



Matching of asymptotic expansions for the wave propagation in media with thin slot

Sébastien Tordeux, Patrick Joly

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Matching of asymptotic expansions for the wave propagation in media with thin slot

Sébastien Tordeux and Patrick Joly

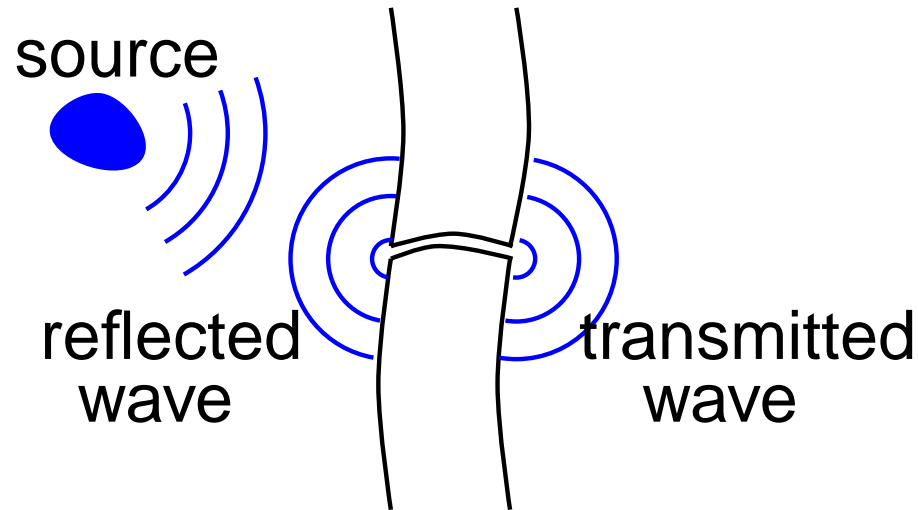
AG Analysis und Numerik, January 2005

INRIA-Rocquencourt-Projet POEMS

ETH-SAM

A typical application

How can we study the scattering in media with **thin slot** ?

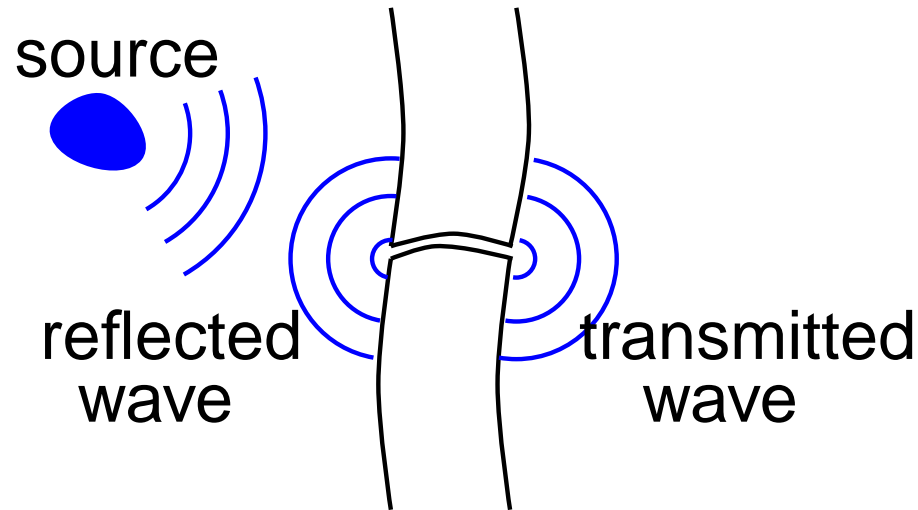


A physical problem with two **characteristical** lengthes

$$\left\{ \begin{array}{l} \text{The } \textbf{wavelength} \lambda \\ \text{The } \textbf{width} \text{ of the slot } \varepsilon \end{array} \right.$$

A typical application

How can we study the scattering in media with **thin slot** ?

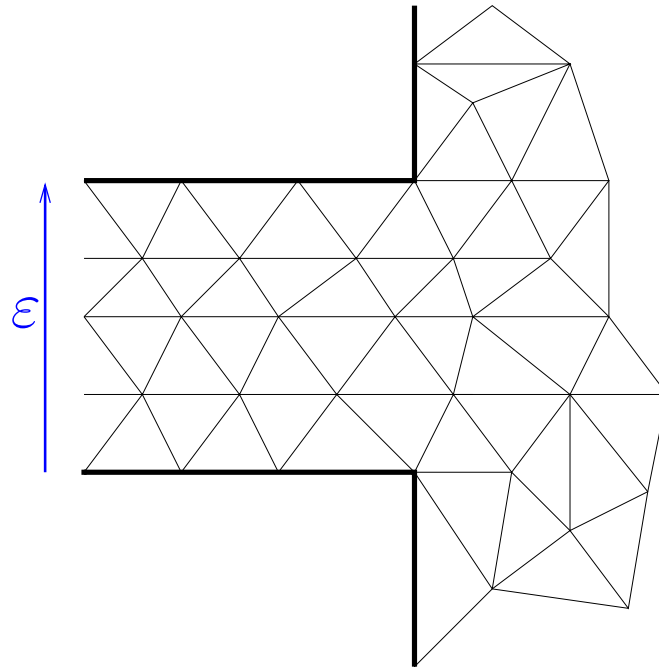


An **asymptotic** case:

$$\varepsilon \ll \lambda$$

The numerical difficulty

A **mesh step** smaller than ε



This leads to **costly** computations

Some references

- Thin slot:
Harrington, Auckland (1980), Tatout (1996).
- Finite differences:
Taflove (1995).
- Thin plates and junction theory,...
Ciarlet, Le Dret, Dauge-Costabel.
- Matching of asymptotic expansions:
McIver, Rawlins (1993), Il'in (1992).
- multiscale analysis
Maz'ya, Nazarov, Plamenevskii (1991)
Oleinik, Shamaev, Yosifian (1992)

A simple problem

Scalar wave equation:

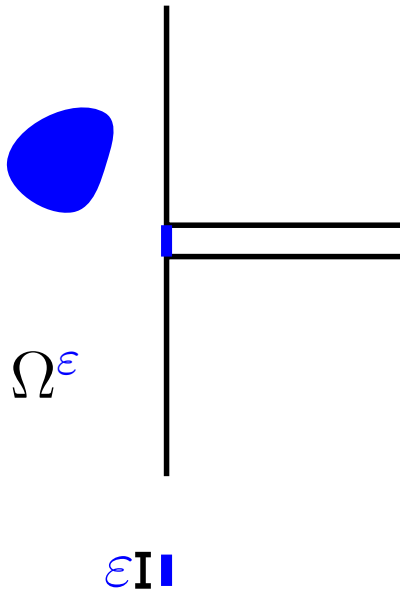
$$\frac{\partial^2 p^\varepsilon}{\partial t^2} - \Delta p^\varepsilon = f$$

Harmonic solution:

$$p^\varepsilon(x, y, t) = \exp(-i\omega t) u^\varepsilon(x, y)$$

Helmholtz Equation:

$$\Delta u^\varepsilon + \omega^2 u^\varepsilon = -f \quad \text{in } \Omega^\varepsilon$$



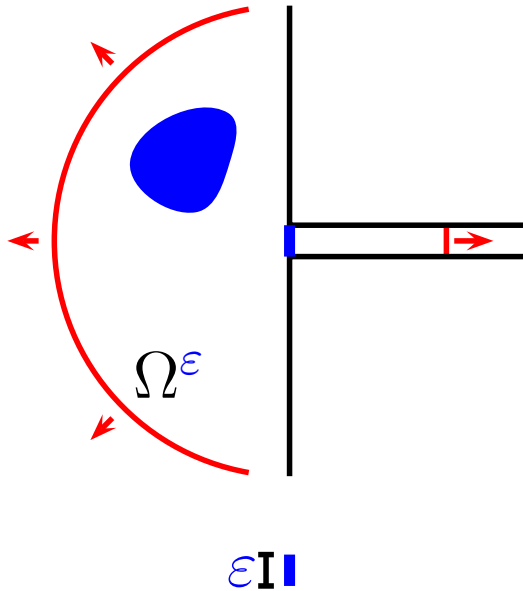
A simple problem

Outgoing solution at infinity:

$$\frac{\partial u^\varepsilon}{\partial n} - i\omega u^\varepsilon \leq \frac{C}{r^2}, \quad \text{for } r \text{ large,}$$

Neumann limit condition
(rigid wall)

$$\frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega^\varepsilon$$



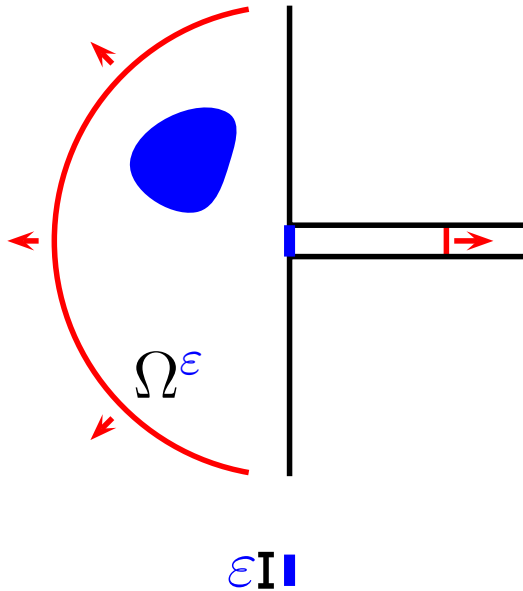
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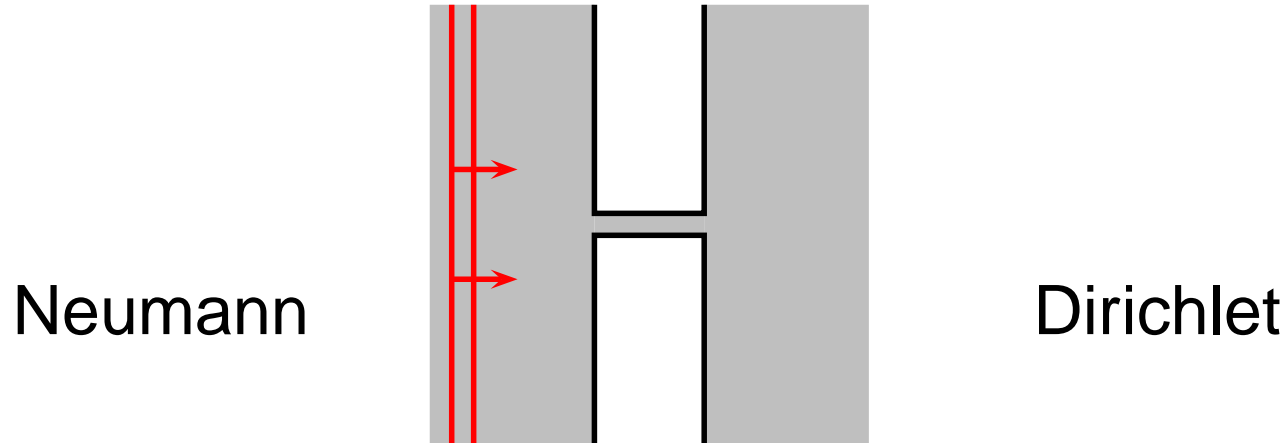
Neumann limit condition
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$$\frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega^\varepsilon$$



With the Dirichlet limit condition, the transmission inside the slot is negligible ($o(\varepsilon^\infty)$).

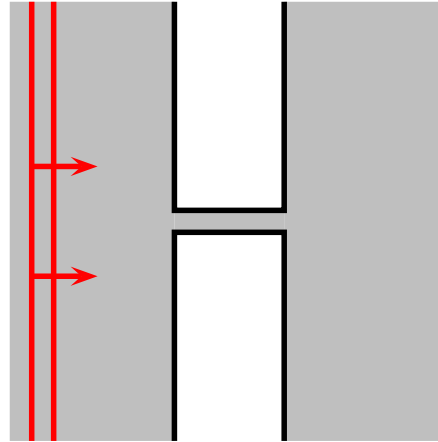
A numerical computation



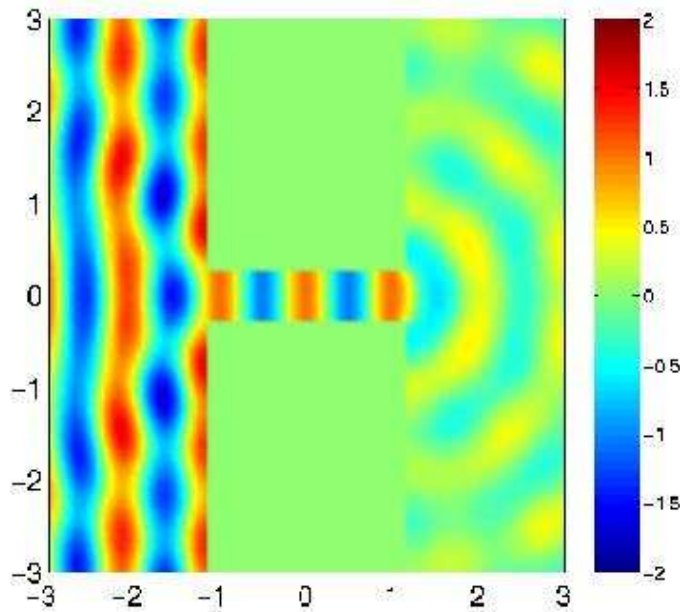
Numerical computation done with the **high order finite elements code** of (M. Duruflé, INRIA)

A numerical computation

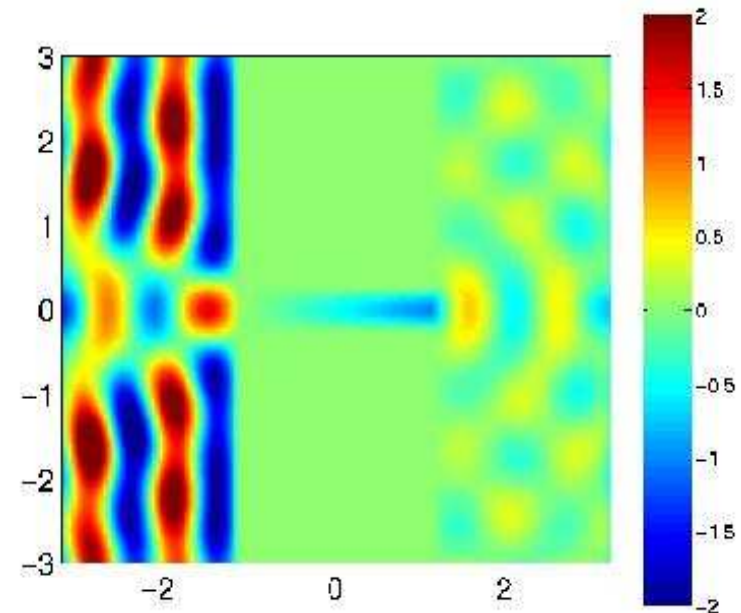
Neumann



Dirichlet

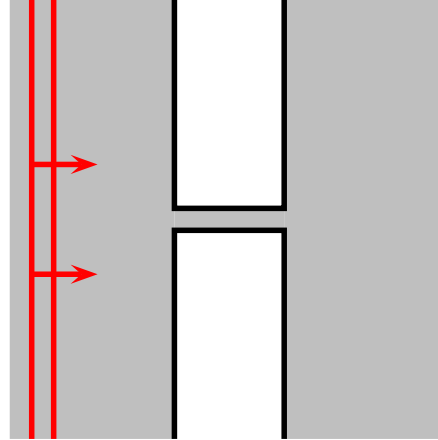


$$\frac{\varepsilon}{\lambda} = 0.5$$

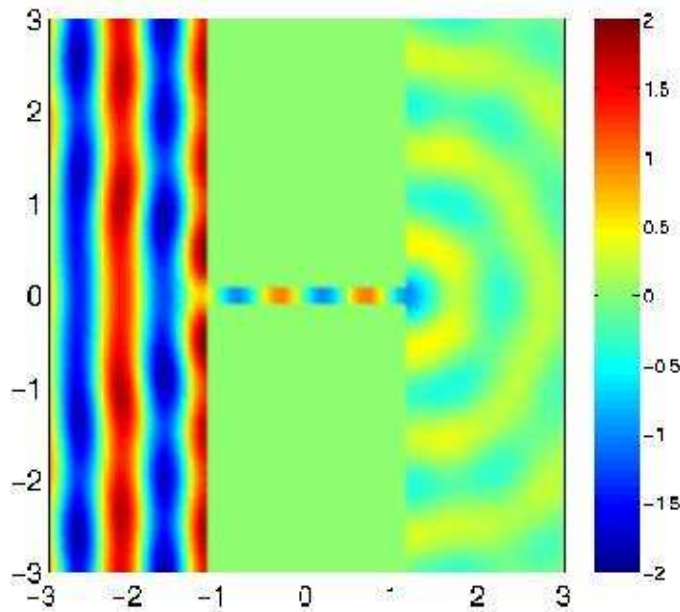


A numerical computation

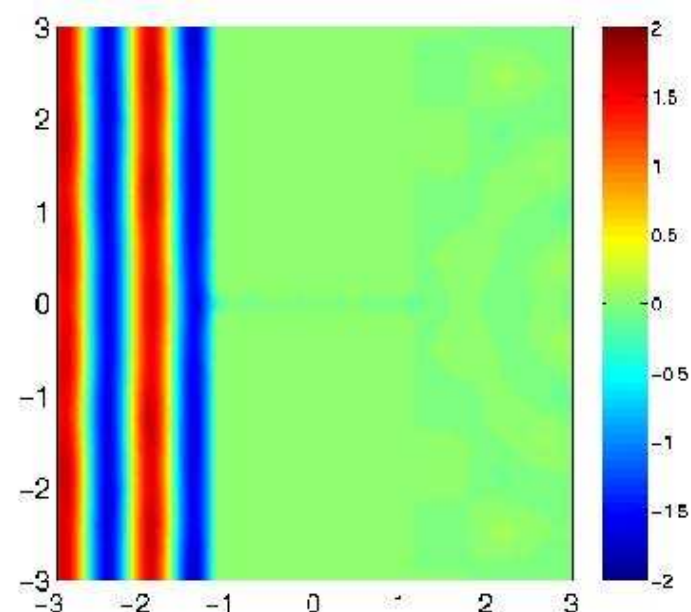
Neumann



Dirichlet

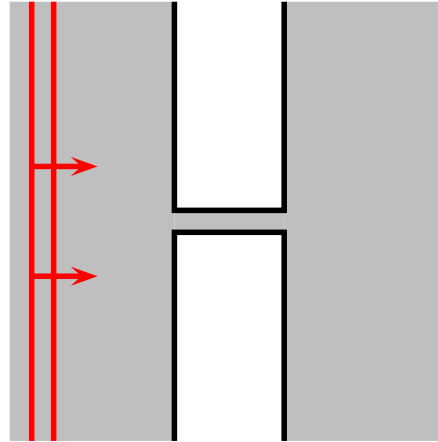


$$\frac{\varepsilon}{\lambda} = 0.2$$

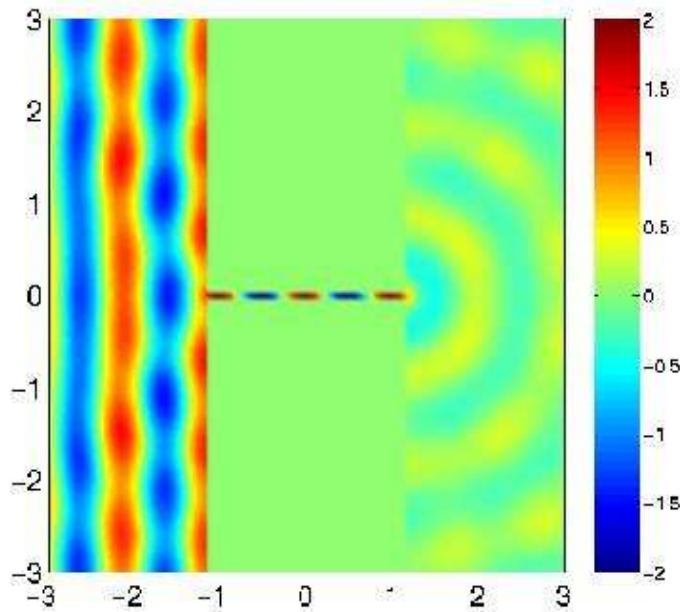


A numerical computation

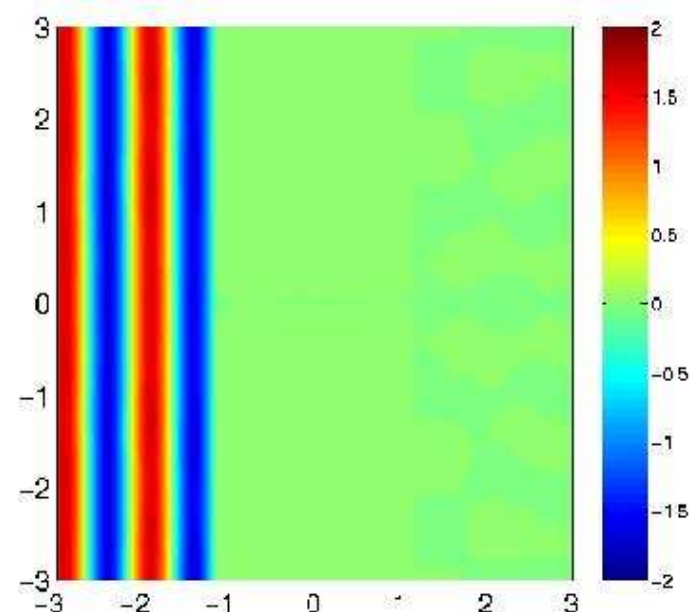
Neumann



Dirichlet



$$\frac{\varepsilon}{\lambda} = 0.1$$

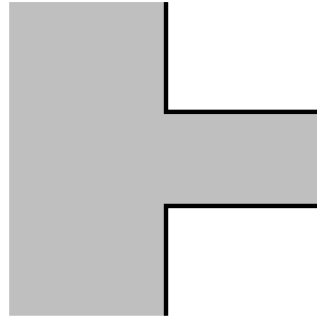


Objectives

- Introduce **accurate** numerical methods

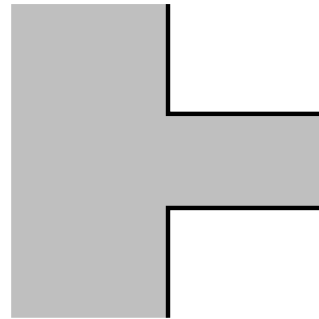
Objectives

- Introduce **accurate** numerical methods
- We need an **intermediate zone**



Objectives

- Introduce **accurate** numerical methods
- We need an **intermediate zone**



- A technique **the matching of asymptotic expansions**
 - Define **new approximate models** to compute the solution.
 - Use effectively “universal” technique of numerical computation (mesh reffinement).

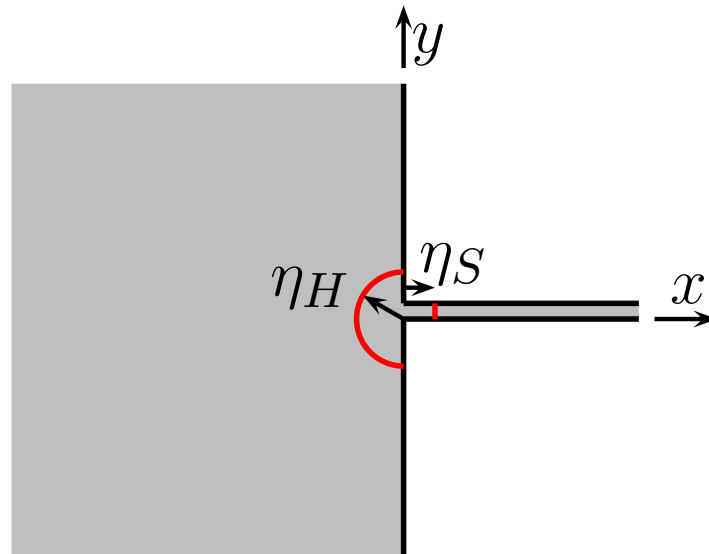
Contributions to the match. of as. exp.

- A new presentation of the **matching principle** (not allways clear) postulated by the english school.

Contributions to the match. of as. exp.

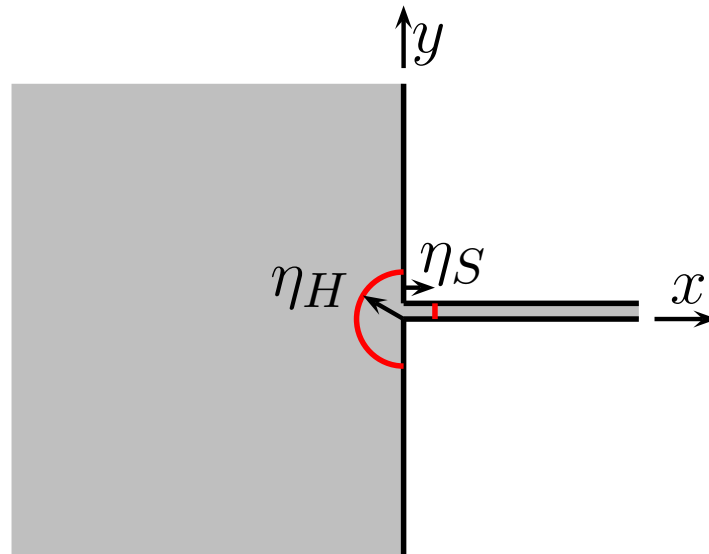
- A new presentation of the **matching principle** (not allways clear) postulated by the english school.
- The **mathematical justification** of this technique.
 - The proof are **inspired** by the multiscale technique
 - **Existence and unicity** of the terms of the expansions.
 - Specific technique: **error estimates**.

Three zones



- Far field (2D field)
- Near field (boundary layer)
- Slot field (1D field)

Three zones

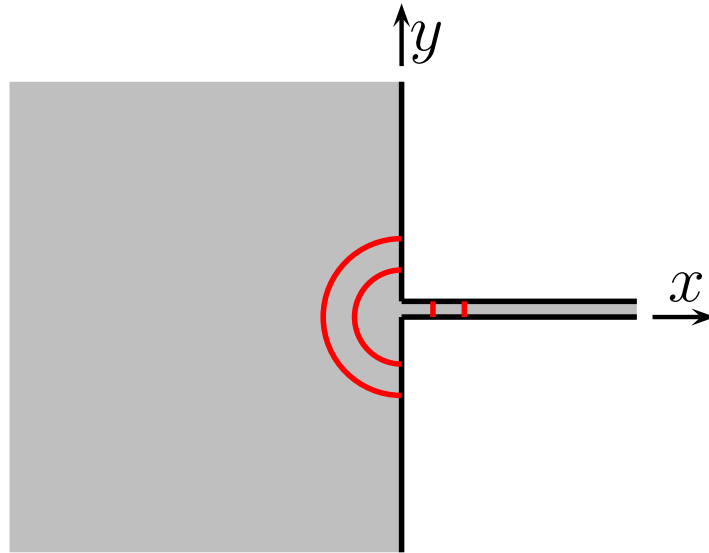


The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones

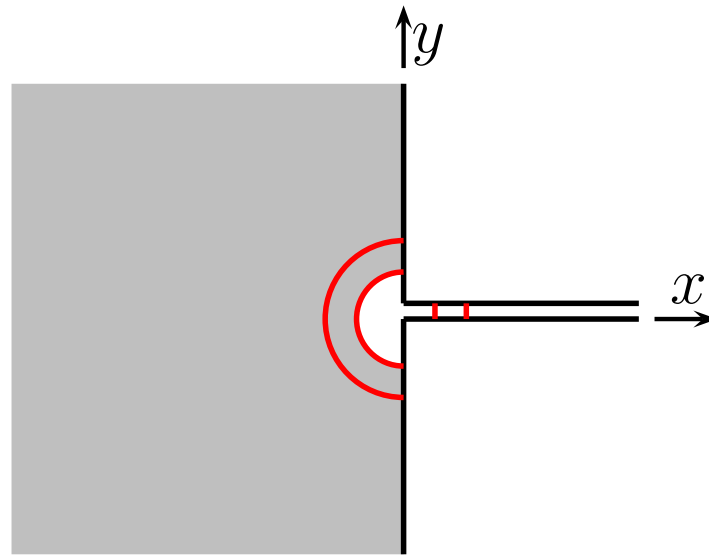


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Three zones



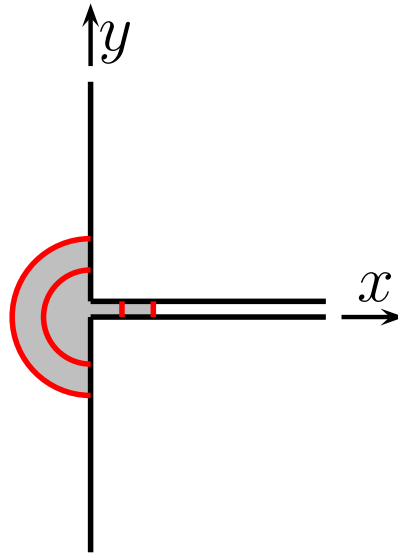
Far field

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Three zones



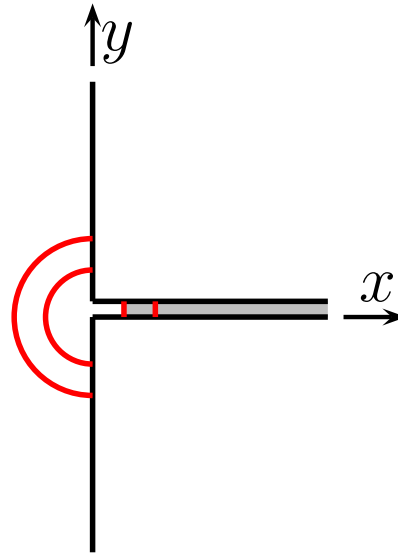
Near field

The **asymptotic assumptions**:

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Three zones



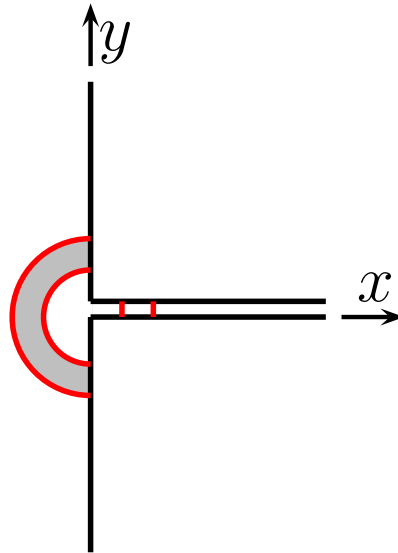
Slot field

The **asymptotic assumptions**:

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$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



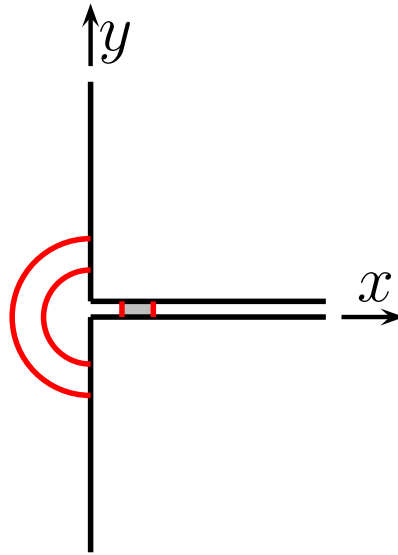
Far and near

The **asymptotic assumptions**:

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$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

Three zones



Slot and near

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

The different steps of the method

- **Derivate** the asymptotic expansions:
 - **Formal** part
 - Several presentations are possible

The different steps of the method

- **Derivate** the asymptotic expansions:
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 - **Definition** of the terms of the asymptotic expansions

The different steps of the method

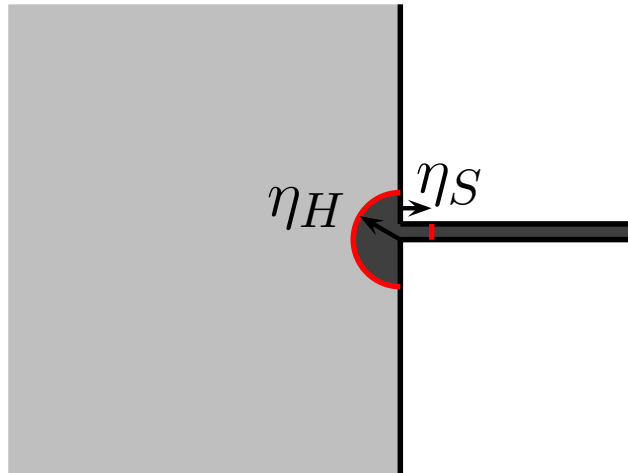
- **Derivate** the asymptotic expansions:
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 - **Rigorous** part
 - **Error estimates**

The different steps of the method

- 2 **Derivate** the asymptotic expansions:
 - **Formal** part
 - Several presentations are possible
- 1 **Describe** the asymptotic expansions
 - **Rigorous** part
 - **Definition** of the terms of the asymptotic expansions
- 3 **Mathematical validation** of the asymptotic expansions
 - **Rigorous** part
 - **Error estimates**

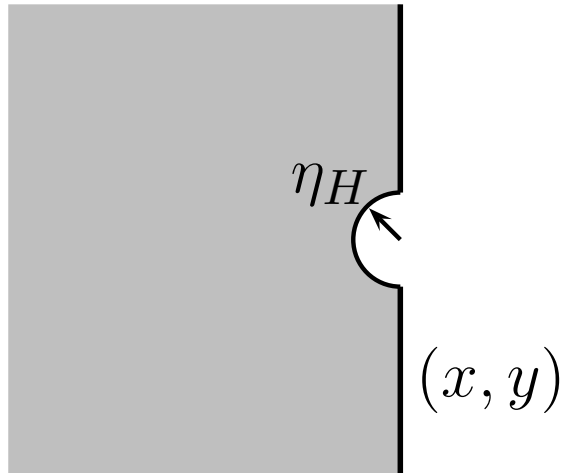
Far field

Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda.$



Far field

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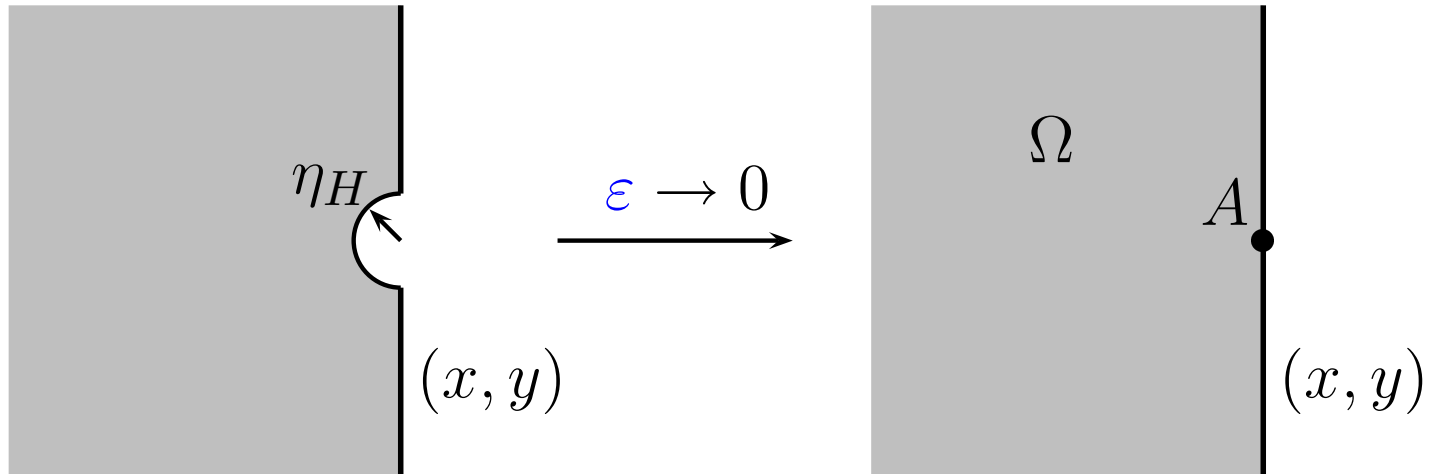


No **normalization**:

$$X = x, \quad Y = y.$$

Far field

Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda.$



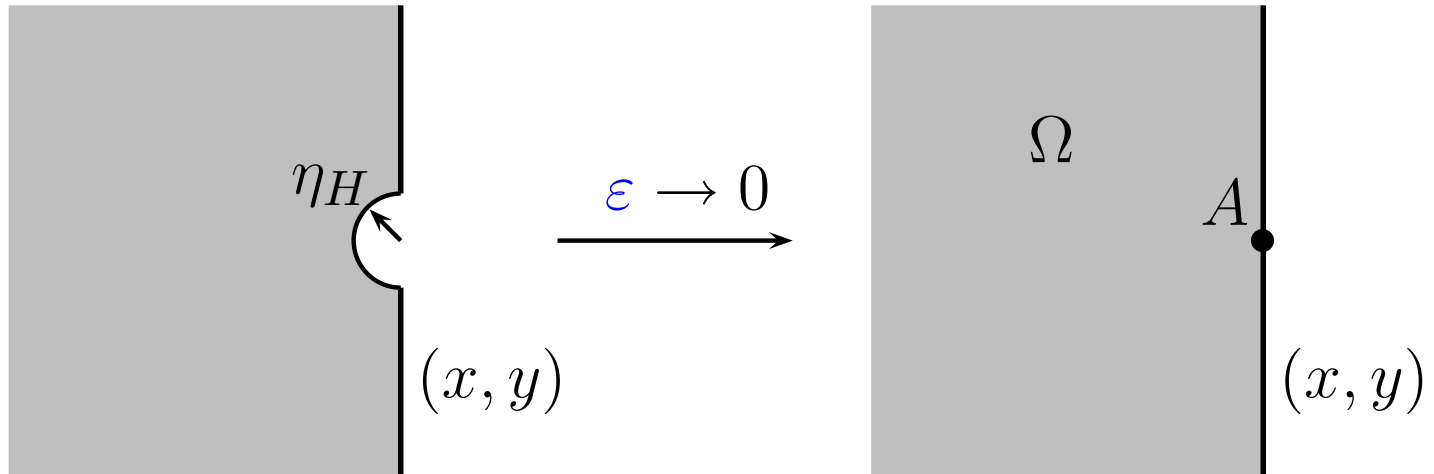
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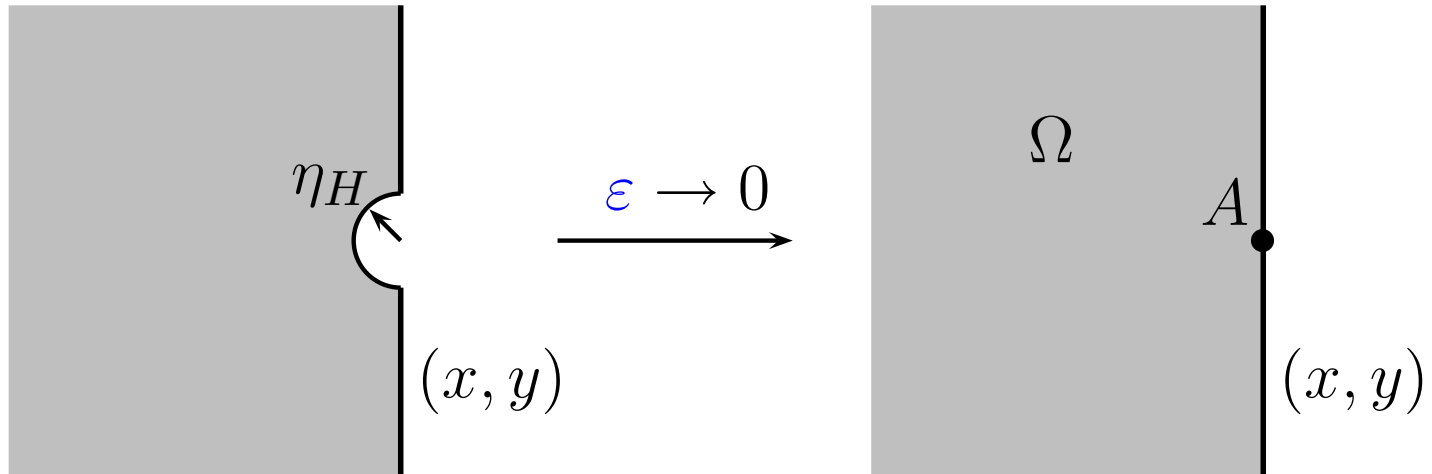
Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda.$



$$u^\varepsilon = u^0 + \sum_{i=1}^{+\infty} \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k + o(\varepsilon^\infty), \quad \text{in } \Omega.$$

Far field

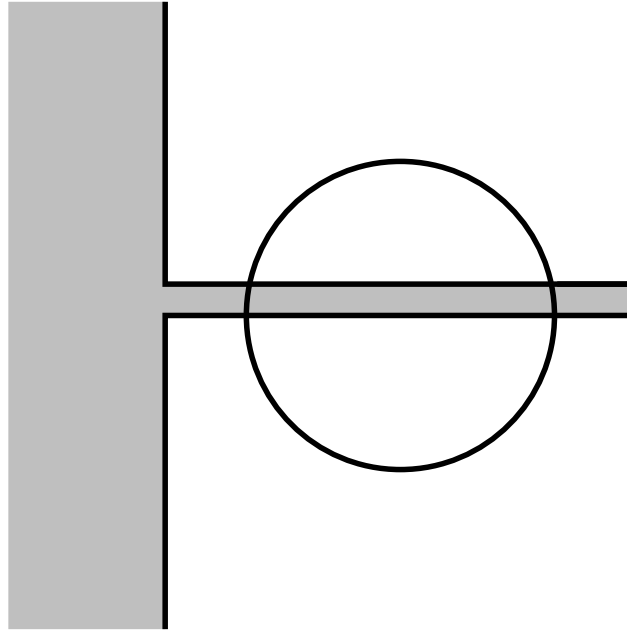
Asymptotic context: $\varepsilon \ll \eta_H \ll \lambda$.



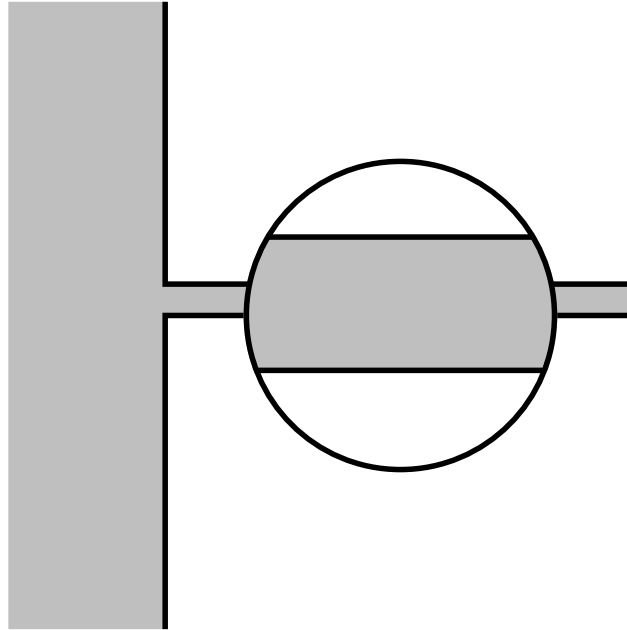
where the u_i^k satisfy the **homogeneous Helmholtz** equation

$$\Delta u_i^k + \omega^2 u_i^k = 0.$$

Slot field

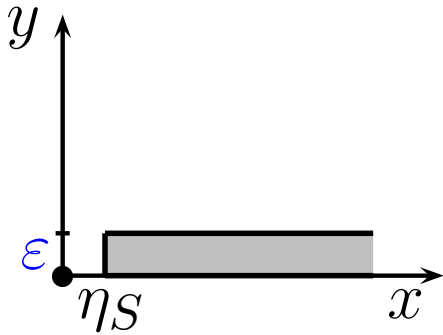


Slot field



$$u^\varepsilon(x, y) = U^\varepsilon\left(x, \frac{y}{\varepsilon}\right)$$

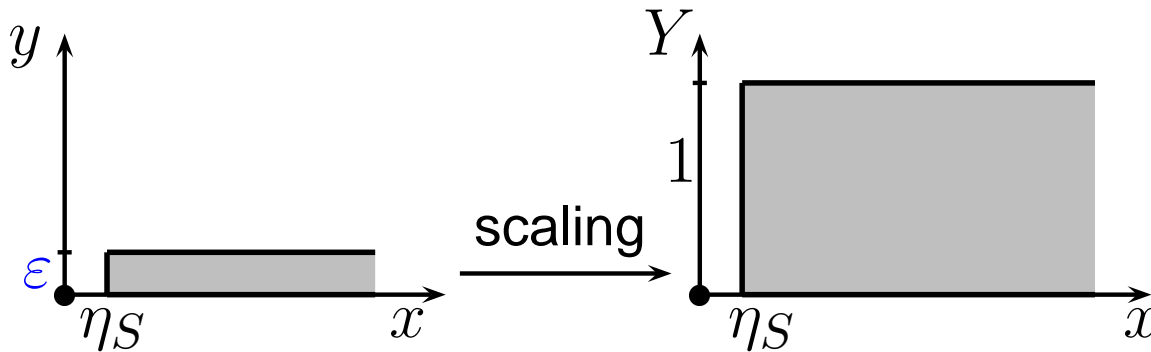
Slot field



The **asymptotic** context: $\varepsilon \ll \eta_S \ll \lambda$.

The **normalization**: $X = x, \quad Y = \frac{y}{\varepsilon}$

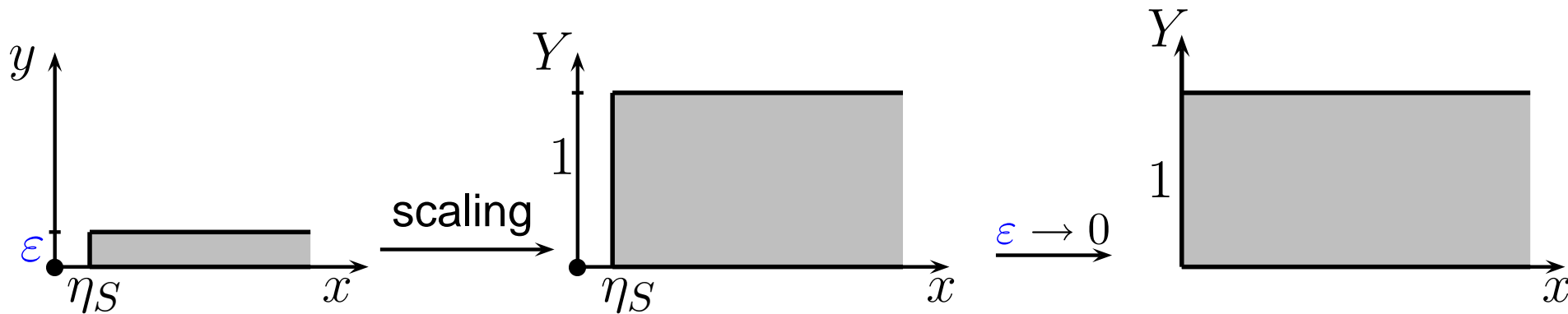
Slot field



The **asymptotic** context: $\varepsilon \ll \eta_S \ll \lambda$.

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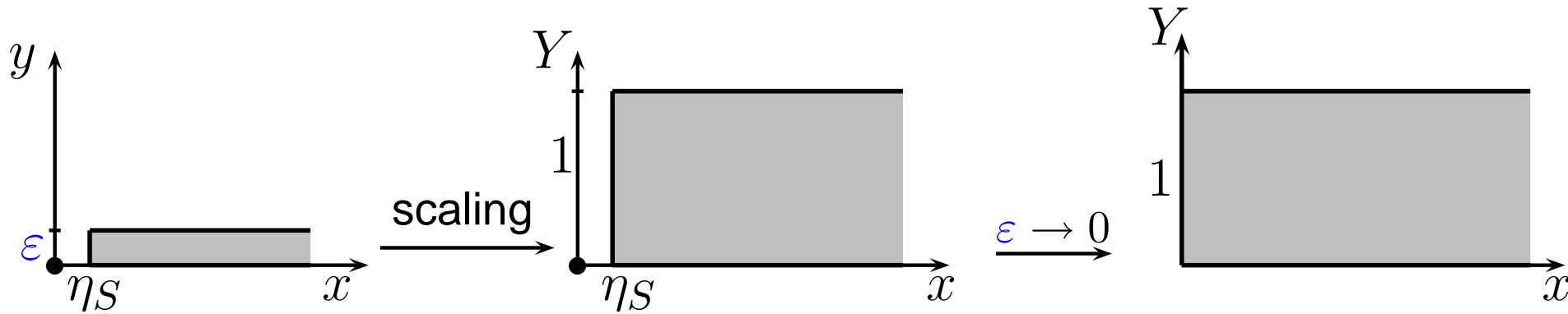
Slot field



The **asymptotic** context: $\varepsilon \ll \eta_S \ll \lambda$.

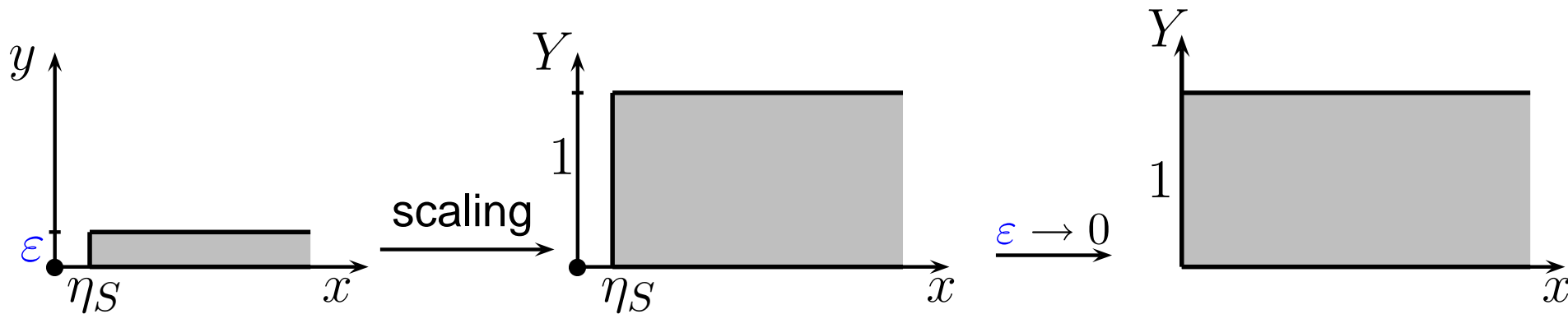
The **normalization**: $X = x, \quad Y = \frac{y}{\varepsilon}$

Slot field



$$u^\varepsilon(x, Y\varepsilon) = U^\varepsilon(x, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k U_i^k(x, Y) + o(\varepsilon^\infty),$$

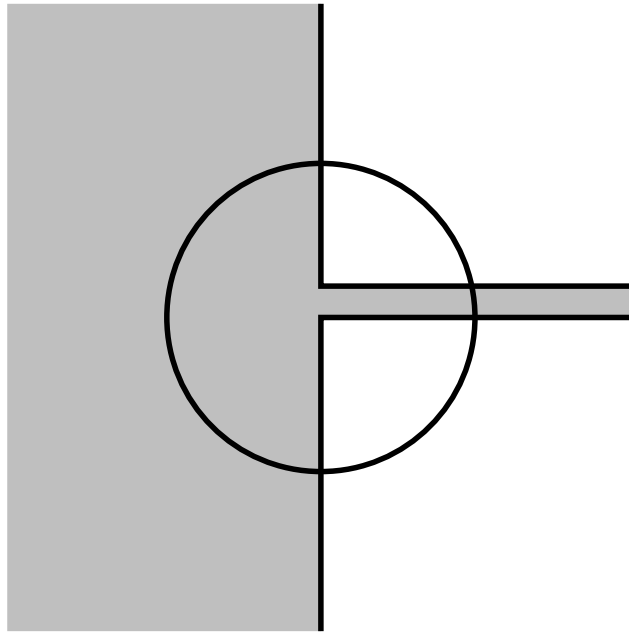
Slot field



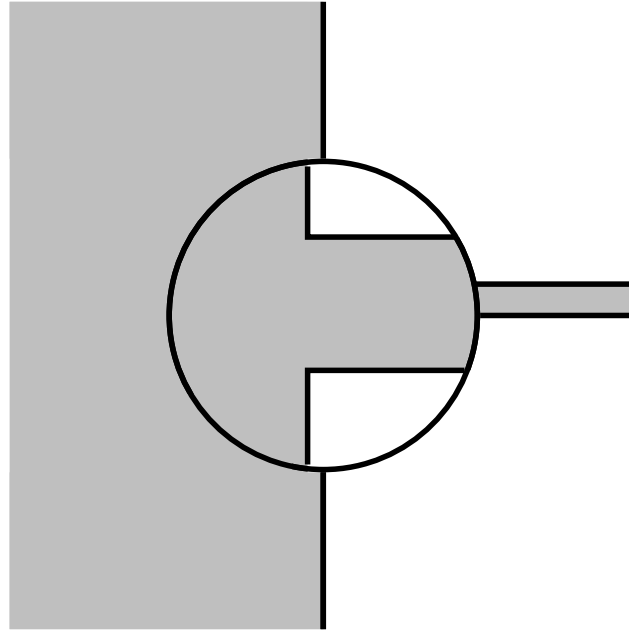
where the U_i^k satisfy the **1D Helmholtz** equation:

$$\frac{d^2 U_i^k}{dx^2} + \omega^2 U_i^k = 0$$

Near field

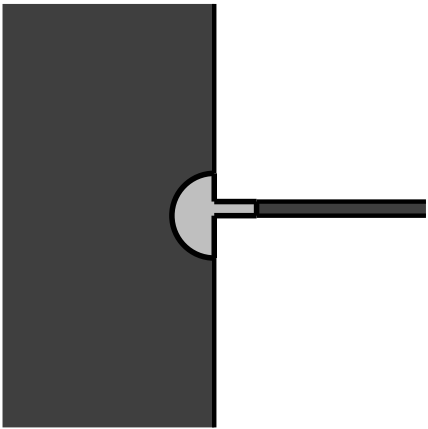


Near field



$$u^\varepsilon(x, y) = u_p^\varepsilon\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

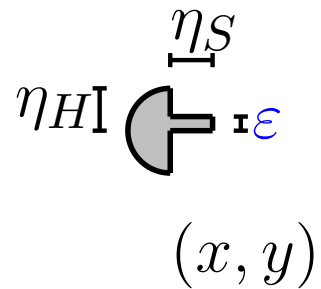
Near field



The **Asymptotic** context: $\varepsilon \ll \eta_H \ll \lambda$, $\varepsilon \ll \eta_S \ll \lambda$.

The **normalization**: $X = \frac{x}{\varepsilon}$, $Y = \frac{y}{\varepsilon}$

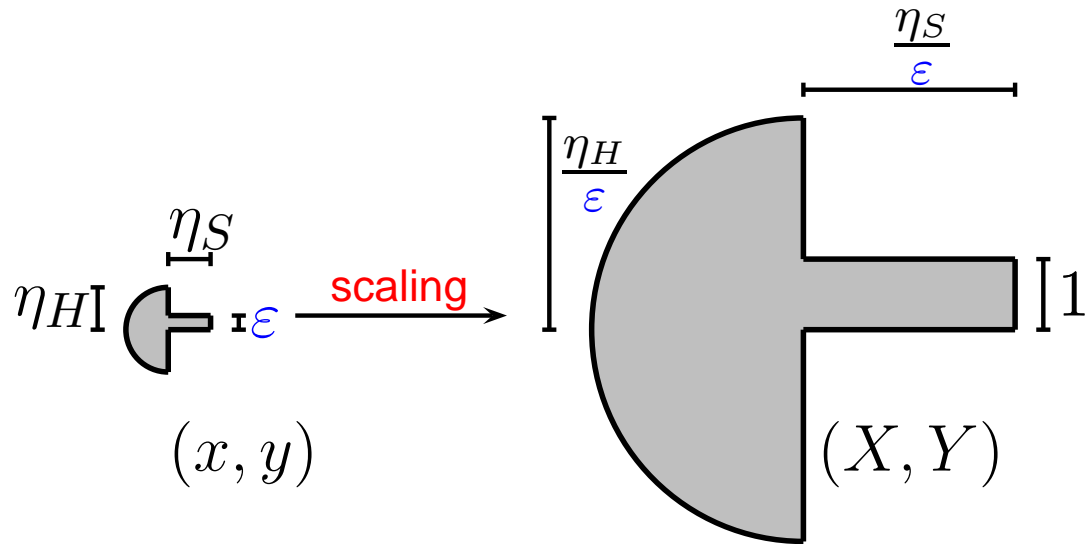
Near field



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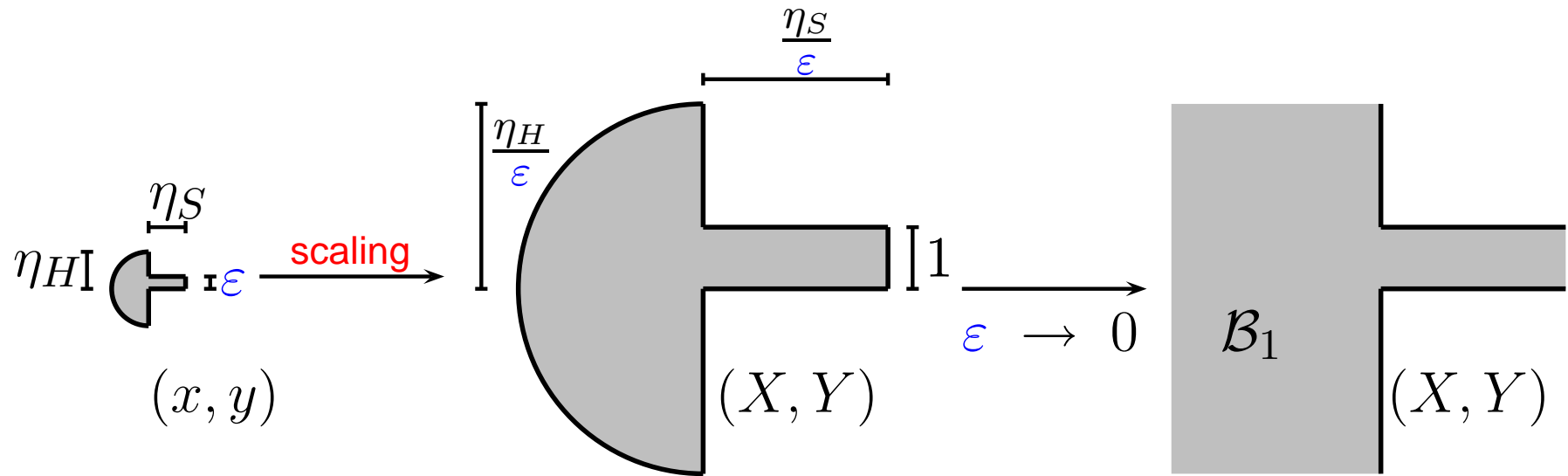
Near field



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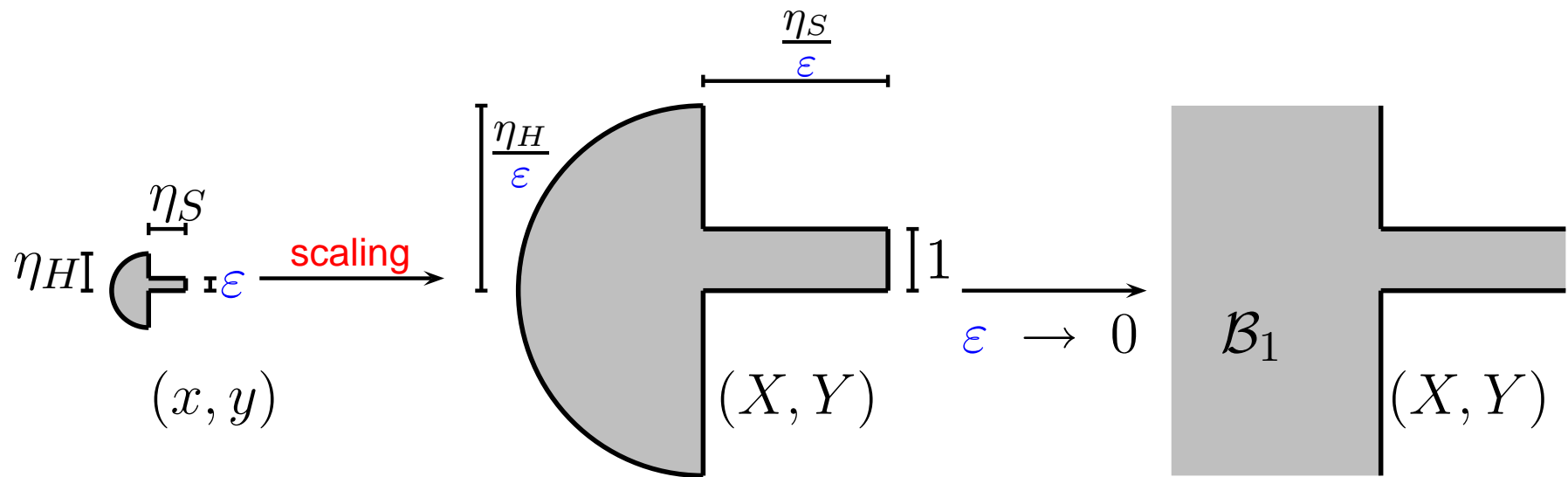
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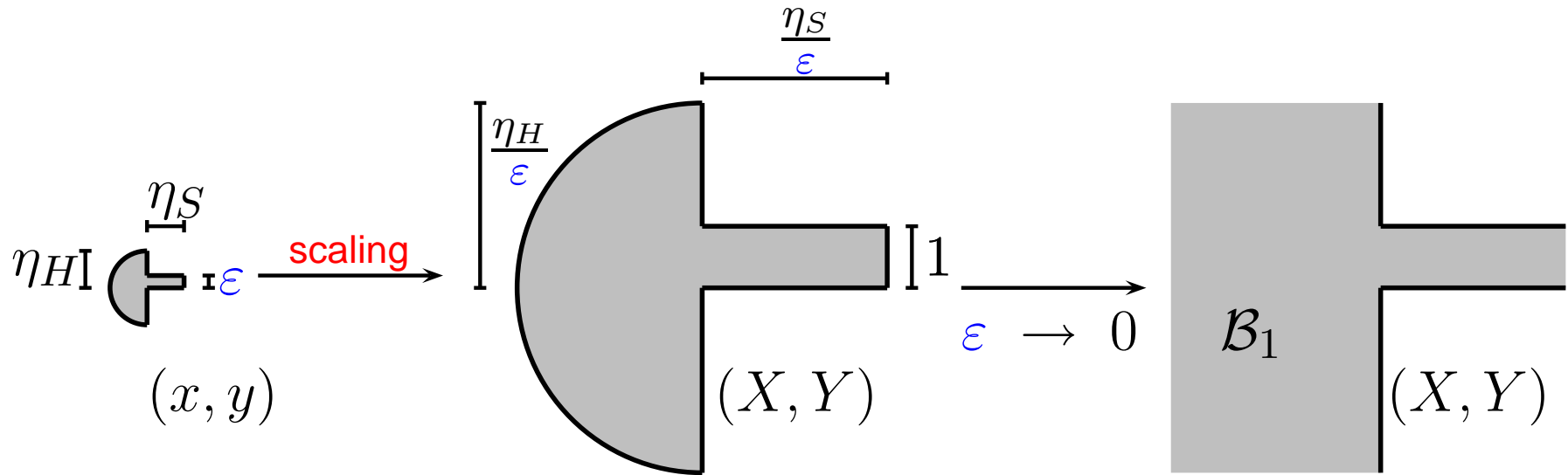
The **normalization**: $X = \frac{x}{\epsilon}, \quad Y = \frac{y}{\epsilon}$

Near field



$$u^\varepsilon(\varepsilon X, \varepsilon Y) = u_p^\varepsilon(X, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k(X, Y) + o(\varepsilon^\infty)$$

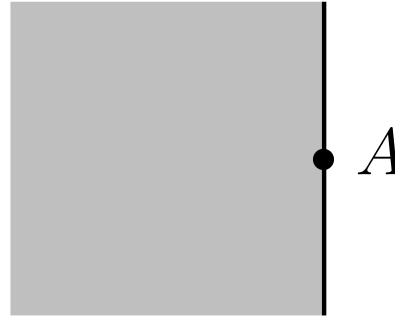
Near field



where the $(u_p)_i^k$ satisfy the (in)-homogeneous Laplace equation.

$$\begin{cases} \Delta(u_p)_i^k = 0, & \text{if } i = k \text{ or } k + 1, \\ \Delta(u_p)_i^k = -\omega^2 (u_p)_{i-2}^k, & \text{else.} \end{cases}$$

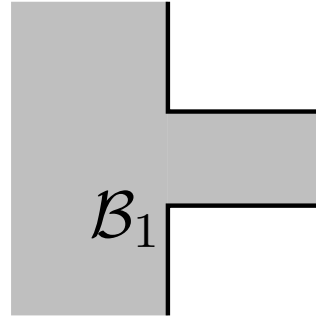
Order 0 : \underline{u}^0 , $(u_p)_0^0$, U_0^0



Far field:

$$\left\{ \begin{array}{ll} \text{Find } u^0 \in H_{loc}^1(\Omega) \text{ such that :} & \\ -\Delta u^0 - \omega^2 u^0 = f, & \text{in } \Omega, \\ \frac{\partial u^0}{\partial n} = 0, & \text{on } \partial\Omega, \\ u^0 \text{ is outgoing.} & \end{array} \right.$$

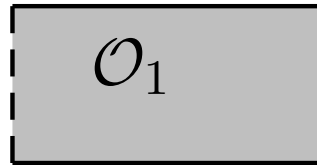
Order 0 : u^0 , $\underline{(u_p)_0^0}$, U_0^0



Near field:

$$(u_p)_0^0(X, Y) = u^0(A), \quad \text{in } \mathcal{B}_1.$$

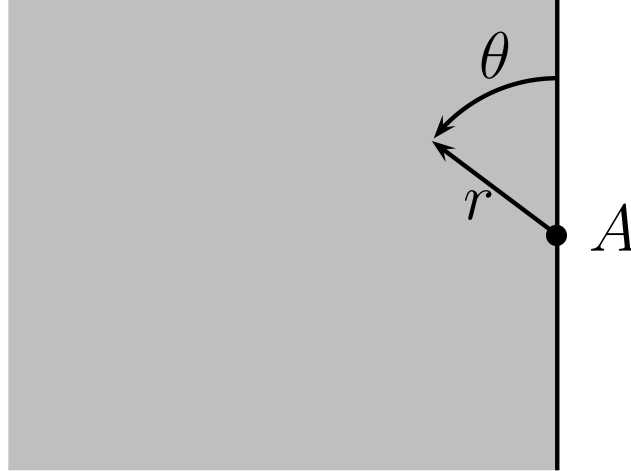
Order 0 : u^0 , $(u_p)_0^0$, U_0^0



Slot field:

$$U_0^0(x, Y) = u^0(A) \exp(\mathbf{i}\omega x), \quad \text{in } \mathcal{O}_1.$$

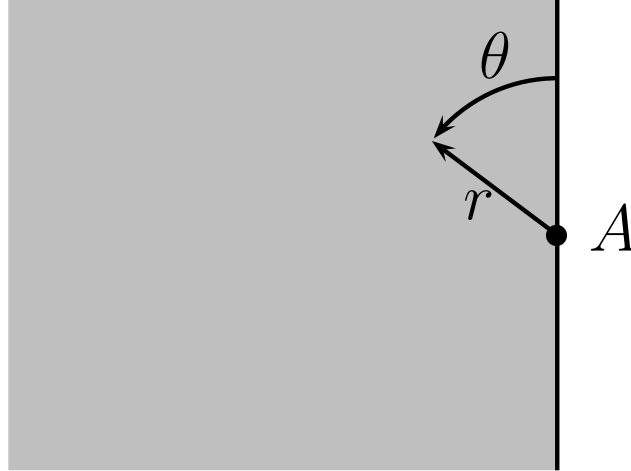
Order 1 : $\underline{u}_1^0, (u_p)_1^0, (u_p)_1^1, U_1^0, U_1^1$



Approximation of the exact Solution:

$$u^\epsilon \simeq u^0 + \epsilon u_1^0$$

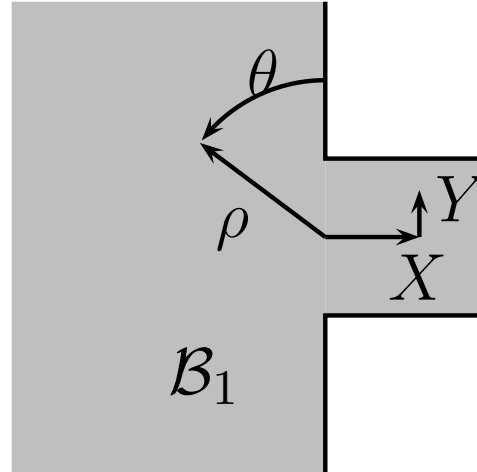
Order 1 : $\underline{u}_1^0, (u_p)_1^0, (u_p)_1^1, U_1^0, U_1^1$



explicit form of u_1^0

$$u_1^0(r, \theta) = -\frac{\omega}{2} u^0(A) H_0^{(1)}(\omega r).$$

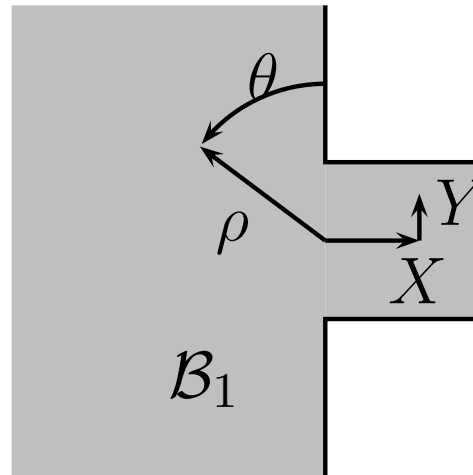
Order 1 : $u_1^0, \underline{(u_p)_1^0}, \underline{(u_p)_1^1}, U_1^0, U_1^1$



Approximation of the exact solution:

$$\begin{cases} u^\varepsilon(\varepsilon X, \varepsilon Y) = u_p^\varepsilon(X, Y), \\ u_p^\varepsilon \simeq (u_p)_0^0 + \varepsilon (u_p)_1^0 + \varepsilon \log \varepsilon (u_p)_1^1. \end{cases}$$

Order 1 : $u_1^0, \underline{(u_p)_1^0}, (u_p)_1^1, U_1^0, U_1^1$

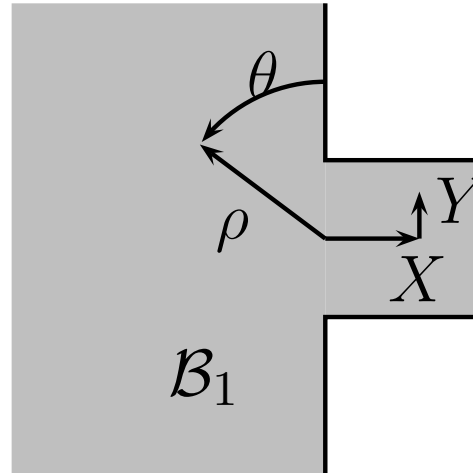


Near field:

Find $(u_p)_1^0 \in H_{loc}^1(\mathcal{B}_1)$ such that:

$$\begin{cases} \Delta(u_p)_1^0 = 0, & \text{in } \mathcal{B}_1 \\ \frac{\partial(u_p)_1^0}{\partial n} = 0, & \text{on } \partial\mathcal{B}_1. \end{cases}$$

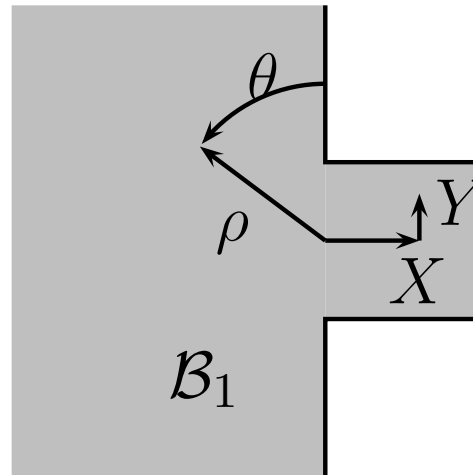
Order 1 : $u_1^0, \underline{(u_p)_1^0}, (u_p)_1^1, U_1^0, U_1^1$



Behavior at infinity in the half-space:

$$(u_p)_1^0(\rho, \theta) - \frac{\partial u^0}{\partial y}(A) \rho \cos \theta + \frac{\omega}{2} u^0(A) \left[1 + \frac{2i}{\pi} (\log \rho + \gamma) \right] = O\left(\frac{1}{\rho}\right).$$

Order 1 : $u_1^0, \underline{(u_p)_1^0}, (u_p)_1^1, U_1^0, U_1^1$



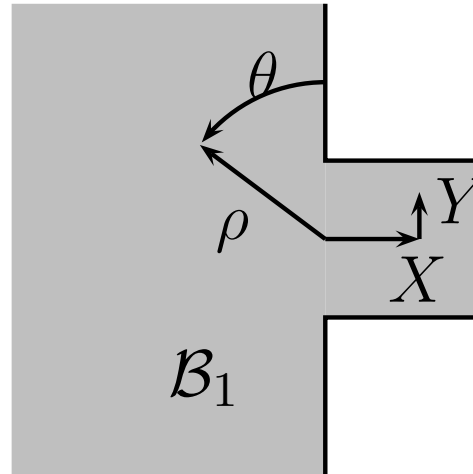
Behavior at infinity in the half-space:

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Behavior at infinity in the slot:

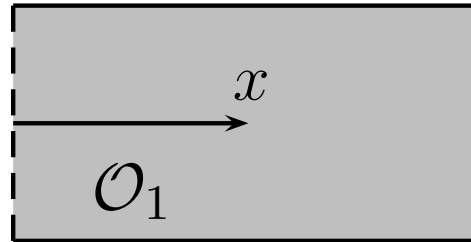
$$(u_p)_1^0(X, Y) - i \omega u^0(A) X = O(1).$$

Order 1 : u_1^0 , $(u_p)_1^0$, $\underline{(u_p)_1^1}$, U_1^0 , U_1^1



$$(u_p)_1^1 = -\frac{\mathbf{i}\omega}{\pi} u^0(A)$$

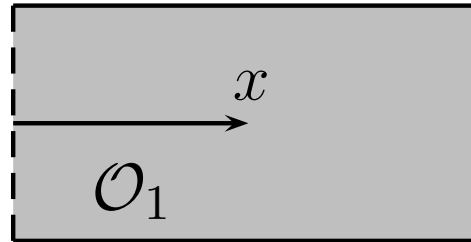
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Approximation of the exact solution:

$$\begin{cases} u^\varepsilon(x, \varepsilon Y) = U^\varepsilon(x, Y), \\ U^\varepsilon \simeq U_0^0 + \varepsilon U_1^0 + \varepsilon \log \varepsilon U_1^1. \end{cases}$$

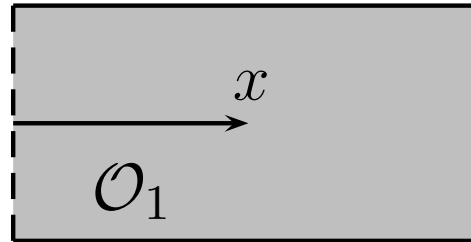
Order 1 : u_1^0 , $(u_p)_1^0$, $(u_p)_1^1$, U_1^0 , U_1^1



The slot field:

$$U_1^0(x) = \int_0^1 \mathcal{U}_1^0(0, Y) dY \exp(\mathbf{i}\omega x),$$

Order 1 : u_1^0 , $(u_p)_1^0$, $(u_p)_1^1$, U_1^0 , U_1^1

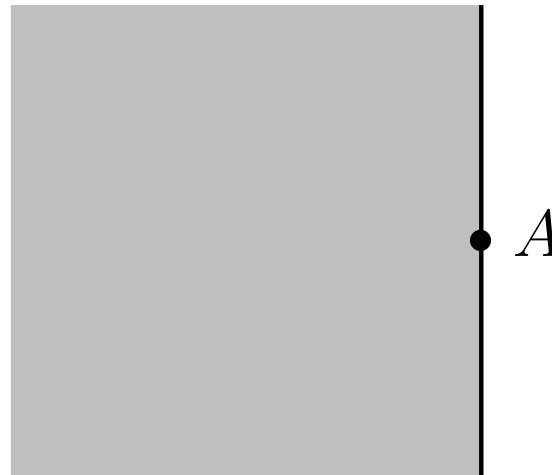


The slot field:

$$U_1^1(x) = -\frac{\mathbf{i}\omega}{\pi} u^0(A) \exp(\mathbf{i}\omega x).$$

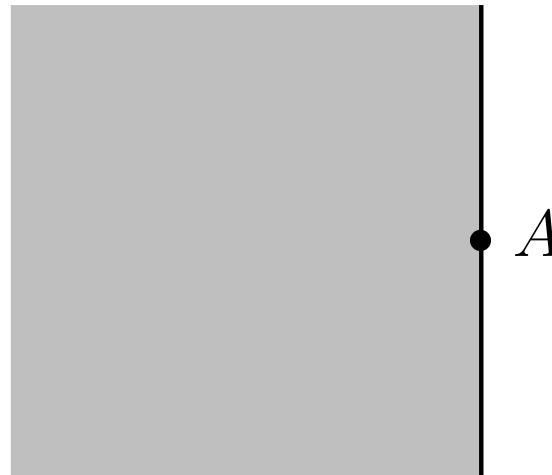
The far field of order $i > 1$

- The field u_i^k are defined in the half space:



The far field of order $i > 1$

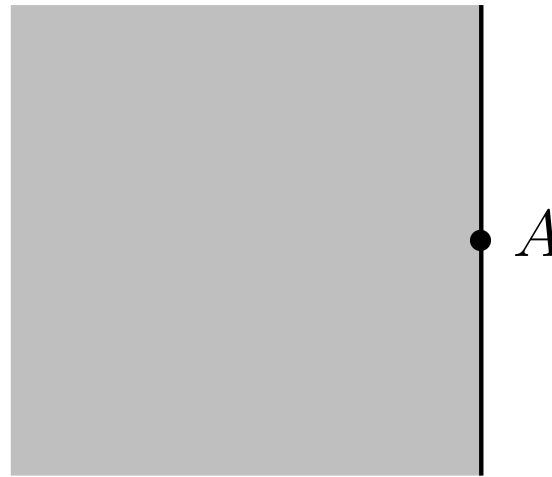
- The field u_i^k are defined in the **half space**:



- The far fields u_i^k
 - satisfy the **homogeneous Helmholtz** equation
 - are **singular** at the neighborhood of the origin
 - are outgoing at infinity

The far field of order $i > 1$

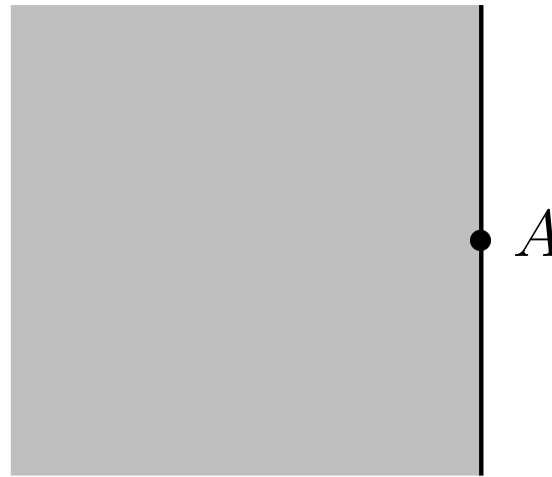
- The field u_i^k are defined in the half space:



- $$u_i^k = \sum_{p=0}^{+\infty} a_p H_p^{(1)}(\omega r) \cos p\theta$$

The far field of order $i > 1$

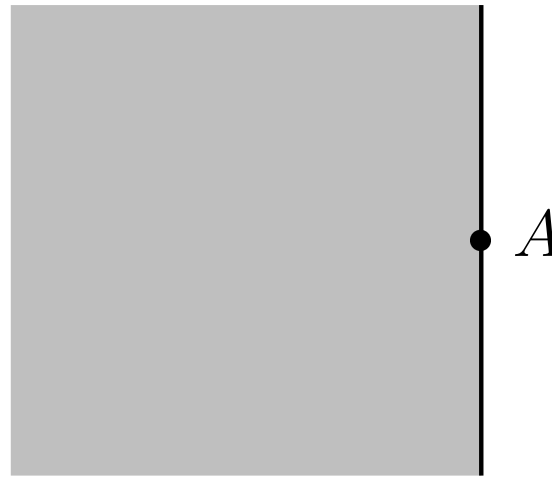
- The field u_i^k are defined in the half space:



- $$u_i^k = \sum_{p=0}^{i-k-1} a_p H_p^{(1)}(\omega r) \cos p\theta$$

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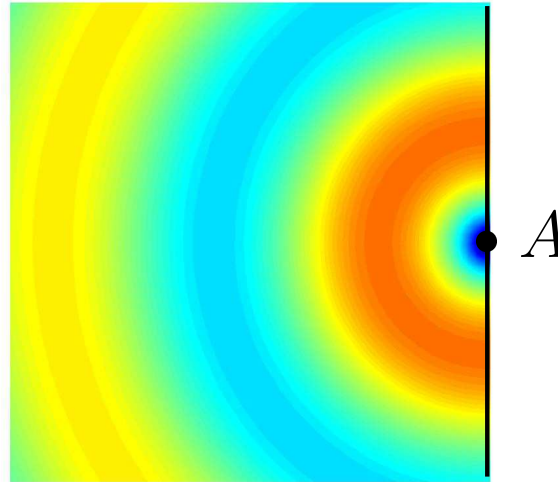
- $$u_i^k = \sum_{p=0}^{i-k-1} a_p H_p^{(1)}(\omega r) \cos p\theta$$

The a_p are functions of **lower order** terms

The far field of order $i > 1$

- The field u_i^k are defined in the **half space**:

$$\text{Im}(H_0^{(1)}(\omega r))$$



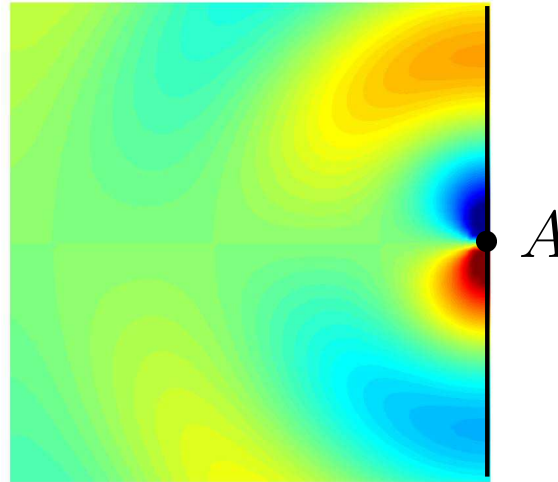
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The far field of order $i > 1$

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$$\text{Im}(H_1^{(1)}(\omega r) \cos \theta)$$

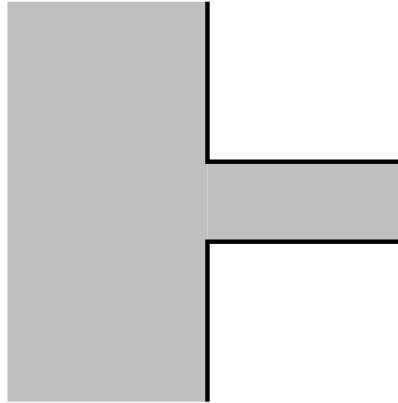


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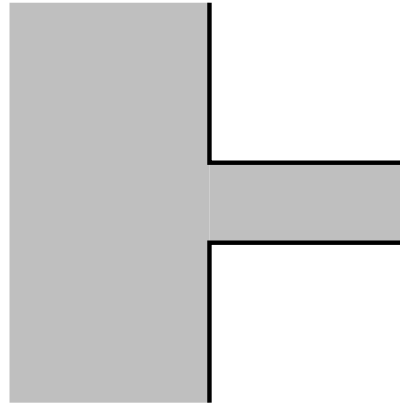
The near fields of order $i > 1$

- The $(u_p)_i^k(X, Y)$ are defined in the **canonical** domain:



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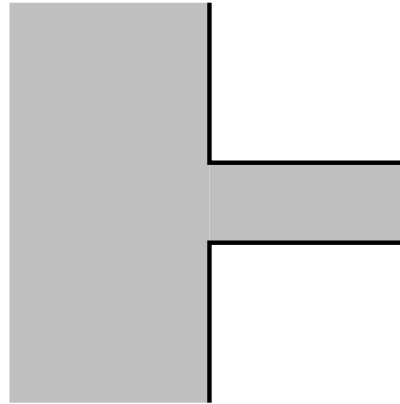
- by **Laplace** equation:

$$\Delta(u_p)_i^k = 0, \quad (i = k \text{ ou } k + 1),$$

$$\Delta(u_p)_i^k = -\omega^2 (u_p)_{i-2}^k, \quad (i \geq k + 2),$$

The near fields of order $i > 1$

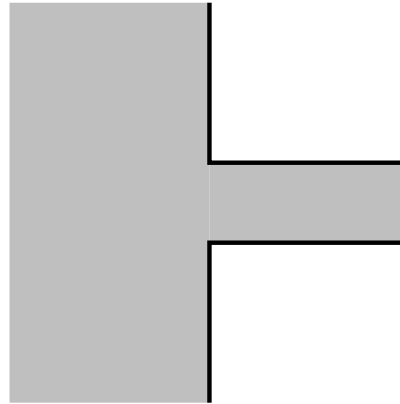
- The $(u_p)_i^k(X, Y)$ are defined in the **canonical** domain:



- by **Laplace** equation:
- by polynomial **growings** at infinity:
 - The **growings** in the half space are functions of **far field of lower (or equal) order**
 - The **growings** in the slot are functions of the slot fields **of lower order**

The near fields of order $i > 1$

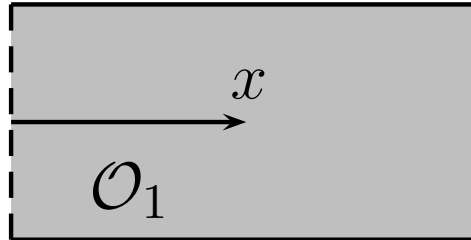
- The $(u_p)_i^k(X, Y)$ are defined in the **canonical** domain:



- Proof of the **existence-unicity**:
 - with truncature functions, we subtract the growing behavior at infinity of the $(u_p)_i^k$
 - We use the “classical” **variational theory** (wheighted Sobolev spaces, Leroux, Hardy,...)

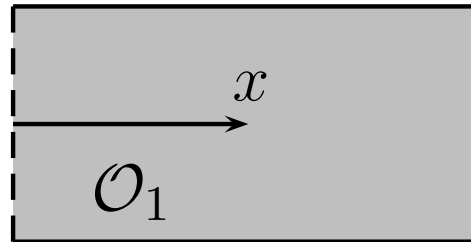
The slot field of order $i > 1$

- The U_i^k are defined on the canonical domain:



The slot field of order $i > 1$

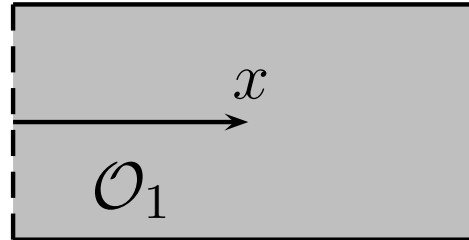
- The U_i^k are defined on the **canonical** domain:



- The U_i^k does not depend on y .

The slot field of order $i > 1$

- The U_i^k are defined on the **canonical** domain:



- The U_i^k does not depend on y .
- $U_i^k(x) = \int_0^1 (u_p)_i^k(0, Y) dY \exp \mathbf{i} \omega x$

Some properties

We see that:

- More $i - k$ is **large** more u_i^k is **singular** at the origin:

$$r^{-p} \text{ terms, } p = 0, \dots, i - k - 1$$

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$$\begin{cases} \rho^p \text{ terms, } & p = 0, \dots, i - k, \\ X^p \text{ terms, } & p = 0, \dots, i - k, \end{cases}$$

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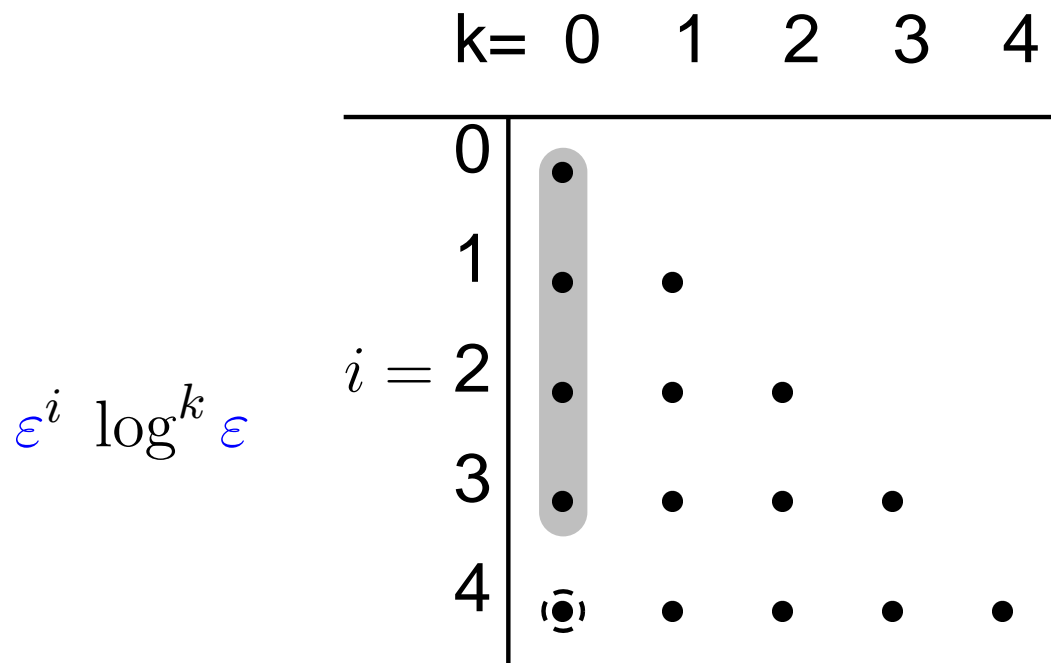
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- More $i - k$ is **large** more $(u_p)_i^k$ is **growing**:

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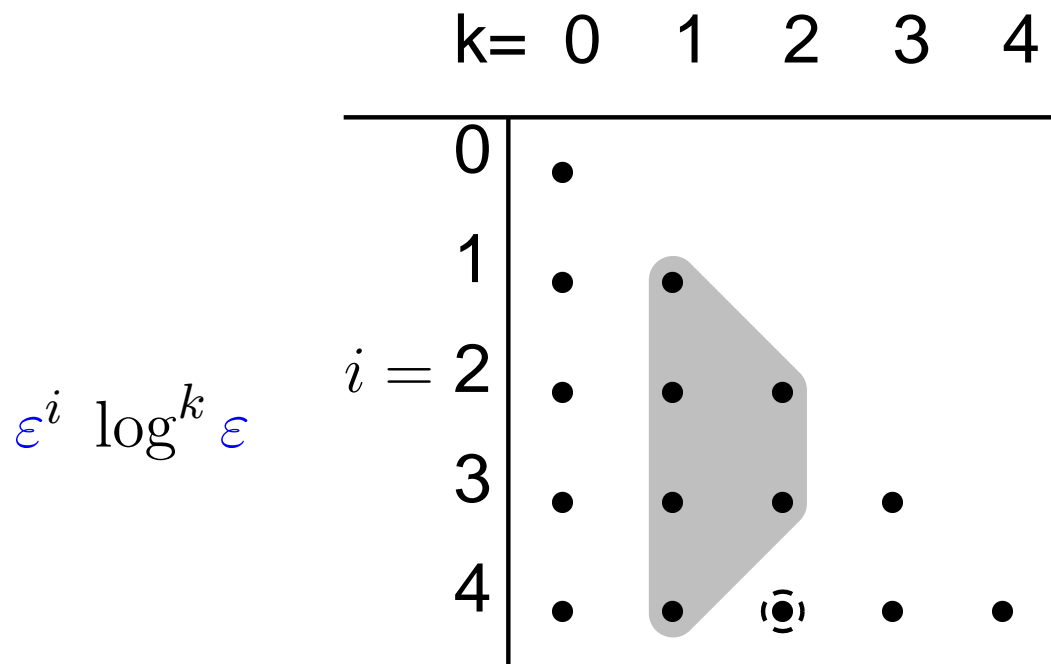
- When the **order** i grows, one has $O(\frac{i^2}{2})$ ($\times 3$) terms to compute...

Dependence diagram of the asymp. terms



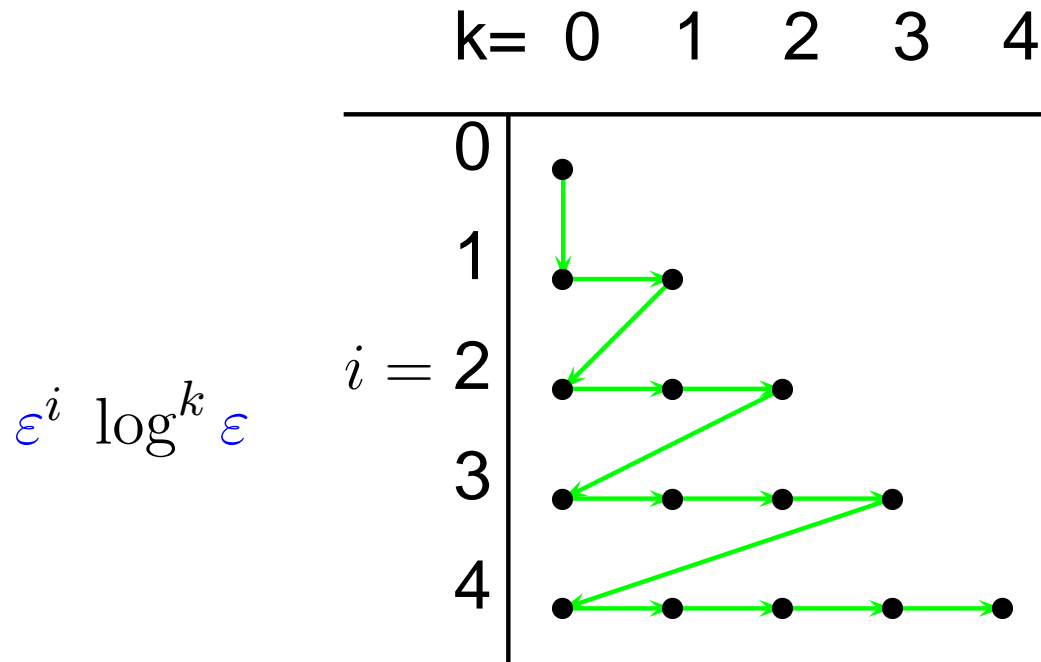
Any point corresponds to the 3 functions $(u_i^k, (u_p)_i^k, U_i^k)$.

Dependence diagram of the asymp. terms



Any point corresponds to the 3 functions $(u_i^k, (u_p)_i^k, U_i^k)$.

Natural scheduling of the computations



Any point corresponds to the three functions $(u_i^k, (u_p)_i^k, U_i^k)$.

Devirvate the terms of the as. exp.

- We search for solutions of the form:

$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k \textcolor{red}{u}_i^k \quad (\text{far field})$$

$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k (\textcolor{red}{u}_p)_i^k \quad (\text{near field})$$

$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k \textcolor{red}{U}_i^k \quad (\text{slot field})$$

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- We **inject** the equations (Helmholtz, Neumann)

Devirvate the terms of the as. exp.

- We search for solutions of the form:

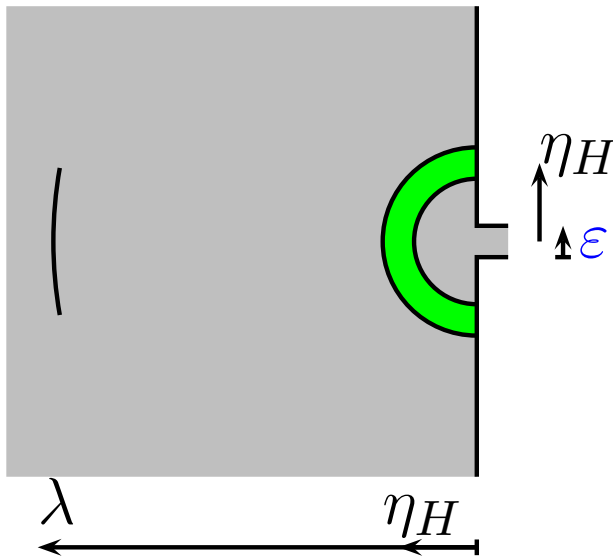
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$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k \mathbf{U}_i^k \quad (\text{slot field})$$

- We **inject** the equations (Helmholtz, Neumann)
- We obtain the **coupling** conditions: (**the difficulty**)

Far-Near coupling



In a **thick zone**:

$$\varepsilon \ll \eta_H \ll \lambda.$$

We write the coupling condition:

$$u^\varepsilon(\eta_H, \theta) = (u_p)^\varepsilon\left(\frac{\eta_H}{\varepsilon}, \theta\right).$$

$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k u_i^k(\eta_H, \theta) \simeq \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k (u_p)_i^k\left(\frac{\eta_H}{\varepsilon}, \theta\right)$$

$\eta_H \rightarrow 0$
 $\frac{\eta_H}{\varepsilon} \rightarrow +\infty$

Far-Near coupling

$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k \mathbf{u}_i^k(\eta_H, \theta) \simeq \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k (\mathbf{u}_p)_i^k\left(\frac{\eta_H}{\varepsilon}, \theta\right)$$

$$\eta_H \rightarrow 0 \qquad \frac{\eta_H}{\varepsilon} \rightarrow +\infty$$

We **expand**

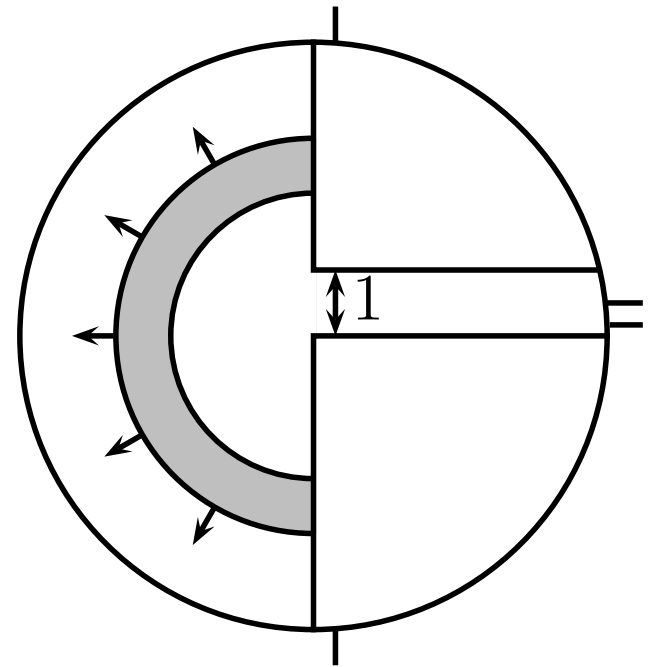
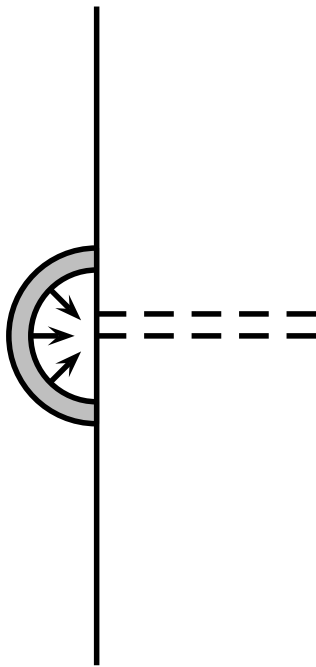
- the left serie according to η_H near **0**
- The right serie according to η_H/ε tending ot **infinity**

We **identify** all the terms of the two series.

The conclusion of the coupling

- The **far** field-**near** field coupling:

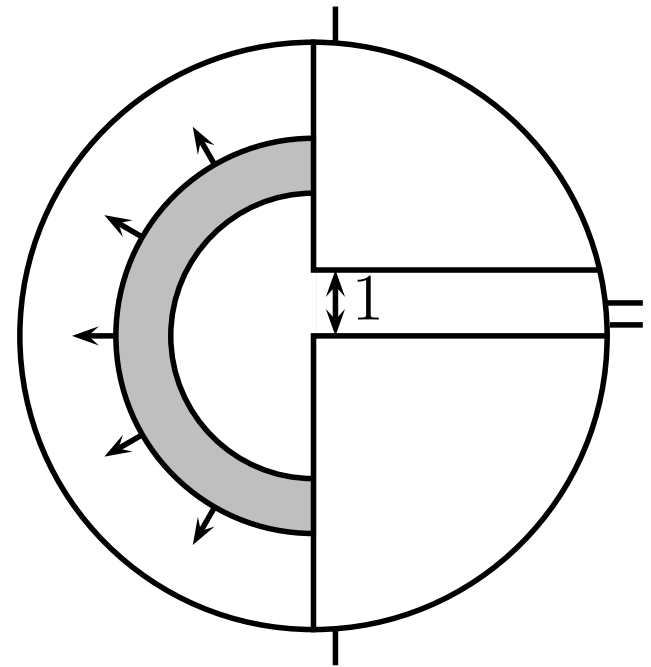
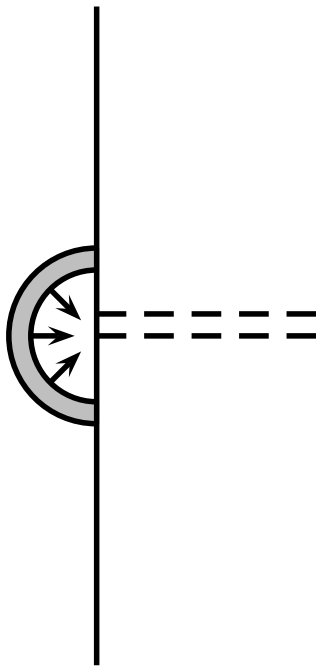
The **singular** behavior of the **far field** is coupled with the **none growing** behavior of the **near field at infinity**.



The conclusion of the coupling

- The far field-near field coupling:

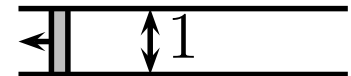
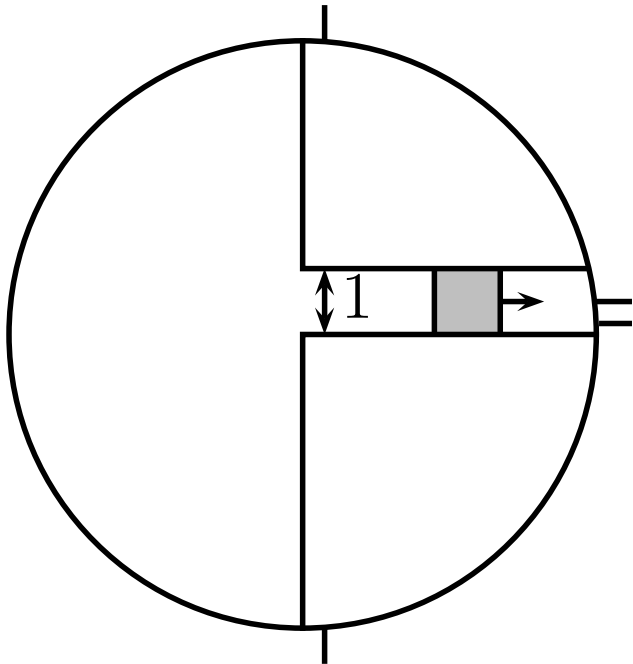
The growing behavior of the near field at infinity is coupled with the none singular behavior of the far field .



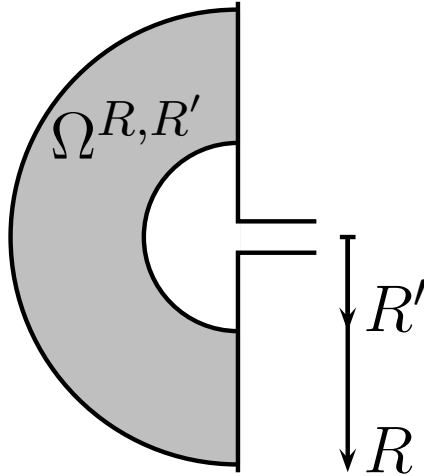
The conclusion of the coupling

- The **far** field-**near** field coupling:
- The **near** field-**slot** field coupling:

The **growing** behavior of the **near field** is coupled with the **none growing** behavior of the **slot field** (derivative values)

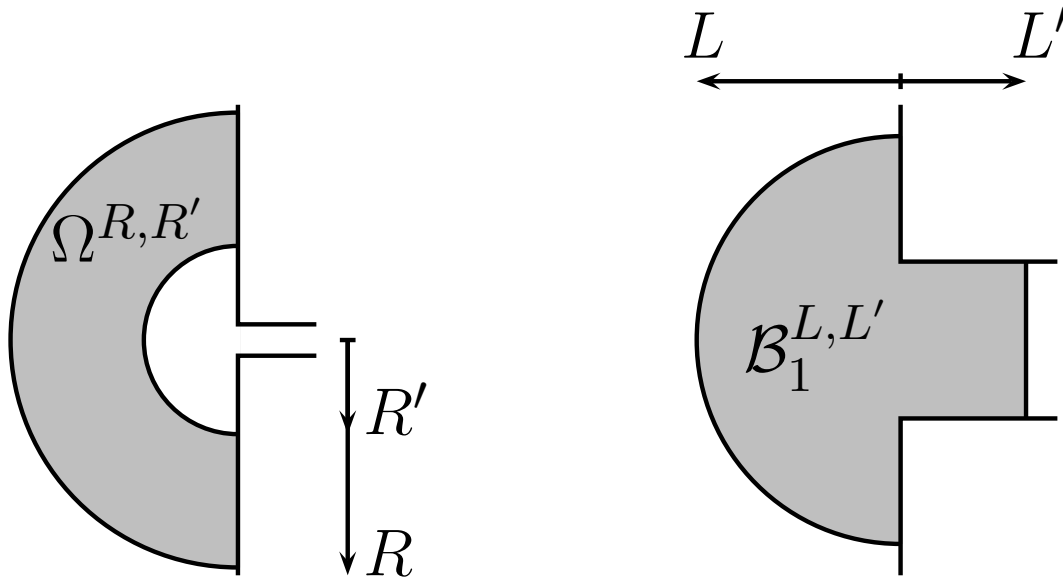


Mathematical analysis



$$\left\| u^\epsilon - u^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \epsilon^i (\log \epsilon)^k u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \epsilon^{n+1} (\log \epsilon)^n \|f\|_{L^2(\Omega)}.$$

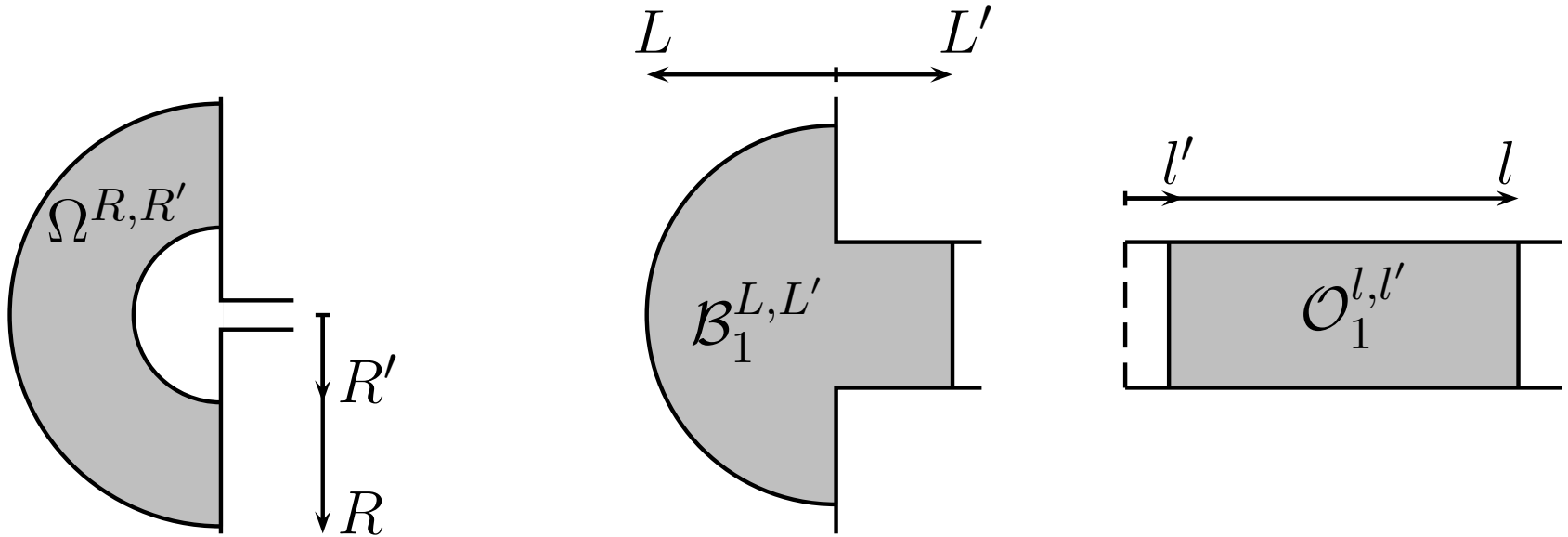
Mathematical analysis



$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^n \|f\|_{L^2(\Omega)}.$$

$$\left\| u_p^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

Mathematical analysis



$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^n \|f\|_{L^2(\Omega)}.$$

$$\left\| u_p^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

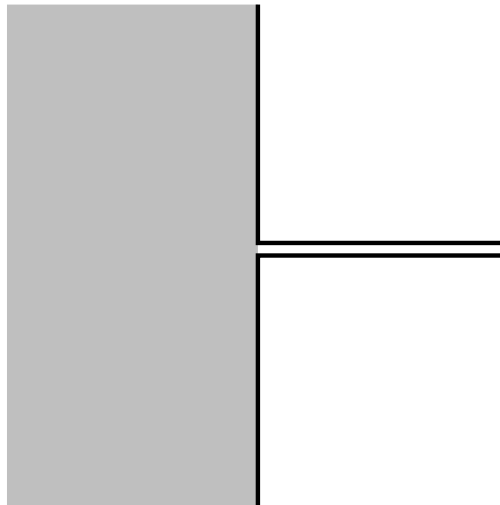
$$\left\| U^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k U_i^k \right\|_{H^1(\mathcal{O}_1^{l,l'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

Idea of the proof

We want to define an **approximation** \tilde{u}_n^ε of the exact solution which **coincide** with:

- the **truncated** expansion of the **far field** away from the slot in the half space.

$$u_n^{H,\varepsilon}(x,y) = u^0(x,y) + \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k(x,y)$$

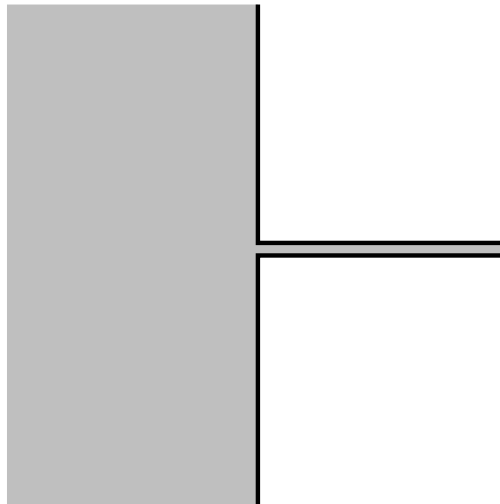


Idea of the proof

We want to define an **approximation** \tilde{u}_n^ε of the exact solution which **coincide** with:

- The **truncated** expansion of the **near field** in the neighbourhood of the end of the slot

$$u_n^{N,\varepsilon}(x,y) = \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right)$$

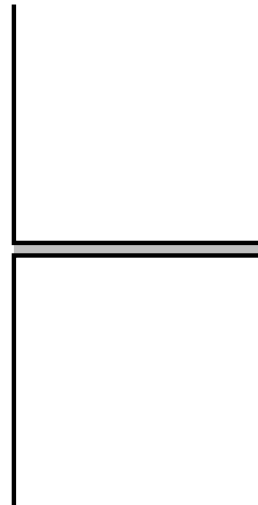


Idea of the proof

We want to define an **approximation** \tilde{u}_n^ε of the exact solution which **coincide** with:

- the **truncated** expansion of the **slot field** far away in the slot

$$u_n^{S,\varepsilon}(x,y) = \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k U_i^k(x, \frac{y}{\varepsilon})$$



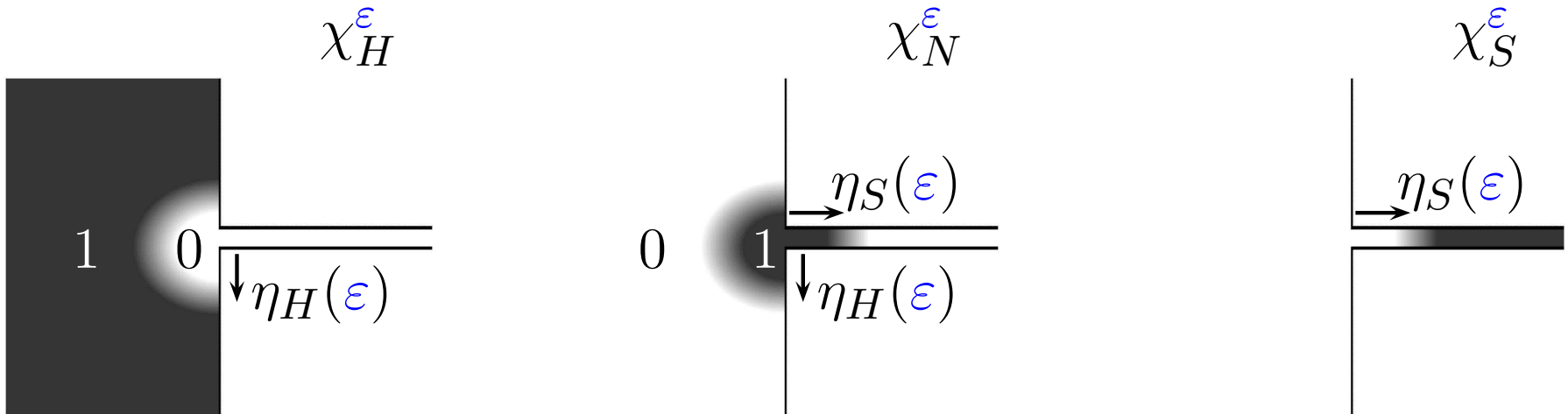
Idea of the proof

Introduce a partition of unity

$$\tilde{u}_n^\varepsilon(r, \theta) = \chi_H^\varepsilon u_n^{H,\varepsilon} + \chi_N^\varepsilon u_n^{N,\varepsilon} + \chi_S^\varepsilon u_n^{S,\varepsilon}$$

with

$$\chi_H^\varepsilon + \chi_N^\varepsilon + \chi_S^\varepsilon = 1.$$

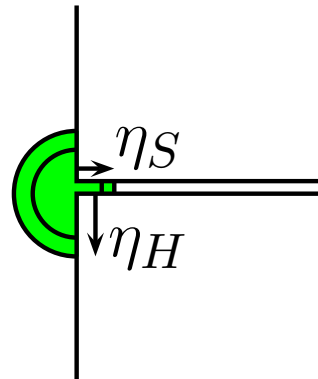


Idea of the proof

The **error** equation: $e_n^\varepsilon = \tilde{u}_n^\varepsilon - u^\varepsilon$

$$\begin{cases} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)_n^\varepsilon + (\delta_{H-N})_n^\varepsilon + (\delta_{S-N})_n^\varepsilon, & \text{in } \Omega_\varepsilon, \\ \frac{\partial e_n^\varepsilon}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ is outgoing.} \end{cases}$$

$(\delta_N)_n^\varepsilon$ is related to the **approximation** of the **Helmholtz** equation by the **near** field

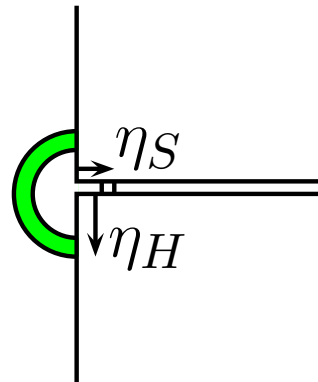


Idea of the proof

The **error** equation $e_n^\varepsilon = \tilde{u}_n^\varepsilon - u^\varepsilon$

$$\begin{cases} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)_n^\varepsilon + (\delta_{H-N})_n^\varepsilon + (\delta_{S-N})_n^\varepsilon, & \text{dans } \Omega_\varepsilon, \\ \frac{\partial e_n^\varepsilon}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ is outgoing.} \end{cases}$$

$(\delta_{H-N})_n^\varepsilon$ is related to the **matching error** between the **far** field and the **near** field

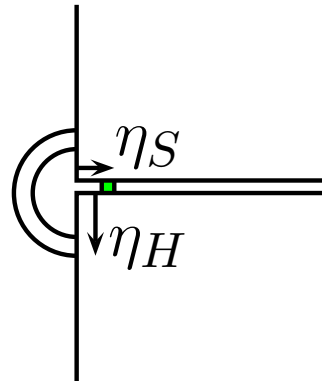


Idea of the proof

The **error** equation $e_n^\varepsilon = \tilde{u}_n^\varepsilon - u^\varepsilon$

$$\left\{ \begin{array}{ll} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)_n^\varepsilon + (\delta_{H-N})_n^\varepsilon + (\delta_{S-N})_n^\varepsilon, & \text{dans } \Omega_\varepsilon, \\ \frac{\partial e_n^\varepsilon}{\partial n} = 0, & \text{sur } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ is outgoing.} \end{array} \right.$$

$(\delta_{S-N})_n^\varepsilon$ is related to the **matching error** between the **slot** field and the **near** champ



Idea of the proof

The **error** equation: $e_n^\varepsilon = \tilde{u}_n^\varepsilon - u^\varepsilon$

$$\left\{ \begin{array}{ll} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)_n^\varepsilon + (\delta_{H-N})_n^\varepsilon + (\delta_{S-N})_n^\varepsilon, & \text{in } \Omega_\varepsilon, \\ \frac{\partial e_n^\varepsilon}{\partial n} = 0, & \text{on } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ os outgoing.} \end{array} \right.$$

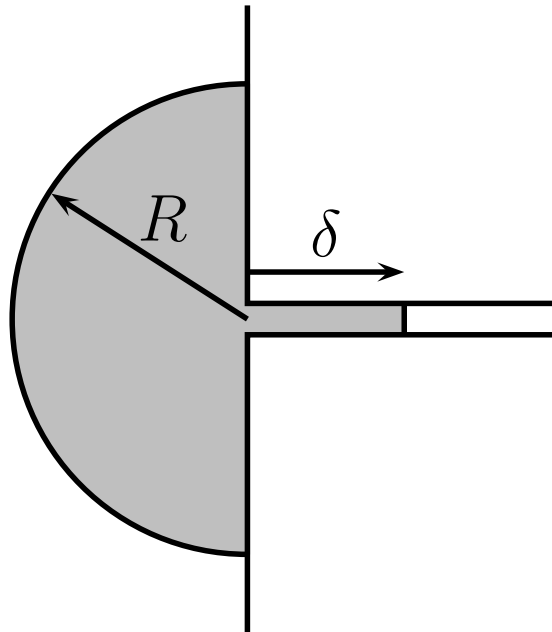
classical asymptotic techniques:

- **Stability**: proof by **contradiction** (Helmholtz)
- **Consistency**: A little bit more difficult (study of the singularities and of the growings by separation of variable)

Idea of the proof

Global error estimates

$$\left\{ \begin{aligned} \|u^\varepsilon - \tilde{u}_n^\varepsilon\|_{H^1(\Omega_\varepsilon^{R,\delta})} &\leq C \left[\left(\eta_H(\varepsilon) \right)^n + \left(\frac{\varepsilon}{\eta_H(\varepsilon)} \right)^n \right] \\ &+ C \left[\left(\eta_S(\varepsilon) \right)^n + \left(\frac{\varepsilon}{\eta_S(\varepsilon)} \right)^n \right]. \end{aligned} \right.$$



Idea of the proof

Global error estimate

$$\left\{ \begin{array}{l} \|u^\varepsilon - \tilde{u}_n^\varepsilon\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \left[\left(\eta_H(\varepsilon) \right)^n + \left(\frac{\varepsilon}{\eta_H(\varepsilon)} \right)^n \right] \\ \quad + C \left[\left(\eta_S(\varepsilon) \right)^n + \left(\frac{\varepsilon}{\eta_S(\varepsilon)} \right)^n \right]. \end{array} \right.$$

One can choose $\eta_H(\varepsilon)$ and $\eta_S(\varepsilon)$ to **optimize** this relation

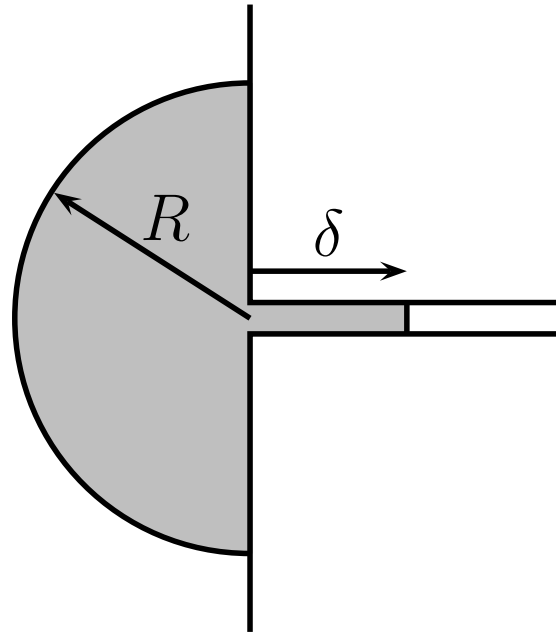
$$\eta_H(\varepsilon) = \eta_S(\varepsilon) = \sqrt{\varepsilon}$$

This leads to

$$\|u^\varepsilon - \tilde{u}_n^\varepsilon\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \varepsilon^{\frac{n}{2}}$$

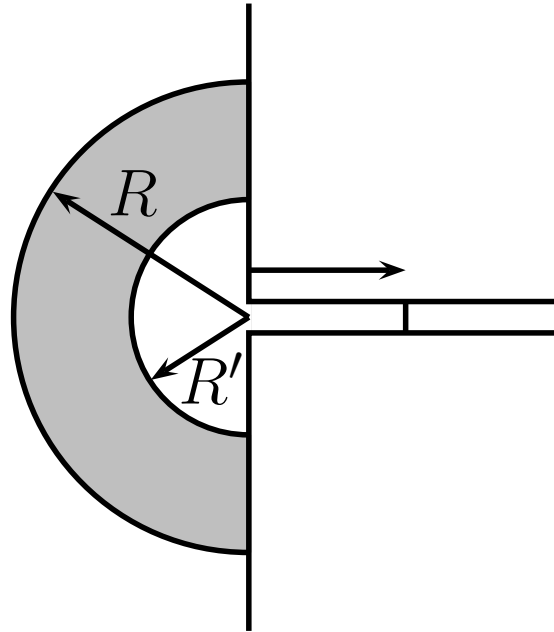
Idea of the proof

$$\|u^\varepsilon - \tilde{u}_n^\varepsilon\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \varepsilon^{\frac{n}{2}}$$



Idea of the proof

$$\left\| u^\varepsilon - \tilde{u}_n^\varepsilon \right\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \varepsilon^{\frac{n}{2}} \implies \left\| u^\varepsilon - \tilde{u}_n^\varepsilon \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{\frac{n}{2}}$$



Idea of the proof

$$\left\| u^\varepsilon - \tilde{u}_n^\varepsilon \right\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \varepsilon^{\frac{n}{2}} \quad \Longrightarrow \quad \left\| u^\varepsilon - \tilde{u}_n^\varepsilon \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{\frac{n}{2}}$$

In the **far field** zone:

$$\tilde{u}_n^\varepsilon = u_n^{H,\varepsilon} = u^0 + \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k$$

Idea of the proof

$$\left\| u^\varepsilon - \tilde{u}_n^\varepsilon \right\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \varepsilon^{\frac{n}{2}} \quad \Longrightarrow \quad \left\| u^\varepsilon - \tilde{u}_n^\varepsilon \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{\frac{n}{2}}$$

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$$\tilde{u}_n^\varepsilon = u_n^{H,\varepsilon} = u^0 + \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k$$

$$\left\{ \begin{array}{l} \left\| u^\varepsilon - u_{3n}^{H,\varepsilon} \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{\frac{3n}{2}} \\ \left\| u_{3n}^{H,\varepsilon} - u_n^{H,\varepsilon} \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} \log^n \varepsilon \end{array} \right.$$

Idea of the proof

$$\left\| u^\varepsilon - \tilde{u}_n^\varepsilon \right\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \varepsilon^{\frac{n}{2}} \quad \Longrightarrow \quad \left\| u^\varepsilon - \tilde{u}_n^\varepsilon \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{\frac{n}{2}}$$

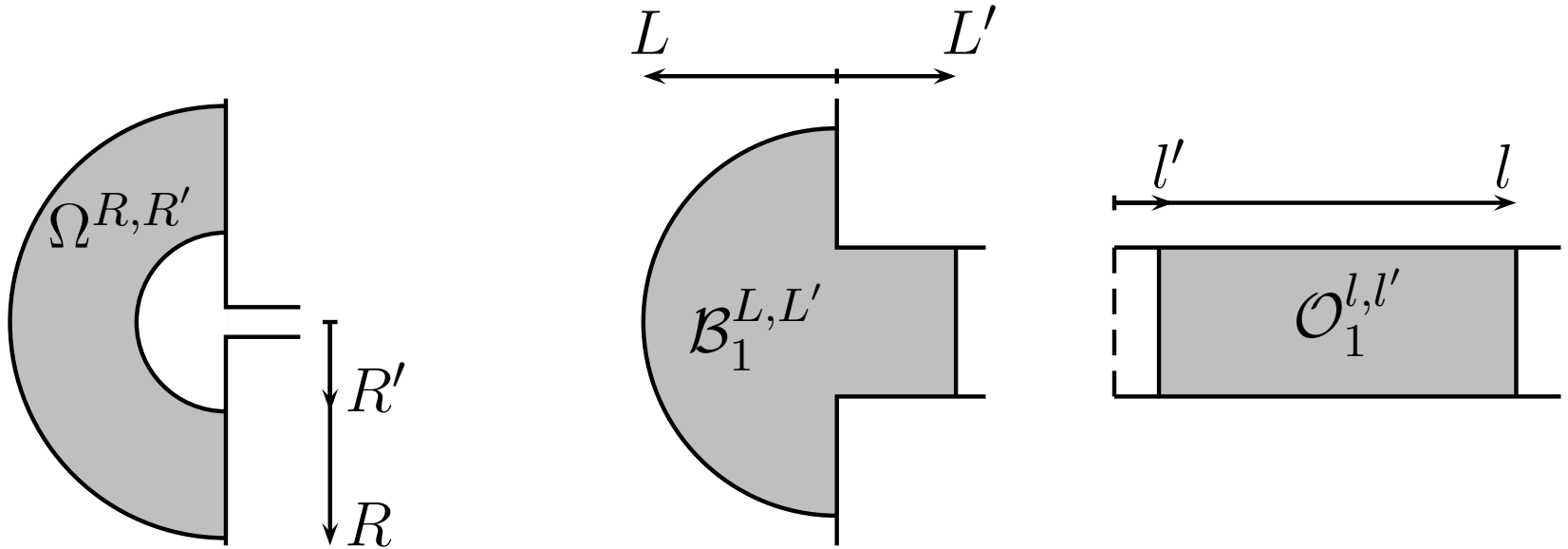
In the **far field** zone:

$$\tilde{u}_n^\varepsilon = u_n^{H,\varepsilon} = u^0 + \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k$$

$$\begin{cases} \left\| u^\varepsilon - u_{3n}^{H,\varepsilon} \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{\frac{3n}{2}} \\ \left\| u_{3n}^{H,\varepsilon} - u_n^{H,\varepsilon} \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} \log^n \varepsilon \end{cases}$$

One can conclude using the **triangular inequality**.

Mathematical analysis



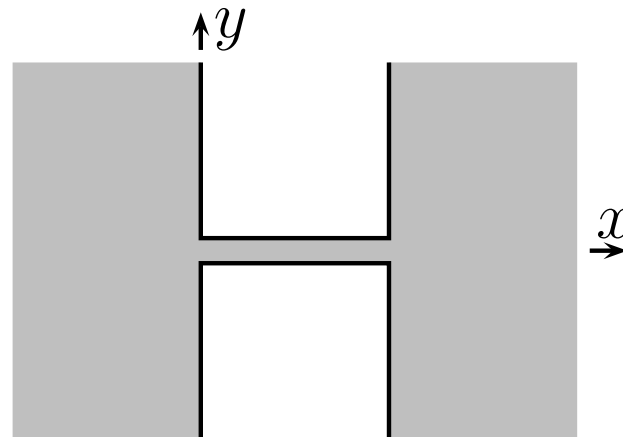
$$\left\| \mathbf{u}^\varepsilon - \mathbf{u}^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k \mathbf{u}_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^n \|f\|_{L^2(\Omega)}.$$

$$\left\| \mathbf{u}_p^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (\mathbf{u}_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

$$\left\| \mathbf{U}^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k \mathbf{U}_i^k \right\|_{H^1(\mathcal{O}_1^{l,l'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

Perspectives

1. Mathematical analysis of the finite slot (**resonance** phenomena)



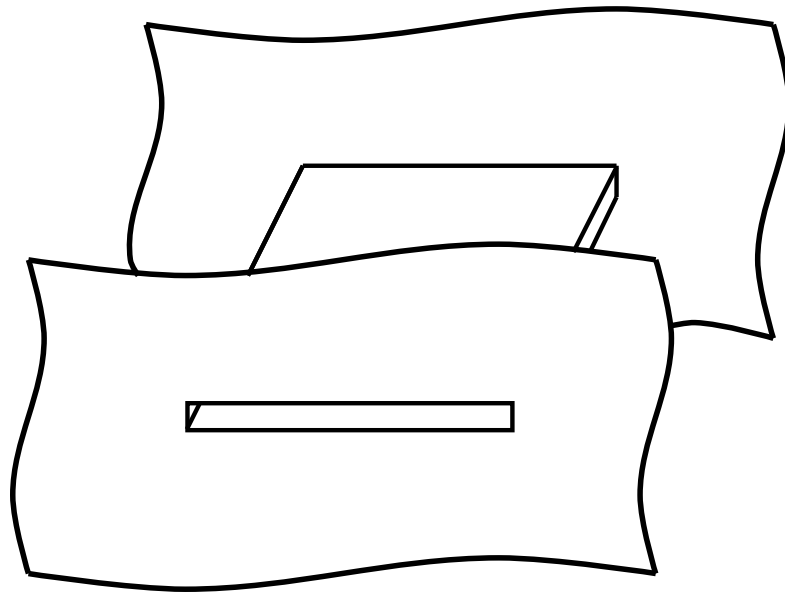
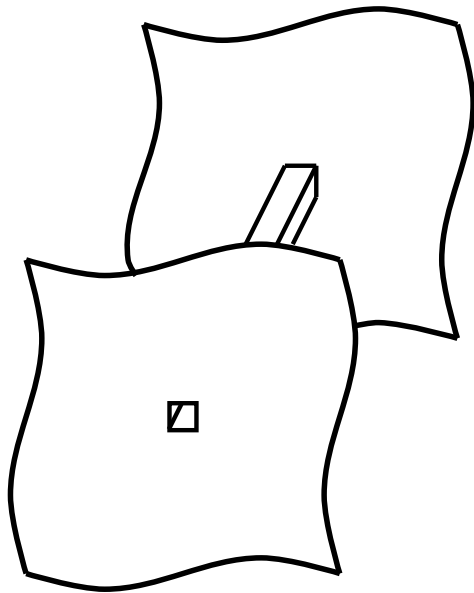
The difficulty: the **stability** result.

Perspectives

1. Mathematical analysis of the finite slot (**resonance** phenomena)
2. Comparison with the **multi-scale** technique

Perspectives

1. Mathematical analysis of the finite slot (**resonance** phenomena)
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3. The 3D **Maxwell** equation



Perspectives

1. Mathematical analysis of the finite slot (**resonance** phenomena)
2. Comparison with the **multi-scale** technique
3. The 3D **Maxwell** equation
4. The **time domain** (evolution equation)

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - c^2 \Delta \boldsymbol{u} = 0.$$