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## Matching of asymptotic expansions for the wave propagation in media with thin slot

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# Matching of asymptotic expansions for the wave propagation in media with thin slot

Sébastien Tordeux and Patrick Joly

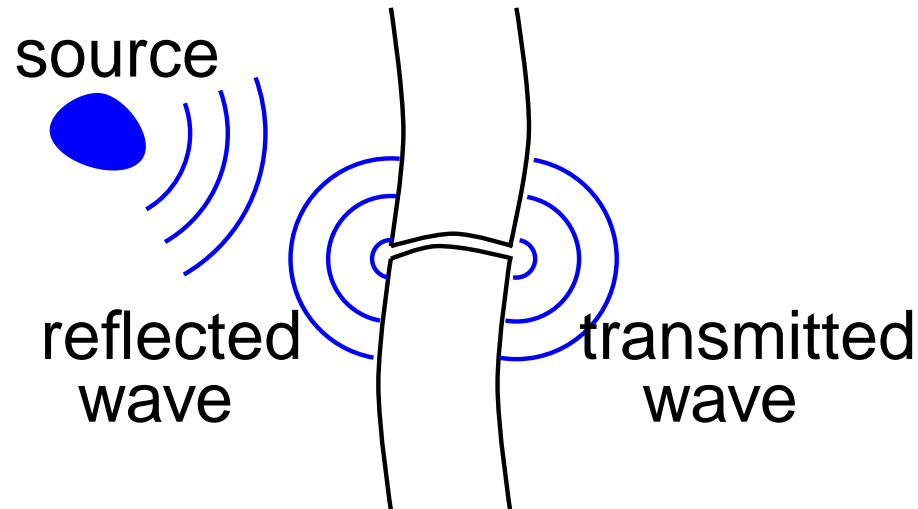
AG Analysis und Numerik, January 2005

INRIA-Rocquencourt-Projet POEMS

ETH-SAM

# A typical application

How can we study the scattering in media with **thin slot** ?

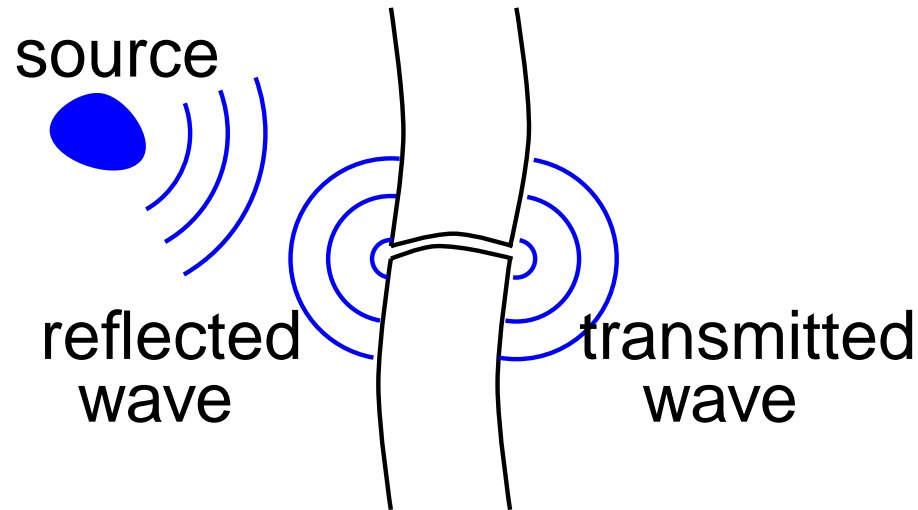


A physical problem with two **characteristical** lengthes

- The **wavelength**  $\lambda$
- The **width** of the slot  $\varepsilon$

# A typical application

How can we study the scattering in media with **thin slot** ?

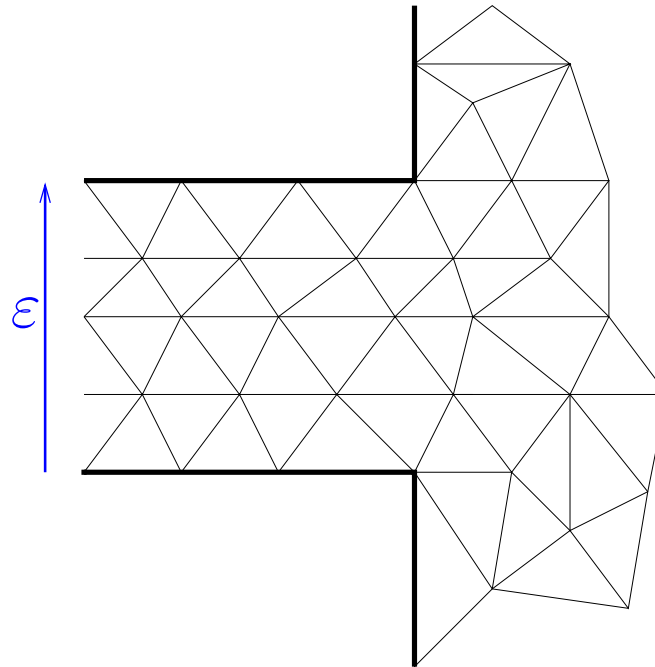


An **asymptotic** case:

$$\varepsilon \ll \lambda$$

# The numerical difficulty

A **mesh step** smaller than  $\varepsilon$



This leads to **costly** computations

# Some references

- Thin slot:  
Harrington, Auckland (1980), Tatout (1996).
- Finite differences:  
Taflove (1995).
- Thin plates and junction theory,...  
Ciarlet, Le Dret, Dauge-Costabel.
- Matching of asymptotic expansions:  
McIver, Rawlins (1993), Il'in (1992).
- multiscale analysis  
Maz'ya, Nazarov, Plamenevskii (1991)  
Oleinik, Shamaev, Yosifian (1992)

# A simple problem

**Scalar** wave equation:

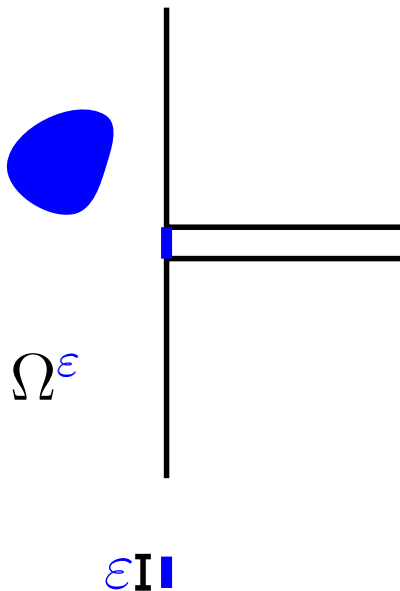
$$\frac{\partial^2 p^\varepsilon}{\partial t^2} - \Delta p^\varepsilon = f$$

**Harmonic** solution:

$$p^\varepsilon(x, y, t) = \exp(-i\omega t) u^\varepsilon(x, y)$$

**Helmholtz** Equation:

$$\Delta u^\varepsilon + \omega^2 u^\varepsilon = -f \quad \text{in } \Omega^\varepsilon$$



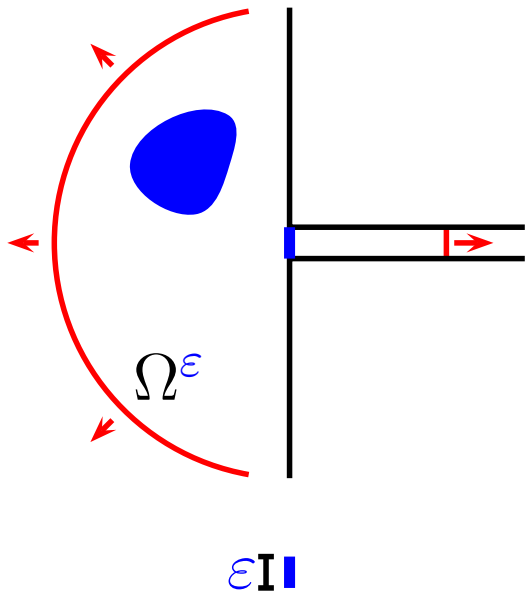
# A simple problem

Outgoing solution at infinity:

$$\frac{\partial u^\varepsilon}{\partial n} - i\omega u^\varepsilon \leq \frac{C}{r^2}, \quad \text{for } r \text{ large,}$$

Neumann limit condition  
(rigid wall)

$$\frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega^\varepsilon$$





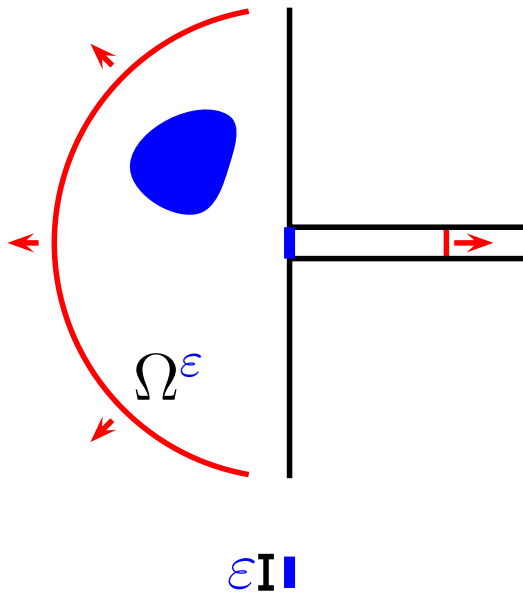
# A simple problem

Outgoing solution at infinity:

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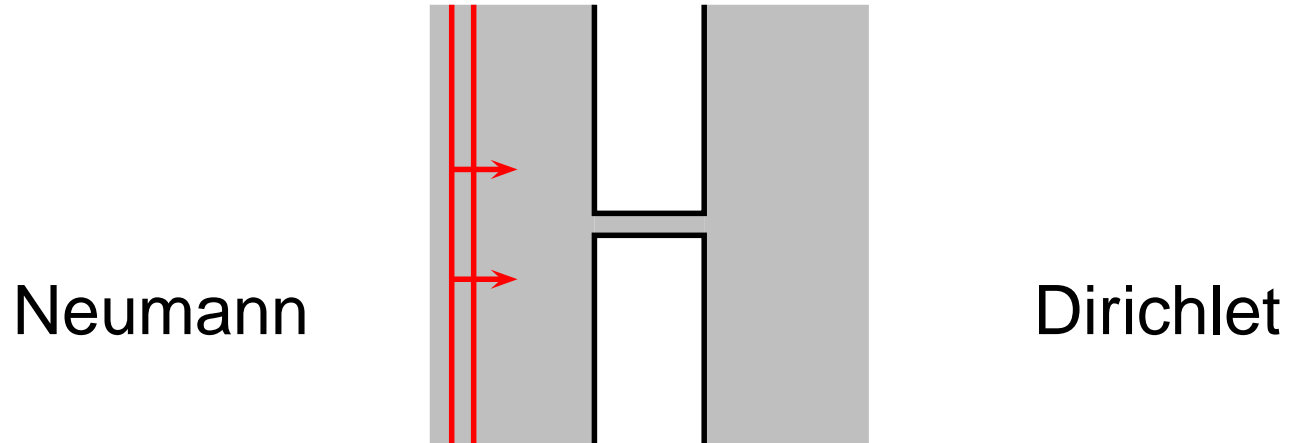
Neumann limit condition  
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$$\frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega^\varepsilon$$



With the **Dirichlet** limit condition, the transmission inside the slot is **negligible** ( $o(\varepsilon^\infty)$ ).

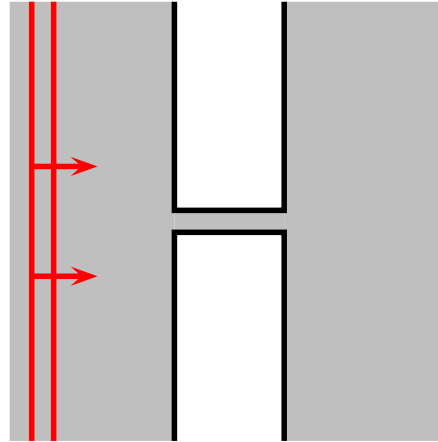
# A numerical computation



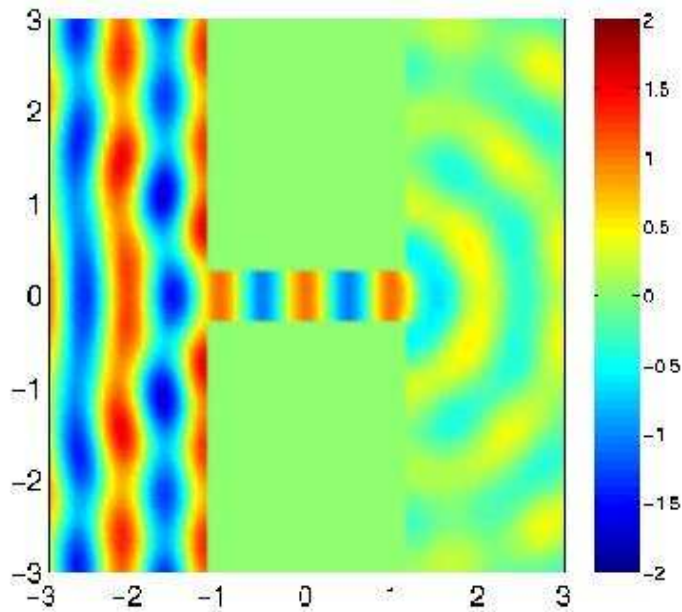
Numerical computation done with the **high order finite elements code** of (M. Duruflé, INRIA)

# A numerical computation

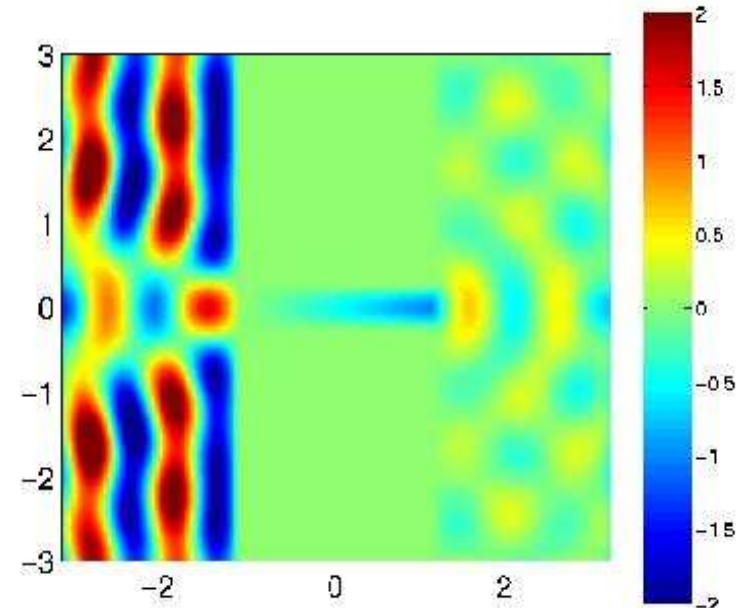
Neumann



Dirichlet

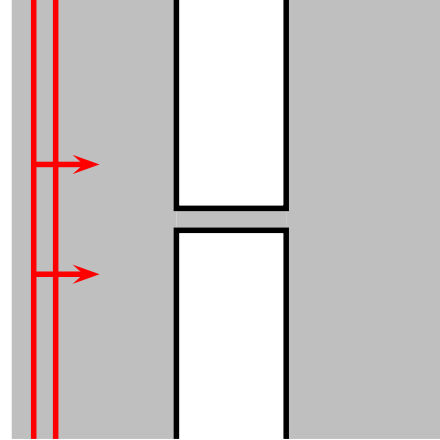


$$\frac{\varepsilon}{\lambda} = 0.5$$

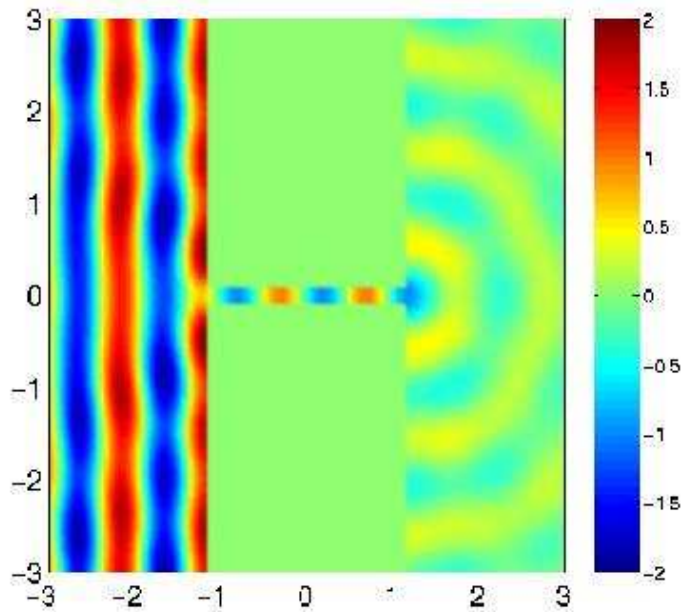


# A numerical computation

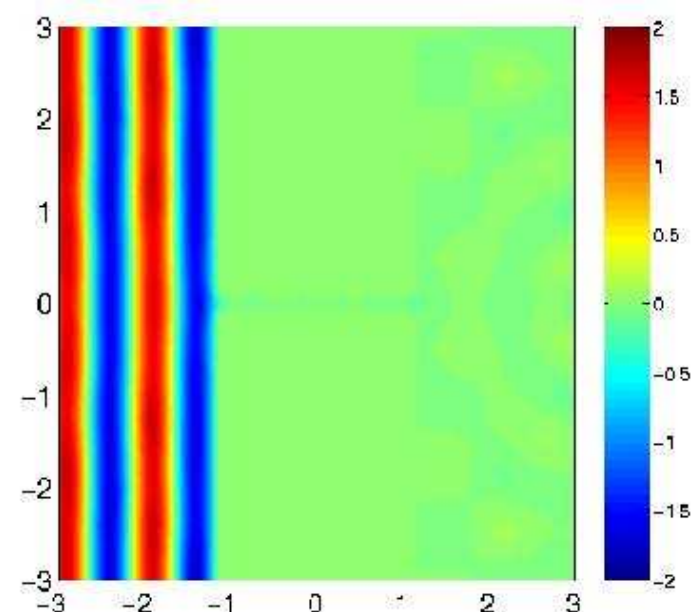
Neumann



Dirichlet

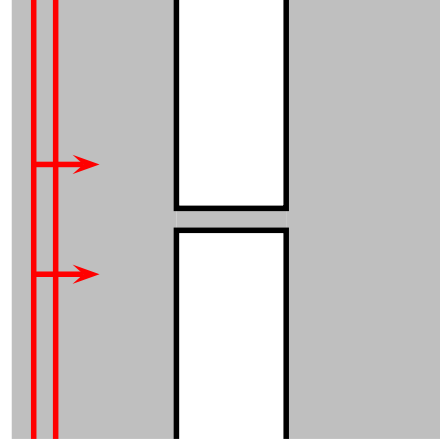


$$\frac{\varepsilon}{\lambda} = 0.2$$

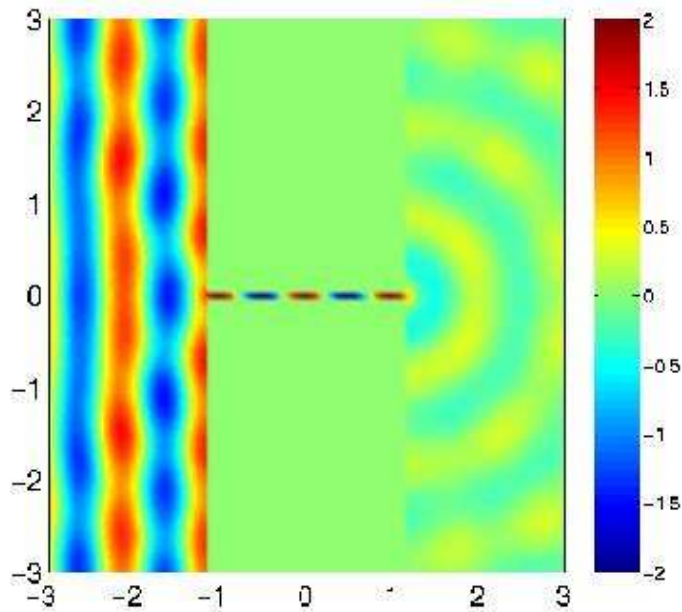


# A numerical computation

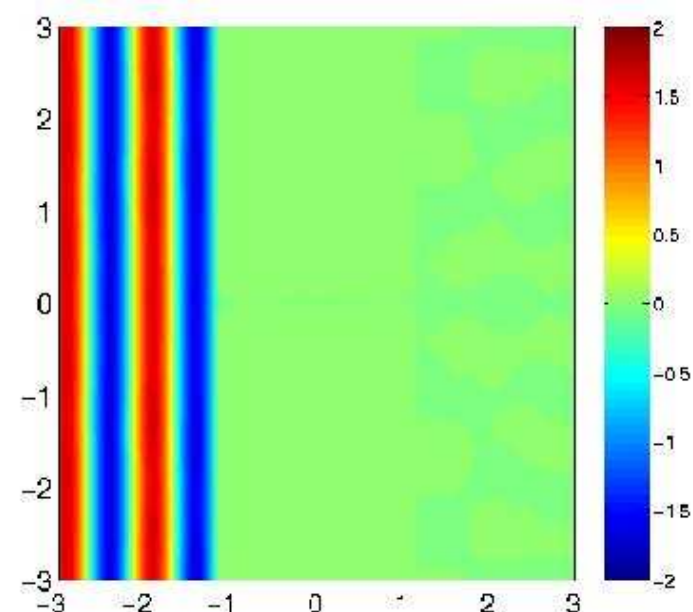
Neumann



Dirichlet



$$\frac{\varepsilon}{\lambda} = 0.1$$

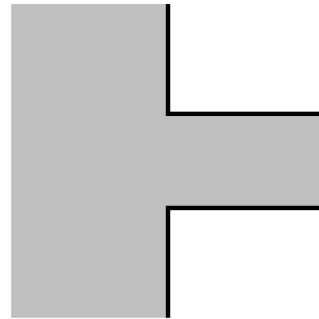


# Objectives

- Introduce **accurate** numerical methods

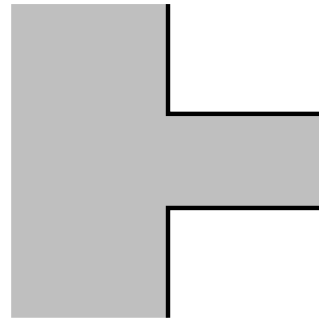
# Objectives

- Introduce **accurate** numerical methods
- We need an **intermediate zone**



# Objectives

- Introduce **accurate** numerical methods
- We need an **intermediate zone**



- A technique **the matching of asymptotic expansions**
  - Define **new approximate models** to compute the solution.
  - Use effectively “universal” technique of numerical computation (mesh reffinement).



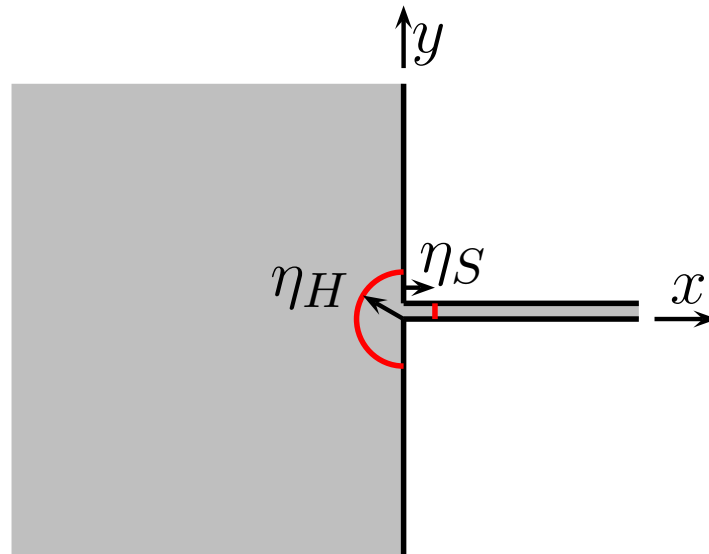
# Contributions to the match. of as. exp.

- A new presentation of the **matching principle** (not allways clear) postulated by the english school.

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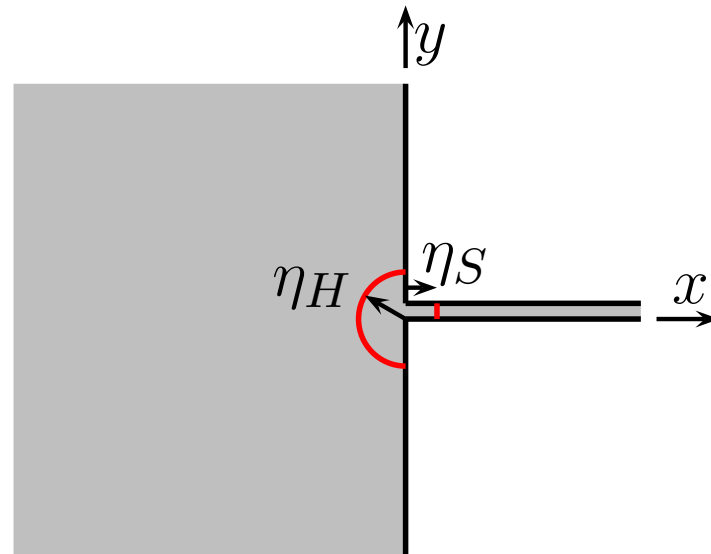
- A new presentation of the **matching principle** (not allways clear) postulated by the english school.
- The **mathematical justification** of this technique.
  - The proof are **inspired** by the multiscale technique
  - **Existence and unicity** of the terms of the expansions.
  - Specific technique: **error estimates**.

# Three zones



- Far field (2D field)
- Near field (boundary layer)
- Slot field (1D field)

# Three zones

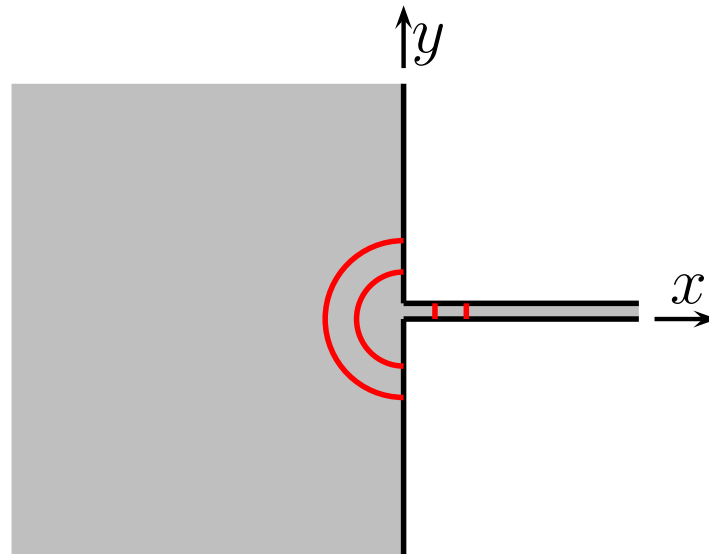


The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

# Three zones

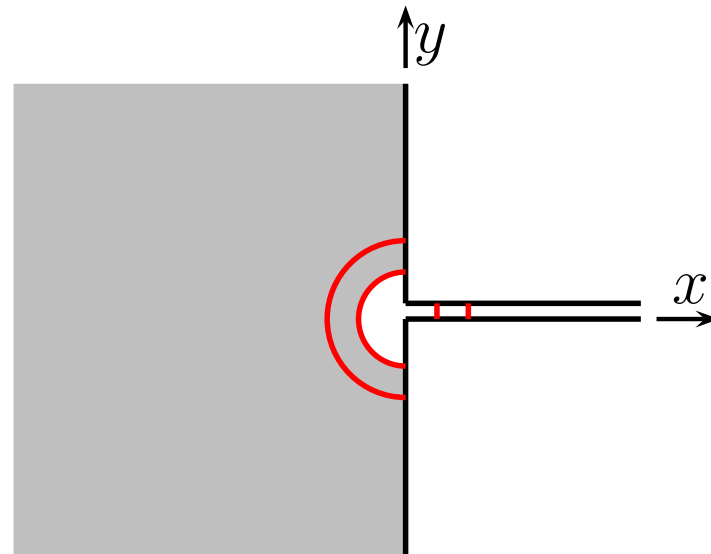


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# Three zones



Far field

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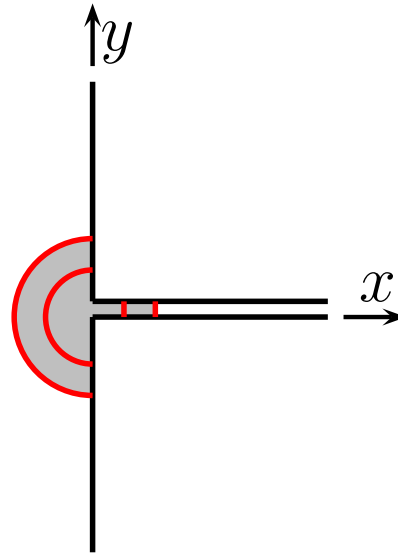
$$\varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

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# Three zones



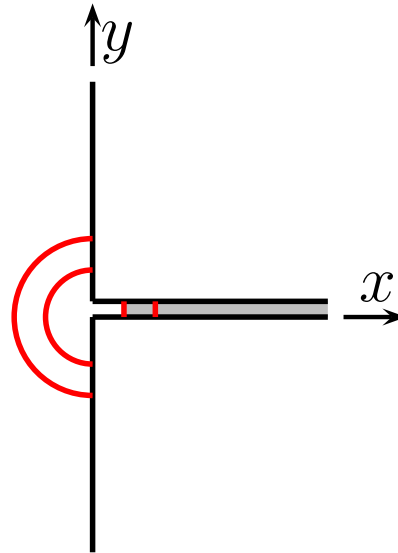
Near field

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

# Three zones



Slot field

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda,$$

$$\varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

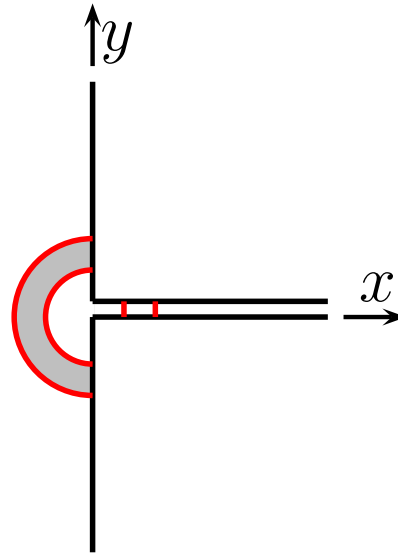
$$\varepsilon \rightarrow 0$$

$$\eta(\varepsilon) \rightarrow 0$$

$$\eta(\varepsilon)/\varepsilon \rightarrow +\infty$$



# Three zones



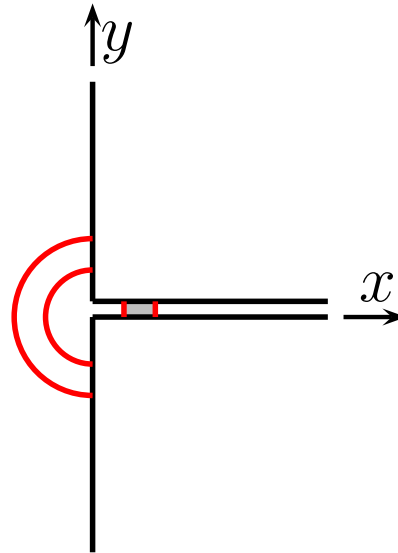
Far and near

The **asymptotic assumptions**:

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$$\varepsilon \rightarrow 0 \quad \eta(\varepsilon) \rightarrow 0 \quad \eta(\varepsilon)/\varepsilon \rightarrow +\infty$$

# Three zones



Slot and near

The **asymptotic assumptions**:

$$\varepsilon \ll \eta_H(\varepsilon) \ll \lambda, \quad \varepsilon \ll \eta_S(\varepsilon) \ll \lambda.$$

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- **Derivate** the asymptotic expansions:
  - **Formal** part
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  - **Definition** of the terms of the asymptotic expansions

# The different steps of the method

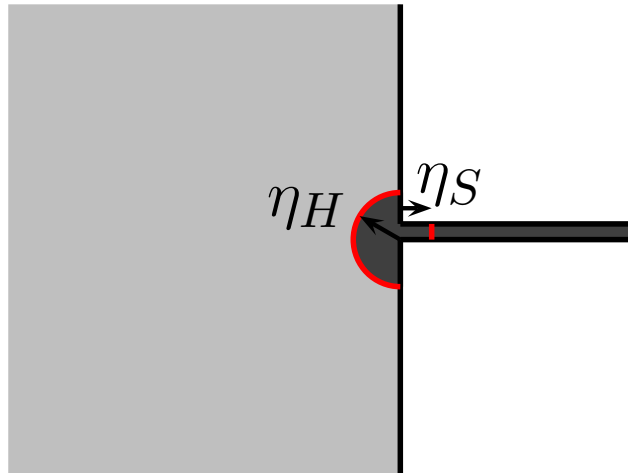
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  - **Error estimates**

# The different steps of the method

- 2 **Derivate** the asymptotic expansions:
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- 1 **Describe** the asymptotic expansions
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- 3 **Mathematical validation** of the asymptotic expansions
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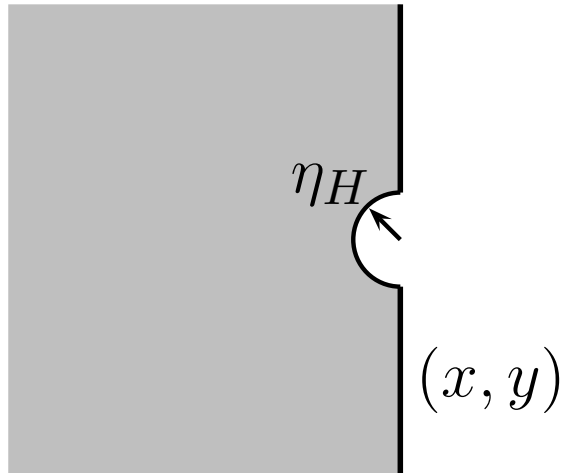
# Far field

Asymptotic context:  $\varepsilon \ll \eta_H \ll \lambda.$



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No **normalization**:

$$X = x,$$

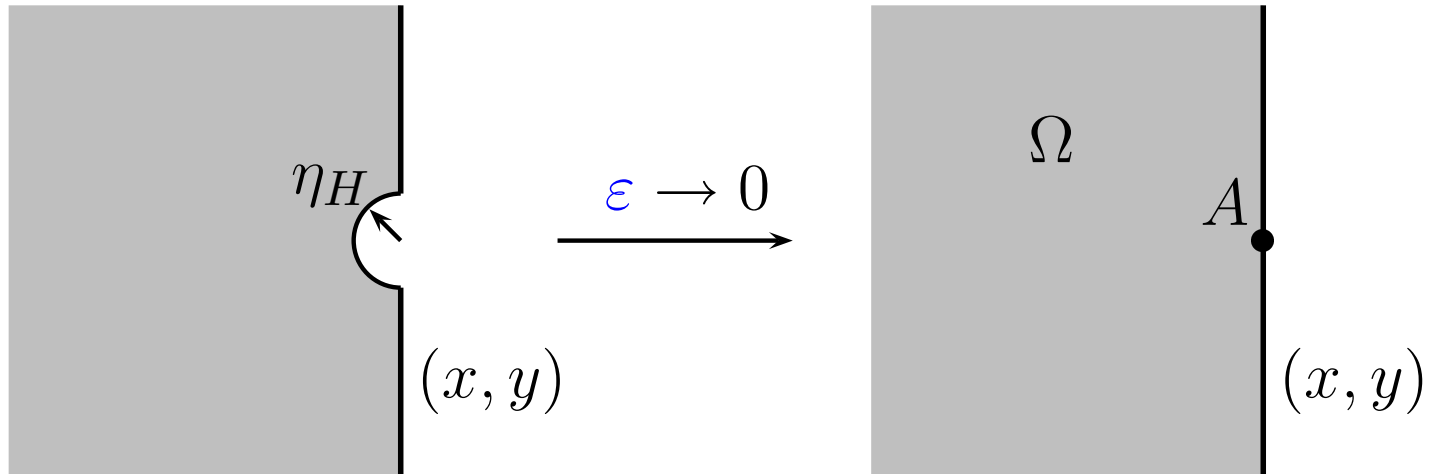
$$Y = y.$$



# Far field

Asymptotic context:

$$\varepsilon \ll \eta_H \ll \lambda.$$



No **normalization**:

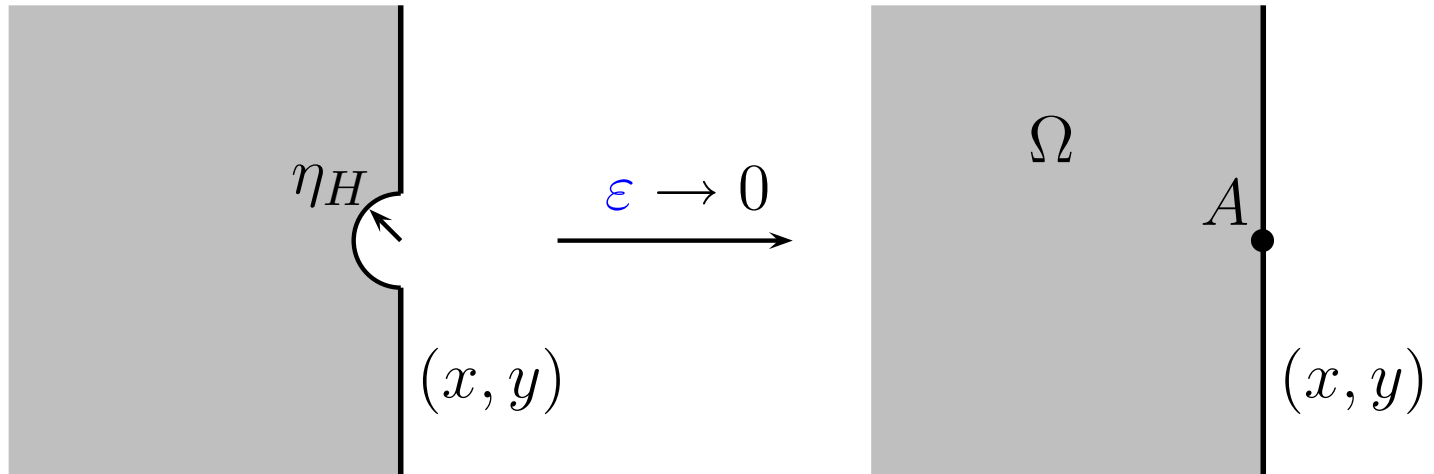
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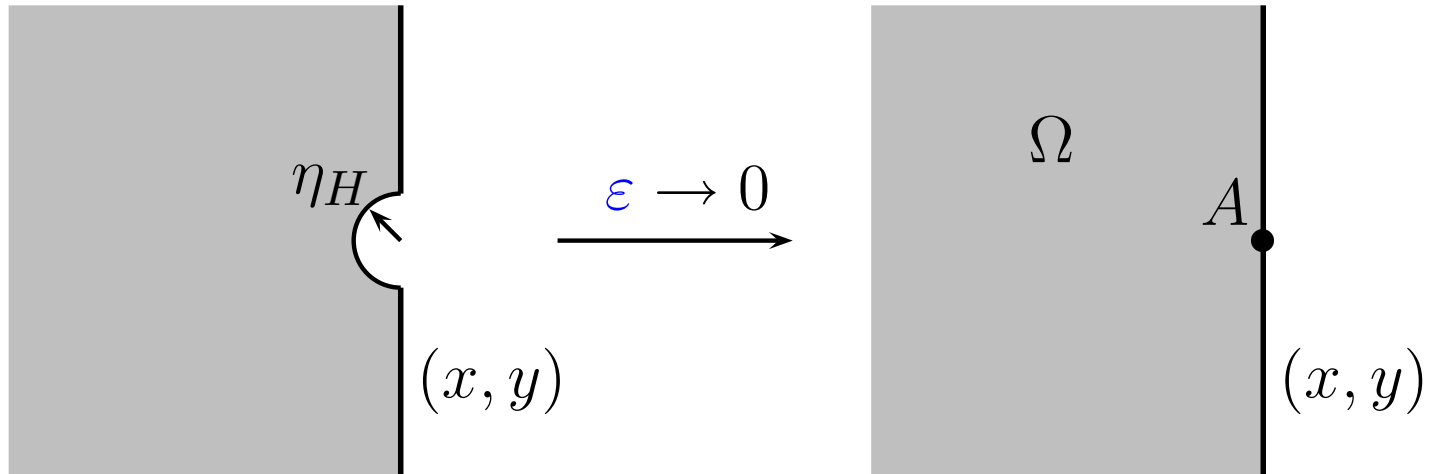


$$u^\varepsilon = u^0 + \sum_{i=1}^{+\infty} \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k + o(\varepsilon^\infty), \quad \text{in } \Omega.$$

# Far field

Asymptotic context:

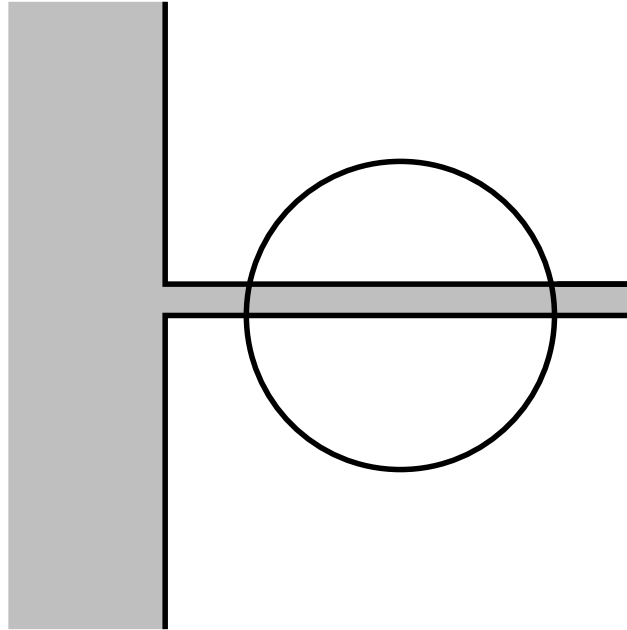
$$\varepsilon \ll \eta_H \ll \lambda.$$



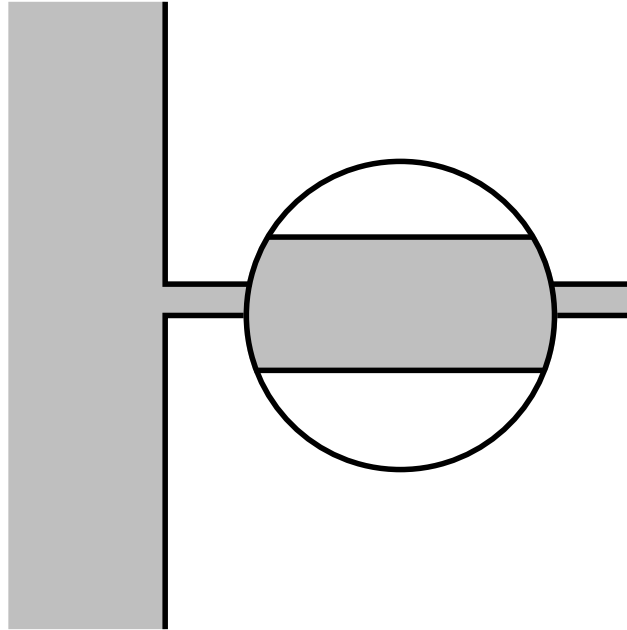
where the  $u_i^k$  satisfy the **homogeneous Helmholtz** equation

$$\Delta u_i^k + \omega^2 u_i^k = 0.$$

# Slot field

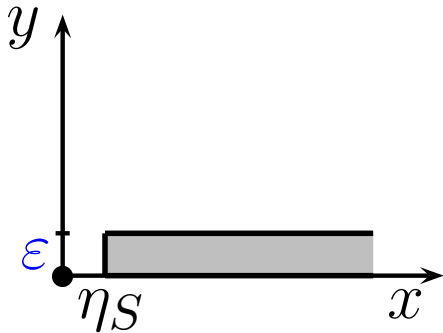


# Slot field



$$u^\varepsilon(x, y) = U^\varepsilon\left(x, \frac{y}{\varepsilon}\right)$$

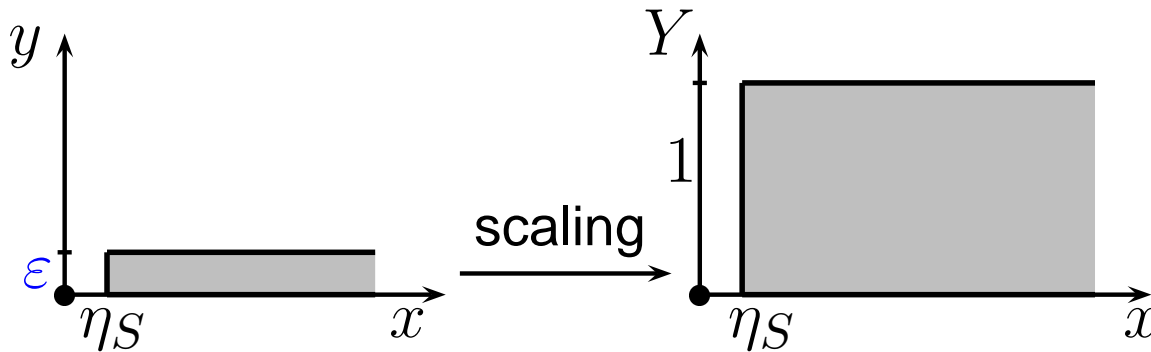
# Slot field



The **asymptotic** context:  $\varepsilon \ll \eta_S \ll \lambda$ .

The **normalization**:  $X = x, \quad Y = \frac{y}{\varepsilon}$

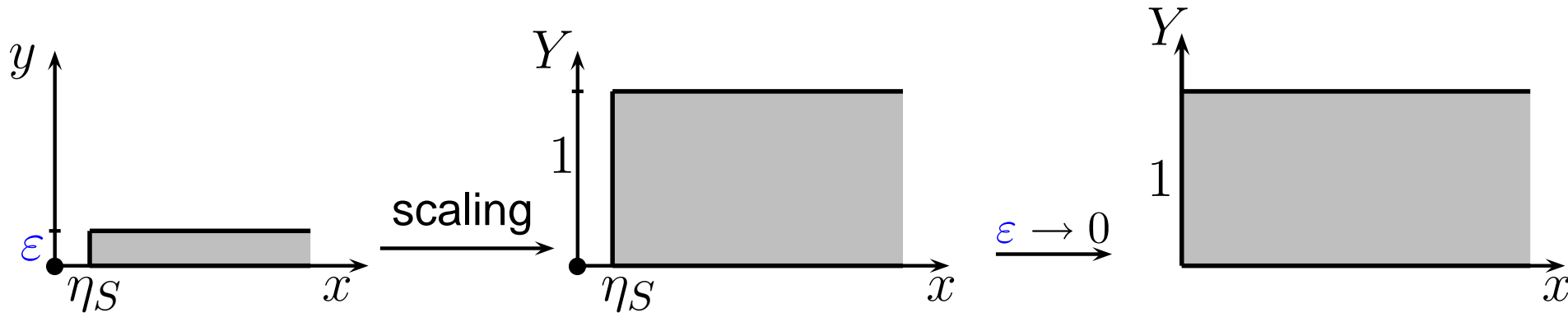
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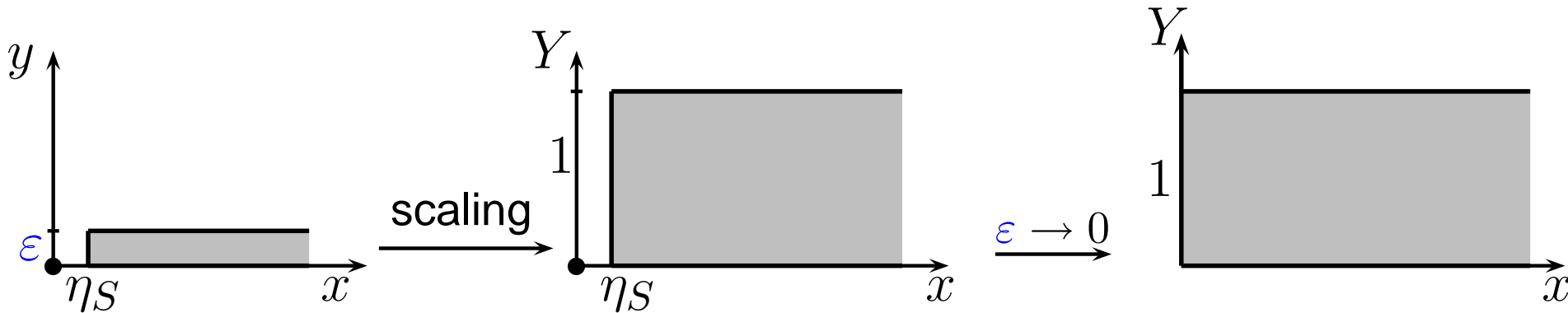


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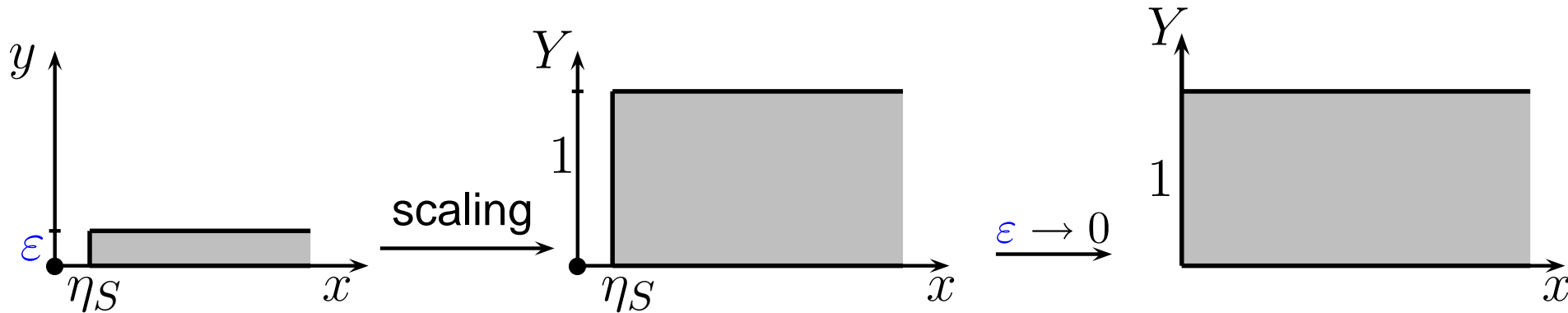


# Slot field



$$u^\varepsilon(x, Y\varepsilon) = U^\varepsilon(x, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k U_i^k(x, Y) + o(\varepsilon^\infty),$$

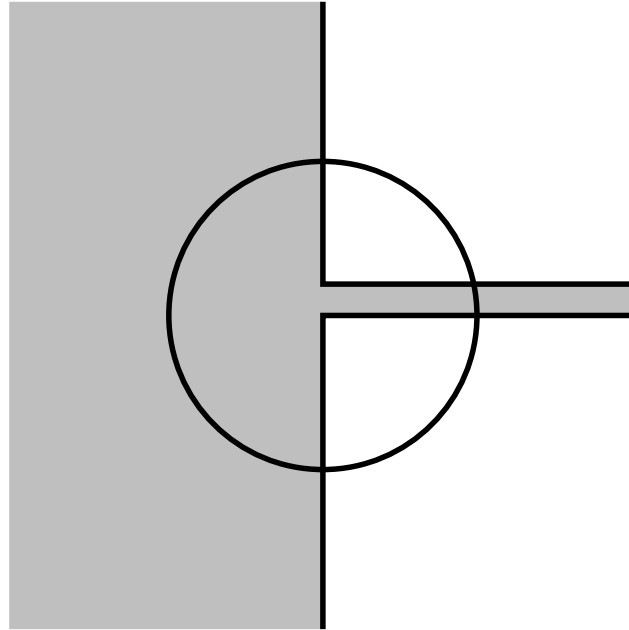
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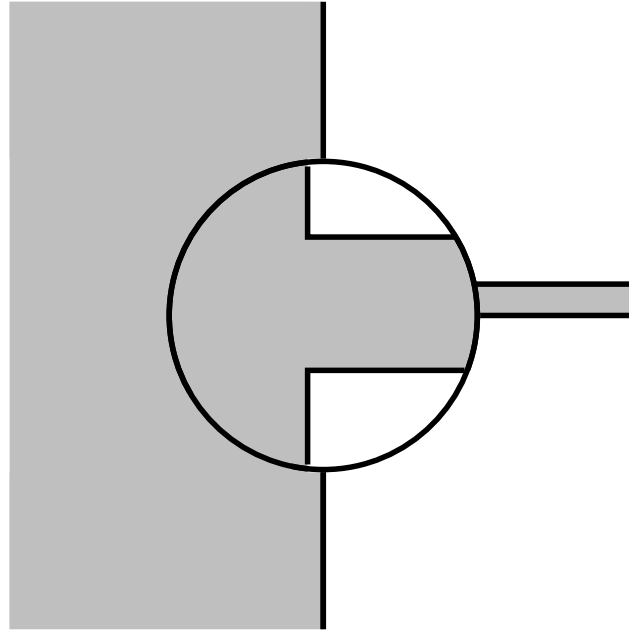
where the  $U_i^k$  satisfy the **1D Helmholtz** equation:

$$\frac{d^2 U_i^k}{dx^2} + \omega^2 U_i^k = 0$$

# Near field

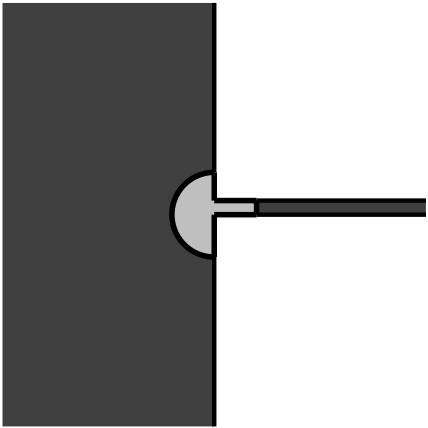


# Near field



$$u^\varepsilon(x, y) = u_p^\varepsilon\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

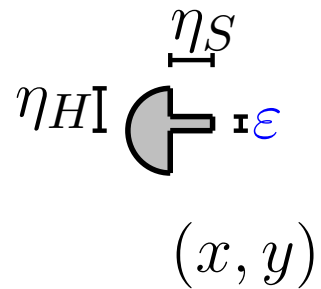
# Near field



The **Asymptotic** context:  $\varepsilon \ll \eta_H \ll \lambda$ ,  $\varepsilon \ll \eta_S \ll \lambda$ .

The **normalization**:  $X = \frac{x}{\varepsilon}$ ,  $Y = \frac{y}{\varepsilon}$

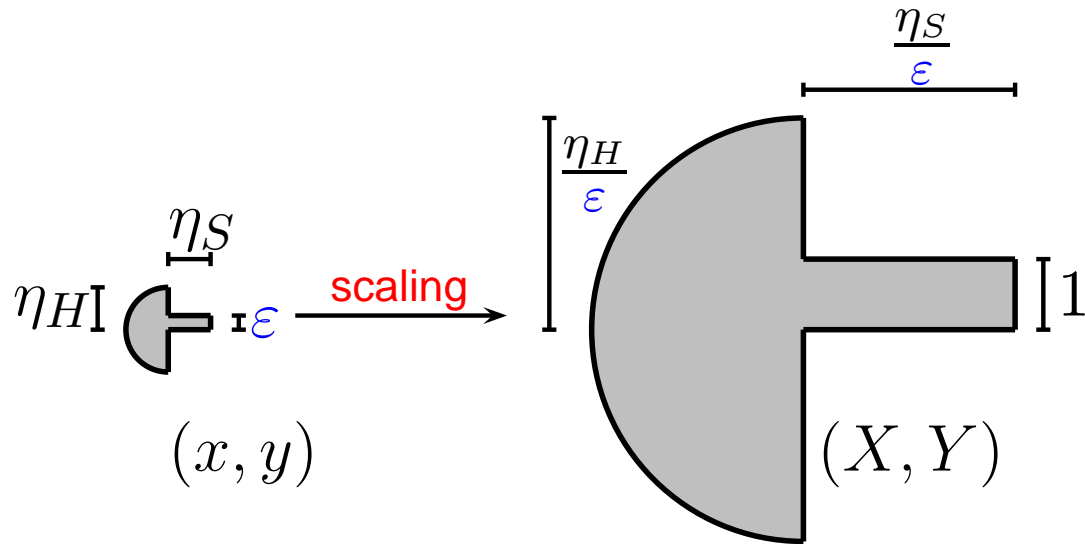
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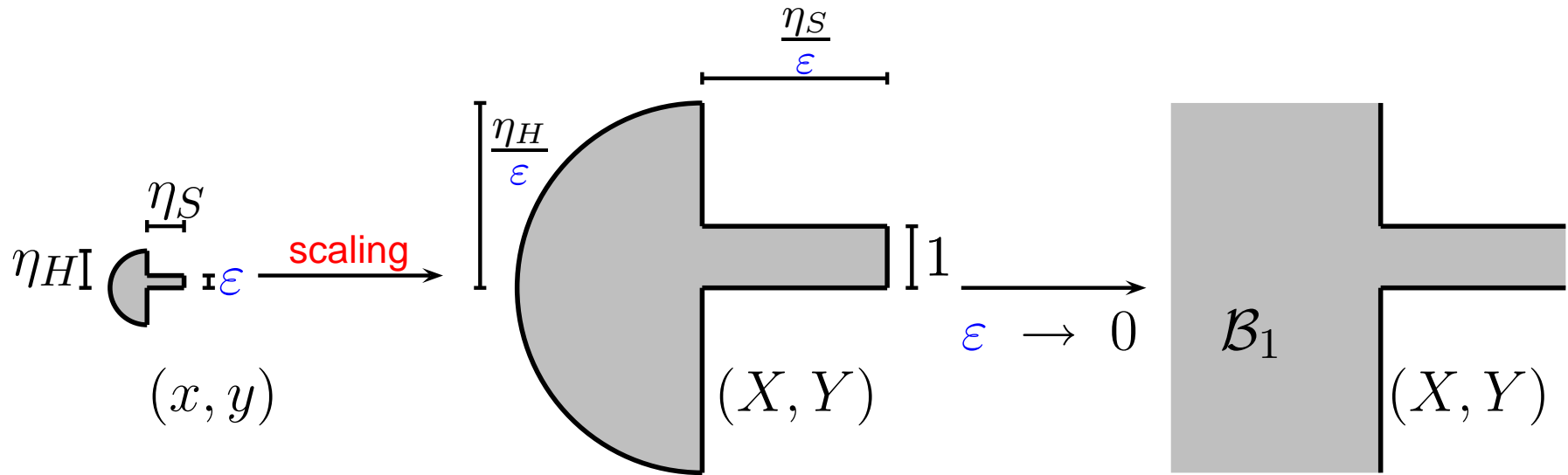
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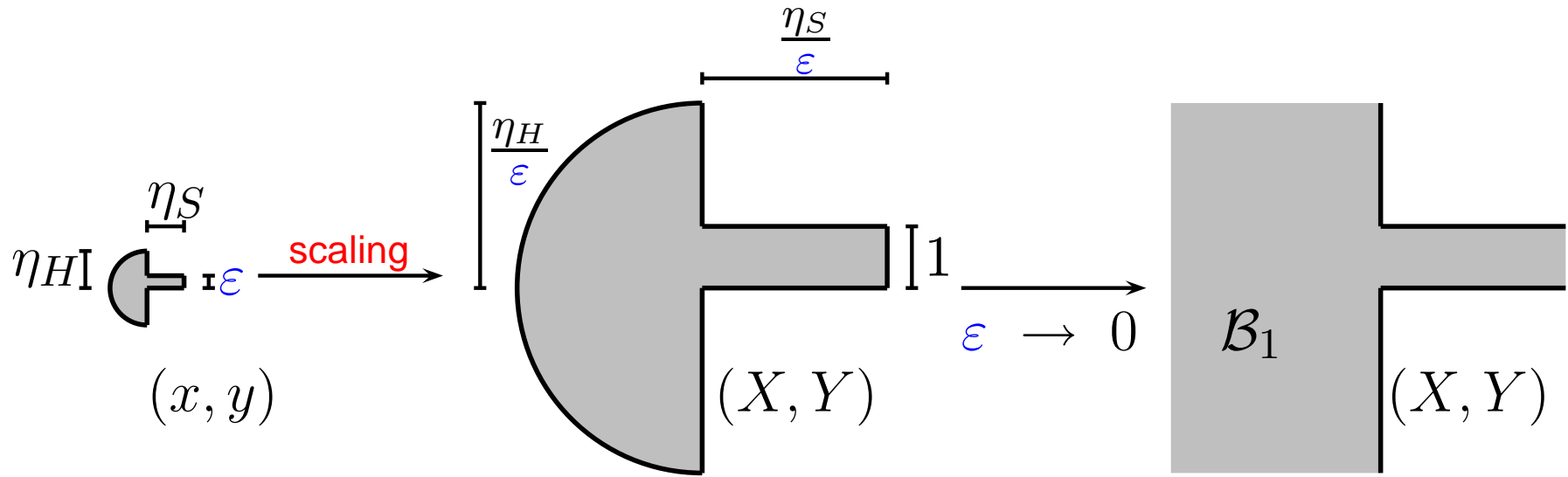


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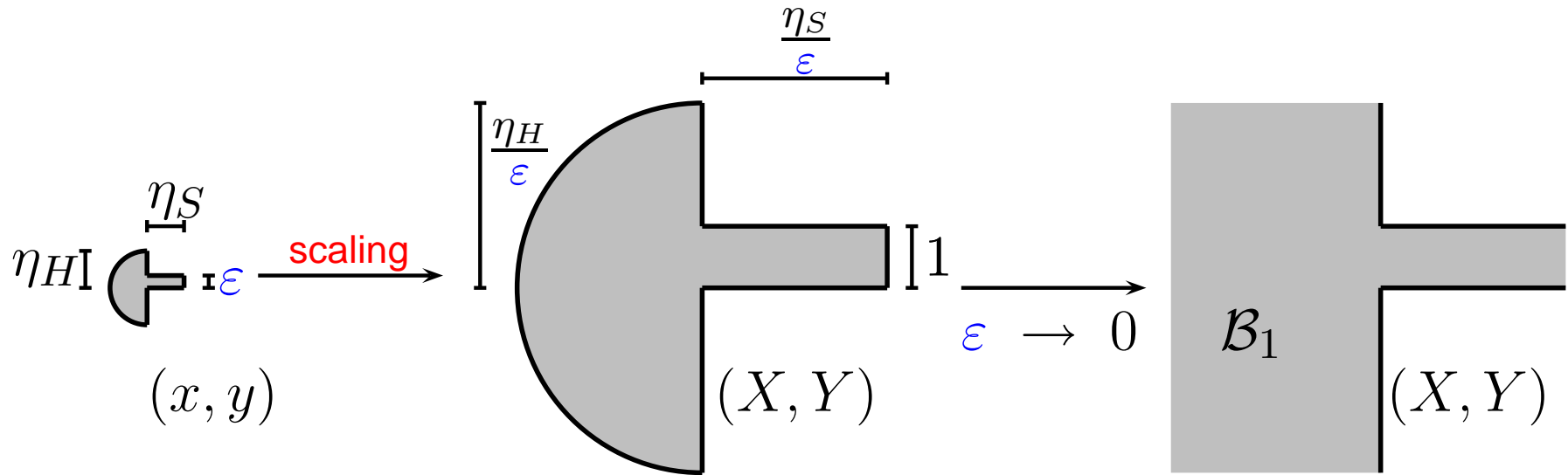


# Near field



$$u^\varepsilon(\varepsilon X, \varepsilon Y) = u_p^\varepsilon(X, Y) = \sum_{i=0}^{+\infty} \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k(X, Y) + o(\varepsilon^\infty)$$

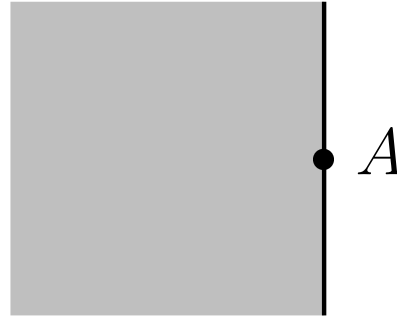
# Near field



where the  $(u_p)_i^k$  satisfy the (in)-homogeneous Laplace equation.

$$\begin{cases} \Delta(u_p)_i^k = 0, & \text{if } i = k \text{ or } k + 1, \\ \Delta(u_p)_i^k = -\omega^2 (u_p)_{i-2}^k, & \text{else.} \end{cases}$$

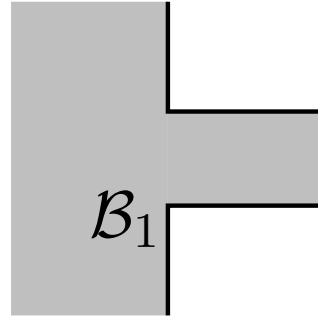
**Order 0 :**  $\underline{u}^0$ ,  $(u_p)_0^0$ ,  $U_0^0$



**Far field:**

$$\left\{ \begin{array}{l} \text{Find } u^0 \in H_{loc}^1(\Omega) \text{ such that :} \\ -\Delta u^0 - \omega^2 u^0 = f, \quad \text{in } \Omega, \\ \frac{\partial u^0}{\partial n} = 0, \quad \text{on } \partial\Omega, \\ u^0 \text{ is outgoing.} \end{array} \right.$$

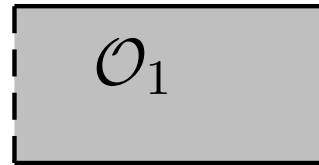
**Order 0** :  $u^0$ ,  $\underline{(u_p)_0^0}$ ,  $U_0^0$



**Near field:**

$$(u_p)_0^0(X, Y) = u^0(A), \quad \text{in } B_1.$$

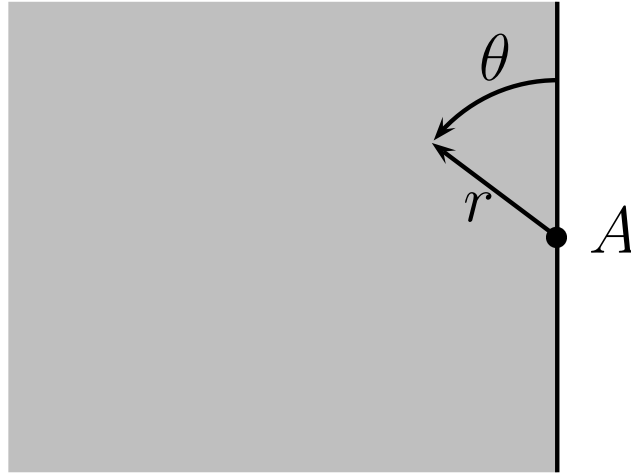
**Order 0** :  $u^0$ ,  $(u_p)_0^0$ ,  $U_0^0$



**Slot field:**

$$U_0^0(x, Y) = u^0(A) \exp(i\omega x), \quad \text{in } \mathcal{O}_1.$$

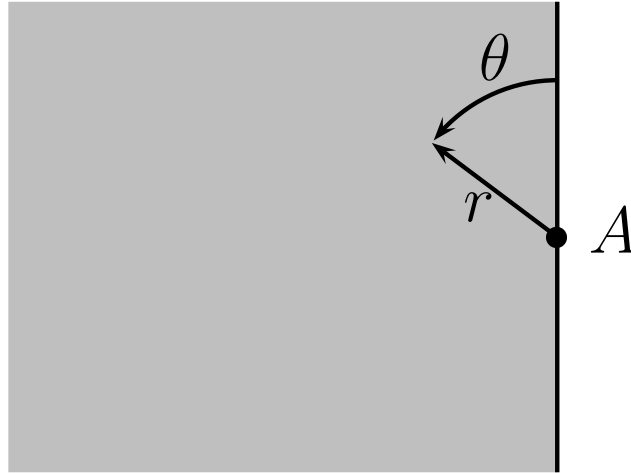
**Order 1 :**  $\underline{u}_1^0$ ,  $(u_p)_1^0$ ,  $(u_p)_1^1$ ,  $U_1^0$ ,  $U_1^1$



**Approximation** of the exact Solution:

$$u^\varepsilon \simeq u^0 + \varepsilon u_1^0$$

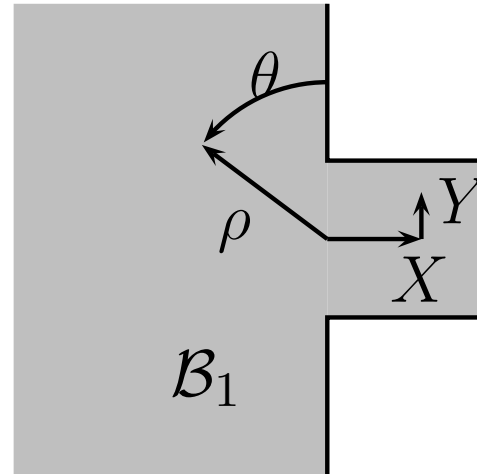
**Order 1** :  $\underline{u}_1^0$ ,  $(u_p)_1^0$ ,  $(u_p)_1^1$ ,  $U_1^0$ ,  $U_1^1$



**explicit form of  $u_1^0$**

$$u_1^0(r, \theta) = -\frac{\omega}{2} u^0(A) H_0^{(1)}(\omega r).$$

**Order 1** :  $u_1^0$ ,  $\underline{(u_p)_1^0}$ ,  $\underline{(u_p)_1^1}$ ,  $U_1^0$ ,  $U_1^1$

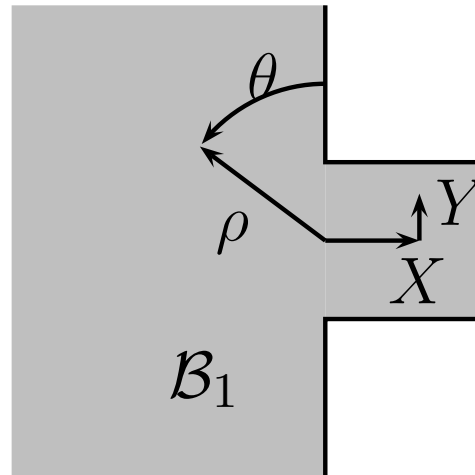


**Approximation** of the exact solution:

$$\begin{cases} u^\varepsilon(\varepsilon X, \varepsilon Y) = u_p^\varepsilon(X, Y), \\ u_p^\varepsilon \simeq (u_p)_0^0 + \varepsilon (u_p)_1^0 + \varepsilon \log \varepsilon (u_p)_1^1. \end{cases}$$



**Order 1 :**  $u_1^0$ ,  $\underline{(u_p)_1^0}$ ,  $(u_p)_1^1$ ,  $U_1^0$ ,  $U_1^1$

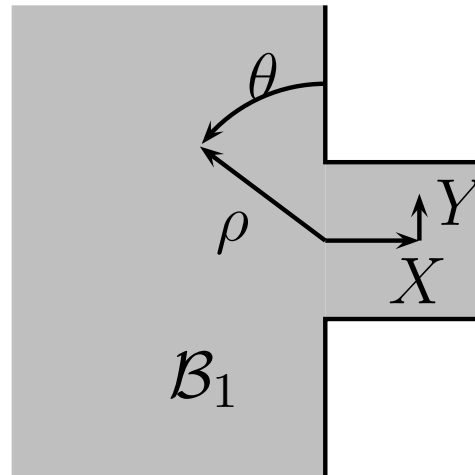


**Near field:**

Find  $(u_p)_1^0 \in H_{loc}^1(\mathcal{B}_1)$  such that:

$$\begin{cases} \Delta(u_p)_1^0 = 0, & \text{in } \mathcal{B}_1 \\ \frac{\partial(u_p)_1^0}{\partial n} = 0, & \text{on } \partial\mathcal{B}_1. \end{cases}$$

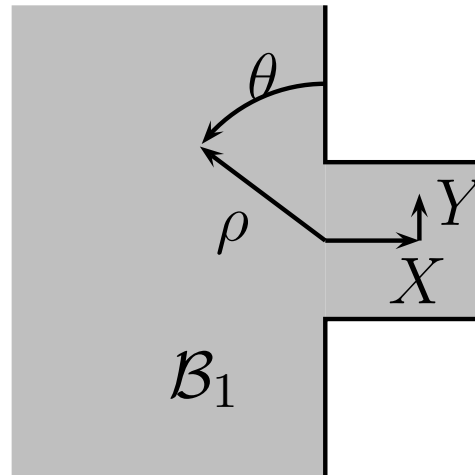
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**Behavior at infinity in the half-space:**

$$(u_p)_1^0(\rho, \theta) - \frac{\partial u^0}{\partial y}(A) \rho \cos \theta + \frac{\omega}{2} u^0(A) \left[ 1 + \frac{2i}{\pi} (\log \rho + \gamma) \right] = O\left(\frac{1}{\rho}\right).$$

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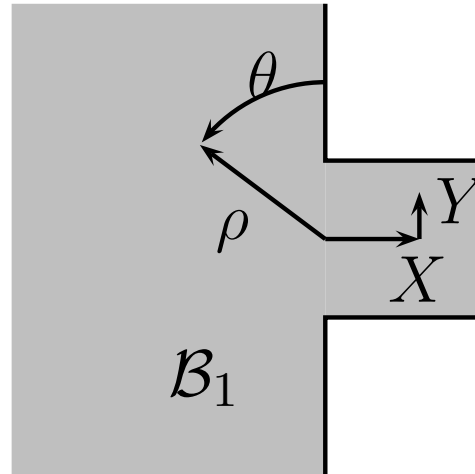
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**Behavior at infinity in the slot:**

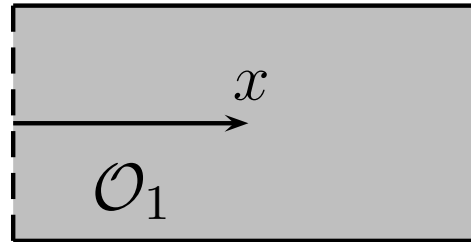
$$(u_p)_1^0(X, Y) - i \omega u^0(A) X = O(1).$$

**Order 1 :**  $u_1^0$ ,  $(u_p)_1^0$ ,  $\underline{(u_p)_1^1}$ ,  $U_1^0$ ,  $U_1^1$



$$(u_p)_1^1 = -\frac{i\omega}{\pi} u^0(A)$$

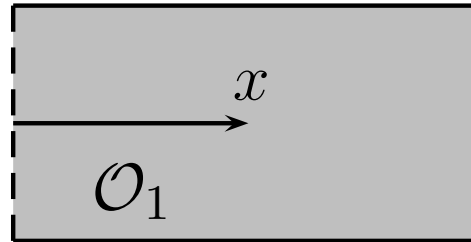
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**Approximation** of the exact solution:

$$\begin{cases} u^\varepsilon(x, \varepsilon Y) = U^\varepsilon(x, Y), \\ U^\varepsilon \simeq U_0^0 + \varepsilon U_1^0 + \varepsilon \log \varepsilon U_1^1. \end{cases}$$

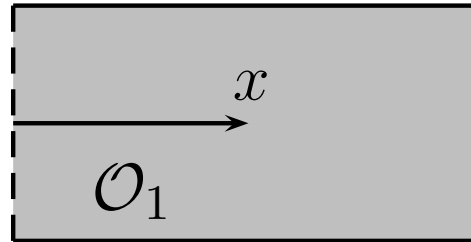
**Order 1 :**  $u_1^0, (u_p)_1^0, (u_p)_1^1, \underline{U_1^0}, U_1^1$



**The slot field:**

$$U_1^0(x) = \int_0^1 \mathcal{U}_1^0(0, Y) dY \exp(\mathbf{i}\omega x),$$

**Order 1 :**  $u_1^0$ ,  $(u_p)_1^0$ ,  $(u_p)_1^1$ ,  $U_1^0$ ,  $U_1^1$

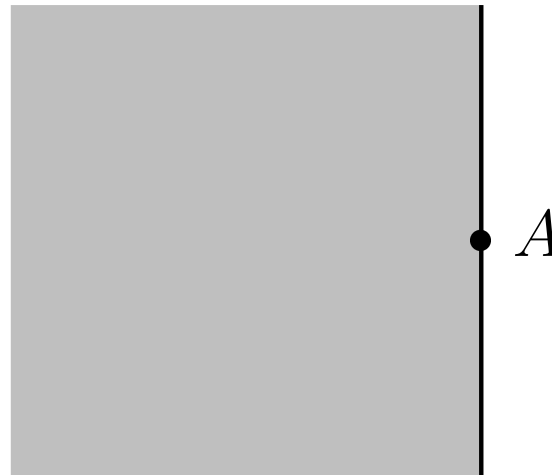


**The slot field:**

$$U_1^1(x) = -\frac{\mathbf{i}\omega}{\pi} u^0(A) \exp(\mathbf{i}\omega x).$$

# The far field of order $i > 1$

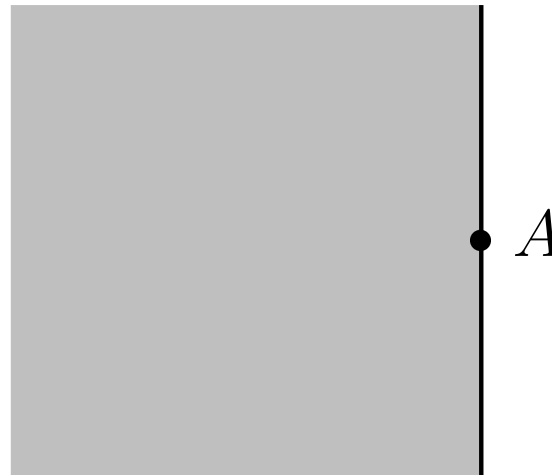
- The field  $u_i^k$  are defined in the **half space**:





# The far field of order $i > 1$

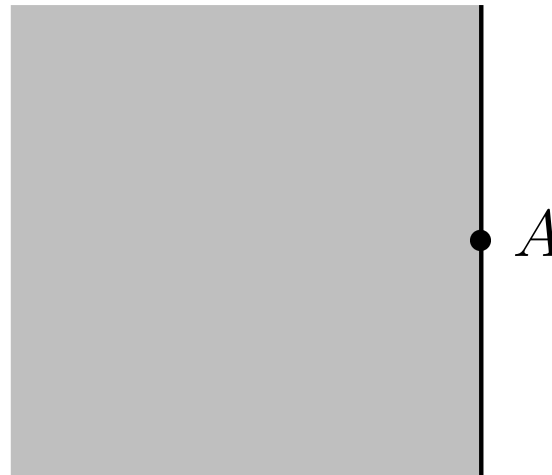
- The field  $u_i^k$  are defined in the **half space**:



- The far fields  $u_i^k$ 
  - satisfy the **homogeneous Helmholtz** equation
  - are **singular** at the neighborhood of the origin
  - are outgoing at infinity

# The far field of order $i > 1$

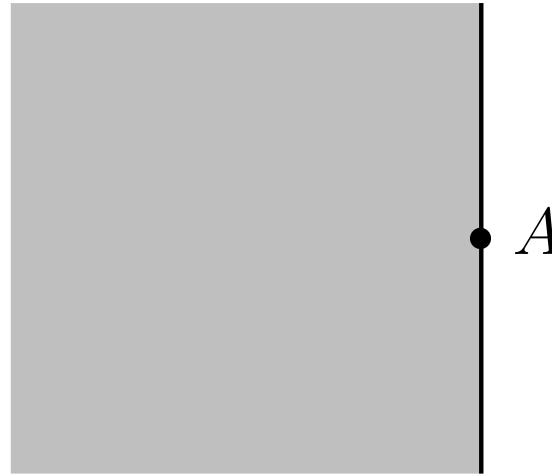
- The field  $u_i^k$  are defined in the **half space**:



- $$u_i^k = \sum_{p=0}^{+\infty} a_p H_p^{(1)}(\omega r) \cos p\theta$$

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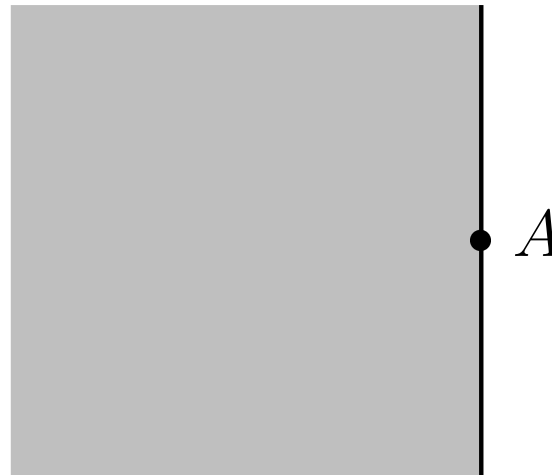
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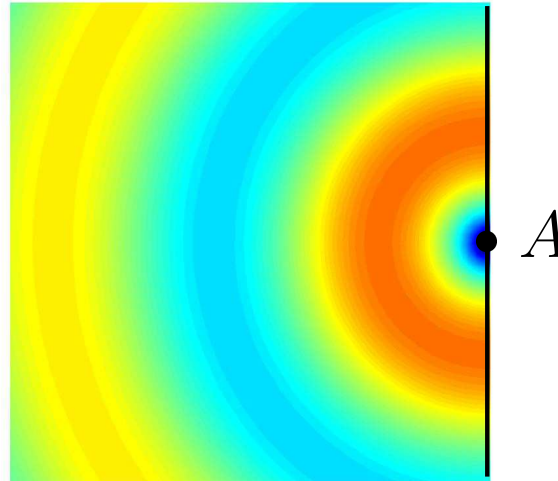
- $$u_i^k = \sum_{p=0}^{i-k-1} a_p H_p^{(1)}(\omega r) \cos p\theta$$

The  $a_p$  are functions of **lower order** terms

# The far field of order $i > 1$

- The field  $u_i^k$  are defined in the **half space**:

$$\text{Im}(H_0^{(1)}(\omega r))$$



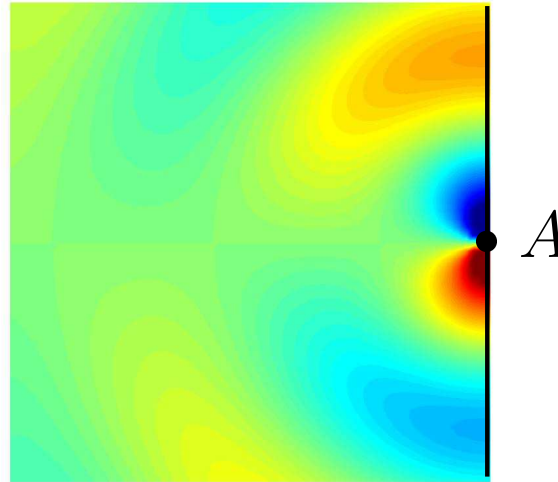
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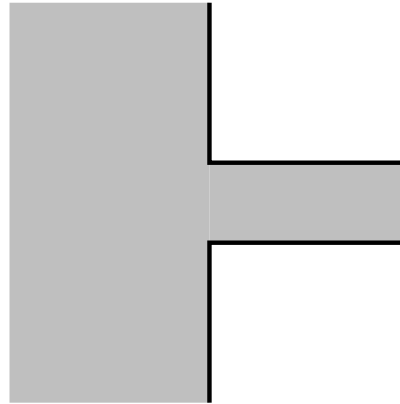


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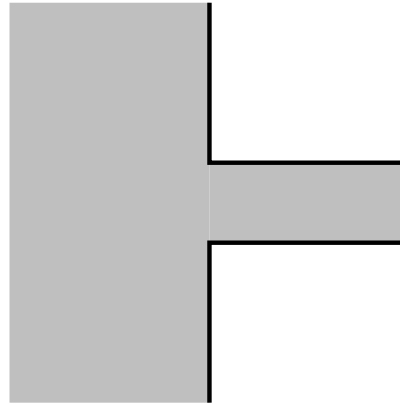
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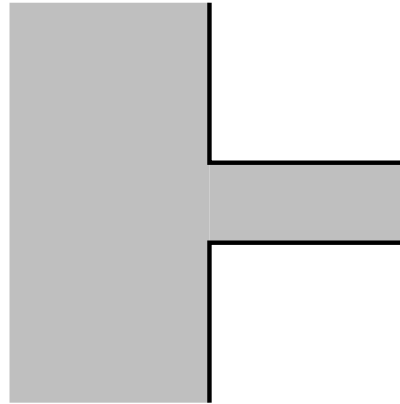
$$\Delta(u_p)_i^k = 0, \quad (i = k \text{ ou } k + 1),$$

$$\Delta(u_p)_i^k = -\omega^2 (u_p)_{i-2}^k, \quad (i \geq k + 2),$$



# The near fields of order $i > 1$

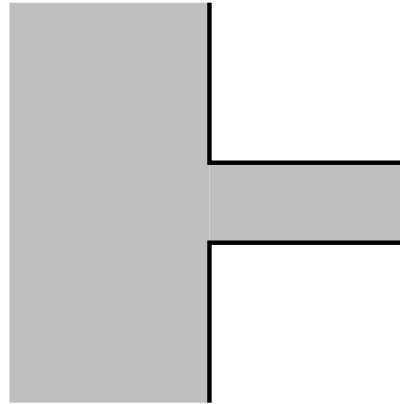
- The  $(u_p)_i^k(X, Y)$  are defined in the **canonical** domain:



- by **Laplace** equation:
- by polynomial **growings** at infinity:
  - The **growings** in the half space are functions of **far field of lower (or equal) order**
  - The **growings** in the slot are functions of the slot fields **of lower order**

# The near fields of order $i > 1$

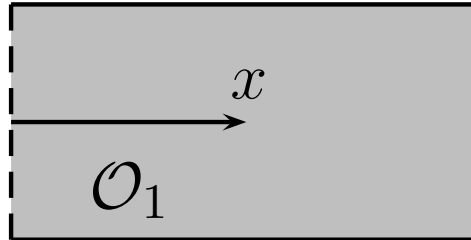
- The  $(u_p)_i^k(X, Y)$  are defined in the **canonical** domain:



- Proof of the **existence-unicity**:
  - with truncature functions, we subtract the growing behavior at infinity of the  $(u_p)_i^k$
  - We use the “classical” **variational theory** (wheighted Sobolev spaces, Leroux, Hardy,...)

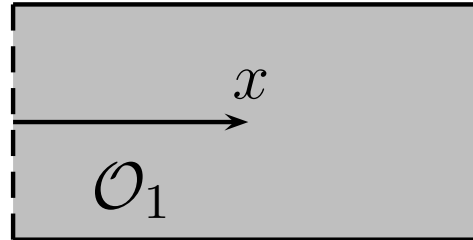
# The slot field of order $i > 1$

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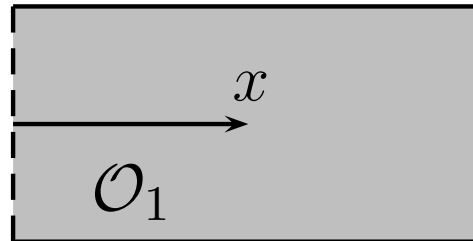
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- $$U_i^k(x) = \int_0^1 (u_p)_i^k(0, Y) dY \exp i\omega x$$

# Some properties

We see that:

- More  $i - k$  is **large** more  $u_i^k$  is **singular** at the origin:

$$r^{-p} \text{ terms, } p = 0, \dots, i - k - 1$$

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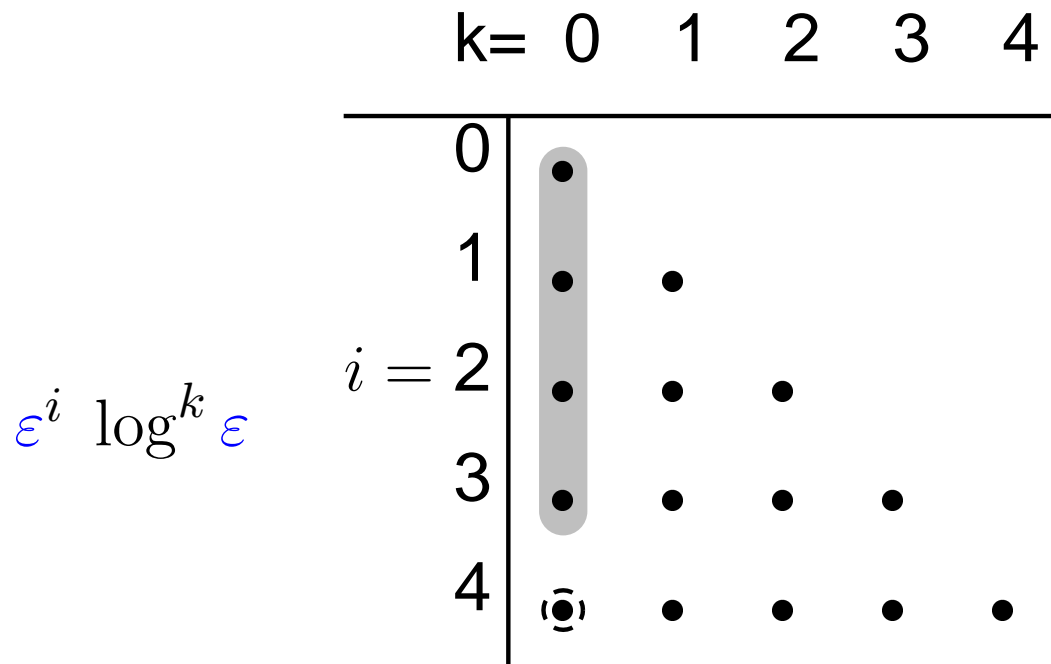
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- When the **order**  $i$  grows, one has  $O(\frac{i^2}{2})$  ( $\times 3$ ) terms to compute...

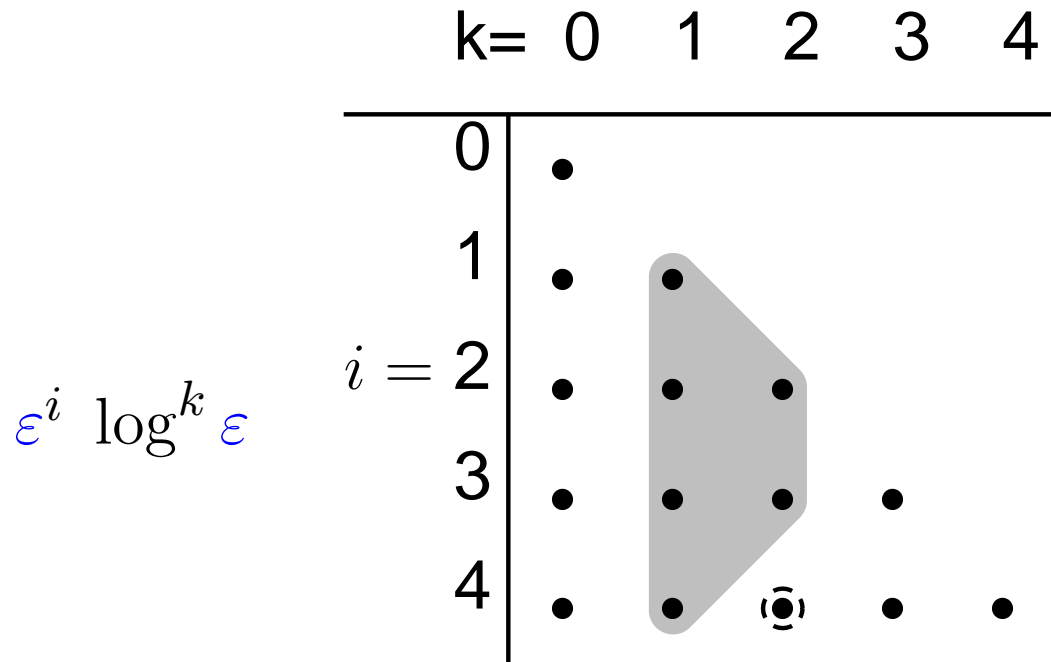


# Dependance diagram of the asymp. terms



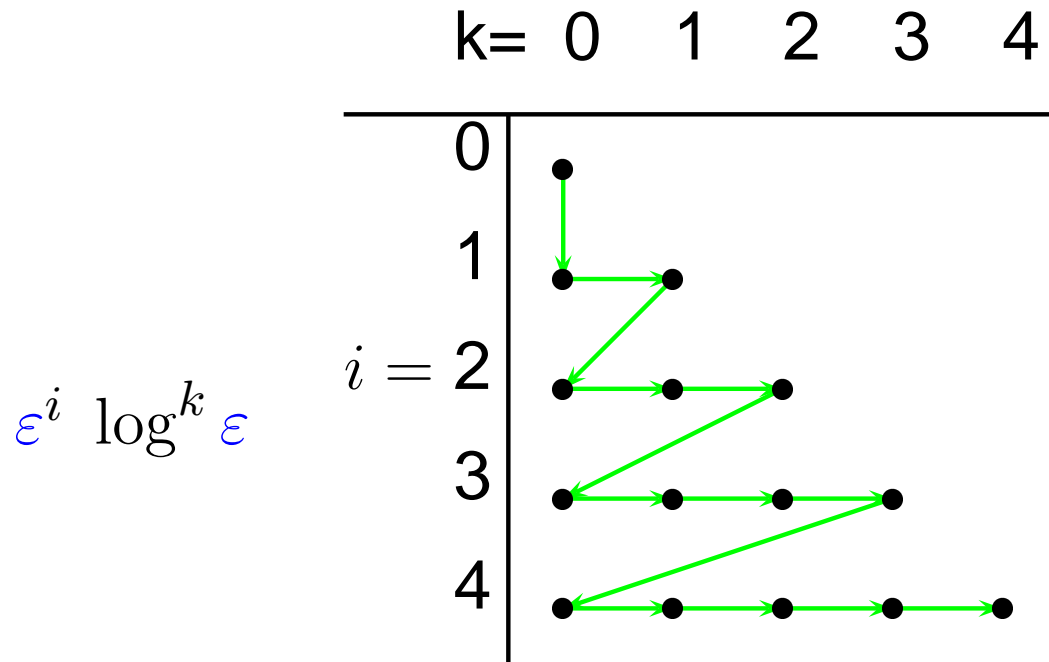
Any point corresponds to the 3 functions  $(u_i^k, (u_p)_i^k, U_i^k)$ .

# Dependance diagram of the asymp. terms



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# Natural scheduling of the computations



Any point corresponds to the three functions  $(u_i^k, (u_p)_i^k, U_i^k)$ .

# Derivate the terms of the as. exp.

- We search for solutions of the form:

$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k u_i^k \quad (\text{far field})$$

$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \quad (\text{near field})$$

$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k U_i^k \quad (\text{slot field})$$

# Devirvate the terms of the as. exp.

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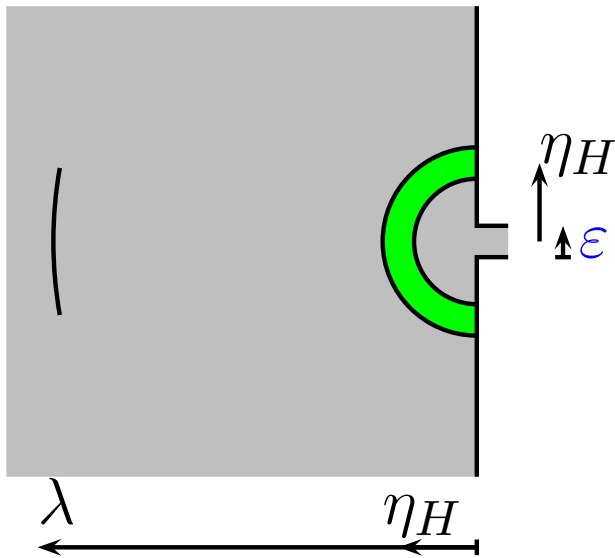
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- We **inject** the equations (Helmholtz, Neumann)
- We obtain the **coupling** conditions: (**the difficulty**)

# Far-Near coupling



In a **thick zone**:

$$\varepsilon \ll \eta_H \ll \lambda.$$

We write the coupling condition:

$$u^\varepsilon(\eta_H, \theta) = (u_p)^\varepsilon\left(\frac{\eta_H}{\varepsilon}, \theta\right).$$

$$\sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k u_i^k(\eta_H, \theta) \simeq \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon^i (\log \varepsilon)^k (u_p)_i^k\left(\frac{\eta_H}{\varepsilon}, \theta\right)$$

$$\eta_H \rightarrow 0 \qquad \frac{\eta_H}{\varepsilon} \rightarrow +\infty$$

# Far-Near coupling

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$\eta_H \rightarrow 0$   $\frac{\eta_H}{\varepsilon} \rightarrow +\infty$

We **expand**

- the left serie according to  $\eta_H$  near **0**
- The right serie according to  $\eta_H/\varepsilon$  tending ot **infinity**

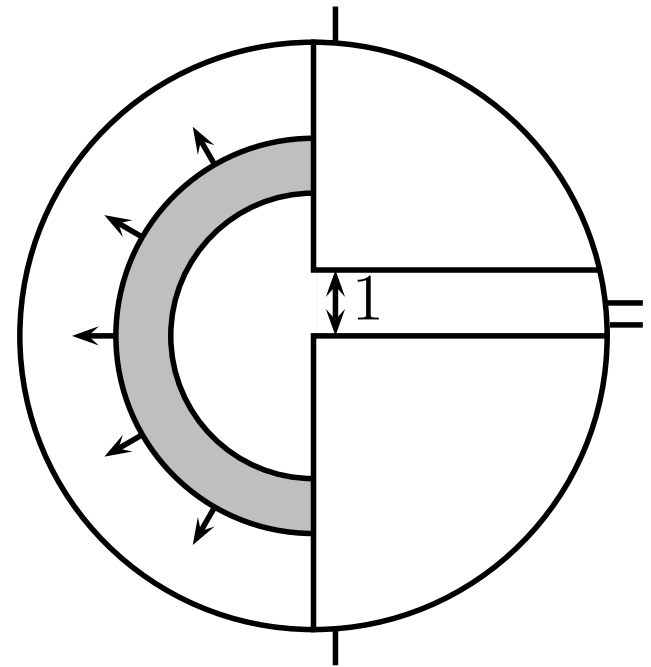
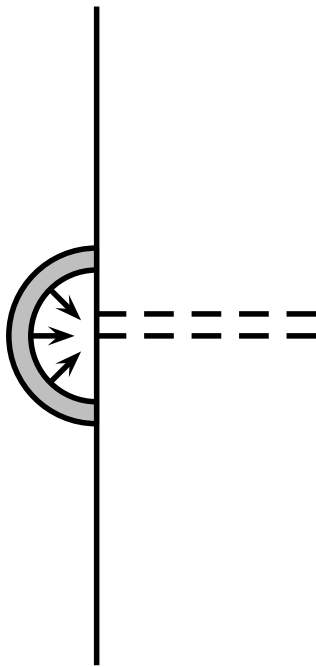
We **identify** all the terms of the two series.



# The conclusion of the coupling

- The **far** field-**near** field coupling:

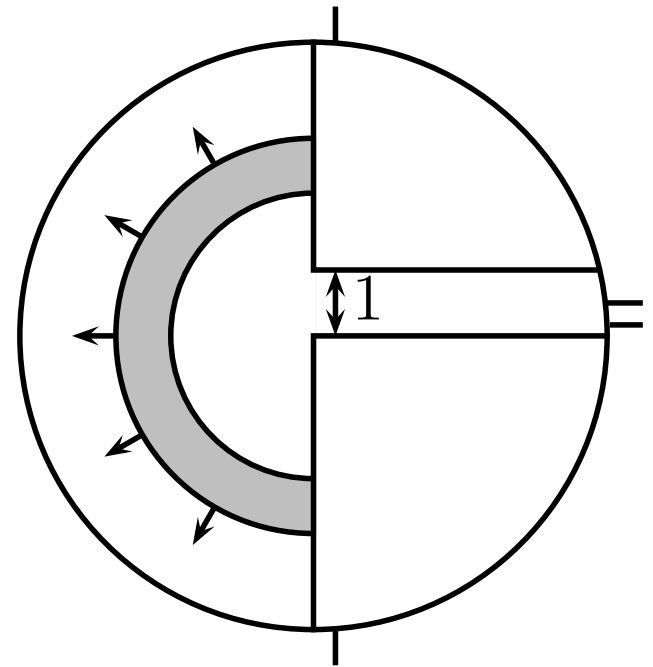
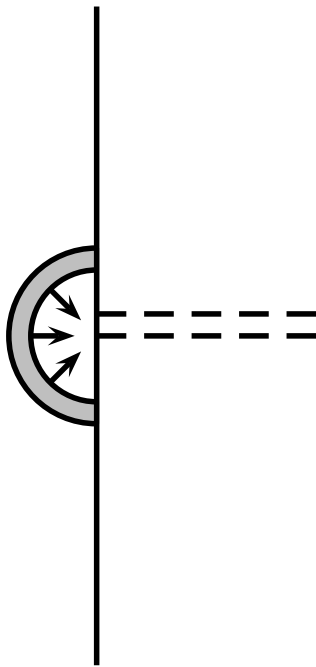
The **singular** behavior of the **far field** is coupled with the **none growing** behavior of the **near field at infinity** .



# The conclusion of the coupling

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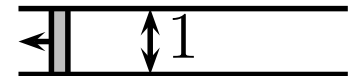
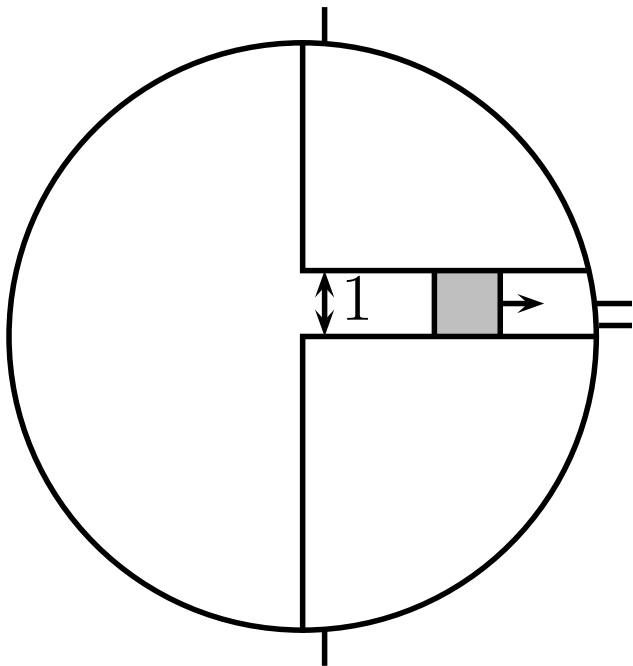
The **growing** behavior of the **near field at infinity** is coupled with the **none singular** behavior of the **far field**.



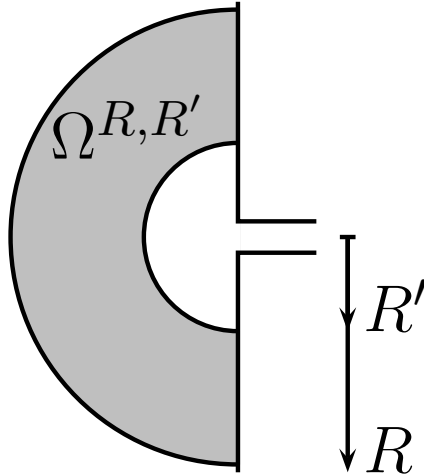
# The conclusion of the coupling

- The **far** field-**near** field coupling:
- The **near** field-**slot** field coupling:

The **growing** behavior of the **near field** is coupled with the **none growing** behavior of the **slot field** (derivative values)

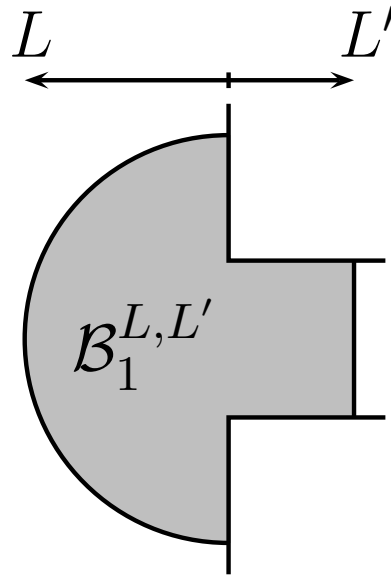
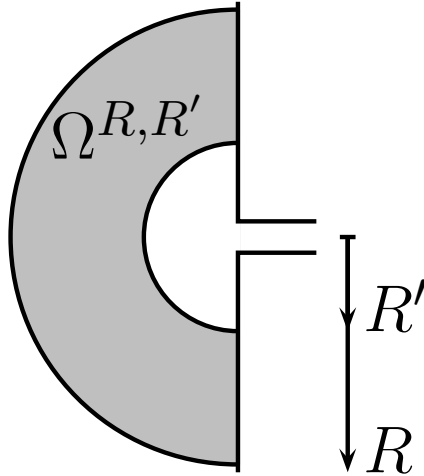


# Mathematical analysis



$$\left\| u^\epsilon - u^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \epsilon^i (\log \epsilon)^k u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \epsilon^{n+1} (\log \epsilon)^n \|f\|_{L^2(\Omega)}.$$

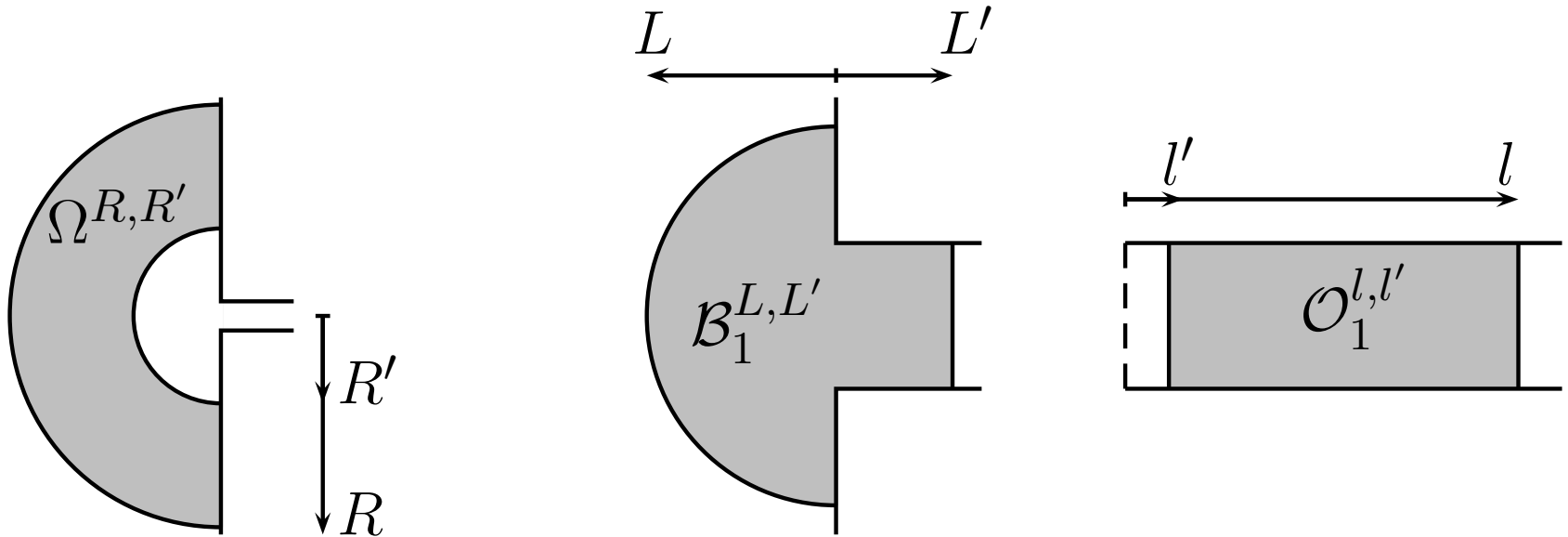
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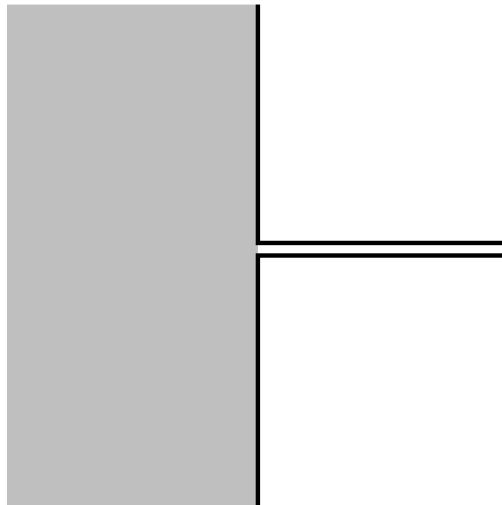
$$\left\| U^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k U_i^k \right\|_{H^1(\mathcal{O}_1^{l,l'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

# Idea of the proof

We want to define an **approximation**  $\tilde{u}_n^\varepsilon$  of the exact solution which **coincide** with:

- the **truncated** expansion of the **far field** away from the slot in the half space.

$$u_n^{H,\varepsilon}(x, y) = u^0(x, y) + \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k(x, y)$$

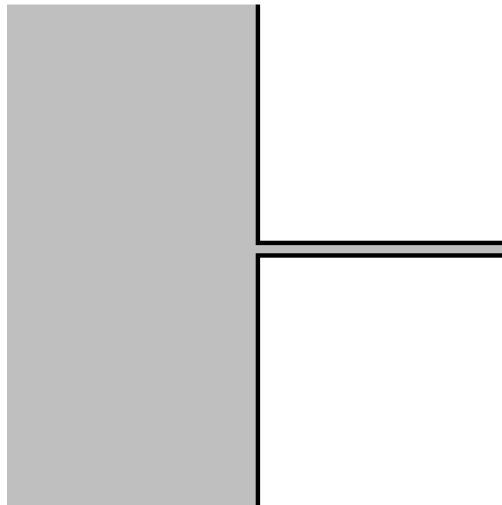


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We want to define an **approximation**  $\tilde{u}_n^\varepsilon$  of the exact solution which **coincide** with:

- The **truncated** expansion of the **near field** in the neighbourhood of the end of the slot

$$u_n^{N,\varepsilon}(x, y) = \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right)$$



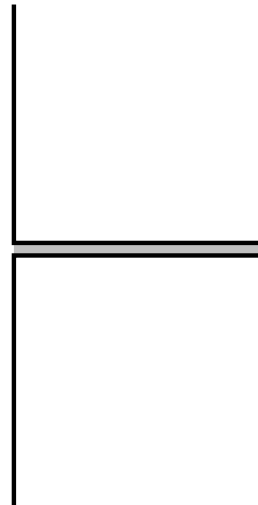


# Idea of the proof

We want to define an **approximation**  $\tilde{u}_n^\varepsilon$  of the exact solution which **coincide** with:

- the **truncated** expansion of the **slot field** far away in the slot

$$u_n^{S,\varepsilon}(x, y) = \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k U_i^k(x, \frac{y}{\varepsilon})$$



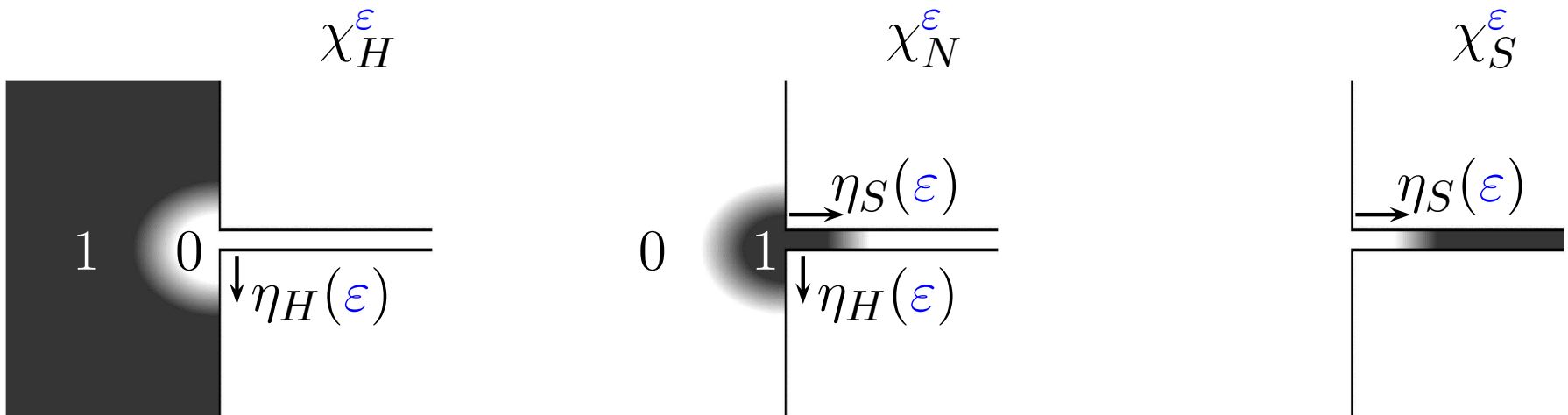
# Idea of the proof

Introduce a partition of unity

$$\tilde{u}_n^\varepsilon(r, \theta) = \chi_H^\varepsilon u_n^{H,\varepsilon} + \chi_N^\varepsilon u_n^{N,\varepsilon} + \chi_S^\varepsilon u_n^{S,\varepsilon}$$

with

$$\chi_H^\varepsilon + \chi_N^\varepsilon + \chi_S^\varepsilon = 1.$$

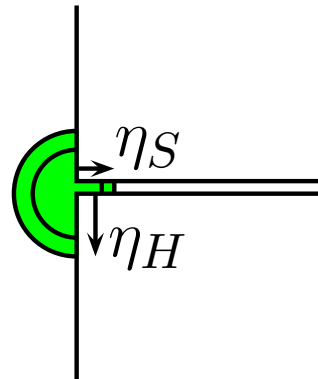


# Idea of the proof

The **error** equation:  $e_n^\varepsilon = \tilde{u}_n^\varepsilon - u^\varepsilon$

$$\left\{ \begin{array}{l} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)_n^\varepsilon + (\delta_{H-N})_n^\varepsilon + (\delta_{S-N})_n^\varepsilon, \quad \text{in } \Omega_\varepsilon, \\ \frac{\partial e_n^\varepsilon}{\partial n} = 0, \quad \text{on } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ is outgoing.} \end{array} \right.$$

$(\delta_N)_n^\varepsilon$  is related to the **approximation** of the **Helmholtz** equation by the **near** field

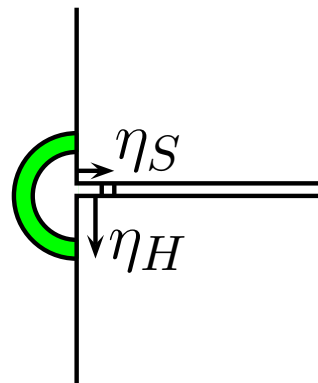


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$$\left\{ \begin{array}{l} \Delta e_n^\varepsilon + \omega^2 e_n^\varepsilon = (\delta_N)^\varepsilon_n + (\delta_{H-N})^\varepsilon_n + (\delta_{S-N})^\varepsilon_n, \quad \text{dans } \Omega_\varepsilon, \\ \frac{\partial e_n^\varepsilon}{\partial n} = 0, \quad \text{on } \partial\Omega_\varepsilon, \\ e_n^\varepsilon \text{ is outgoing.} \end{array} \right.$$

$(\delta_{H-N})^\varepsilon_n$  is related to the **matching error** between the **far** field and the **near** field

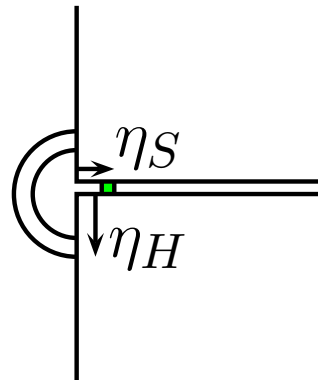


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$(\delta_{S-N})^\varepsilon_n$  is related to the **matching error** between the **slot** field and the **near** champ



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The **error** equation:  $e_n^\varepsilon = \tilde{u}_n^\varepsilon - u^\varepsilon$

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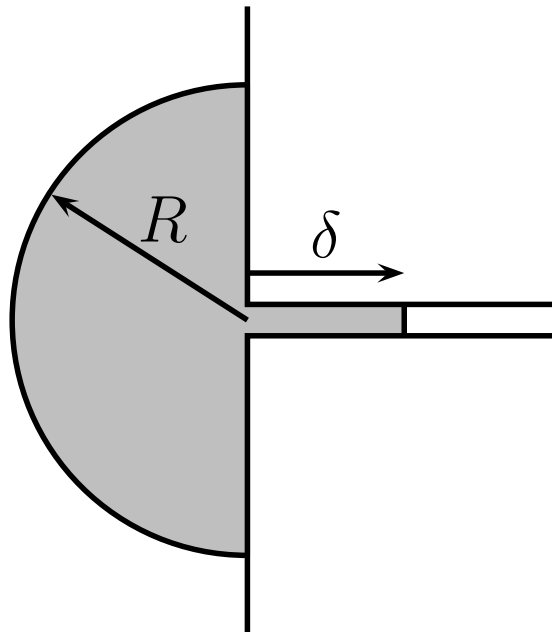
classical asymptotic techniques:

- **Stability**: proof by **contradiction** (Helmholtz)
- **Consistency**: A little bit more difficult (study of the singularities and of the growings by separation of variable)

# Idea of the proof

Global error estimates

$$\left\{ \begin{array}{l} \|u^\varepsilon - \tilde{u}_n^\varepsilon\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \left[ \left(\eta_H(\varepsilon)\right)^n + \left(\frac{\varepsilon}{\eta_H(\varepsilon)}\right)^n \right] \\ + C \left[ \left(\eta_S(\varepsilon)\right)^n + \left(\frac{\varepsilon}{\eta_S(\varepsilon)}\right)^n \right]. \end{array} \right.$$



# Idea of the proof

Global error estimate

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One can choose  $\eta_H(\varepsilon)$  and  $\eta_S(\varepsilon)$  to **optimize** this relation

$$\eta_H(\varepsilon) = \eta_S(\varepsilon) = \sqrt{\varepsilon}$$

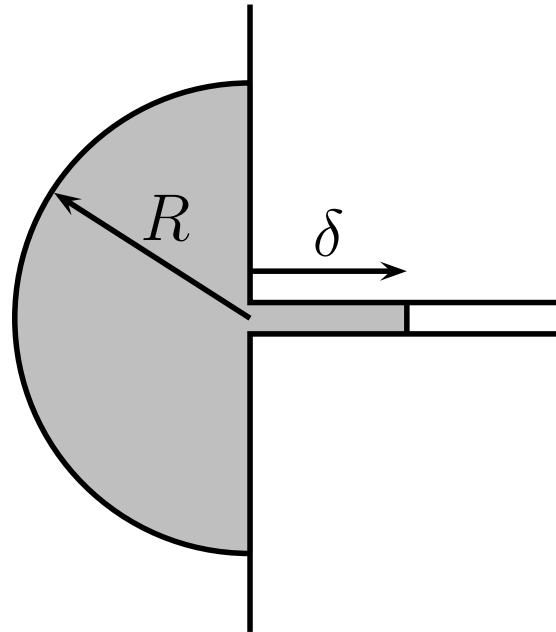
This leads to

$$\|u^\varepsilon - \tilde{u}_n^\varepsilon\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \varepsilon^{\frac{n}{2}}$$



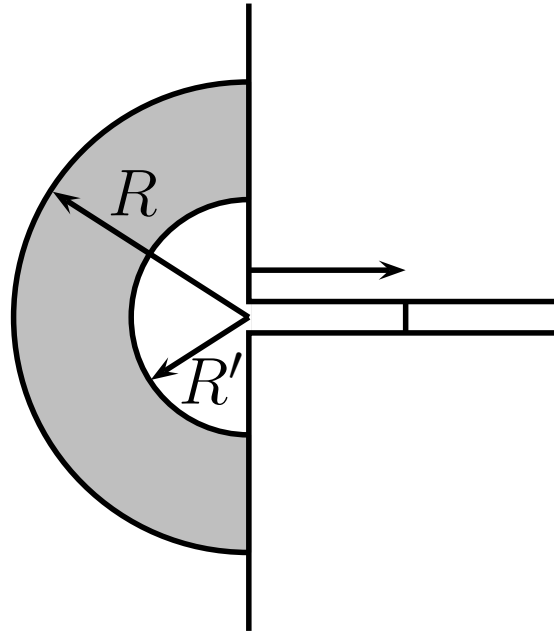
# Idea of the proof

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$$\|u^\varepsilon - \tilde{u}_n^\varepsilon\|_{H^1(\Omega_\varepsilon^{R,\delta})} \leq C \varepsilon^{\frac{n}{2}} \implies \|u^\varepsilon - \tilde{u}_n^\varepsilon\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{\frac{n}{2}}$$



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In the **far field** zone:

$$\tilde{u}_n^\varepsilon = u_n^{H,\varepsilon} = u^0 + \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k$$

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$$\left\{ \begin{array}{l} \|u^\varepsilon - u_{3n}^{H,\varepsilon}\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{\frac{3n}{2}} \\ \|u_{3n}^{H,\varepsilon} - u_n^{H,\varepsilon}\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} \log^n \varepsilon \end{array} \right.$$

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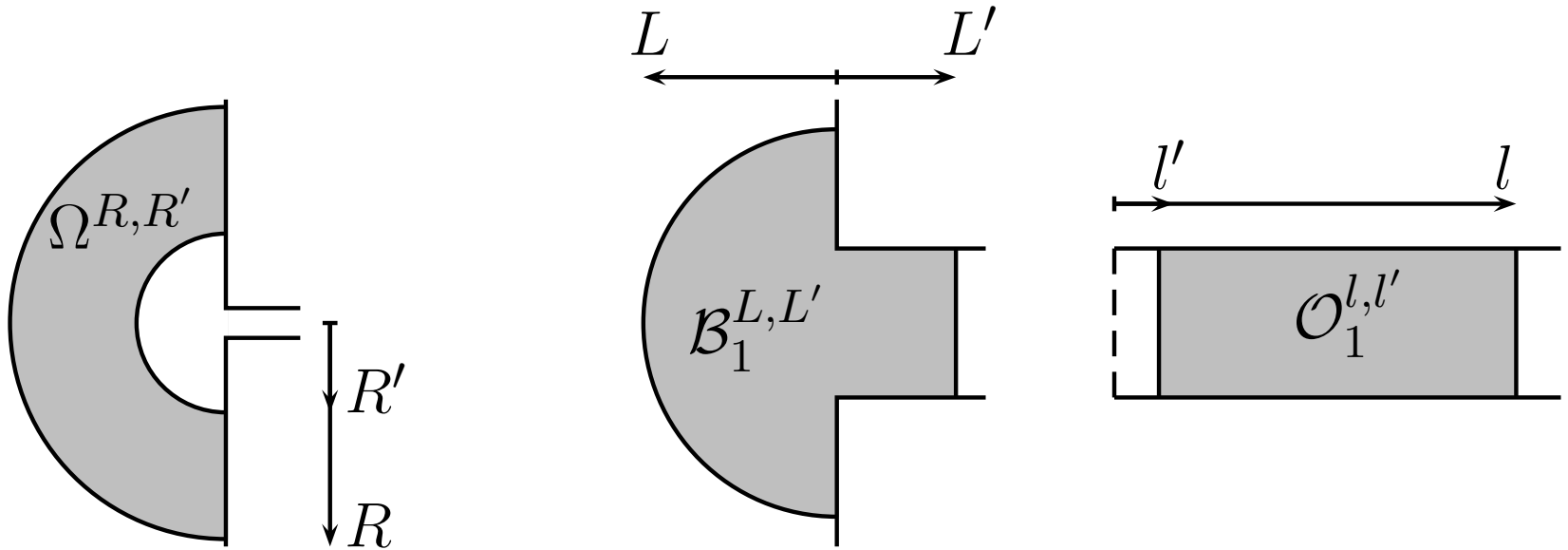
In the **far field** zone:

$$\tilde{u}_n^\varepsilon = u_n^{H,\varepsilon} = u^0 + \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k$$

$$\begin{cases} \|u^\varepsilon - u_{3n}^{H,\varepsilon}\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{\frac{3n}{2}} \\ \|u_{3n}^{H,\varepsilon} - u_n^{H,\varepsilon}\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} \log^n \varepsilon \end{cases}$$

One can conclude using the **triangular inequality**.

# Mathematical analysis



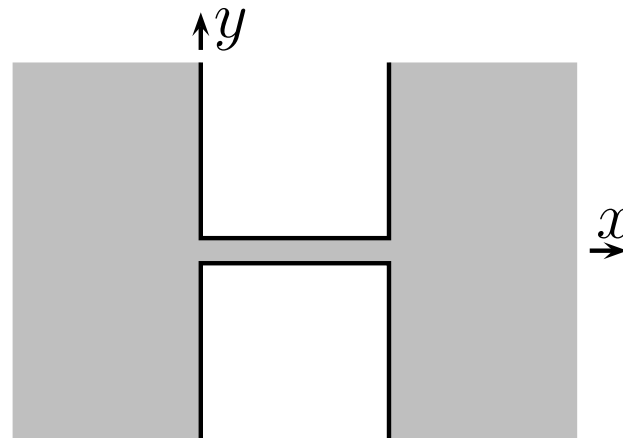
$$\left\| u^\varepsilon - u^0 - \sum_{i=1}^n \sum_{k=0}^{i-1} \varepsilon^i (\log \varepsilon)^k u_i^k \right\|_{H^1(\Omega^{R,R'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^n \|f\|_{L^2(\Omega)}.$$

$$\left\| u_p^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k (u_p)_i^k \right\|_{H^1(\mathcal{B}_1^{L,L'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

$$\left\| U^\varepsilon - \sum_{i=0}^n \sum_{k=0}^i \varepsilon^i (\log \varepsilon)^k U_i^k \right\|_{H^1(\mathcal{O}_1^{l,l'})} \leq C \varepsilon^{n+1} (\log \varepsilon)^{n+1} \|f\|_{L^2(\Omega)}.$$

# Perspectives

1. Mathematical analysis of the finite slot (**resonance** phenomena)



The difficulty: the **stability** result.

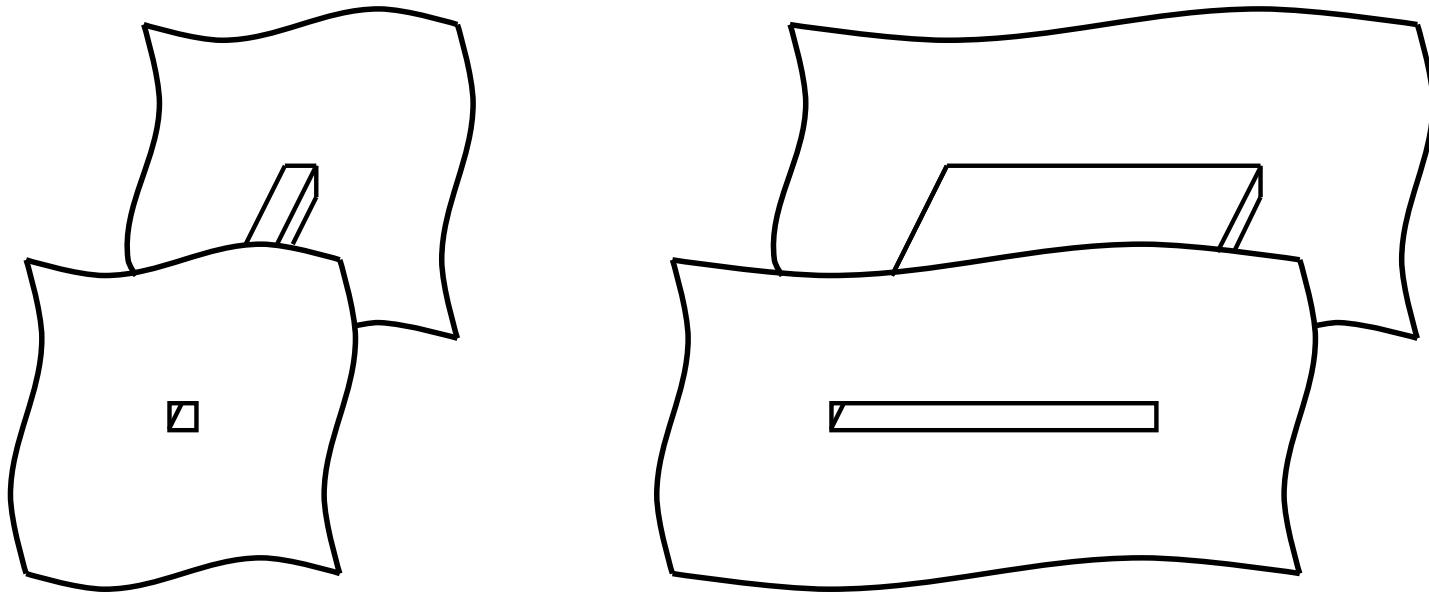
# Perspectives

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2. Comparison with the **multi-scale** technique



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2. Comparison with the **multi-scale** technique
3. The 3D **Maxwell** equation



# Perspectives

1. Mathematical analysis of the finite slot (**resonance** phenomena)
2. Comparison with the **multi-scale** technique
3. The 3D **Maxwell** equation
4. The **time domain** (evolution equation)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0.$$