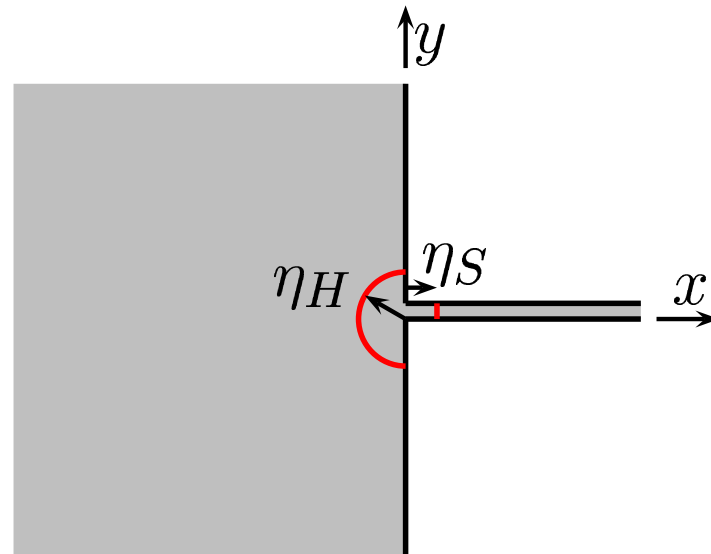
- A technique **the matching of asymptotic expansions**
 - Define **new approximate models** to compute the solution.
 - Use effectively “universal” technique of numerical computation (mesh reffinement).

Three zones



- Far field (2D field)
- Near field (boundary layer)
- Slot field (1D field)

Three zones

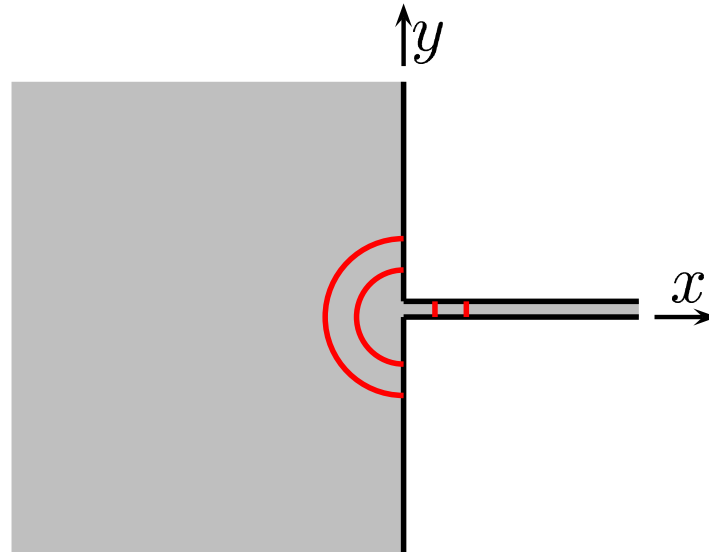


The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones

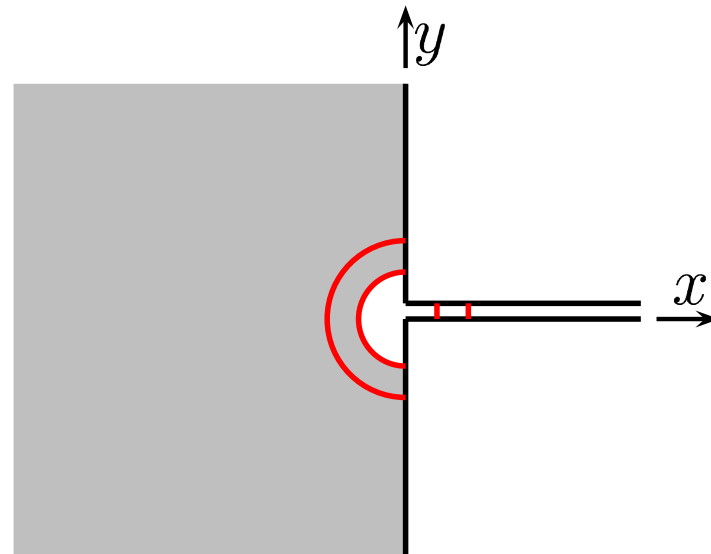


The asymptotic assumptions:

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Three zones



Far field

The **asymptotic assumptions**:

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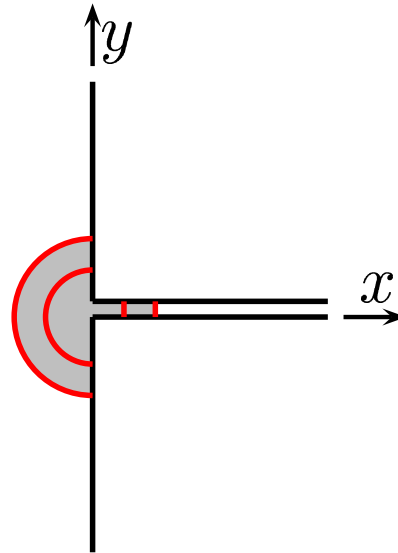
$$\varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0$$

$$\eta(\varepsilon) \rightarrow 0$$

$$\eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



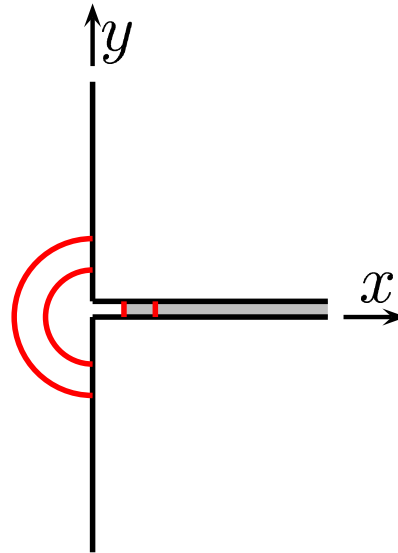
Near field

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



Slot field

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda,$$

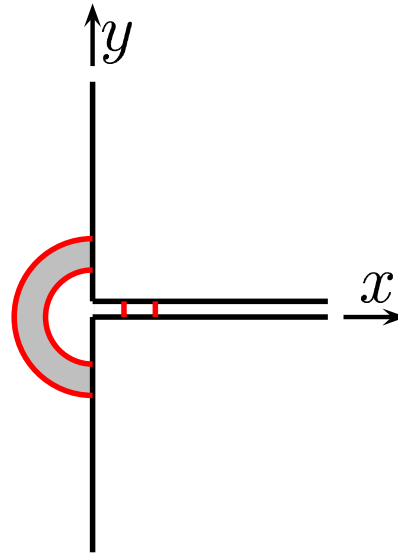
$$\varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0$$

$$\eta(\varepsilon) \rightarrow 0$$

$$\eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



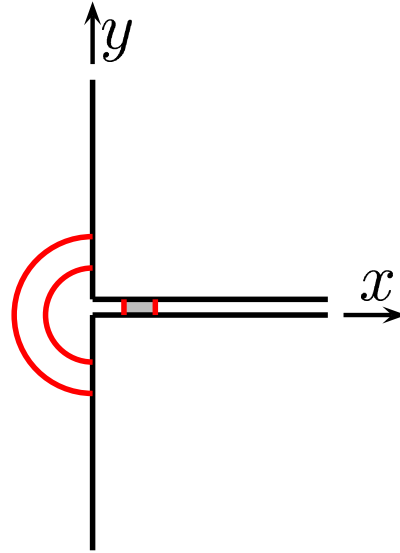
Far and near

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



Slot and near

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

The different steps of the method

- **Derivate** the asymptotic expansions:
 - **Formal** part
 - Several presentations are possible

The different steps of the method

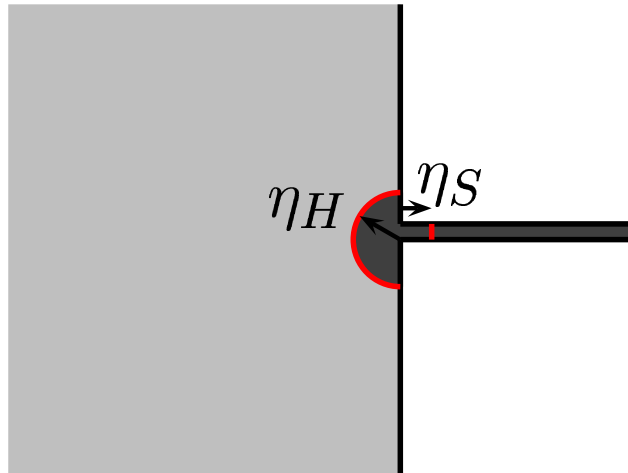
- **Derivate** the asymptotic expansions:
 - **Formal** part
 - Several presentations are possible
- **Describe** the asymptotic expansions
 - **Rigorous** part
 - **Definition** of the terms of the asymptotic expansions

The different steps of the method

- **Derivate** the asymptotic expansions:
 - **Formal** part
 - Several presentations are possible
- **Describe** the asymptotic expansions
 - **Rigorous** part
 - **Definition** of the terms of the asymptotic expansions
- **Mathematical validation** of the asymptotic expansions
 - **Rigorous** part
 - **Error estimates**

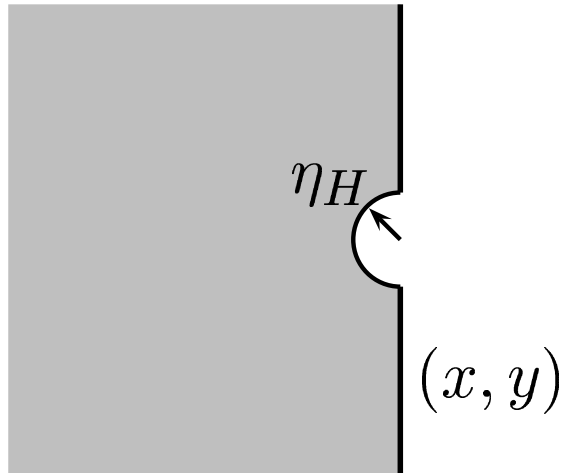
Far field

Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda.$



Far field

Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda.$



No **normalization**:

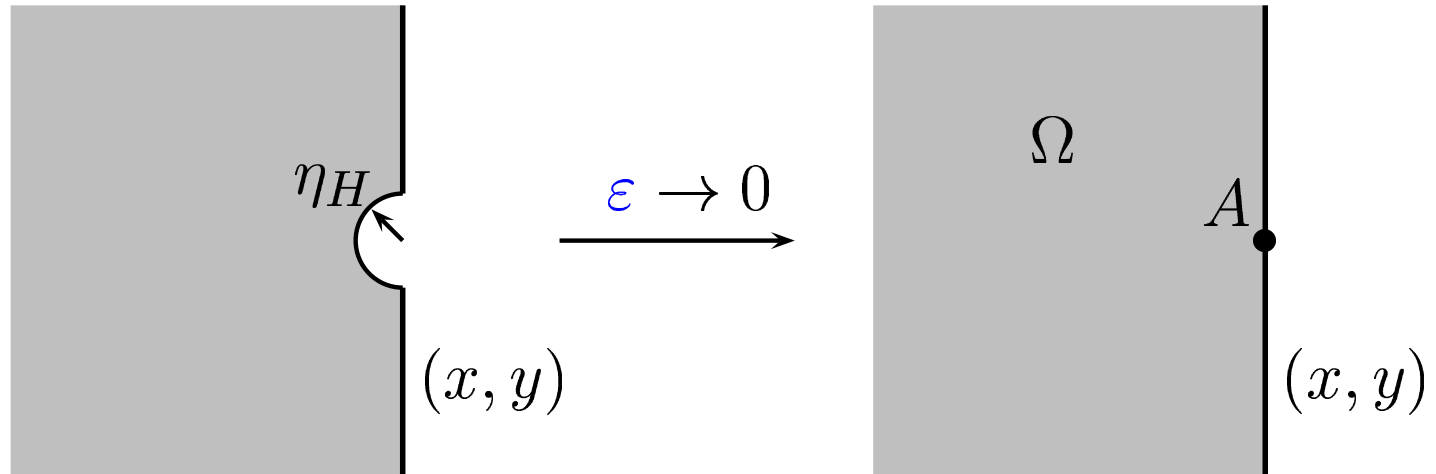
$$X = x,$$

$$Y = y.$$

Far field

Asymptotic context:

$$\varepsilon \ll \eta_H \ll \lambda.$$



No **normalization**:

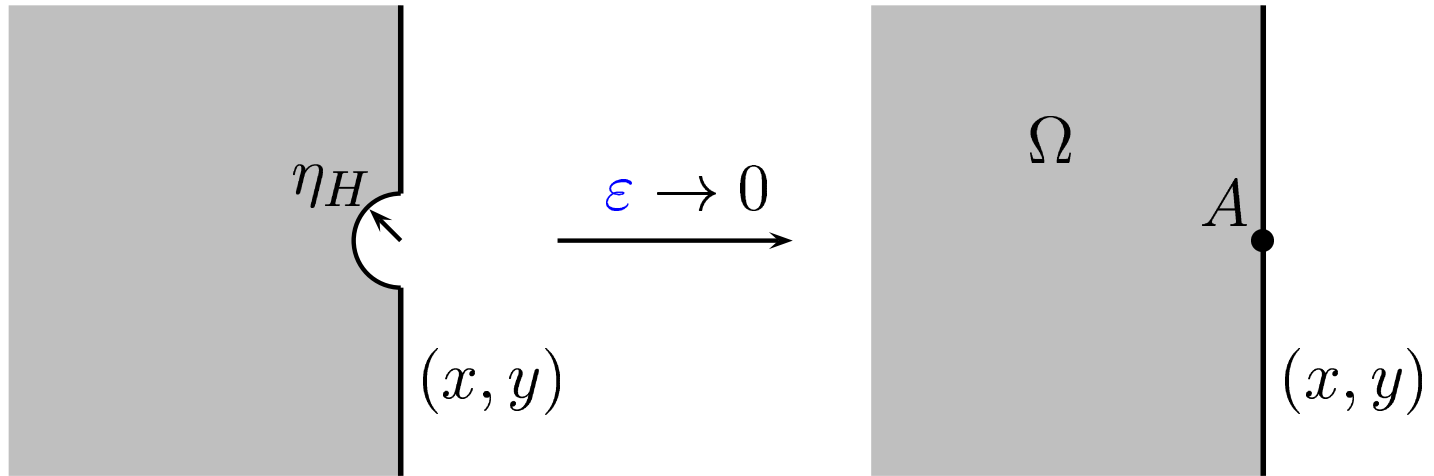
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Asymptotic context:

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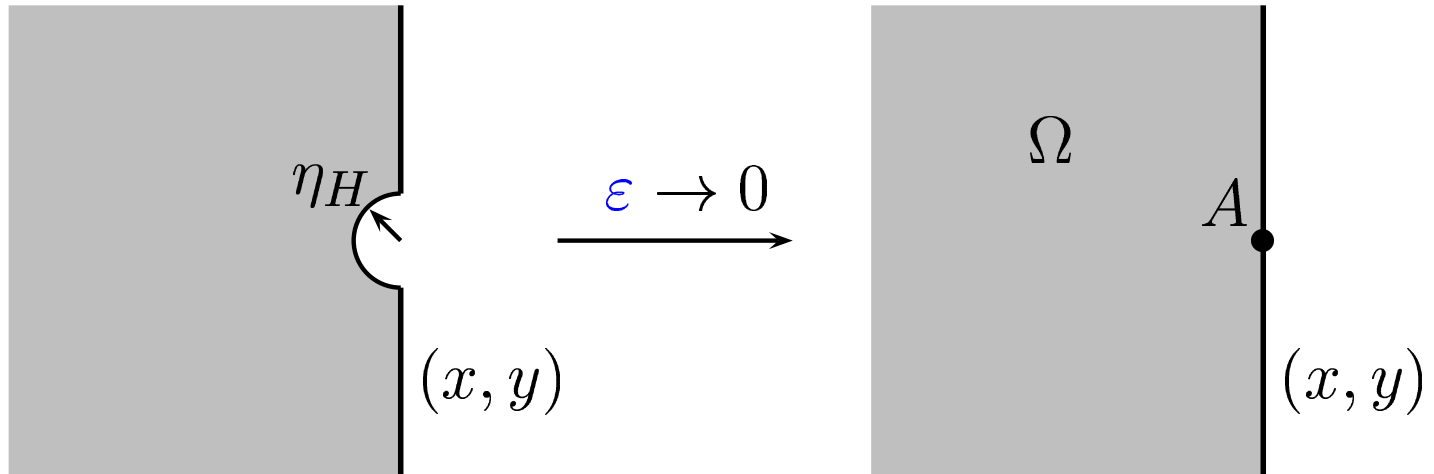


$$u^\varepsilon = u^0 + \sum_{i=1}^{+\infty} \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k + o(\varepsilon^\infty), \quad \text{in } \Omega.$$

Far field

Asymptotic context:

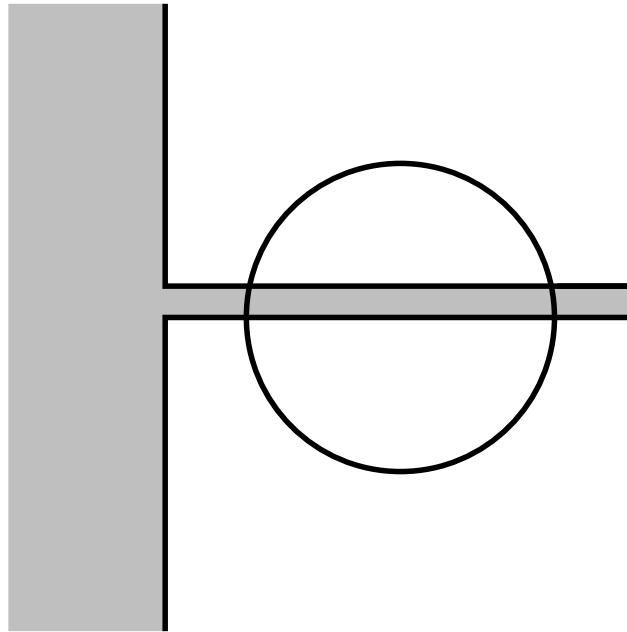
$$\varepsilon \ll \eta_H \ll \lambda.$$



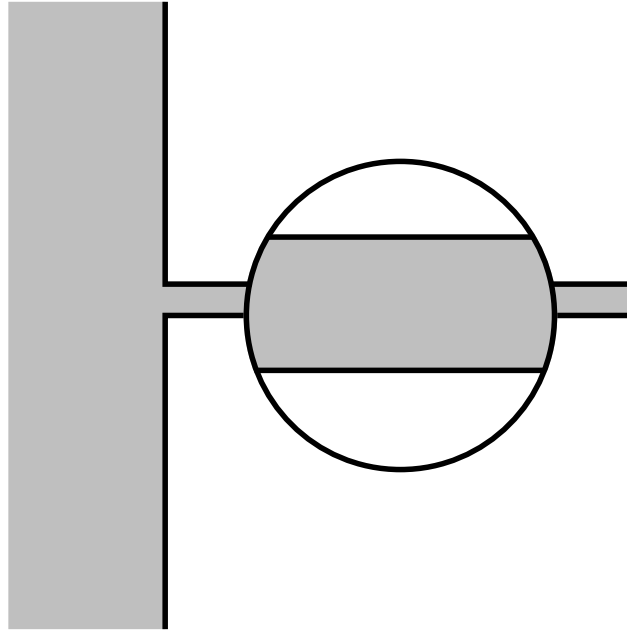
where the u_i^k satisfy the **homogeneous Helmholtz** equation

$$\Delta u_i^k + \omega^2 u_i^k = 0.$$

Slot field

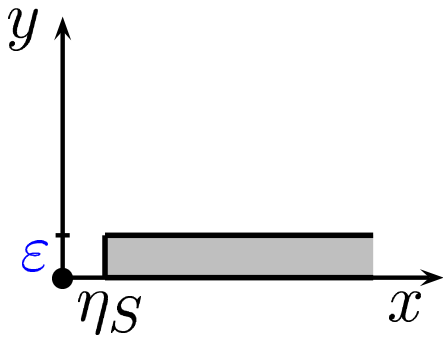


Slot field



$$u^\varepsilon(x, y) = U^\varepsilon\left(x, \frac{y}{\varepsilon}\right)$$

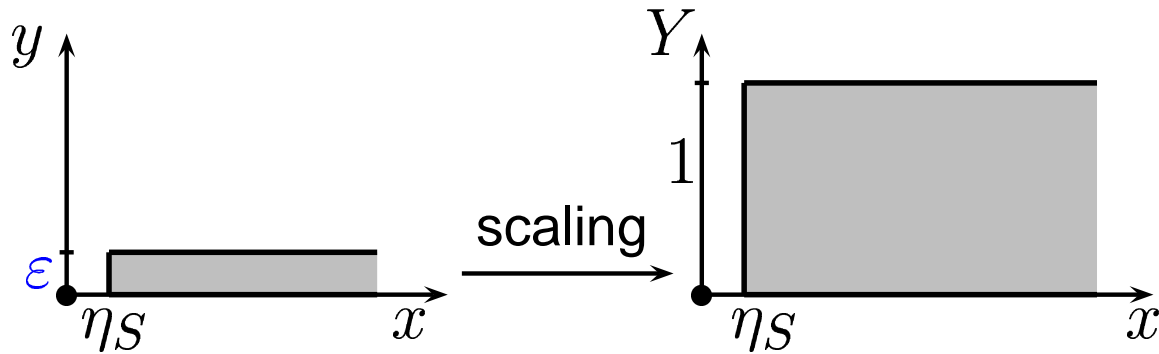
Slot field



The **asymptotic** context: $\varepsilon \ll \eta_S \ll \lambda$.

The **normalization**: $X = x, \quad Y = \frac{y}{\varepsilon}$

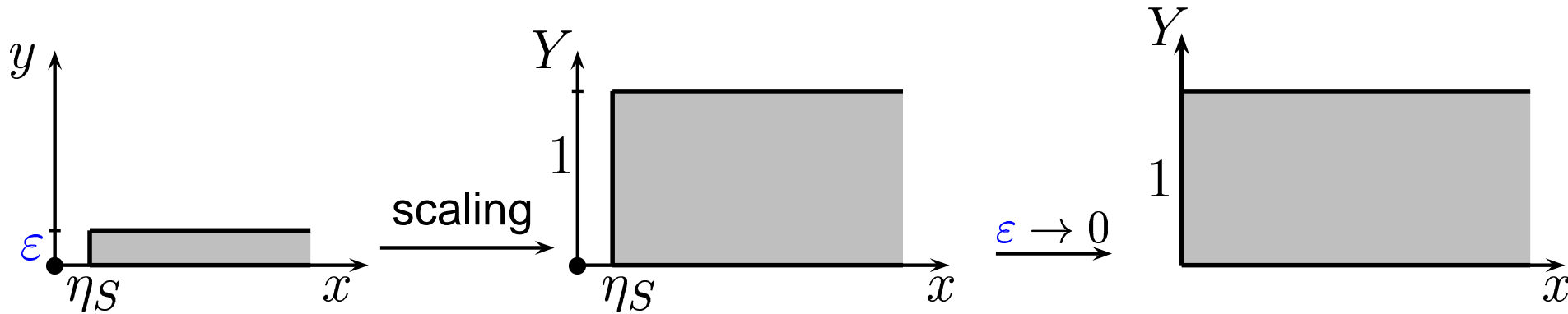
Slot field



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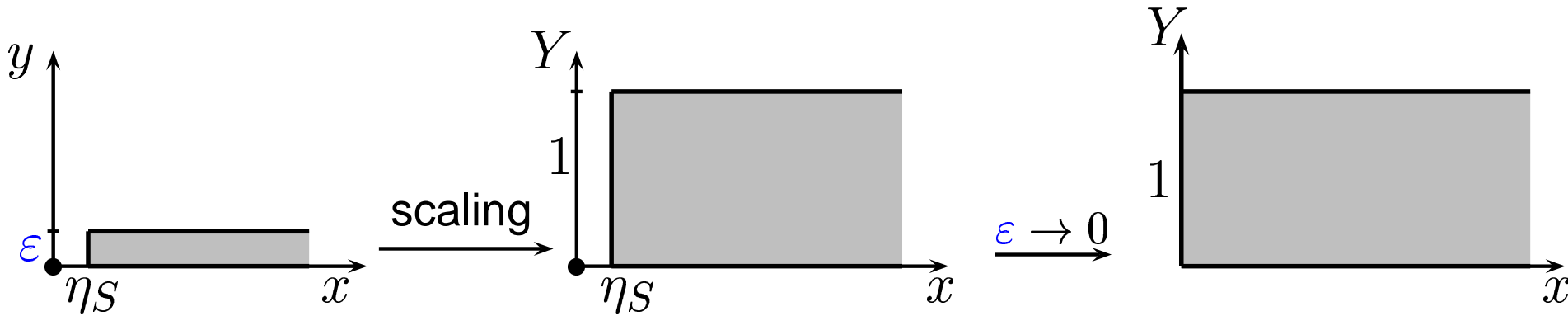
Slot field



The **asymptotic** context: $\varepsilon \ll \eta_S \ll \lambda$.

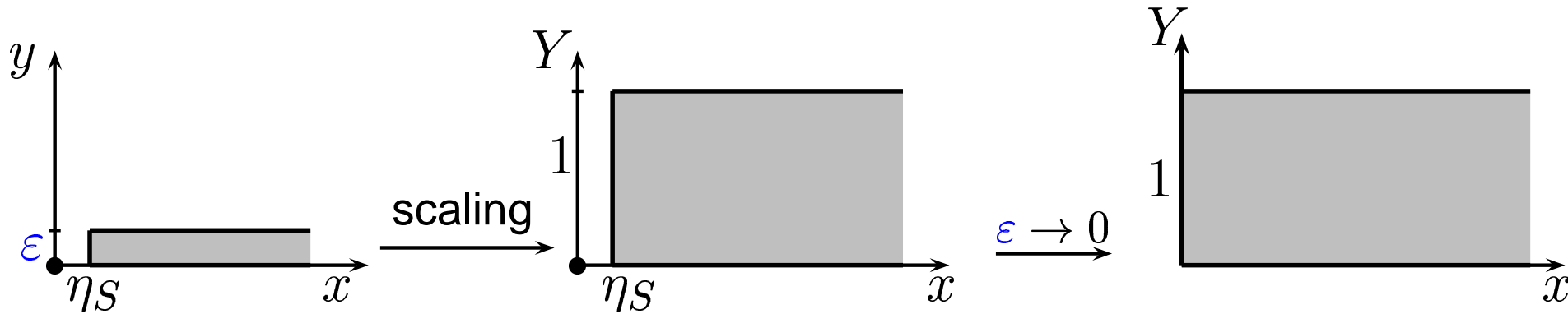
The **normalization**: $X = x, \quad Y = \frac{y}{\varepsilon}$

Slot field



$$u^\varepsilon(x, Y_\varepsilon) = U^\varepsilon(x, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k U_i^k(x, Y) + o(\varepsilon^\infty),$$

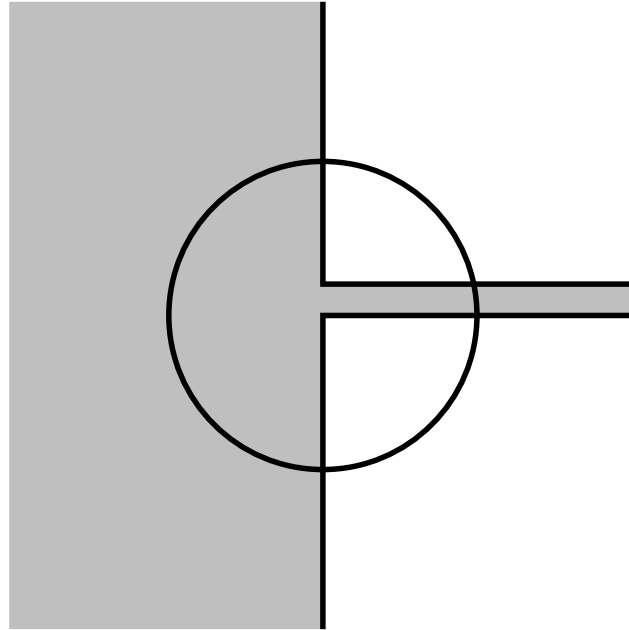
Slot field



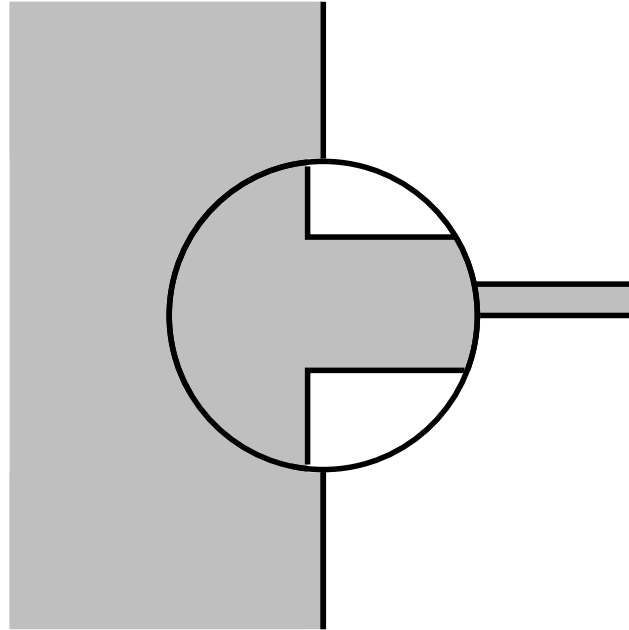
where the U_i^k satisfy the **1D Helmholtz** equation:

$$\frac{d^2 U_i^k}{dx^2} + \omega^2 U_i^k = 0$$

Near field

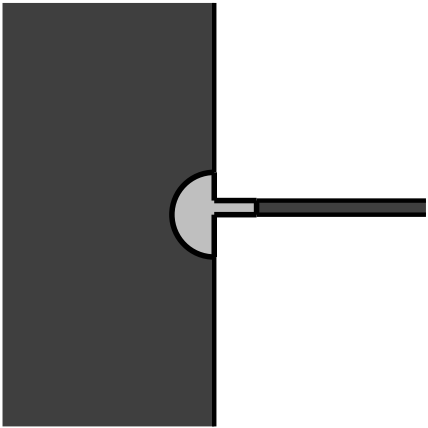


Near field



$$u^\varepsilon(x, y) = u_p^\varepsilon\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

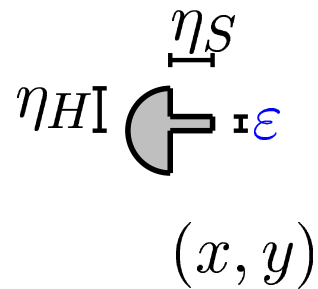
Near field



The **Asymptotic** context: $\varepsilon \ll \eta_H \ll \lambda$, $\varepsilon \ll \eta_S \ll \lambda$.

The **normalization**: $X = \frac{x}{\varepsilon}$, $Y = \frac{y}{\varepsilon}$

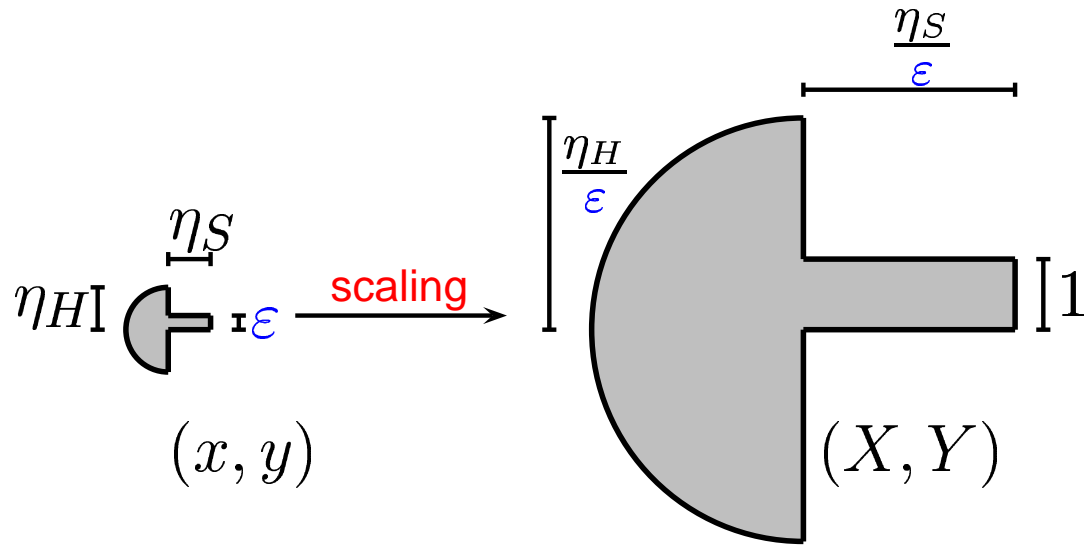
Near field



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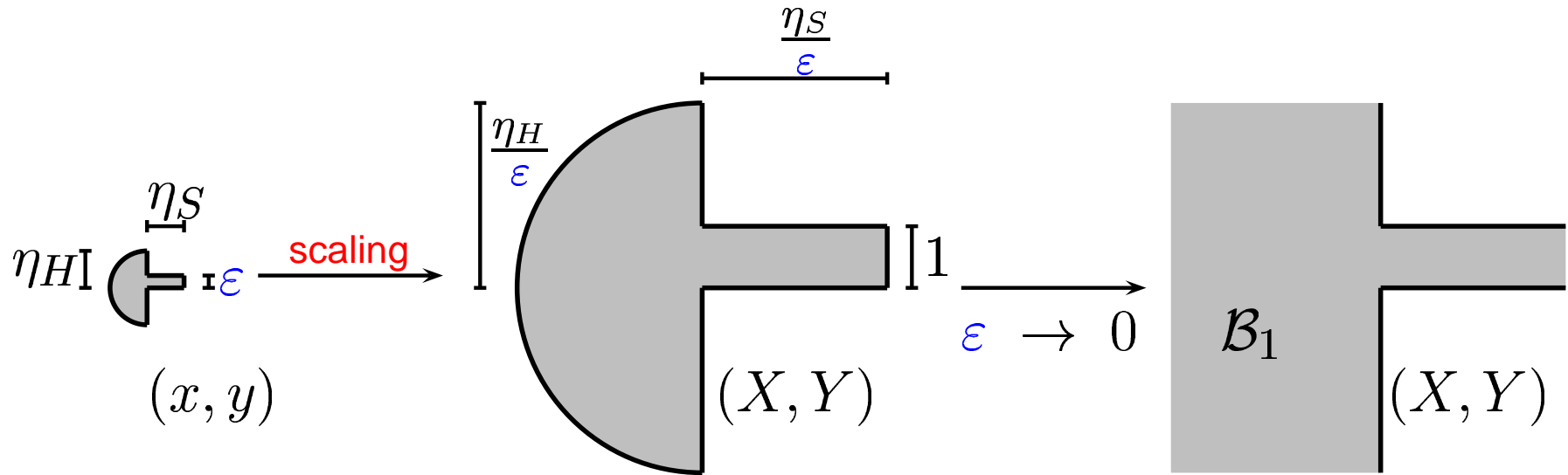
Near field



The **Asymptotic** context: $\epsilon \ll \eta_H \ll \lambda$, $\epsilon \ll \eta_S \ll \lambda$.

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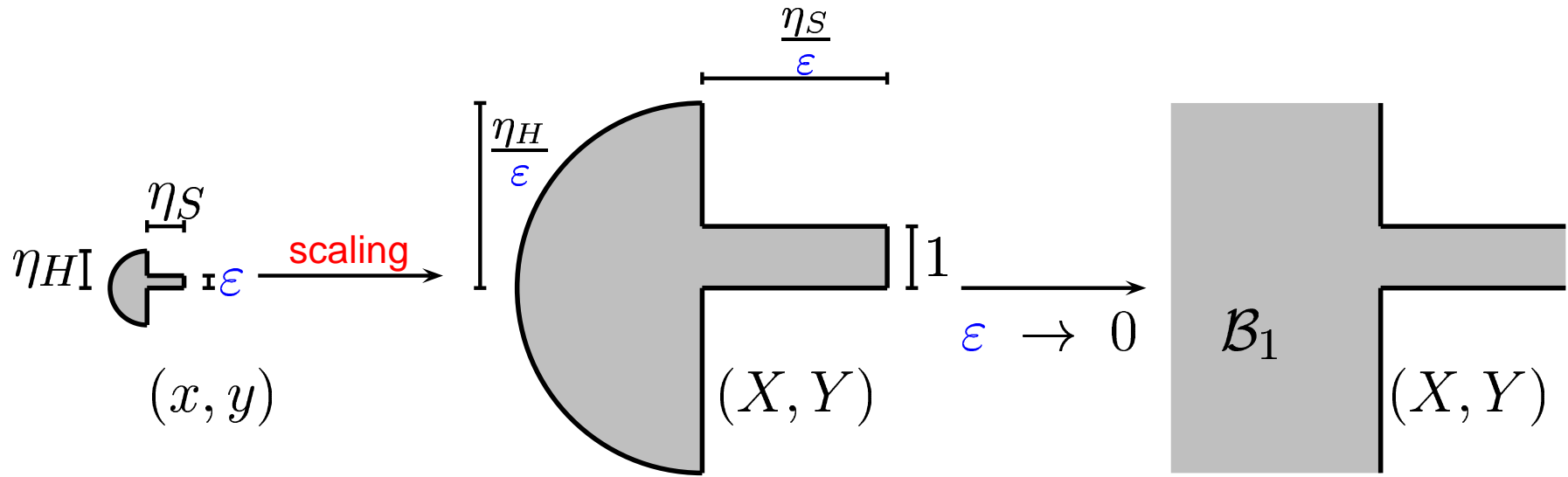
Near field



The **Asymptotic** context: $\varepsilon \ll \eta_H \ll \lambda$, $\varepsilon \ll \eta_S \ll \lambda$.

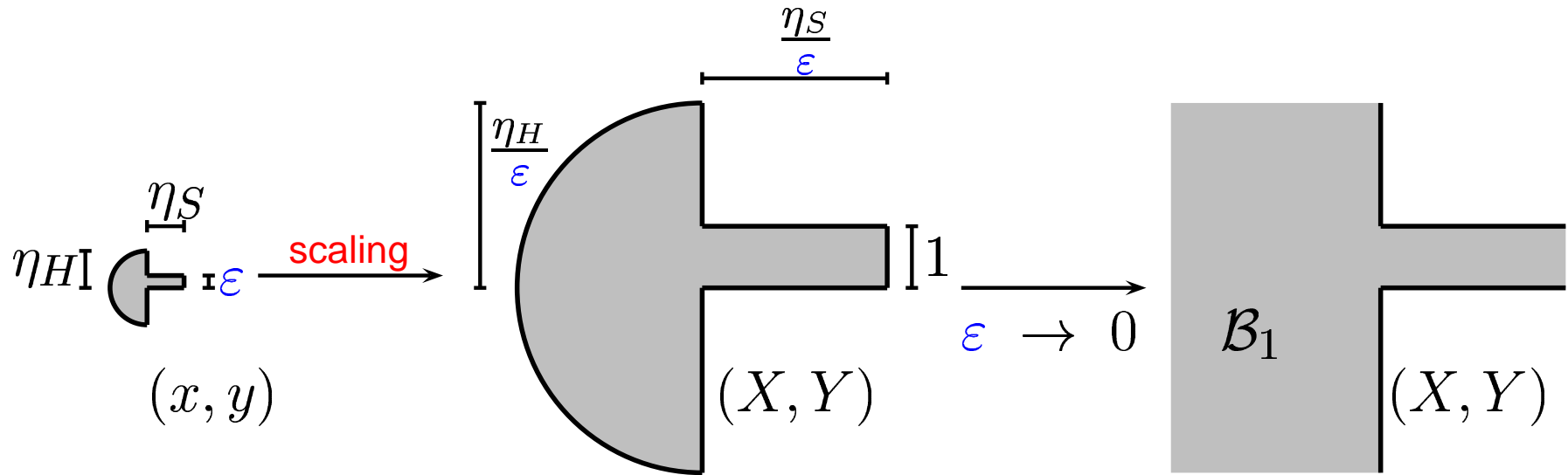
The **normalization**: $X = \frac{x}{\varepsilon}$, $Y = \frac{y}{\varepsilon}$

Near field



$$u^\varepsilon(\varepsilon X, \varepsilon Y) = u_p^\varepsilon(X, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k(X, Y) + o(\varepsilon^\infty)$$

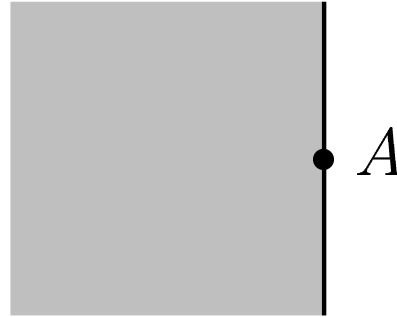
Near field



where the $(u_p)_i^k$ satisfy the (in)-homogeneous Laplace equation.

$$\begin{cases} \Delta(u_p)_i^k = 0, & \text{if } i = k \text{ or } k + 1, \\ \Delta(u_p)_i^k = -\omega^2 (u_p)_{i-2}^k, & \text{else.} \end{cases}$$

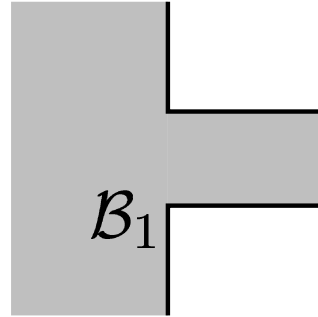
Order 0 : \underline{u}^0 , $(u_p)_0^0$, U_0^0



Far field:

$$\left\{ \begin{array}{l} \text{Find } u^0 \in H_{loc}^1(\Omega) \text{ such that :} \\ -\Delta u^0 - \omega^2 u^0 = f, \quad \text{in } \Omega, \\ \frac{\partial u^0}{\partial n} = 0, \quad \text{on } \partial\Omega, \\ u^0 \text{ is outgoing.} \end{array} \right.$$

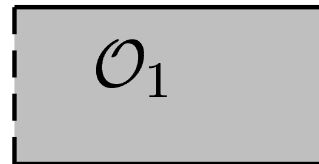
Order 0 : u^0 , $\underline{(u_p)_0^0}$, U_0^0



Near field:

$$(u_p)_0^0(X, Y) = u^0(A), \quad \text{in } B_1.$$

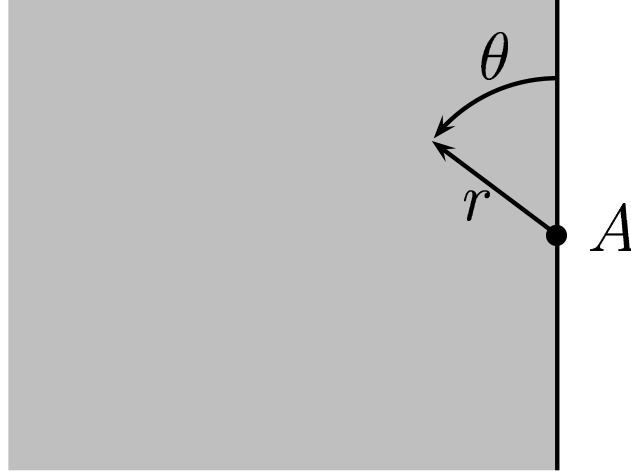
Order 0 : u^0 , $(u_p)_0^0$, U_0^0



Slot field:

$$U_0^0(x, Y) = u^0(A) \exp(i\omega x), \quad \text{in } \mathcal{O}_1.$$

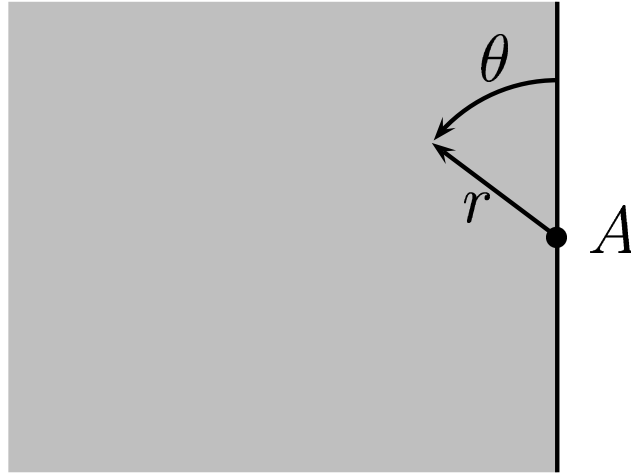
Order 1 : \underline{u}_1^0 , $(u_p)_1^0$, $(u_p)_1^1$, U_1^0 , U_1^1



Approximation of the exact Solution:

$$u^\varepsilon \simeq u^0 + \varepsilon u_1^0$$

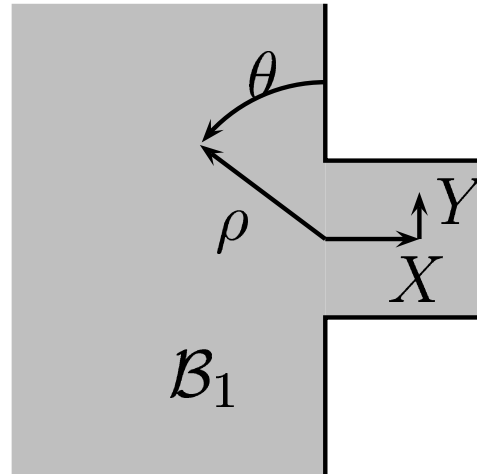
Order 1 : \underline{u}_1^0 , $(u_p)_1^0$, $(u_p)_1^1$, U_1^0 , U_1^1



explicit form of u_1^0

$$u_1^0(r, \theta) = -\frac{\omega}{2} u^0(A) H_0^{(1)}(\omega r).$$

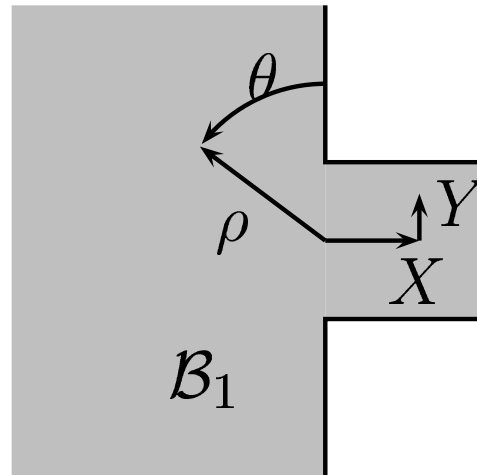
Order 1 : u_1^0 , $\underline{(u_p)_1^0}$, $\underline{(u_p)_1^1}$, U_1^0 , U_1^1



Approximation of the exact solution:

$$\begin{cases} u^\varepsilon(\varepsilon X, \varepsilon Y) = u_p^\varepsilon(X, Y), \\ u_p^\varepsilon \simeq (u_p)_0^0 + \varepsilon (u_p)_1^0 + \varepsilon \log \varepsilon (u_p)_1^1. \end{cases}$$

Order 1 : u_1^0 , $\underline{(u_p)_1^0}$, $(u_p)_1^1$, U_1^0 , U_1^1

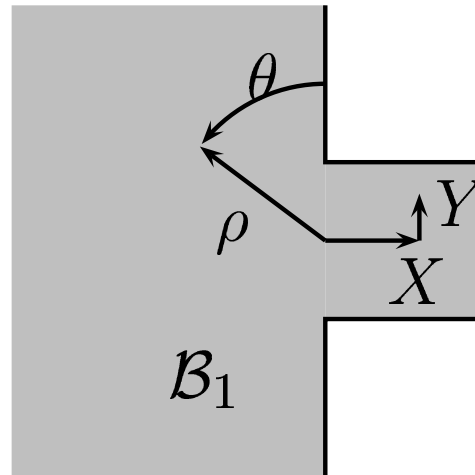


Near field:

Find $(u_p)_1^0 \in H_{loc}^1(\mathcal{B}_1)$ such that:

$$\begin{cases} \Delta(u_p)_1^0 = 0, & \text{in } \mathcal{B}_1 \\ \frac{\partial(u_p)_1^0}{\partial n} = 0, & \text{on } \partial\mathcal{B}_1. \end{cases}$$

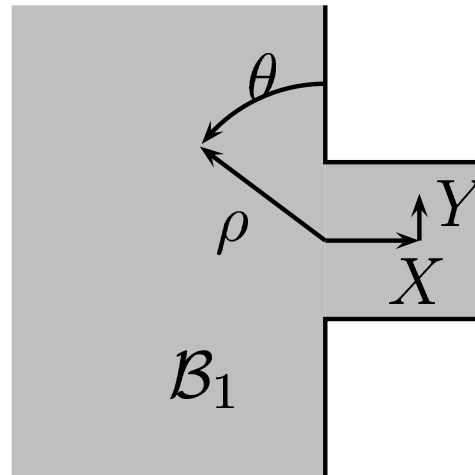
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Behavior at infinity in the half-space:

$$(u_p)_1^0(\rho, \theta) - \frac{\partial u^0}{\partial y}(A) \rho \cos \theta + \frac{\omega}{2} u^0(A) \left[1 + \frac{2i}{\pi} (\log \rho + \gamma) \right] = O\left(\frac{1}{\rho}\right).$$

Order 1 : u_1^0 , $\underline{(u_p)_1^0}$, $(u_p)_1^1$, U_1^0 , U_1^1



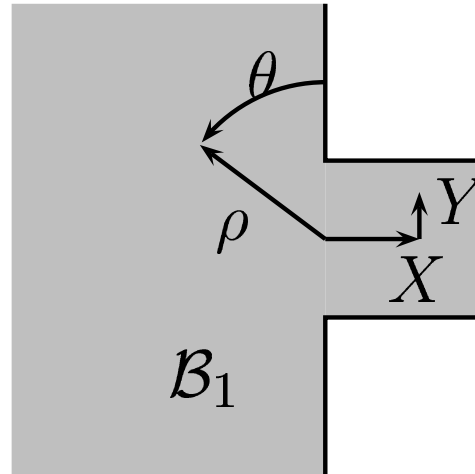
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Behavior at infinity in the slot:

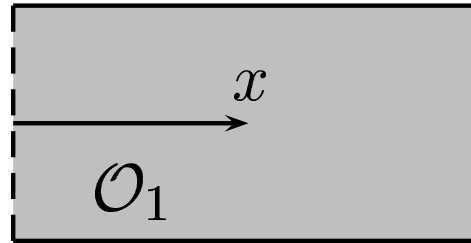
$$(u_p)_1^0(X, Y) - i \omega u^0(A) X = O(1).$$

Order 1 : u_1^0 , $(u_p)_1^0$, $\underline{(u_p)_1^1}$, U_1^0 , U_1^1



$$(u_p)_1^1 = -\frac{\mathbf{i}\omega}{\pi} u^0(A)$$

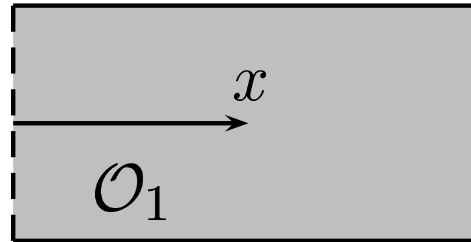
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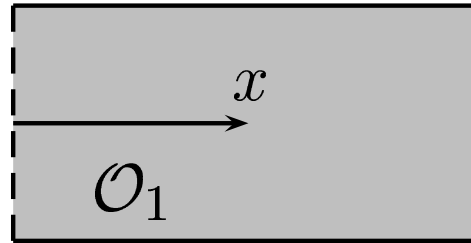
Order 1 : u_1^0 , $(u_p)_1^0$, $(u_p)_1^1$, U_1^0 , U_1^1



The slot field:

$$U_1^0(x) = \int_0^1 (u_p)_1^0(0, Y) dY \exp(i\omega x),$$

Order 1 : $u_1^0, (u_p)_1^0, (u_p)_1^1, U_1^0, \underline{U_1^1}$

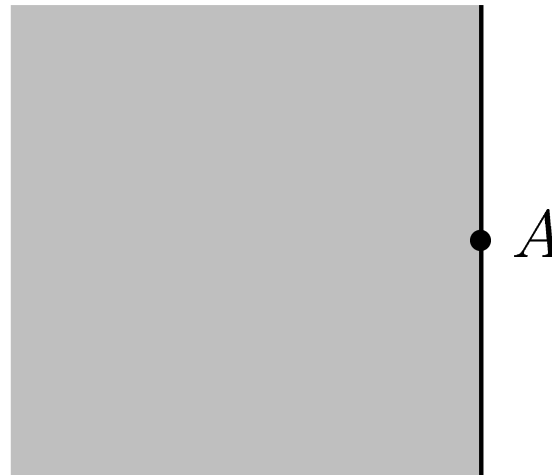


The slot field:

$$U_1^1(x) = -\frac{\mathbf{i}\omega}{\pi} u^0(A) \exp(\mathbf{i}\omega x).$$

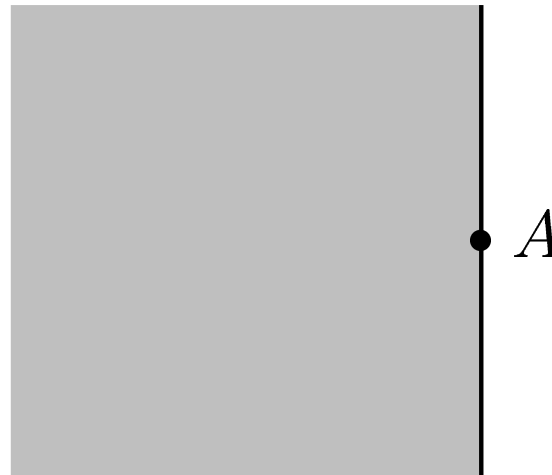
The far field of order $i > 1$

- The fields u_i^k are defined in the **half space**:



The far field of order $i > 1$

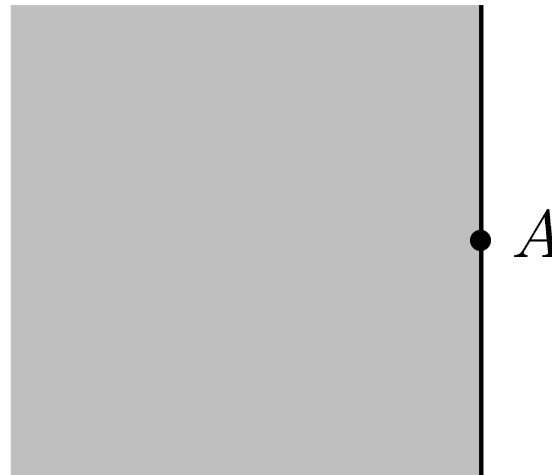
- The fields u_i^k are defined in the **half space**:



- The far fields u_i^k
 - satisfy the **homogeneous Helmholtz** equation
 - are **singular** at the neighborhood of the origin
 - are outgoing at infinity

The far field of order $i > 1$

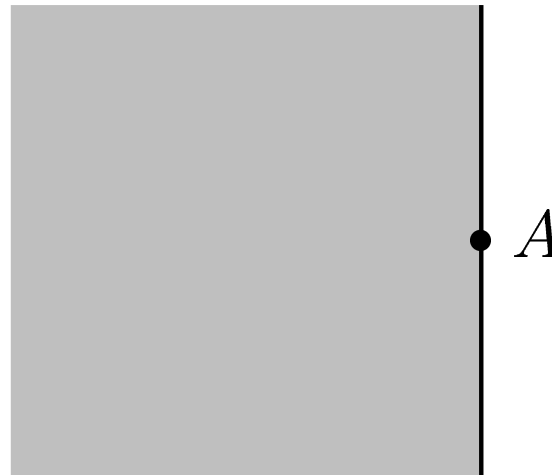
- The fields u_i^k are defined in the **half space**:



- $$u_i^k = \sum_{p=0}^{+\infty} a_p H_p^{(1)}(\omega r) \cos p\theta$$

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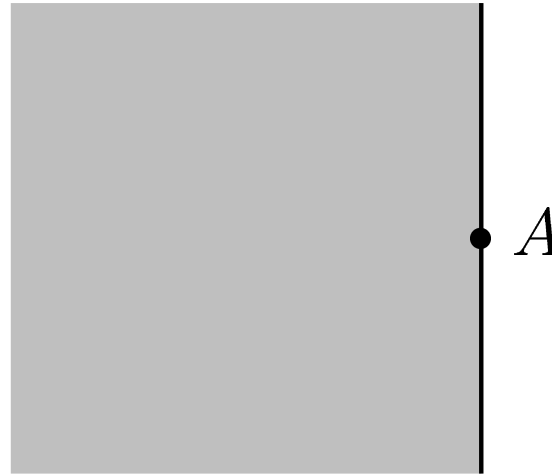
- The fields u_i^k are defined in the **half space**:



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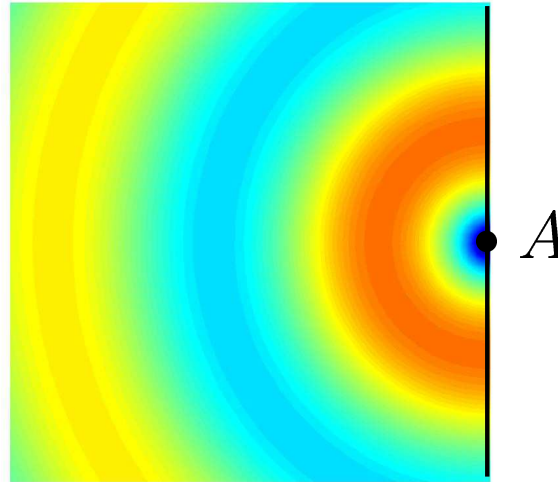
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The a_p are functions of **lower order** terms

The far field of order $i > 1$

- The fields u_i^k are defined in the **half space**:

$$\text{Im}(H_0^{(1)}(\omega r))$$



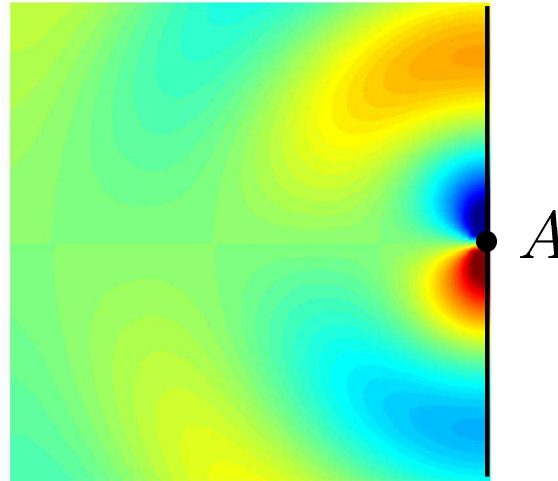
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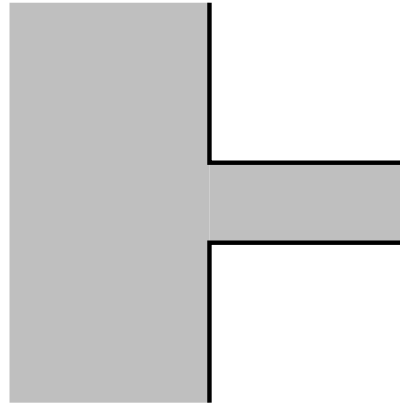


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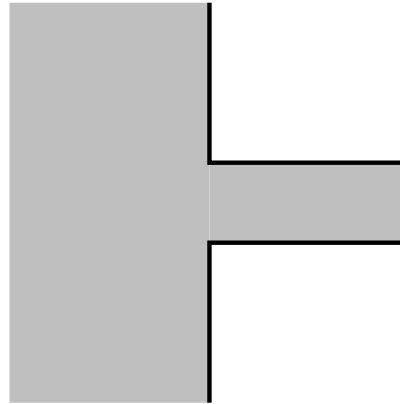
The near fields of order $i > 1$

- The $(u_p)_i^k(X, Y)$ are defined in the **canonical** domain:



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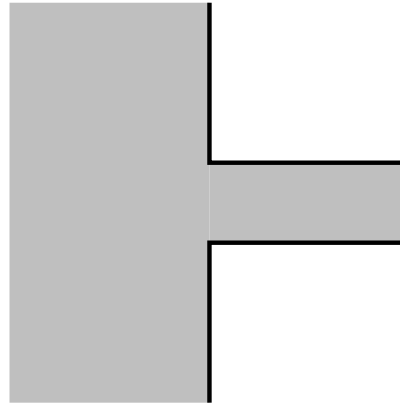
- by **Laplace** equation:

$$\Delta(u_p)_i^k = 0, \quad (i = k \text{ ou } k + 1),$$

$$\Delta(u_p)_i^k = -\omega^2 (u_p)_{i-2}^k, \quad (i \geq k + 2),$$

The near fields of order $i > 1$

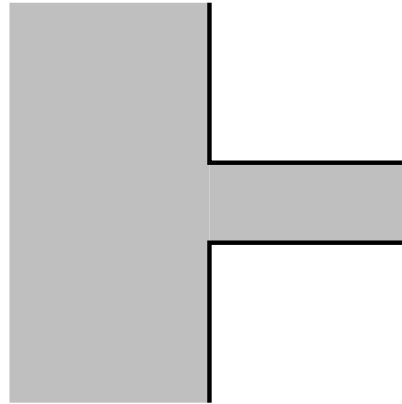
- The $(u_p)_i^k(X, Y)$ are defined in the **canonical** domain:



- by **Laplace** equation:
- by polynomial **growings** at infinity:
 - The **growings** in the half space are functions of **far field of lower (or equal) order**
 - The **growings** in the slot are functions of the slot fields **of lower order**

The near fields of order $i > 1$

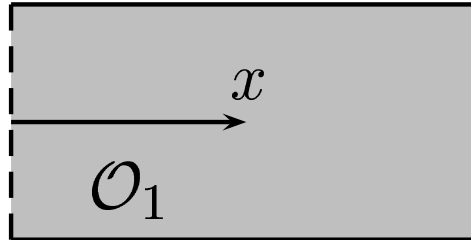
- The $(u_p)_i^k(X, Y)$ are defined in the **canonical** domain:



- Proof of the **existence-unicity**:
 - with truncature functions, we subtract the growing behavior at infinity of the $(u_p)_i^k$
 - We use the “classical” **variational theory** (wheighted Sobolev spaces, Leroux, Hardy,...)

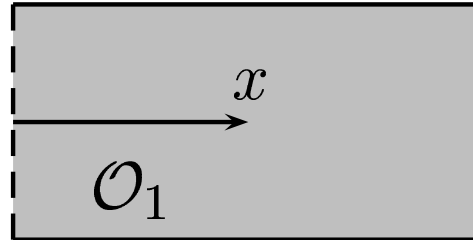
The slot field of order $i > 1$

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The slot field of order $i > 1$

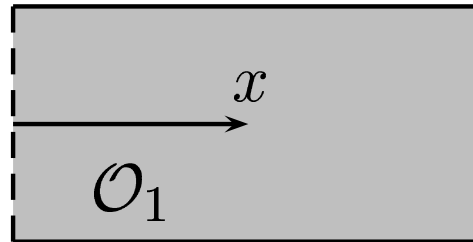
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- The U_i^k does not depend on y .
- $$U_i^k(x) = \left(\int_0^1 (u_p)_i^k(0, Y) dY \right) \exp i\omega x$$

Some properties

We see that:

- More $i - k$ is **large** more u_i^k is **singular** at the origin:

$$r^{-p} \text{ terms, } p = 0, \dots, i - k - 1$$

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$$\begin{cases} \rho^p \text{ terms, } & p = 0, \dots, i - k, \\ X^p \text{ terms, } & p = 0, \dots, i - k, \end{cases}$$

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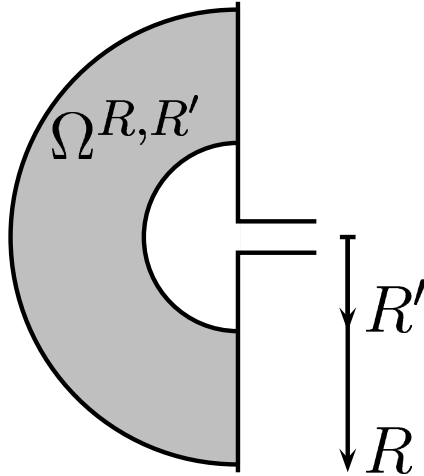
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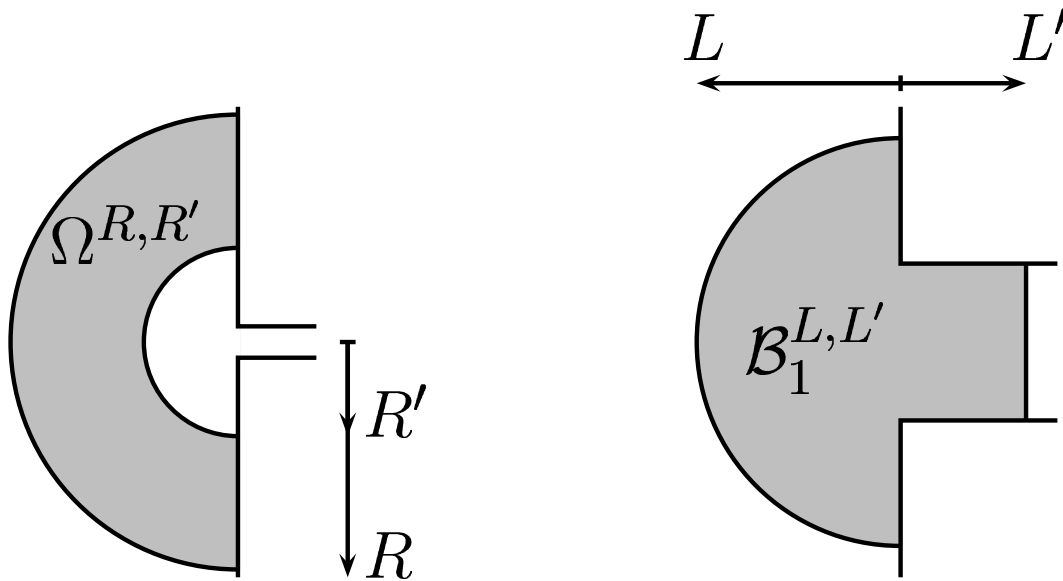
- When the **order** i grows, one has $O(\frac{i^2}{2})$ ($\times 3$) terms to compute...

Mathematical analysis



$$\left\| u^\epsilon - u^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \epsilon^i (\log \epsilon)^k u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \epsilon^{n+1} (\log \epsilon)^n \|f\|_{L^2(\Omega)}.$$

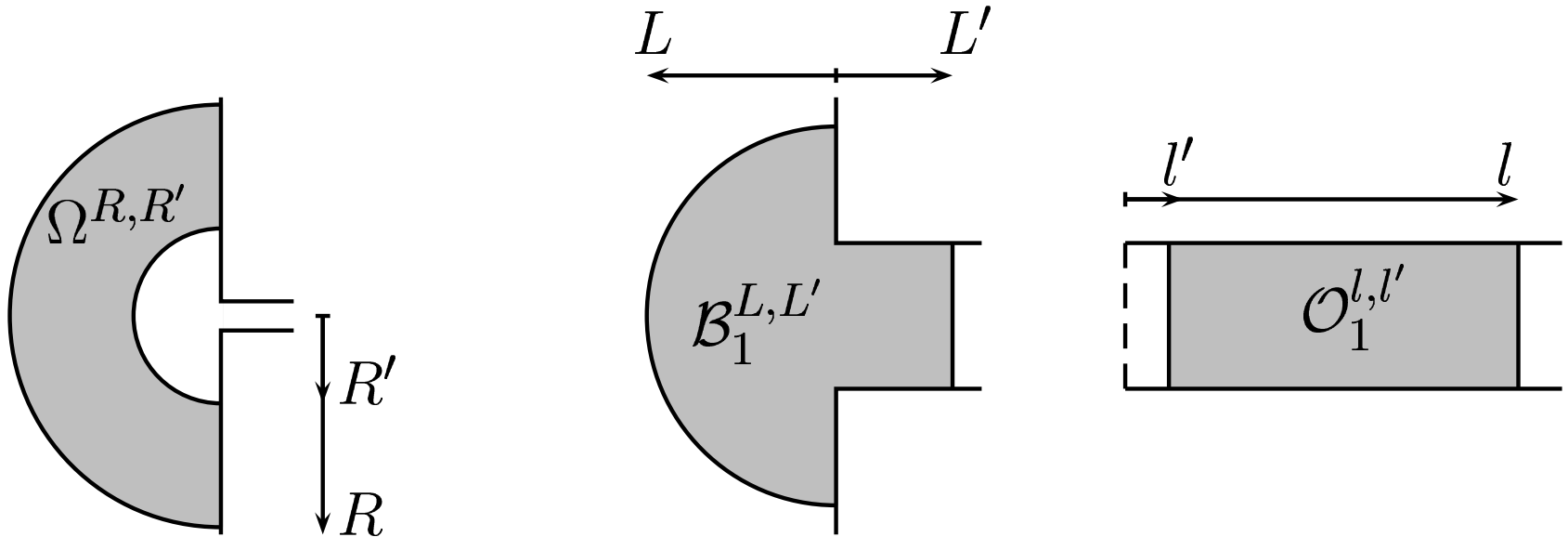
Mathematical analysis



$$\left\| \mathbf{u}^\varepsilon - \mathbf{u}^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k \mathbf{u}_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^n \|f\|_{L^2(\Omega)}.$$

$$\left\| \mathbf{u}_p^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (\mathbf{u}_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

Mathematical analysis



$$\left\| \mathbf{u}^\varepsilon - \mathbf{u}^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k \mathbf{u}_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^n \|f\|_{L^2(\Omega)}.$$

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$$\left\| \mathbf{U}^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k \mathbf{U}_i^k \right\|_{H^1(\mathcal{O}_1^{l,l'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$