Matching of asymptotic expansions for the wave propagation in media with thin slot

Sébastien Tordeux, Patrick Joly

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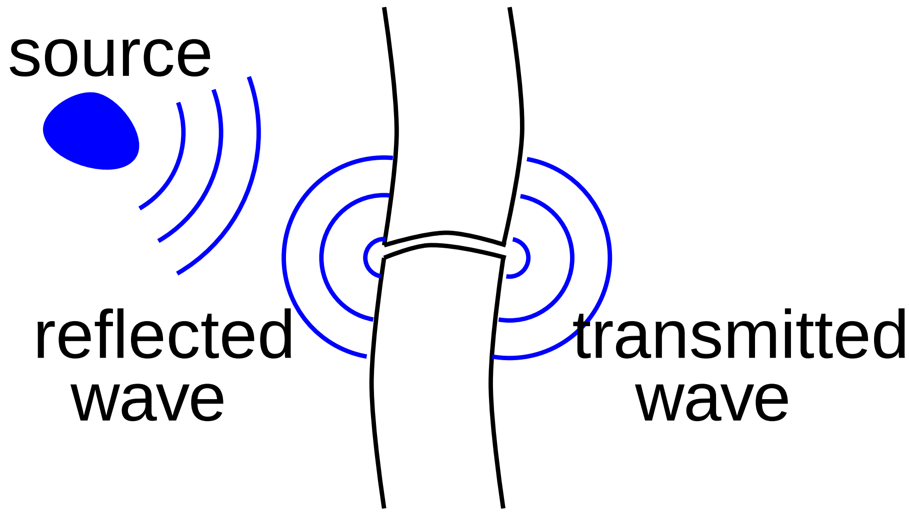
INRIA-Rocquencourt-Projet POEMS ETH-SAM

A typical application

How can we study the scattering in media with thin slot ?

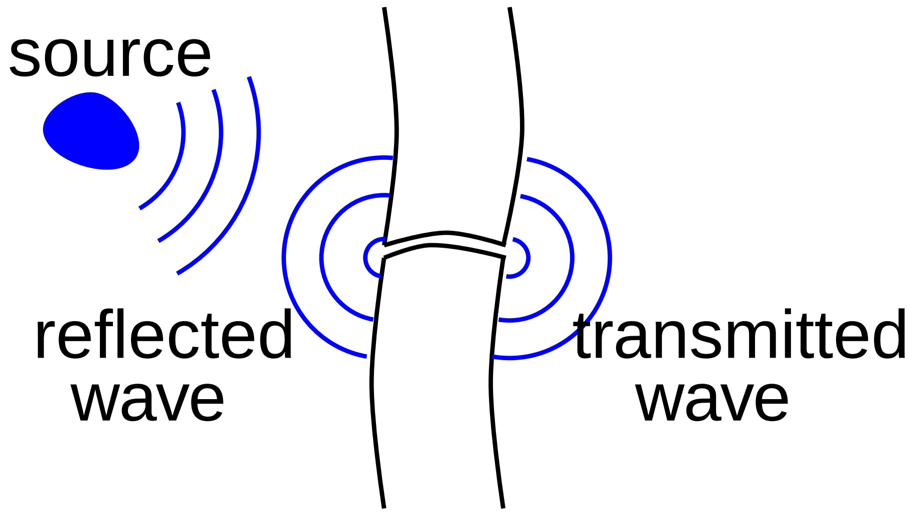


A physical problem with two caracteristical lenghtes

$$\begin{cases} \text{The wavelength } \lambda \\ \text{The width of the slot } \varepsilon \end{cases}$$

A typical application

How can we study the scattering in media with thin slot ?

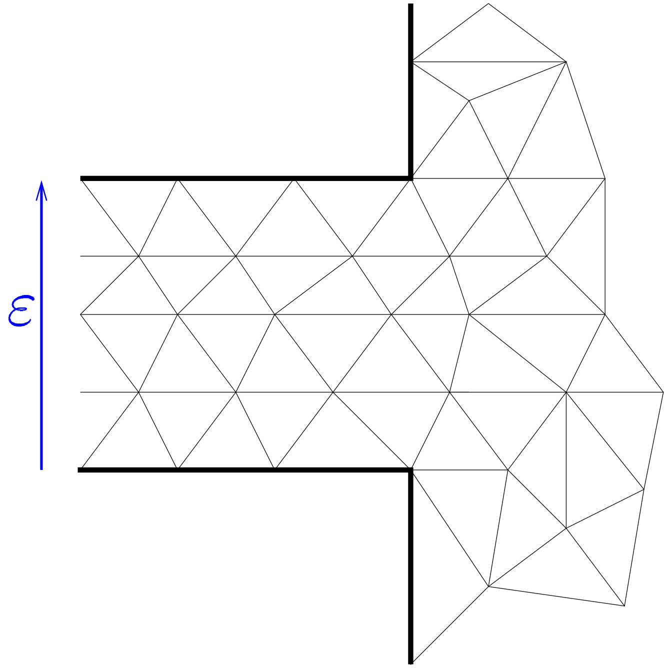


An asymptotic case:

$$\varepsilon \ll \lambda$$

The numerical difficulty

A mesh step smaller than ε



This leads to costly computations

Some references

- Thin slot:

  Harrington, Auckland (1980), Tatout (1996).

- Finite differences:

  Taflove (1995).

- Thin plates and junction theory,...

  Ciarlet, Le Dret, Dauge-Costabel.

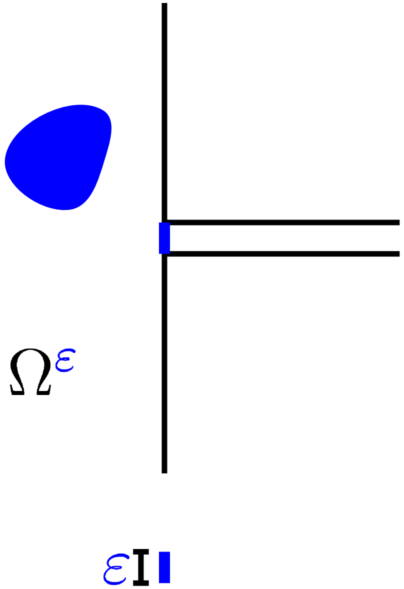
- Matching of asymptotic expansions:

  McIver, Rawlins (1993), Il'in (1992).

- multiscale analysis

  Maz'ya, Nazarov, Plamenevskii (1991).
  Oleinik, Shamaev, Yosifian (1992).

A simple problem



Scalar wave equation:

$$\frac{\partial^2 p^\varepsilon}{\partial t^2} - \Delta p^\varepsilon = f$$

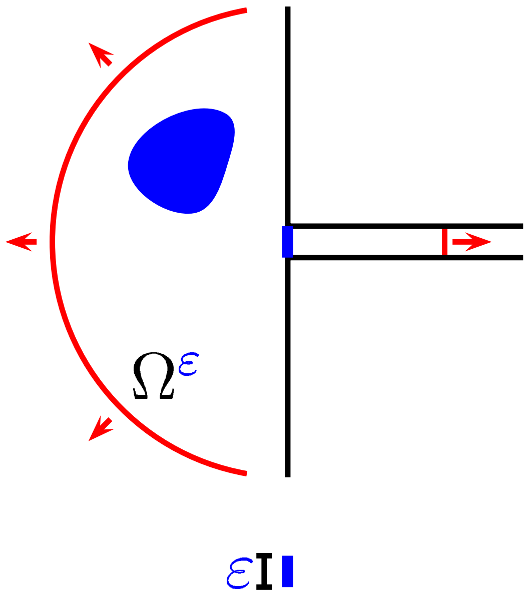
Harmonic solution:

$$p^\varepsilon(x,y,t) = exp(-\mathbf{i}\omega t)\ u^\varepsilon(x,y)$$

Helmholtz Equation:

$$\Delta u^\varepsilon + \omega^2 u^\varepsilon = -f \quad \text{in } \Omega^\varepsilon$$

A simple problem



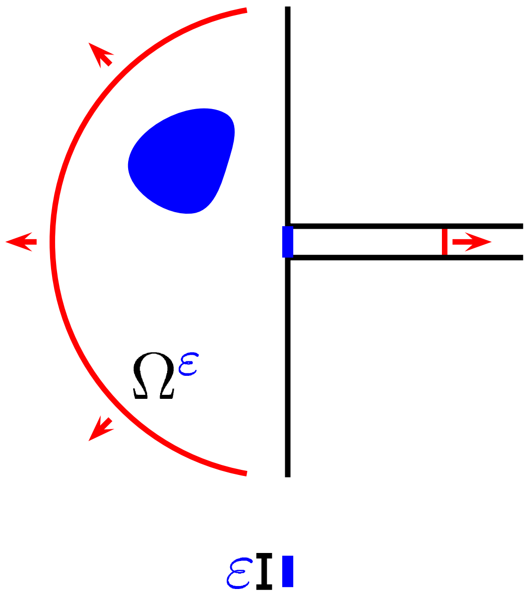
Outgoing solution at infinity:

$$\frac{\partial u^{\varepsilon}}{\partial n} - \mathbf{i}\omega u^{\varepsilon} \leq \frac{C}{r^2}, \qquad \text{for } r \text{ large,}$$

Neumann limit condition (rigid wall)

$$\frac{\partial u^{\varepsilon}}{\partial n} = 0 \quad \text{on } \partial\Omega^{\varepsilon}$$

A simple problem



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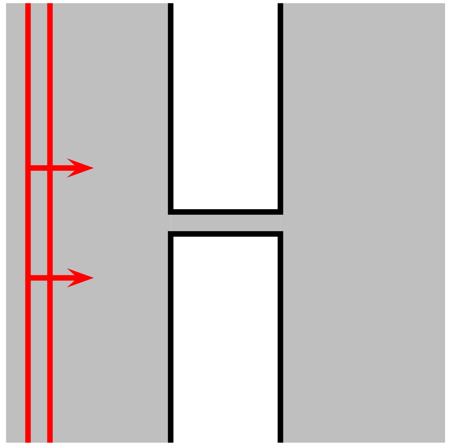
With the Dirichlet limit condition, the transmission inside the slot is negligible ($o(\varepsilon^\infty)$).

A numerical computation

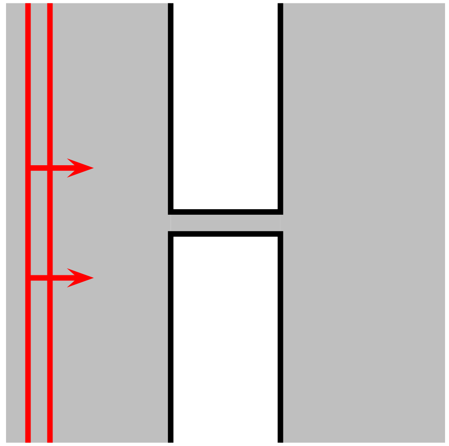
Neumann

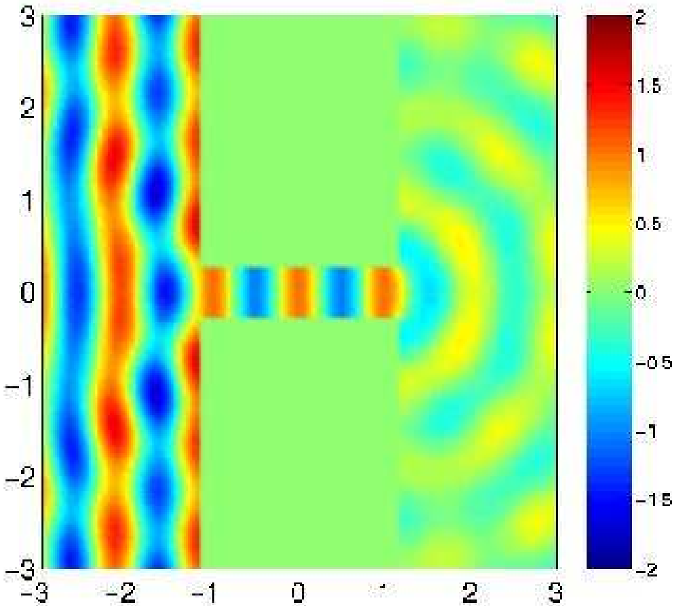


Dirichlet

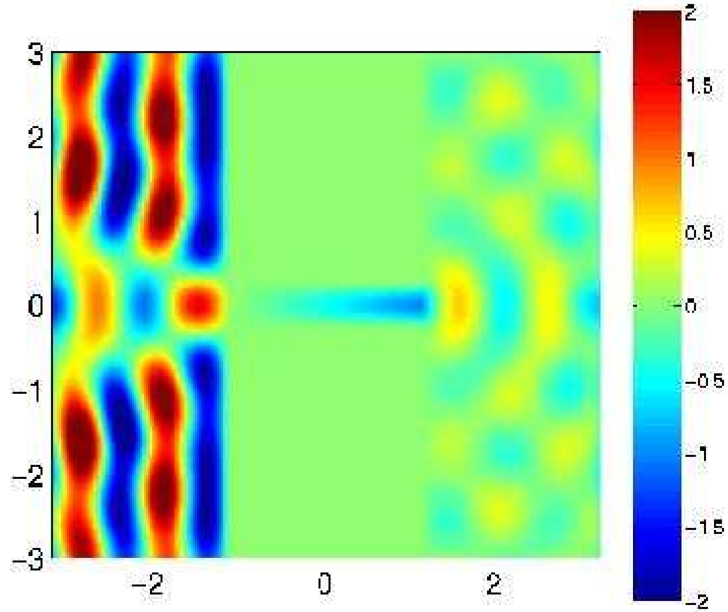
Numerical computation done with the high order finite elements code of (M. Duruflé, INRIA)

A numerical computation

Neumann  Dirichlet

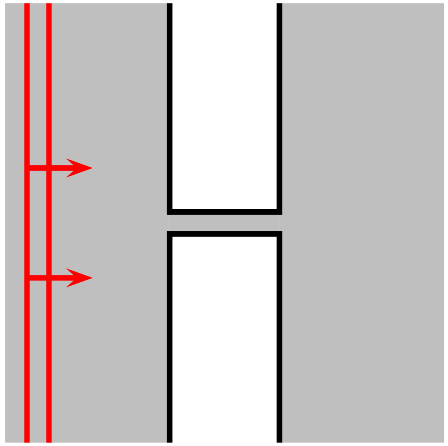


$$\frac{\varepsilon}{\lambda} = 0.5$$

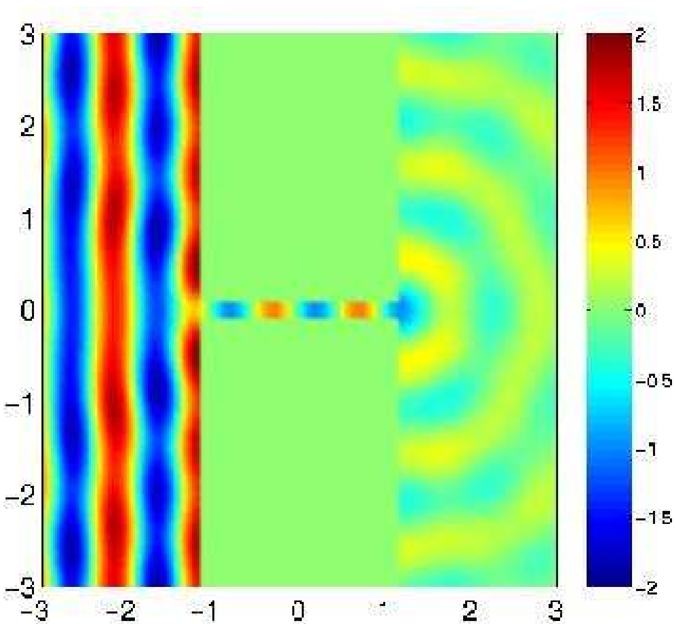


A numerical computation

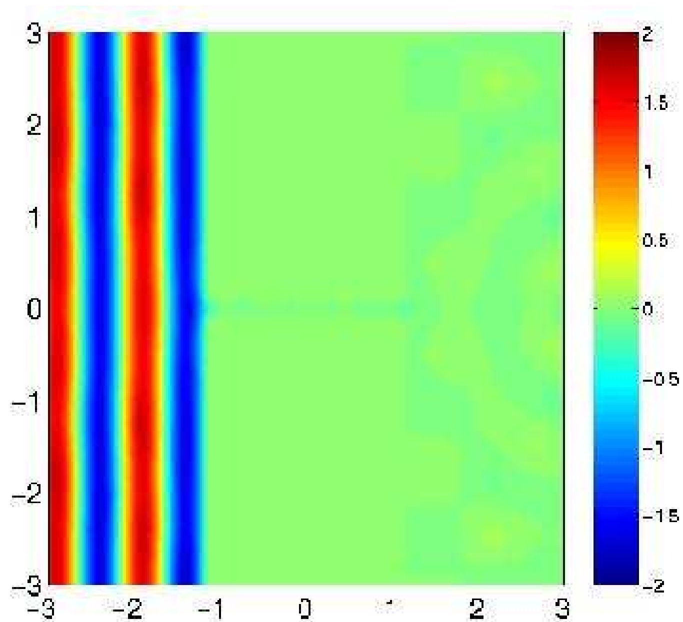
Neumann



Dirichlet

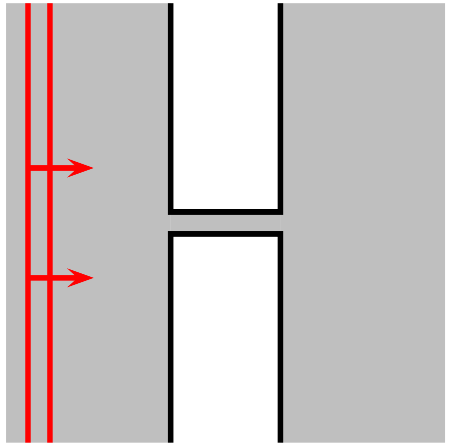


$$\frac{\varepsilon}{\lambda} = 0.2$$

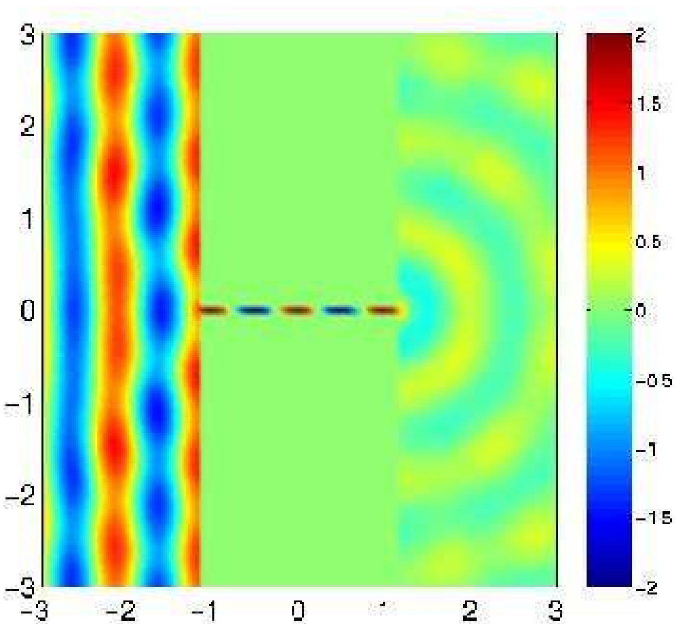


A numerical computation

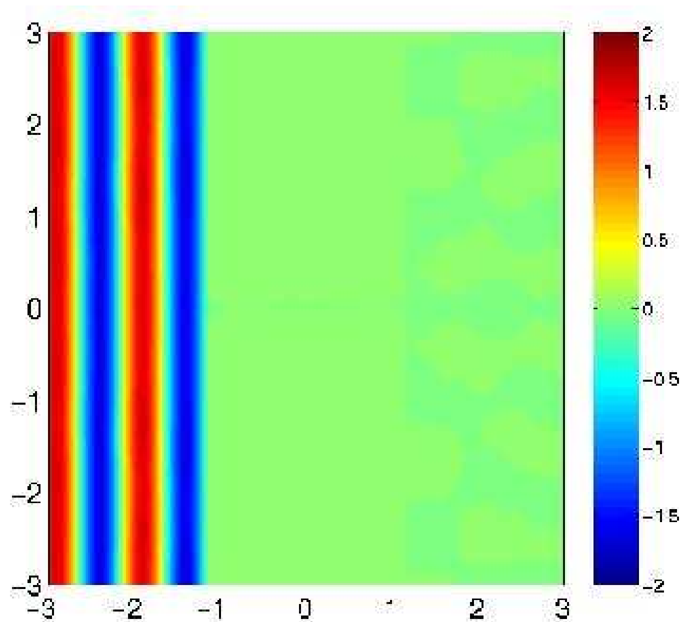
Neumann



Dirichlet



$$\frac{\varepsilon}{\lambda} = 0.1$$



Objectives

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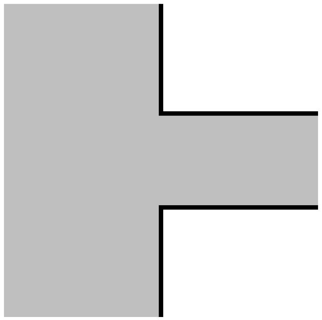
- We need an intermediate zone



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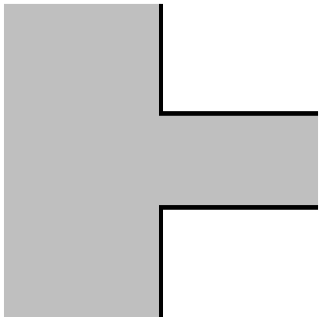
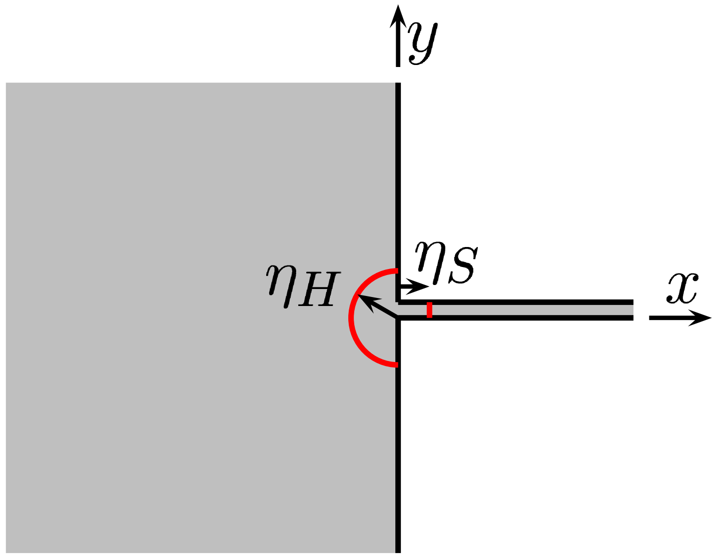


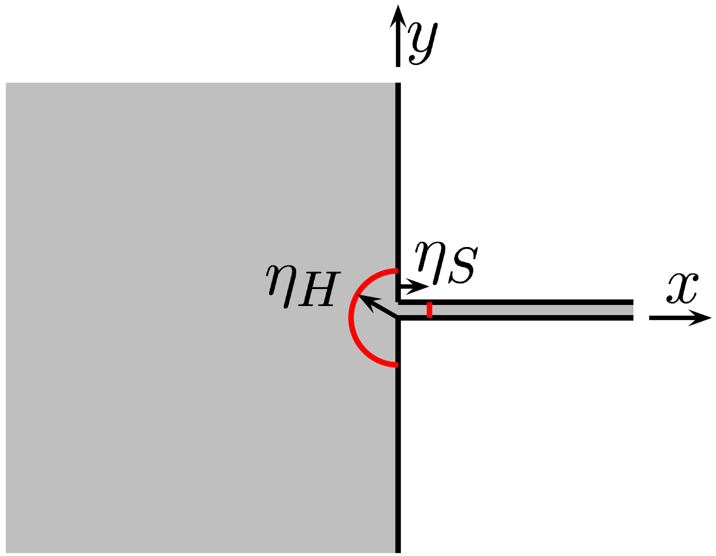
- A technique the matching of asymptotic expansions
  - Define new approximate models to compute the solution.
  - Use effectively "universal" technique of numerical computation (mesh reffinement).

Three zones



- Far field (2D field)
- Near field (boundary layer)
- Slot field (1D field)

Three zones

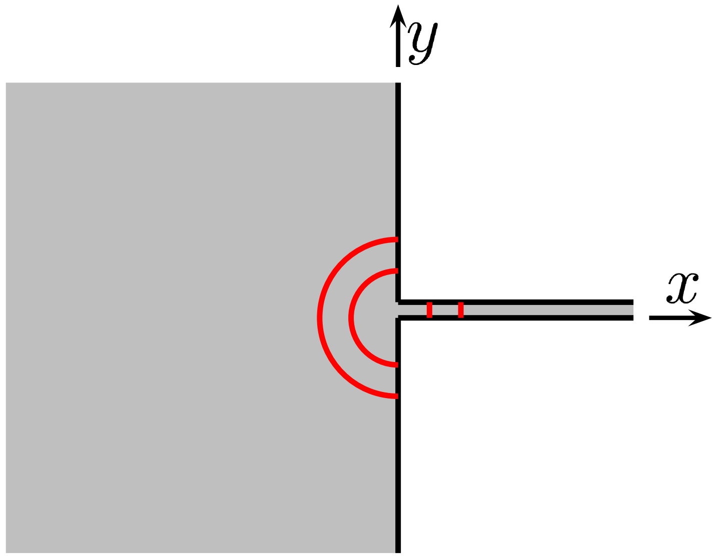


The asymptotic assumptions:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \qquad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \to 0 \qquad \eta(\varepsilon) \to 0 \qquad \eta(\varepsilon)/\varepsilon \to +\infty$$

Three zones



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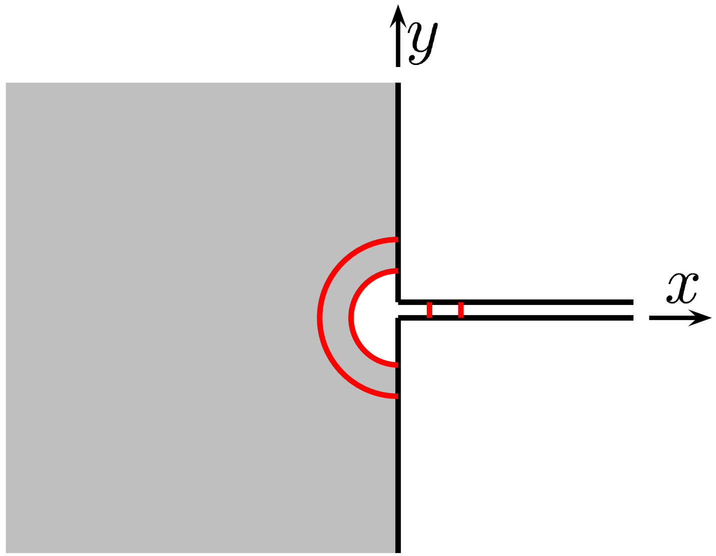

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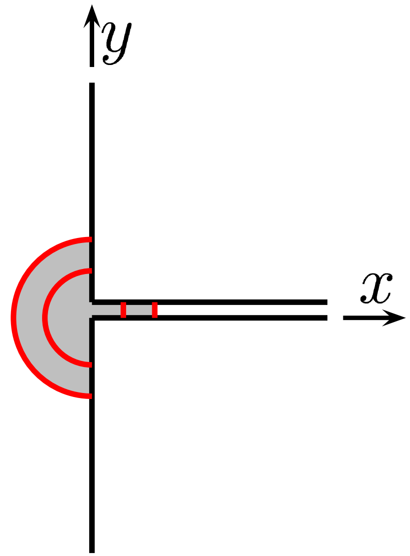

Near field

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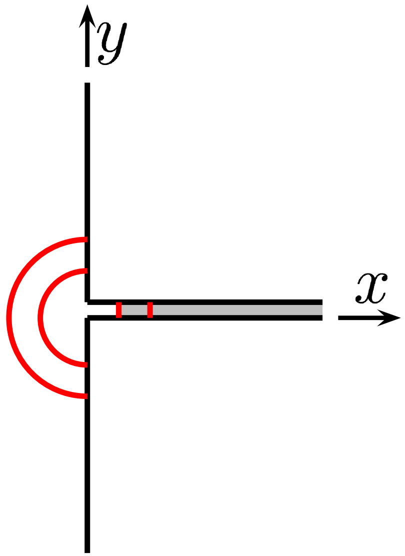

Slot field

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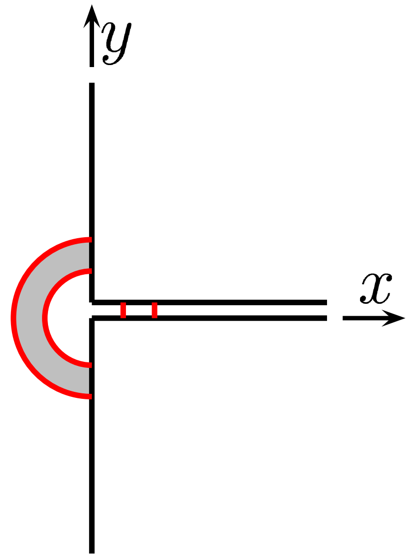

Far and near

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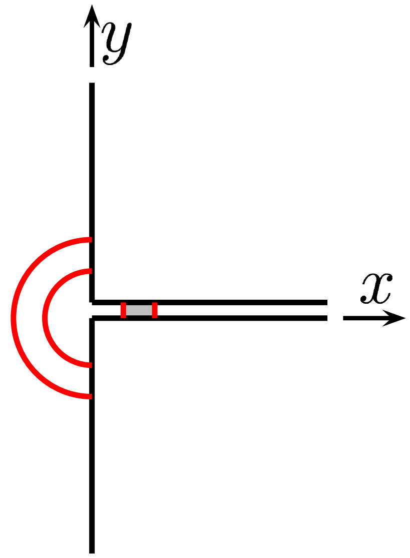

Slot and near

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The different steps of the method

- Derivate the asymptotic expansions:
  - Formal part
  - Several presentations are possible

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- Describe the asymptotic expansions
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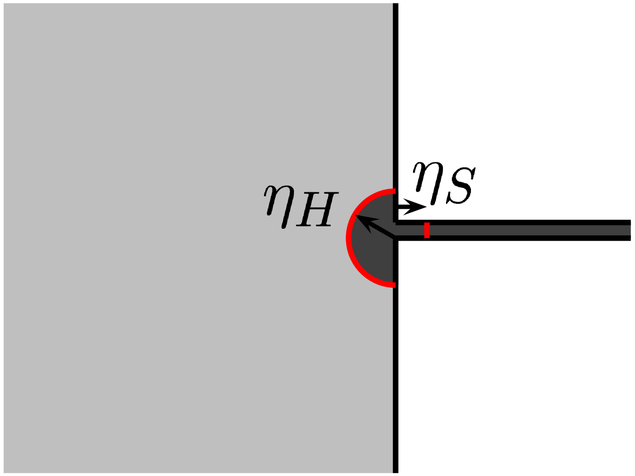
- Derivate the asymptotic expansions:
  - Formal part
  - Several presentations are possible
- Describe the asymptotic expansions
  - Rigorous part
  - Definition of the terms of the asymptotic expansions
- Mathematical validation of the asymptotic expansions
  - Rigorous part
  - Error estimates

Far field

Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda$.



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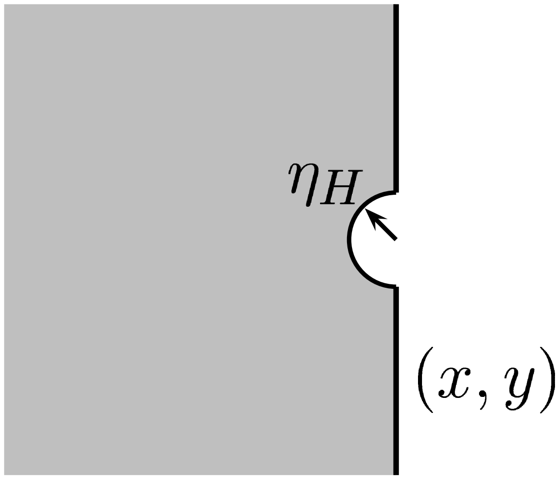

No normalization:

$$X = x, \qquad Y = y.$$

Far field

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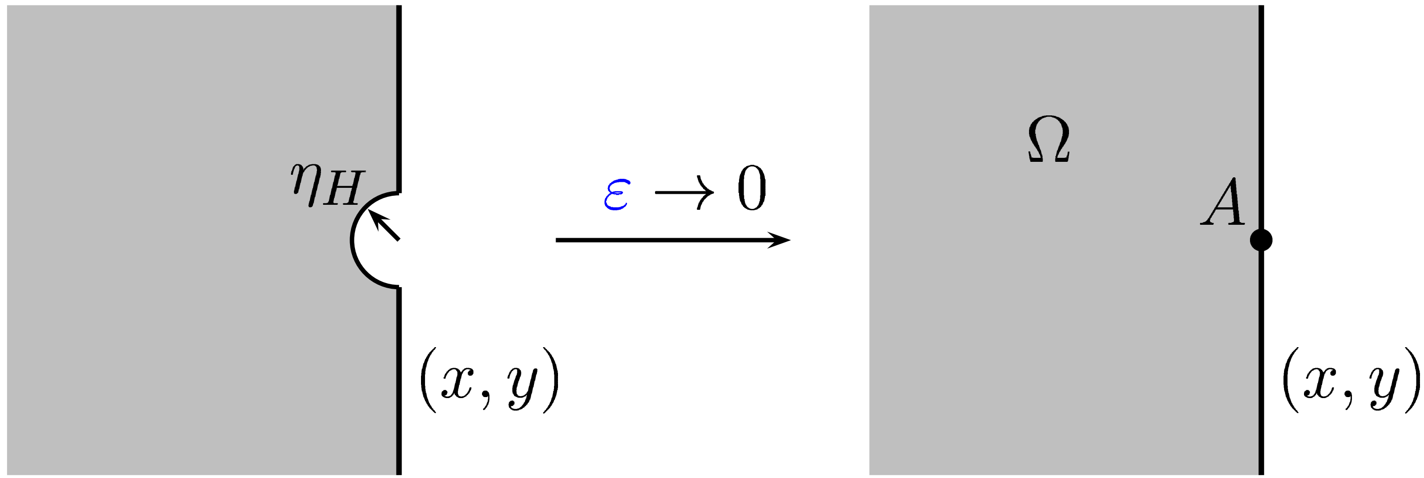

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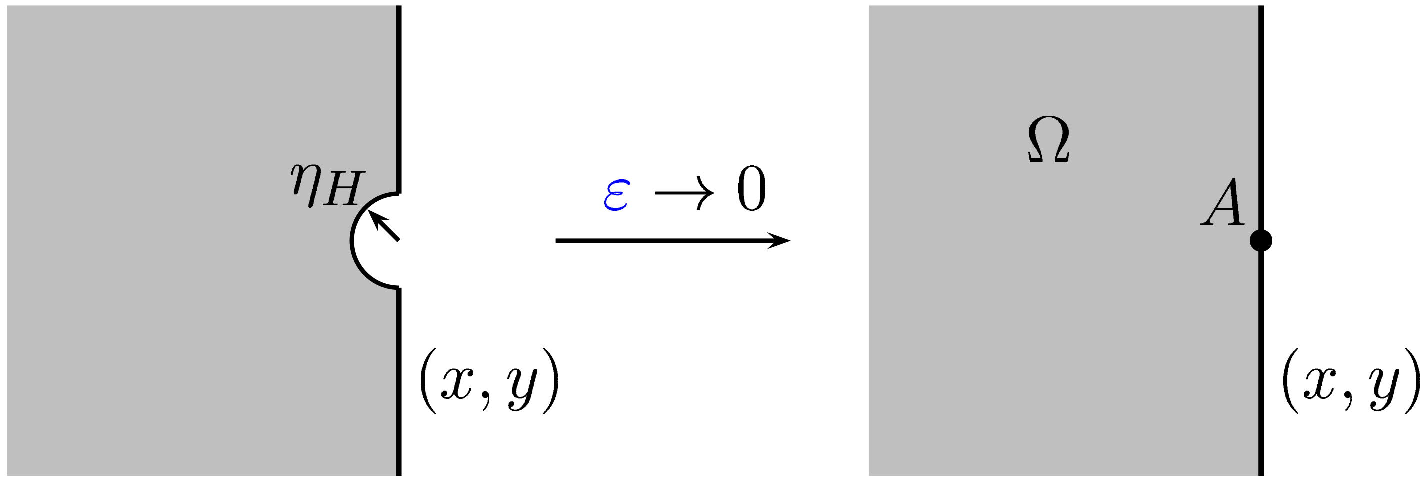
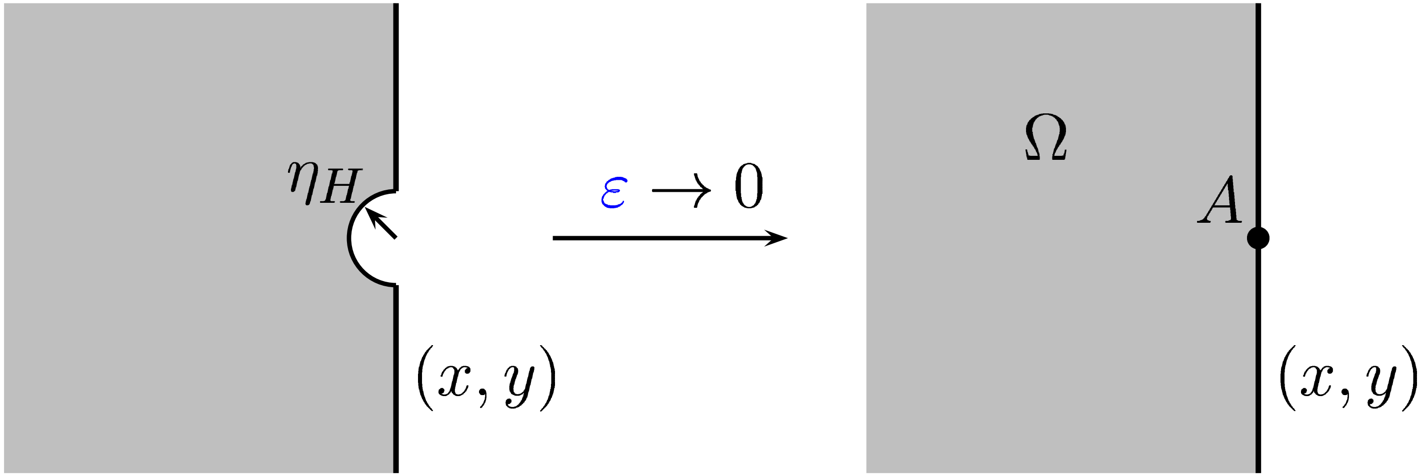


$$u^\varepsilon = u^0 + \sum_{i=1}^{+\infty} \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k + o(\varepsilon^\infty), \qquad \text{in } \Omega.$$

Far field

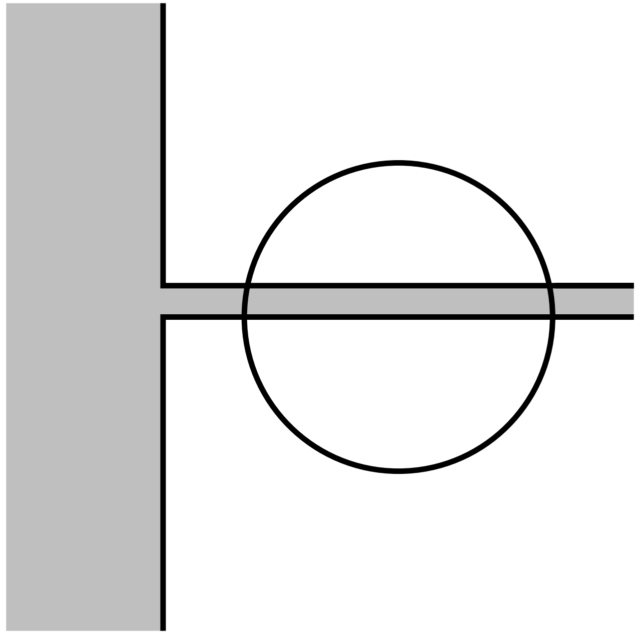
Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda$.



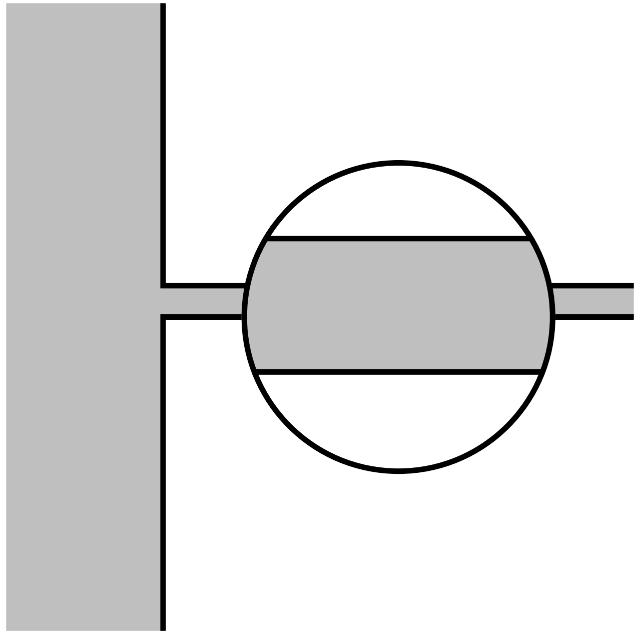
where the u_i^k satisfy the homogeneous Helmholtz equation

$$\Delta u_i^k + \omega^2 u_i^k = 0.$$

Slot field

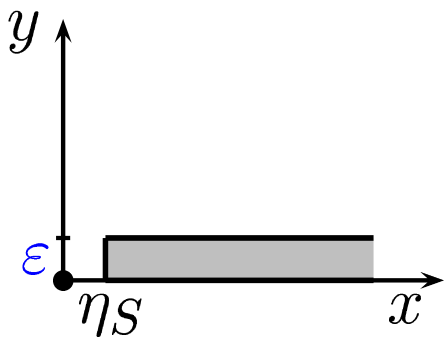


Slot field



$$u^{\varepsilon}(x,y) \;=\; U^{\varepsilon}(x,\frac{y}{\varepsilon})$$

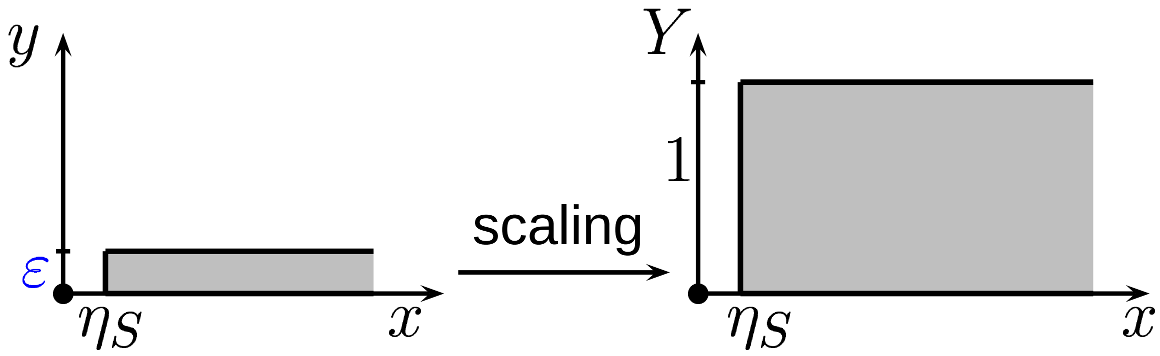
Slot field



The asymptotic context: $\varepsilon \ll \eta_S \ll \lambda$.

The normalization: $X = x, \quad Y = \dfrac{y}{\varepsilon}$

Slot field



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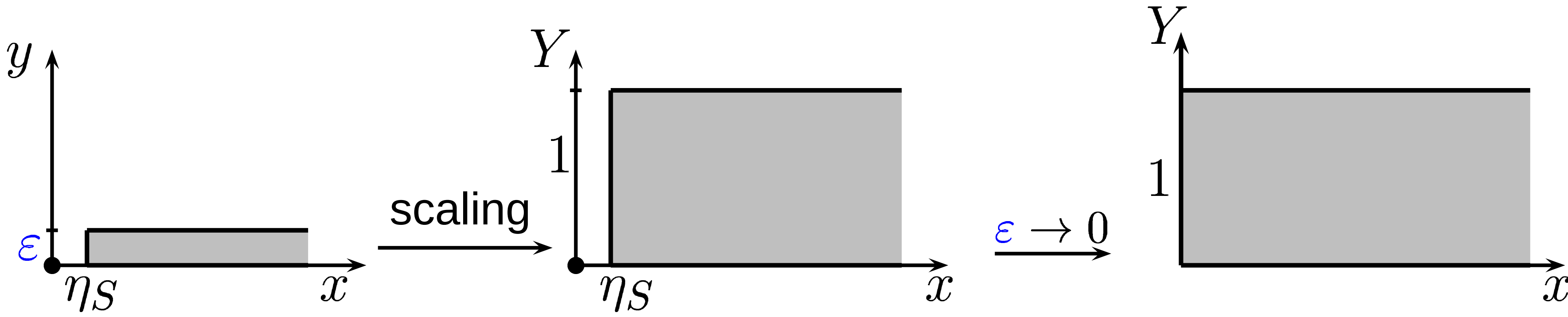

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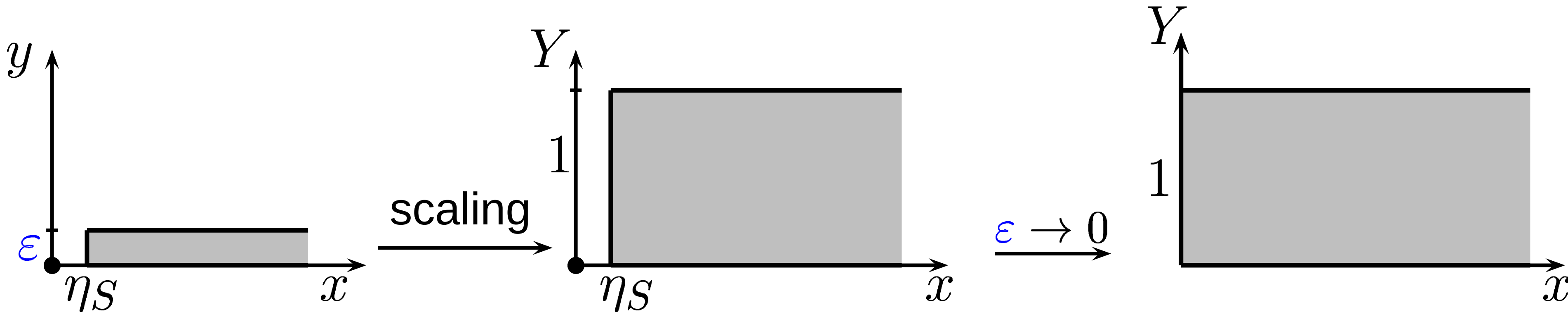
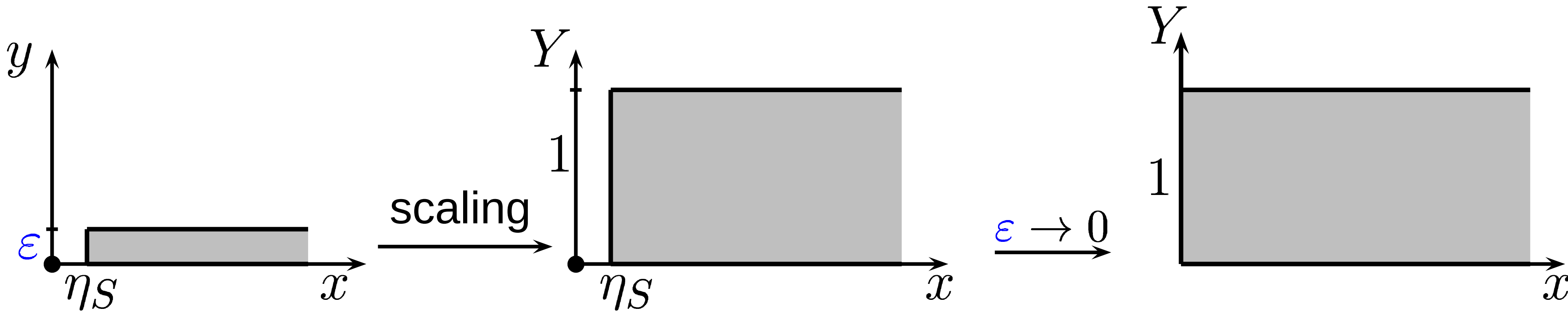


$$u^{\varepsilon}(x, Y\varepsilon) \;=\; U^{\varepsilon}(x, Y) \;=\; \sum_{i=0}^{+\infty} \sum_{k=0}^{i} \varepsilon^{i} \, (\log \varepsilon)^{k} \, U_i^k(x, Y) + o(\varepsilon^{\infty}),$$

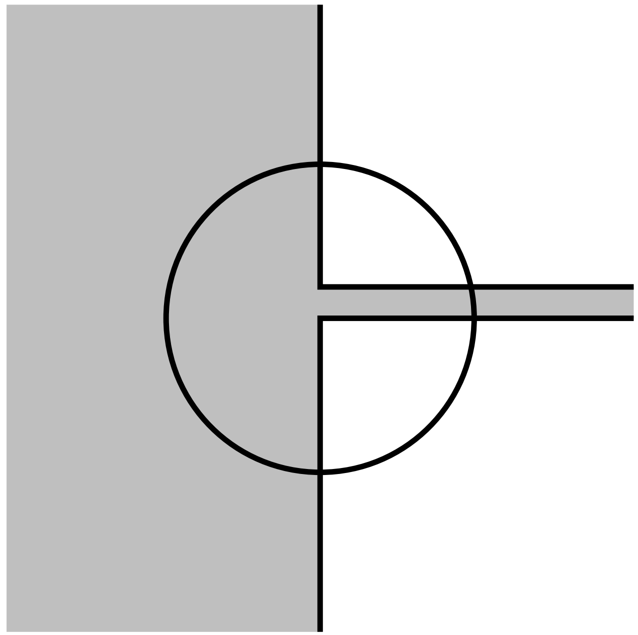
Slot field



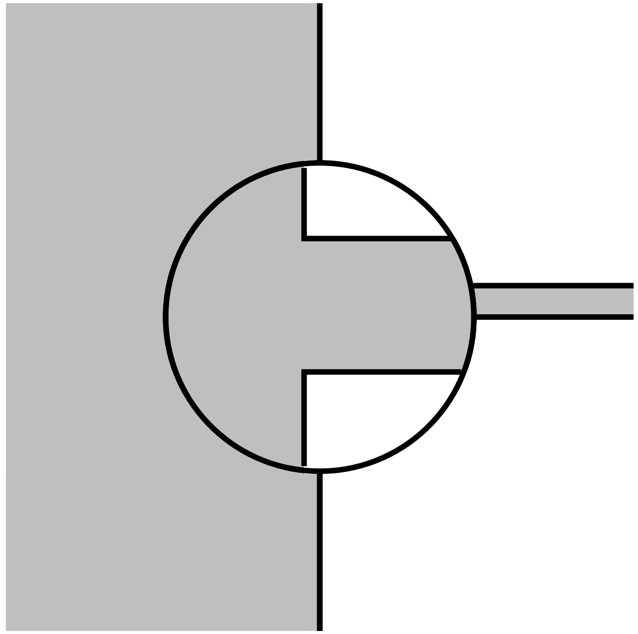
where the U_i^k satisfy the 1D Helmholtz equation:

$$\frac{d^2 U_i^k}{dx^2} + \omega^2 U_i^k = 0$$

Near field

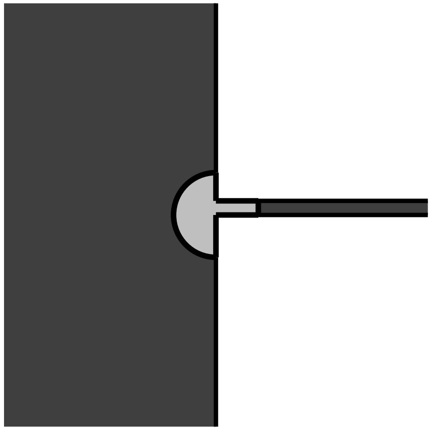


Near field



$$u^{\varepsilon}(x,y) = u_p^{\varepsilon}\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

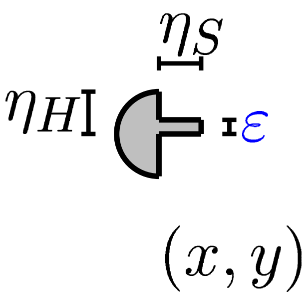
Near field



The Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda, \quad \varepsilon \ll \eta_S \ll \lambda.$

The normalization: $X = \dfrac{x}{\varepsilon}, \quad Y = \dfrac{y}{\varepsilon}$

Near field



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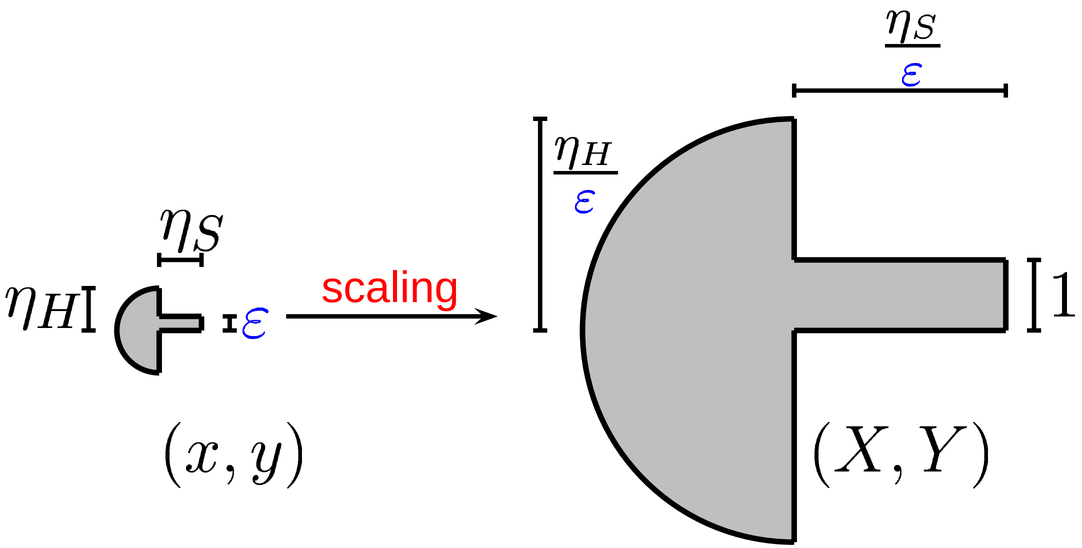

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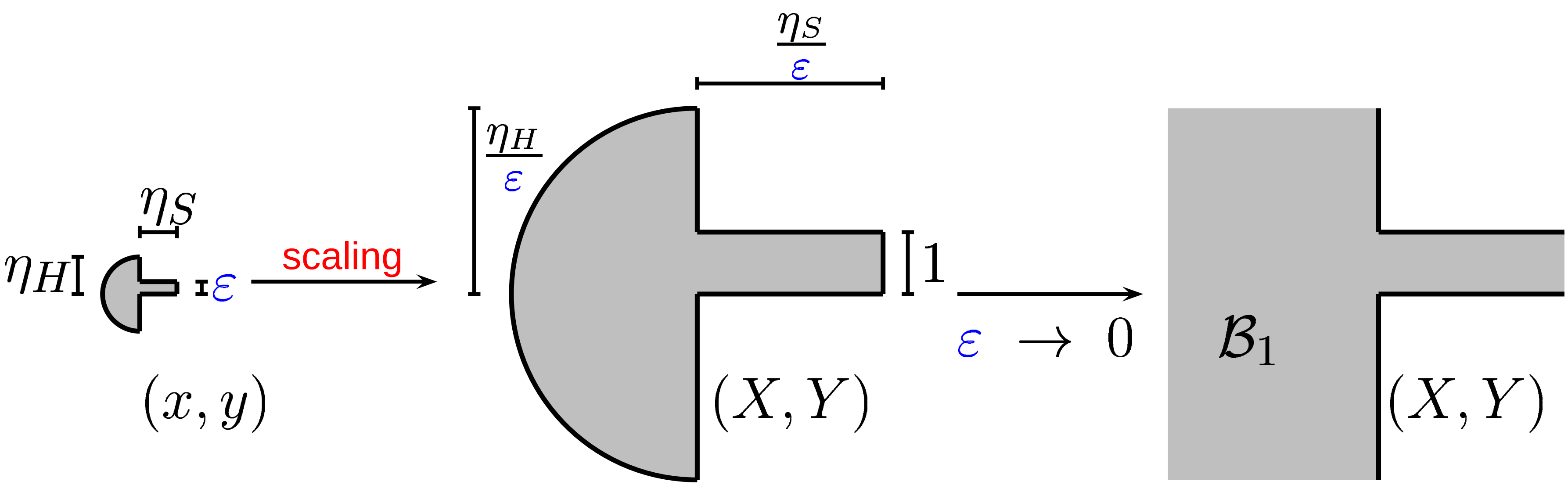

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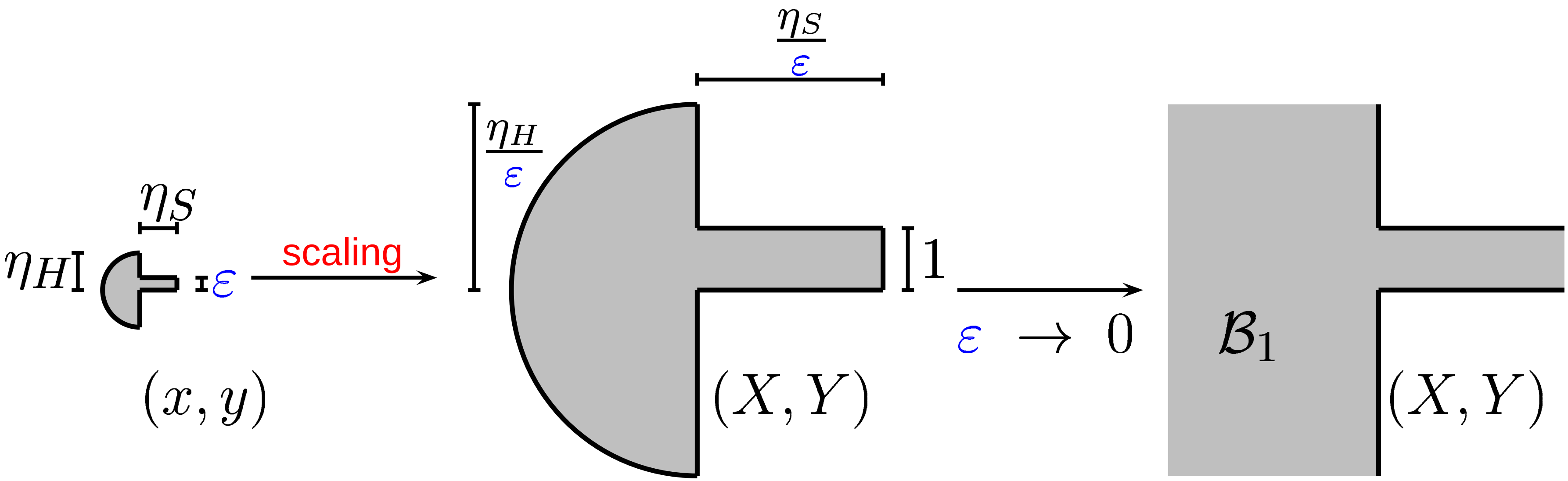
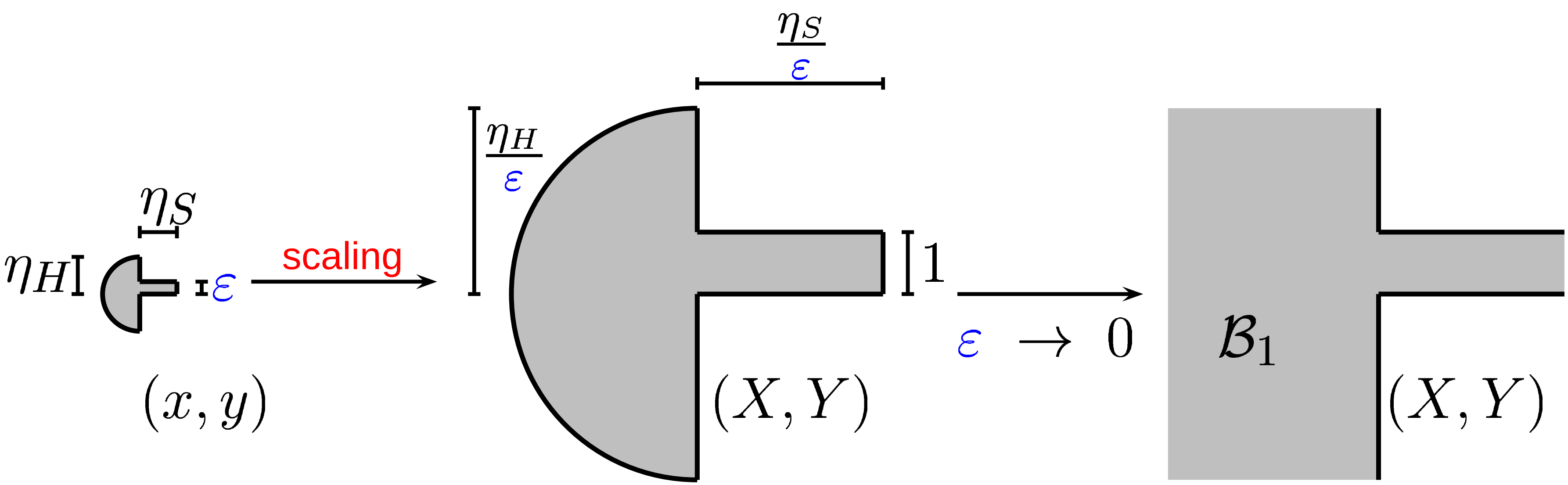


$$u^{\varepsilon}(\varepsilon X, \varepsilon Y) = u_p^{\varepsilon}(X,Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^{i} \varepsilon^i (\log \varepsilon)^k \, (u_p)_i^k(X,Y) + o(\varepsilon^{\infty})$$

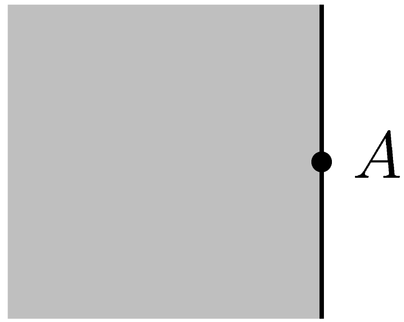
Near field



where the $(u_p)_i^k$ satisfy the (in)-homogeneous Laplace equation.

$$\begin{cases} \Delta(u_p)_i^k = 0, & \text{if } i = k \text{ or } k+1, \\ \Delta(u_p)_i^k = -\,\omega^2\,(u_p)_{i-2}^k, & \text{else.} \end{cases}$$

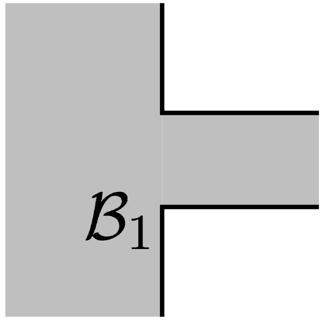
Order 0 : $\underline{u^0}$, $(u_p)_0^0$, U_0^0



Far field:

$$
\begin{cases}
\text{Find } u^0 \in H^1_{loc}(\Omega) \text{ such that :} & \\
-\Delta u^0 - \omega^2 u^0 = f, & \text{in } \Omega, \\
\dfrac{\partial u^0}{\partial n} = 0, & \text{on } \partial\Omega, \\
u^0 \text{ is outgoing.} &
\end{cases}
$$

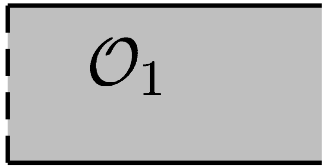
Order 0 : u^0, $\underline{(u_p)_0^0}$, U_0^0



Near field:

$$(u_p)_0^0\,(X,Y) \;=\; u^0(A), \qquad \text{in } \mathcal{B}_1.$$

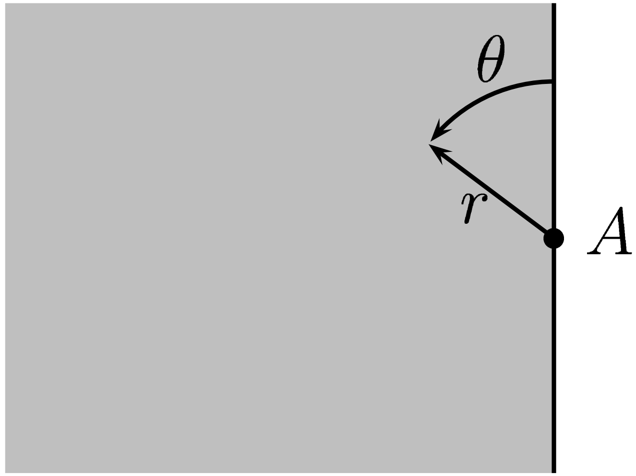
Order 0 : u^0, $(u_p)^0_0$, $\underline{U^0_0}$



Slot field:

$$U^0_0(x, Y) \;=\; u^0(A)\,\mathrm{exp}(\mathbf{i}\omega\, x), \qquad \text{in } \mathcal{O}_1.$$

Order 1 : $\underline{u_1^0}$, $(u_p)_1^0$, $(u_p)_1^1$, U_1^0, U_1^1



Approximation of the exact Solution:

$$u^\varepsilon \simeq u^0 + \varepsilon\, u_1^0$$

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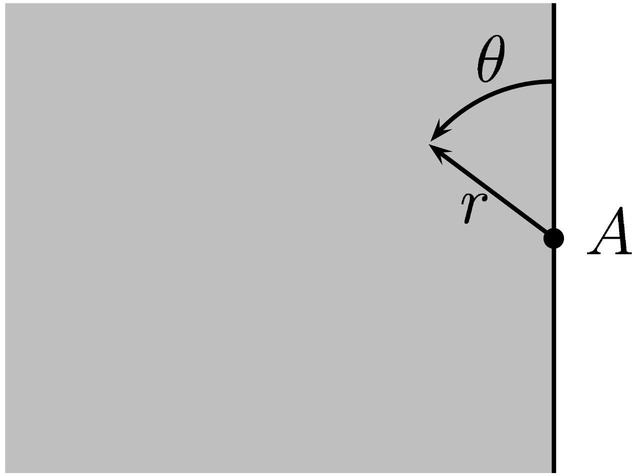

explicit form of u_1^0

$$u_1^0(r,\theta) \;=\; -\,\frac{\omega}{2}\,u^0(A)\,H_0^{(1)}(\omega r).$$

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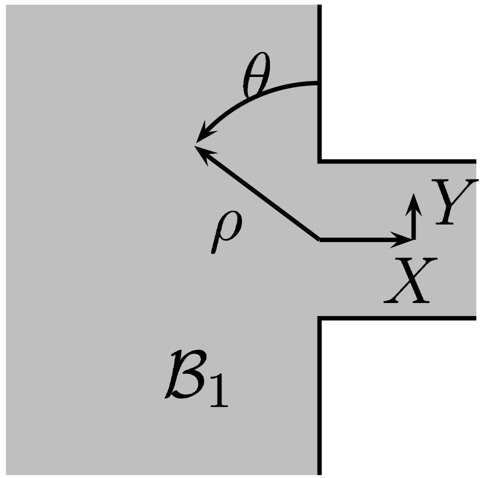

Approximation of the exact solution:

$$\begin{cases} u^\varepsilon(\varepsilon X, \varepsilon Y) \; = \; u_p^\varepsilon(X, Y), \\ u_p^\varepsilon \simeq (u_p)_0^0 \; + \; \varepsilon \; (u_p)_1^0 \; + \; \varepsilon \; \log \varepsilon \; (u_p)_1^1. \end{cases}$$

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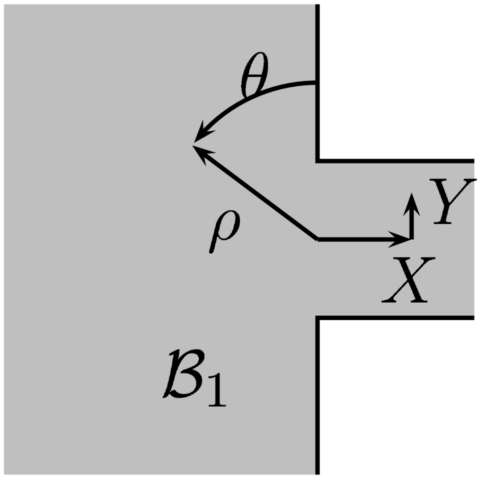
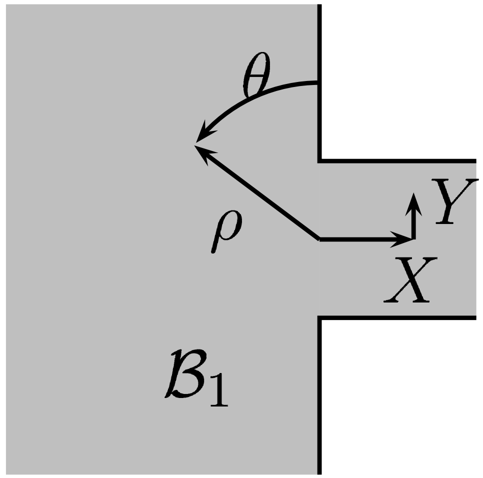


Near field:

Find $(u_p)_1^0 \in H^1_{loc}(\mathcal{B}_1)$ such that:

$$\begin{cases} \Delta (u_p)_1^0 = 0, & \text{in } \mathcal{B}_1 \\ \dfrac{\partial (u_p)_1^0}{\partial n} = 0, & \text{on } \partial\mathcal{B}_1. \end{cases}$$

Order 1 : u_1^0, $\underline{(u_p)_1^0}$, $(u_p)_1^1$, U_1^0, U_1^1



Behavior at infinity in the half-space:

$$(u_p)_1^0(\rho,\theta) - \frac{\partial u^0}{\partial y}(A)\,\rho\,\cos\theta + \frac{\omega}{2}\,u^0(A)\left[1 + \frac{2\mathbf{i}}{\pi}\left(\log\rho + \gamma\right)\right] = O(\frac{1}{\rho}).$$

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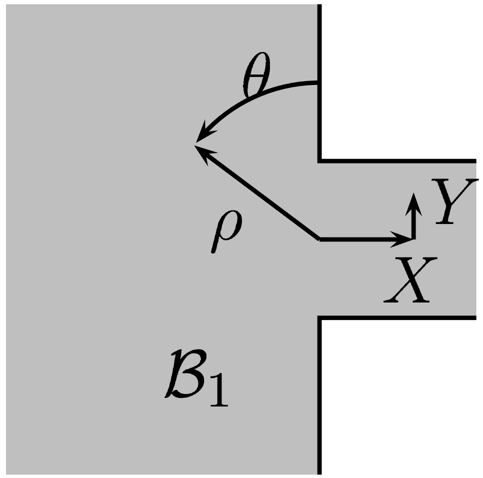

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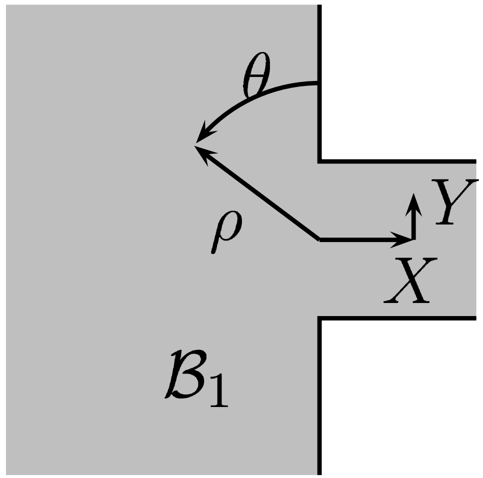
Behavior at infity in the slot:

$$(u_p)_1^0\,(X,Y) - i\,\omega\,u^0(A)\,X = O(1).$$

Order 1 : u_1^0, $(u_p)_1^0$, $\underline{(u_p)_1^1}$, U_1^0, U_1^1



$$(u_p)_1^1 \;=\; -\frac{\mathbf{i}\omega}{\pi}\, u^0(A)$$

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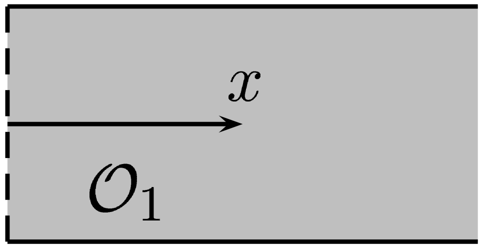

Approximation of the exact solution:

$$\begin{cases} u^\varepsilon(x, \varepsilon Y) \;=\; U^\varepsilon(x, Y), \\ U^\varepsilon \;\simeq\; U_0^0 + \varepsilon\, U_1^0 + \varepsilon\, \log \varepsilon\, U_1^1. \end{cases}$$

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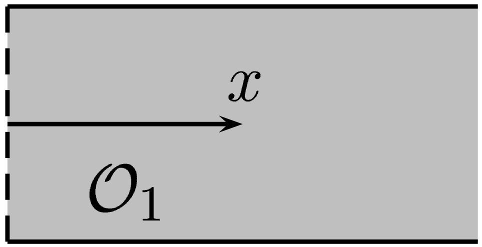

The slot field:

$$U_1^0(x) \;=\; \int_0^1 (u_p)_1^0(0,Y)\; dY \;\; \exp\left(\mathbf{i}\omega x\right),$$

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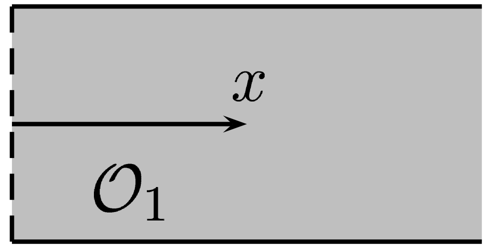
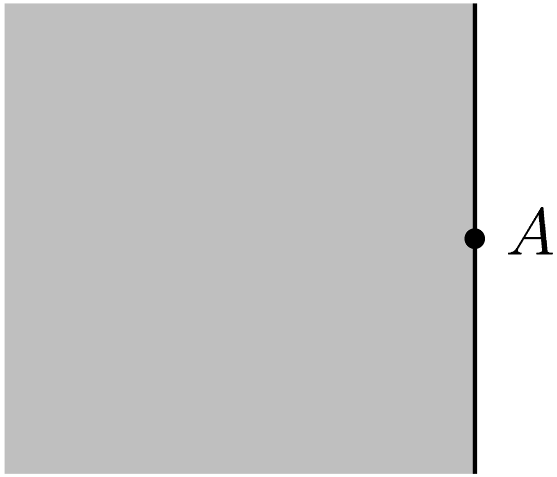


The slot field:

$$U_1^1(x) \; = \; -\frac{\mathbf{i}\omega}{\pi}\, u^0(A) \; \exp\left(\mathbf{i}\omega x\right).$$

The far field of order $i > 1$

- The fields u_i^k are defined in the half space:



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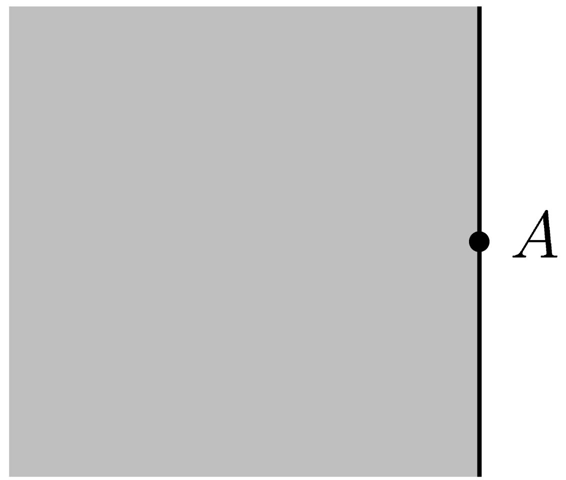

- The far fields u_i^k
  - satisfy the homogeneous Helmholtz equation
  - are singular at the neighborhood of the origin
  - are outgoing at infinity

The far field of order $i > 1$

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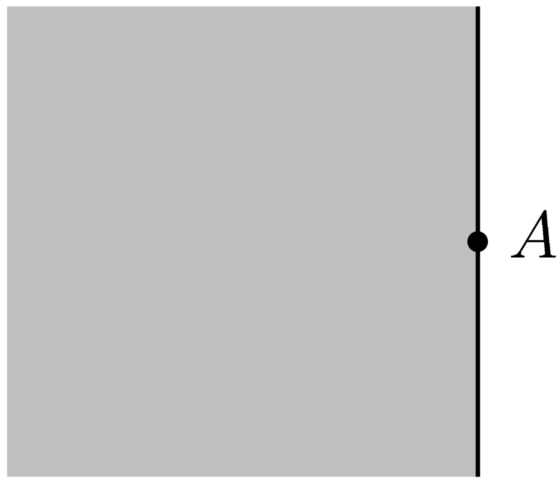

- $$u_i^k = \sum_{p=0}^{+\infty} a_p \, H_p^{(1)}(\omega r) \; \cos p\theta$$

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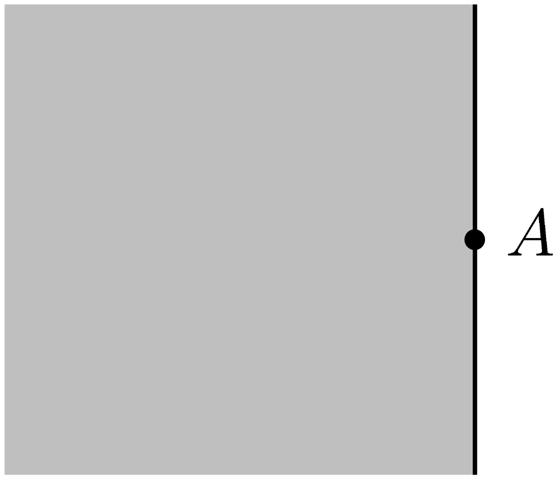

- $$u_i^k = \sum_{p=0}^{i-k-1} a_p \, H_p^{(1)}(\omega r) \, \cos p\theta$$

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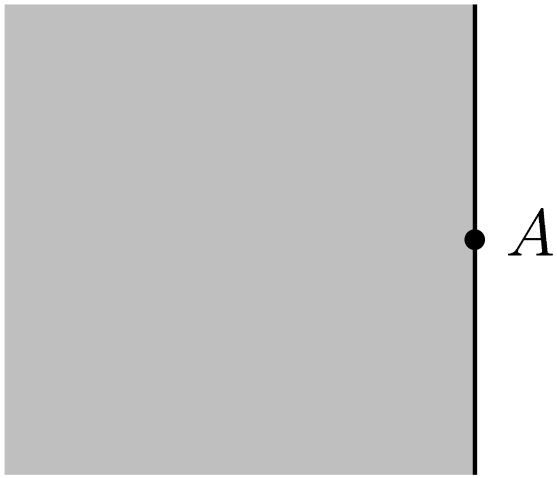

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The a_p are functions of lower order terms

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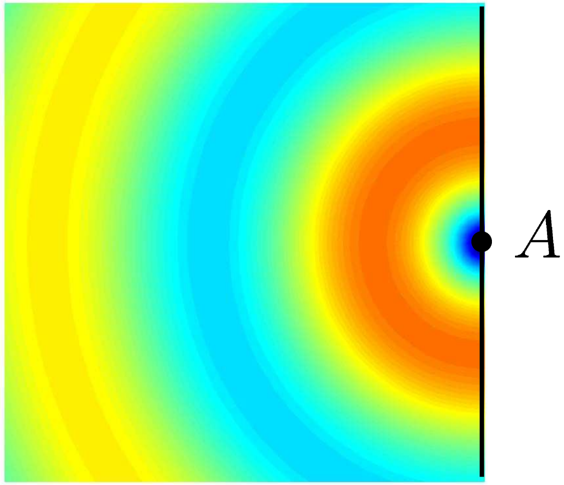
$\mathrm{Im}(H_0^{(1)}(\omega r))$



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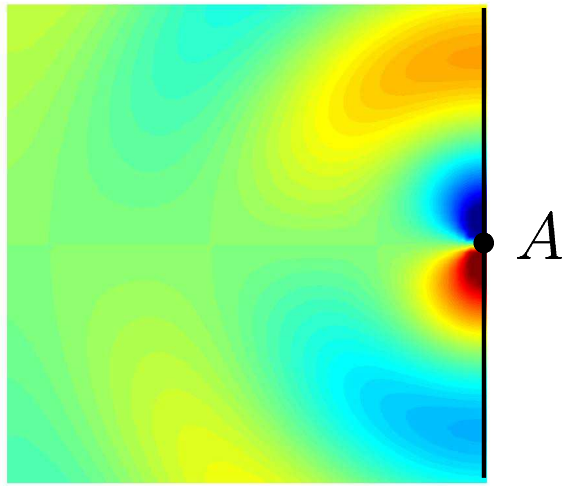

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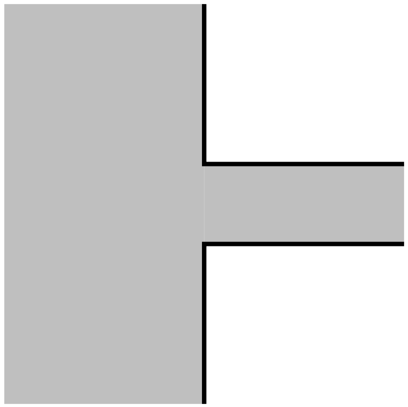
The near fields of order $i > 1$

- The $(u_p)_i^k(X, Y)$ are defined in the canonical domain:



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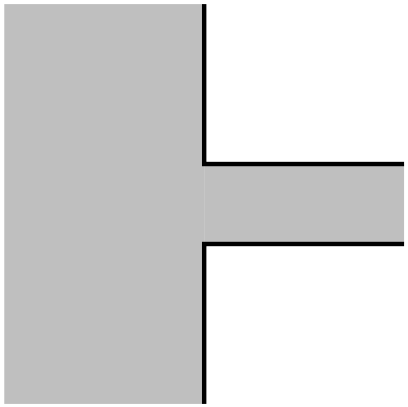

- by Laplace equation:

$$\Delta (u_p)_i^k = 0, \qquad (i = k \text{ ou } k+1),$$

$$\Delta (u_p)_i^k = -\omega^2 \, (u_p)_{i-2}^k, \qquad (i \geqslant k+2),$$

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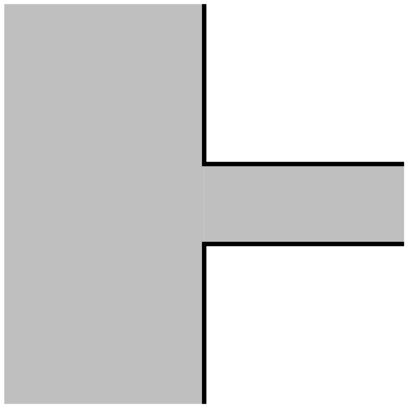

- by Laplace equation:
- by polynomial growings at infinity:
  - The growings in the half space are functions of far field of lower (or equal) order
  - The growings in the slot are functions of the slot fields of lower order

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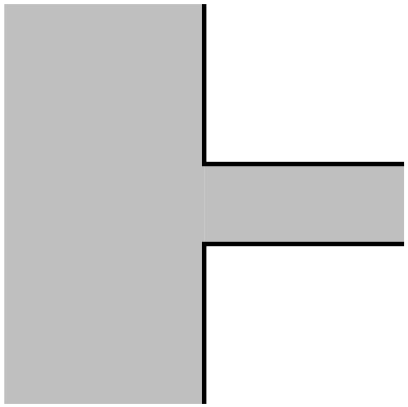

- Proof of the existence-unicity:
  - with truncature functions, we subtract the growing behavior at infinity of the $(u_p)_i^k$
  - We use the “classical” variational theory (wheighted Sobolev spaces, Leroux, Hardy,...)

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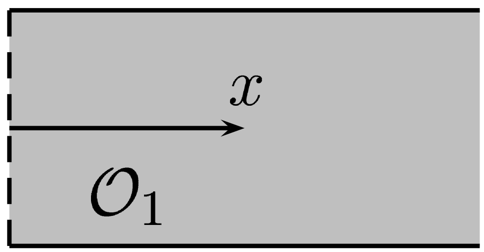

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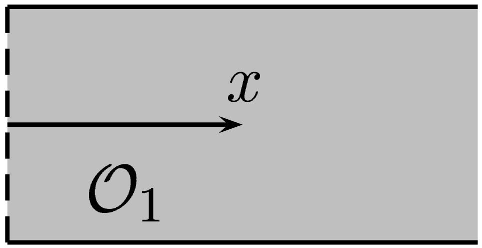

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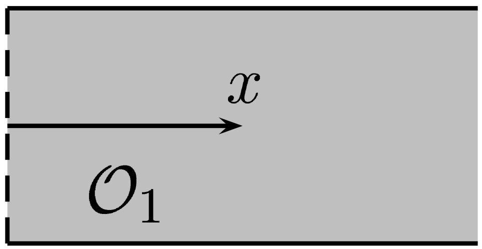

- The U_i^k does not depend on y.
- $U_i^k(x) \;=\; \left(\int_0^1 (u_p)_i^k(0,Y)dY\right)\; \exp \mathbf{i}\omega x$

Some properties

We see that:

- More $i - k$ is large more u_i^k is singular at the origin:

$$r^{-p} \text{ terms}, \quad p = 0, ..., i - k - 1$$

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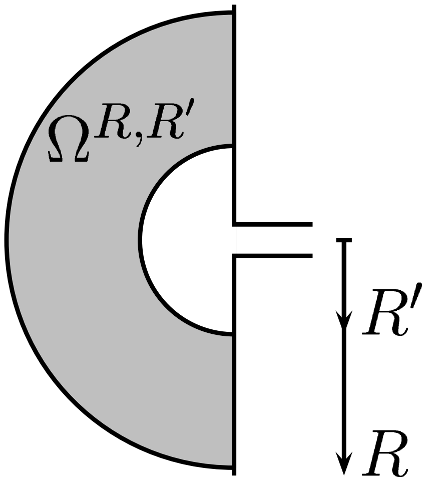
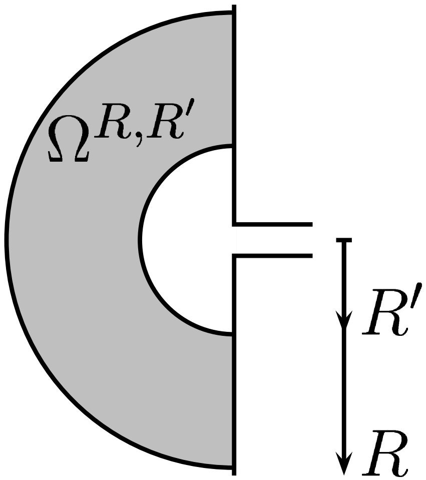
- When the order i grows, one has $O(\frac{i^2}{2})$ ($\times 3$) terms to compute...

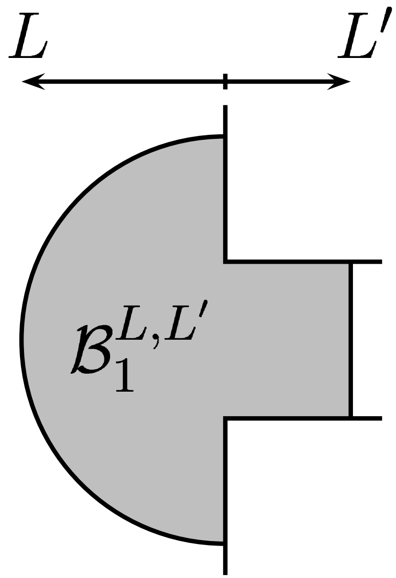
Mathematical analysis



$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^{n} \sum_{k=0}^{i-1} \varepsilon^i \, (\log \varepsilon)^k \, u_i^k \right\|_{H^1(\Omega^{R,R'})} \le C \, \varepsilon^{n+1} \, (\log \varepsilon)^n \, \|f\|_{L^2(\Omega)}.$$

Mathematical analysis

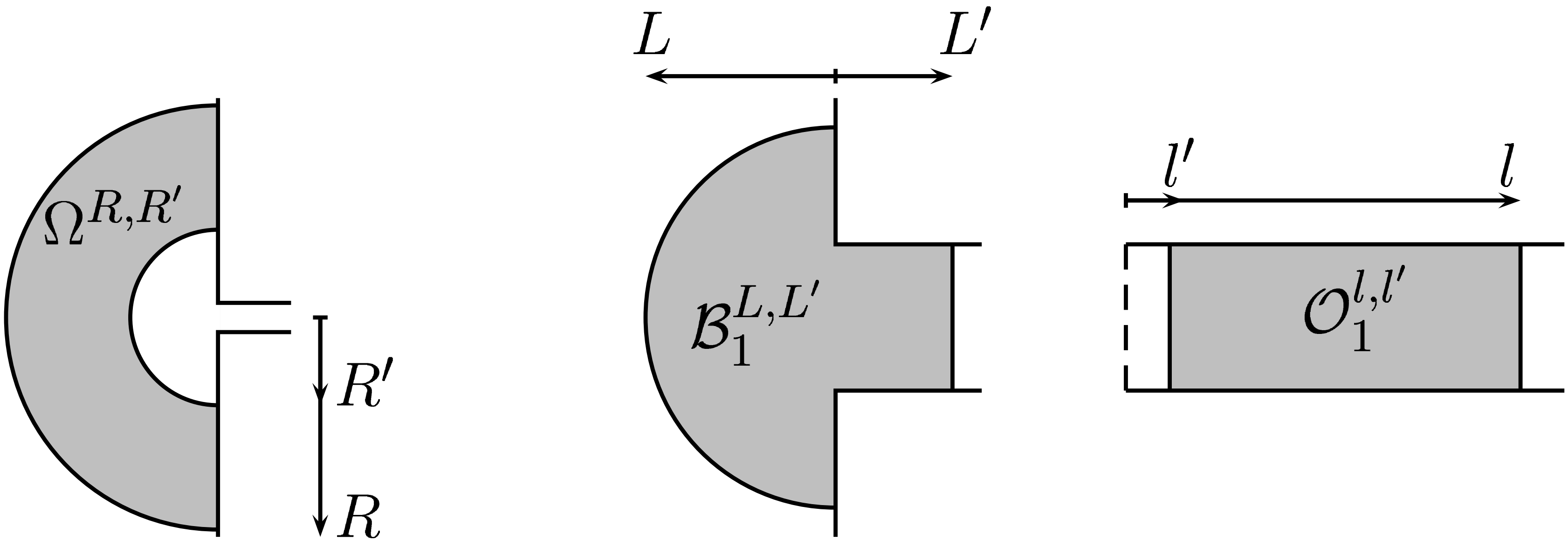




$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^{n} \sum_{k=0}^{i-1} \varepsilon^i \, (\log \varepsilon)^k \; u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \; \varepsilon^{n+1} \, (\log \varepsilon)^n \; \|f\|_{L^2(\Omega)}.$$

$$\left\| u_p^\varepsilon - \sum_{i=0}^{n} \sum_{k=0}^{i} \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \; \varepsilon^{n+1} \, (\log \varepsilon)^{n+1} \; \|f\|_{L^2(\Omega)}.$$

Mathematical analysis



$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^{n} \sum_{k=0}^{i-1} \varepsilon^i \, (\log \varepsilon)^k \, u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \, \varepsilon^{n+1} \, (\log \varepsilon)^n \, \|f\|_{L^2(\Omega)}.$$

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$$\left\| U^\varepsilon - \sum_{i=0}^{n} \sum_{k=0}^{i} \varepsilon^i \, (\log \varepsilon)^k \, U_i^k \right\|_{H^1(\mathcal{O}_1^{l,l'})} \leq C \, \varepsilon^{n+1} \, (\log \varepsilon)^{n+1} \, \|f\|_{L^2(\Omega)}.$$