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# Matching of Asymptotic Expansions for an eigenvalue problem with two cavities linked by a thin hole 

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## Motivation

- The casing of a combustion chamber of a Turbo engine of Turbo Meca (SAFRAN Group, - Acknowledgment -).

- Goal: Study the effects of small holes on resonance frequencies using Matching of Asymptotic Expansions (MAE).


## A non Exhaustive Bibliography

- Small holes: Rauch and Taylor (75), Tuck (75), Sanchez-Hubert and Sanchez-Palencia (82), Taflov (88), Bonnet-BenDhia, Drissi and Gmati (04), Mendez and Nicoud (08), Gadyl'shin (92).
- Dumbell problems: Beale (73), Jimbo and Morita (92) Brown, Hislop and Martinez (95), Arrieta (95), Anné (95).
- Matching of Asymptotic Expansions: Van Dyke (75), Il'in (92), Joly and Tordeux $(06,08)$
- quasi-mode and min-max: Bamberger and Bonnet (90), Dauge, Djurdjevic, Faou, and Rössle (99), Bonnaillie-Noël and Dauge (06).


## A Toy Problem

Problem: Find $u_{n}^{\delta} \in H_{0}^{1}\left(\Omega^{\delta}\right)$ and $\lambda_{n}^{\delta} \in \mathbb{R}$ satisfying

$$
\left\{\begin{array}{l}
-\nabla \cdot\left(a(x, y) \nabla u_{n}^{\delta}\right)(x, y)=\lambda_{n}^{\delta} b(x, y) u_{n}^{\delta}(x, y) \text { in } \Omega^{\delta}  \tag{1}\\
u_{n}^{\delta}(x, y)=0 \text { on } \partial \Omega^{\delta}
\end{array}\right.
$$

- with $a$ and $b$ two bounded positive regular functions with two sides

$$
\begin{equation*}
a_{\text {int }}(\mathbf{0}) \neq a_{\text {ext }}(\mathbf{0}) \text { and } a_{\text {int }}(\mathbf{0}) \neq a_{e x t}(\mathbf{0}) \tag{2}
\end{equation*}
$$

- The small parameter $\delta>0$ is the width of the hole in the domain $\Omega^{\delta}$.


Hypothesis: The eigenvalues of $\Omega$ are simple.

## Outline

(1) Definition of the Asymptotic Expansion
(2) Error Estimates
(3) Numerical Simulations

## The Matched Asymptotic Expansions Method

The MAE is based on a domain decomposition with overlapping


Far-field

The solution is described:

- with a far-field.
- with a near-field.


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## The Asymptotic Expansions: The Eigenvalue Expansion

- The second order asymptotic expansion reads

$$
\begin{equation*}
\lambda^{\delta}=\lambda^{0}+\delta \lambda^{1}+\delta^{2} \lambda^{2}+\underset{\delta \rightarrow 0}{o}\left(\delta^{2}\right) \tag{3}
\end{equation*}
$$

- polynomial gauge functions? not trivial: at fourth order

$$
\begin{equation*}
\lambda^{\delta}=\lambda^{0}+\delta \lambda^{1}+\delta^{2} \lambda^{2}+\delta^{3} \lambda^{3}+\delta^{4} \lambda^{4,0}+\delta^{4} \ln \delta \lambda^{4,1}+\underset{\delta \rightarrow 0}{o}\left(\delta^{4}\right) \tag{4}
\end{equation*}
$$

- The coefficients $\lambda^{i} \in \mathbb{R}$ and do not depend on $\delta$
- Necessity of a proof!


## The Asymptotic Expansions: The Far-field Expansion



Far-field (Asymptotic Expansion):

$$
u^{\delta}=u^{0}+\delta u^{1}+\delta^{2} u^{2}+\underset{\delta \rightarrow 0}{o}\left(\delta^{2}\right)
$$

The coefficients of the far-field asymptotic expansion $u^{i}$ will be

- defined in the far-field domain $\Omega$ : The limit of $\Omega^{\delta}$ when $\delta \rightarrow 0$,
- independent of $\delta$.


## Asymptotic Expansion: The Far-Field Expansion

They are solutions of the following problems

$$
\begin{cases}\text { Find } u^{0}: \Omega \rightarrow \mathbb{R} \text { and } \lambda^{0} \in \mathbb{R} \text { such that }  \tag{5}\\ \nabla \cdot\left(a \nabla u^{0}\right)+\lambda^{0} b u^{0}=0, & \text { in } \Omega, \\ u^{0}=0, & \text { on } \partial \Omega \backslash\{\mathbf{0}\} .\end{cases}
$$

$$
\begin{cases}\text { Find } u^{1}: \Omega \rightarrow \mathbb{R} \text { and } \lambda^{1} \in \mathbb{R} \text { such that } &  \tag{6}\\ \nabla \cdot\left(a \nabla u^{1}\right)+\lambda^{0} b u^{1}=-\lambda^{1} b u^{0}, & \text { in } \Omega, \\ u^{1}=0, & \text { on } \partial \Omega \backslash\{\mathbf{0}\} .\end{cases}
$$

$\begin{cases}\text { Find } u^{2}: \Omega \rightarrow \mathbb{R} \text { and } \lambda^{2} \in \mathbb{R} \text { such that } & \\ \nabla \cdot\left(a \nabla u^{2}\right)+\lambda^{0} b u^{2}=-\lambda^{2} b u^{0}-\lambda^{1} b u^{1}, & \text { in } \Omega, \\ u^{2}=0, & \text { on } \partial \Omega \backslash\{\mathbf{0}\} .\end{cases}$

- The problem (5) can be interpreted in the following way: $u^{0}$ is an eigenfunction in $\Omega$ associated to $\lambda^{0}$.
- The coefficients $u^{1}$ and $u^{2}$ are possibly singular at $\mathbf{x}=0$.
- The problems (6) and (7) do not uniquely define $u^{1}$ and $u^{2}$.


## Asymptotic Expansion: The Near-Field Expansion

Let $X=\frac{X}{\delta}, Y=\frac{y}{\delta}$, and we put $\Pi^{\delta}(X, Y)=u^{\delta}(\delta X, \delta Y)$.


Near-field (Asymptotic Expansion):

$$
\begin{equation*}
\Pi^{\delta}(X, Y)=\Pi^{0}(X, Y)+\delta \Pi^{1}(X, Y)+\delta^{2} \Pi^{2}(X, Y)+\underset{\delta \rightarrow 0}{o}\left(\delta^{2}\right) \tag{8}
\end{equation*}
$$

These functions will be

- defined on the near-field domain $\widehat{\Omega}$.
- independent of $\delta$.


## Asymptotic Expansion: The near-field expansion

They are solutions of elliptic problems

- with coefficients of the principal symbol that are two sides constants
- on the near-field domain $\widehat{\Omega}$

$$
\begin{equation*}
\widehat{\Omega}:=\mathbb{R}^{2} \backslash\left(\{0\} \times(]-\infty,-\frac{1}{2}[\cup], \frac{1}{2},+\infty[)\right) \tag{9}
\end{equation*}
$$

- The problems do not uniquely define $\Pi^{0}$, $\Pi^{1}$, and $\Pi^{2}$.


## The Matched Asymptotic Expansions Method: Theoretical context

In order to ensure the uniqueness of $u^{1}, u^{2}, \Pi^{0}, \Pi^{1}$, and $\Pi^{2}$ we derive additional matching conditions. We use the following procedure to obtain these extra conditions.
(1) We consider the far-field approximation of order $m$ written with $\mathbf{x}=\delta \mathbf{X}$

$$
\begin{equation*}
\sum_{i=0}^{m} \delta^{i} u^{i}(\delta \mathbf{X}) \tag{10}
\end{equation*}
$$

(2) Then this sum is expanded up to $o\left(\delta^{m}\right)$. This defines the $U_{m}^{i}$ in the $\mathbf{X}$ coordinates

$$
\begin{equation*}
\sum_{i=0}^{m} \delta^{i} u^{i}(\delta \mathbf{X})=\sum_{i=0}^{m} \delta^{i} U_{m}^{i}(\mathbf{X})+\underset{\delta \rightarrow 0}{o}\left(\delta^{m}\right) \tag{11}
\end{equation*}
$$

(3) The matching conditions are the following

$$
\begin{equation*}
\Pi^{i}-U_{m}^{i}=\underset{R \rightarrow+\infty}{o}\left(\frac{1}{R^{m-i}}\right) \quad \forall i \in \llbracket 0, m \rrbracket . \tag{12}
\end{equation*}
$$

## The Limits

Using the matching condition, we get the problem defining $u^{0}, \lambda^{0}$, and $\Pi^{0}$. Far-field. The function $u^{0}$ is an eigenfunction of the Dirichlet Laplace:

$$
\begin{cases}\text { Find } u^{0} \in H^{1}(\Omega) \text { such that } &  \tag{13}\\ \nabla \cdot\left(a \nabla u^{0}\right)+\lambda^{0} b u^{0}=0, & \text { in } \Omega \\ u^{0}=0, & \text { on } \partial \Omega\end{cases}
$$

To simplify the presentation $u^{0} \equiv 0$ in $\Omega_{\text {ext }}$

## The First Order Asymptotic Expansion

Using the matching condition, we get the problem defining $u^{1}, \lambda^{1}$, and $\Pi^{1}$.
Far-field.

$$
\begin{cases}\text { Find } u^{1} \in H^{1}(\Omega) \text { and } \lambda^{1} \in \mathbb{R} \text { such that }  \tag{14}\\ \nabla \cdot\left(a \nabla u^{1}\right)+\lambda^{0} b u^{1}=-\lambda^{1} u^{0}, & \text { in } \Omega, \\ u^{1}=0, & \text { on } \partial \Omega .\end{cases}
$$

Note that $u^{1}$ is regular. Due to the Fredholm alternative, the second member has to be orthogonal to $u^{0}$

$$
\begin{equation*}
\lambda^{1} \int_{\Omega} b(x, y)\left(u^{0}(x, y)\right)^{2} d x d y=0 \tag{15}
\end{equation*}
$$

We obtain $\lambda^{1}=0$.
The function $u^{1}$ is still defined up to its $u^{0}$-component, which is classical for eigenvalue problems. Then

$$
\begin{equation*}
u^{1}=\gamma u^{0} \text { in } \Omega_{\text {int }} \text { and } u^{1}=0 \text { in } \Omega_{e x t}, \quad \text { with } \gamma \in \mathbb{R} \tag{16}
\end{equation*}
$$

We add the condition $\int_{\Omega} b(x, y) u^{1}(x, y) u^{0}(x, y) d x d y=0 \Rightarrow u^{1} \equiv 0$.

## The Second Order Asymptotic Expansion

The matching procedure leads to the following problem

$$
\left\{\begin{array}{l}
\text { Find } u^{2}: \Omega \rightarrow \mathbb{R} \text { and } \lambda^{2} \in \mathbb{R} \text { such that } \\
\nabla \cdot\left(a \nabla u^{2}\right)+\lambda^{0} b u^{2}=-\lambda^{2} b u^{0}, \text { in } \Omega, \\
u^{2}=0, \text { on } \partial \Omega \backslash\{\mathbf{0}\} . \\
u_{\text {int }}^{2}(\mathbf{x})-\partial_{x} u_{\text {int }}^{0}(\mathbf{0}) \frac{1}{8} \frac{a_{\text {int }}(\mathbf{0})}{a_{\text {int }}(\mathbf{0})+a_{e x t}(\mathbf{0})} \frac{\sin (\theta)}{r}=\underset{r \rightarrow 0}{O}(1),  \tag{17}\\
u_{\text {ext }}^{2}(\mathbf{x})+\partial_{x} u_{\text {int }}^{0}(\mathbf{0}) \frac{1}{8} \frac{a_{e x t}(\mathbf{0})}{a_{\text {int }}(\mathbf{0})+a_{e x t}(\mathbf{0})} \frac{\sin (\theta)}{r}=\underset{r \rightarrow 0}{O}(1),
\end{array}\right.
$$

- Note that $u^{2}$ is singular $u^{2} \notin H^{1}(\Omega)$.
- Due to the singularity of $u^{2}$, the Fredholm alternative theory can not be directly applied to obtain $\lambda^{2}$.

Proposition: The problem (17) has solutions. Moreover if $\left(u^{2}, \lambda^{2}\right)$ and $\left(u_{*}^{2}, \lambda_{*}^{2}\right)$ are solutions, one has

$$
\lambda^{2}=\lambda_{*}^{2}=-\frac{\pi}{8} \frac{\left(a_{\text {int }}(\mathbf{0})\right)^{2}}{a_{\text {int }}(\mathbf{0})+a_{e x t}(\mathbf{0})} \frac{\left|\partial_{x} u_{i n t}^{0}(\mathbf{0})\right|^{2}}{\left\|u^{0}\right\|_{0}^{2}} \quad \text { and } \quad \exists \gamma \in \mathbb{R}: u_{*}^{2}-u^{2}=\gamma u^{0}
$$

## The Second Order Asymptotic Expansion

Sketch of proof: We introduce the auxiliary function $\omega^{2}$

$$
\begin{equation*}
\omega_{i n t, e x t}^{2}=u_{i n t, e x t}^{2}-\chi(r) \partial_{x} u_{i n t, e x t}^{0}(\mathbf{0}) \alpha_{i n t, e x t} \frac{\sin (\theta)}{r} \in H^{1}\left(\Omega_{i n t, e x t}\right) \tag{18}
\end{equation*}
$$

with $\chi$ the regular cut-off function:


Applying Fredholm alternative to $\omega^{2}$ which is regular, we obtain $\lambda^{2}$.

## Asymptotic Expansion of the Eigenvalues

## Theorem:

Let $\lambda^{0}$ be a simple eigenvalue of $\Omega$. For all $\delta>0$, there exists an eigenvalue $\lambda^{\delta}$ of the Dirichlet laplacian in $\Omega^{\delta}$, see (1), satisfying

$$
\begin{equation*}
\left|\lambda^{\delta}-\left(\lambda^{0}+\delta^{2} \lambda^{2}\right)\right| \leq C \delta^{3}|\ln (\delta)| \tag{19}
\end{equation*}
$$

with $\lambda^{2}$ given by

$$
\begin{cases}\lambda^{2}=-\frac{\pi}{8} \frac{\left(a_{\text {int }}(\mathbf{0})\right)^{2}}{a_{\text {int }}(\mathbf{0})+a_{\text {ext }}(\mathbf{0})} \frac{\left|\partial_{x} u_{\text {int }}^{0}(\mathbf{0})\right|^{2}}{\int_{\Omega} b\left(u^{0}\right)^{2}}, & \text { if } u_{\text {ext }}^{0}=0  \tag{20}\\ \lambda^{2}=-\frac{\pi}{8} \frac{\left(a_{\text {ext }}(\mathbf{0})\right)^{2}}{a_{\text {int }}(\mathbf{0})+a_{\text {ext }}(\mathbf{0})} \frac{\left|\partial_{x} u_{\text {ext }}^{0}(\mathbf{0})\right|^{2}}{\int_{\Omega} b\left(u^{0}\right)^{2}}, & \text { if } u_{\text {int }}^{0}=0\end{cases}
$$

Generalization of the result of Gadyl'shin (92) that has looked to the case of constant coefficients.

## Sketch of the Proof: A Quasi-Mode Approximation

- A mode $\left(u^{\delta}, \lambda^{\delta}\right) \in H_{0}^{1}\left(\Omega^{\delta}\right) \times \mathbb{R}$ satisfies $u^{\delta} \neq 0$ and

$$
\int_{\Omega^{\delta}} \nabla u^{\delta} \nabla v-\lambda^{\delta} \int_{\Omega^{\delta}} u^{\delta} v=0, \quad \forall v \in H_{0}^{1}\left(\Omega^{\delta}\right)
$$

- An $\varepsilon$-quasi-mode $\left(w^{\delta}, \mu^{\delta}\right) \in H_{0}^{1}\left(\Omega^{\delta}\right) \times \mathbb{R}$ satisifies $w^{\delta} \neq 0$ and $\left|\int_{\Omega^{\delta}} \nabla w^{\delta} \nabla v-\mu^{\delta} \int_{\Omega^{\delta}} w^{\delta} v\right| \leq \varepsilon\left\|w^{\delta}\right\|_{L^{2}\left(\Omega^{\delta}\right)}\|v\|_{L^{2}\left(\Omega^{\delta}\right)}, \quad \forall v \in H_{0}^{1}\left(\Omega^{\delta}\right)$.


## Lemma:

Let $\left(w^{\delta}, \mu^{\delta}\right)$ be an $\varepsilon$-quasi-mode. There exists a mode $\left(u^{\delta}, \lambda^{\delta}\right)$ :

$$
\left|\lambda^{\delta}-\mu^{\delta}\right| \leq \varepsilon
$$

## Sketch of the Proof: The Uniformly Valid Approximation

In this proof we consider the quasi-mode $\left(\widetilde{w}^{\delta}, \lambda^{0}+\delta^{2} \lambda^{2}\right)$ defined by

$$
\begin{align*}
\widetilde{w}^{\delta}=\chi^{\delta}\left(u^{0}+\delta u^{1}+\delta^{2} u^{2}\right)+\Psi\left(\widehat{\Pi}^{0}+\right. & \left.\delta \widehat{\Pi}^{1}+\delta^{2} \widehat{\Pi}^{2}\right) \\
& -\chi^{\delta} \Psi\left(\widehat{U}_{2}^{0}+\delta \widehat{U}_{2}^{1}+\delta^{2} \widehat{U}_{2}^{2}\right) . \tag{21}
\end{align*}
$$

with

- the cut-off function $\chi^{\delta}$ defined by $\chi^{\delta}(r, \theta):=\chi(r / \delta)$.
- the cut-off function $\Psi$ defined by $\Psi(r, \theta)=1-\chi(r)$;
- $\widehat{\Pi}^{i}$ the near-field terms expressed in the $\mathbf{x}$-coordinates, $\widehat{\Pi}^{i}(\mathbf{x})=\Pi^{i}(\mathbf{x} / \delta)$.
- $\widehat{U}^{i}$ the near-field behaviors expressed in the $\mathbf{x}$-coordinates, $\widehat{U}^{i}(\mathbf{x})=U^{i}(\mathbf{x} / \delta)$.


## Sketch of the Proof: Final Estimate

We obtain

$$
\left|\mathrm{a}\left(\widetilde{w}^{\delta}, v\right)-\left(\lambda^{0}+\delta^{2} \lambda^{2}\right)\left(\widetilde{w}^{\delta}, v\right)_{0}\right| \leq C \delta^{2}\left\|\widetilde{w}^{\delta}\right\|_{0}\|v\|_{0}, \quad \forall v \in H_{0}^{1}\left(\Omega^{\delta}\right)
$$

$\Rightarrow$ Existence of an eigenvalue $\lambda^{\delta}$ of the Dirichlet laplacian in $\Omega^{\delta}$, satisfying

$$
\lambda^{\delta}=\lambda^{0}+\delta^{2} \lambda^{2}+\underset{\delta \rightarrow 0}{O}\left(\delta^{2}\right)
$$

That is a suboptimal result.

To obtain an optimal result: use the third order asymptotic expansion

$$
\lambda^{\delta}=\lambda^{0}+\delta^{2} \lambda^{2}+\underset{\delta \rightarrow 0}{O}\left(\delta^{3} \ln \delta\right)
$$

## Do we have missed some eigenvalues?

- the eigenvalue $\lambda_{n}$
- the eigenvalue $\lambda_{n}^{\delta}$

The last result reveals:

- There exists a $\lambda_{n}^{\delta}$ in a small neighborhood of each $\lambda_{n}$
- $0<\lambda_{\sigma(n)}^{\delta} \leq \lambda_{n}$ for $\delta$ small enough.

Some questions

- only one $\lambda_{n}^{\delta}$ in the neighoborhood of $\lambda_{n}$ ?
- other $\lambda_{n}^{\delta}$ ?



## Do we have missed some eigenvalues?

- the eigenvalue $\lambda_{n}$
- the eigenvalue $\lambda_{n}^{\delta}$

The last result reveals:

- There exists a $\lambda_{n}^{\delta}$ in a small neighborhood of each $\lambda_{n}$
- $0<\lambda_{\sigma(n)}^{\delta} \leq \lambda_{n}$ for $\delta$ small enough

Some answers (with min-max)

- only one $\lambda_{n}^{\delta}$ in the neighoborhood of $\lambda_{n}$ ? no.
- other $\lambda_{n}^{\delta}$ ? no.



## Numerical Simulations: an Experiment

Let $\Omega^{\delta}$ be the domain defined by the following figure with

$$
\begin{equation*}
\left.\Omega_{\text {int }}=\right]-2,0[\times]-2.5,1.5\left[\quad \text { and } \quad \Omega_{\text {ext }}=\right] 0,2.5[\times]-1.5,1[. \tag{22}
\end{equation*}
$$



A computational mesh.
We recall that the eigenmodes of the limit problem in a domain $[0, a] \times[0, b]$ are

$$
\begin{equation*}
\lambda_{n m}=\pi \sqrt{\frac{n^{2}}{a^{2}}+\frac{m^{2}}{b^{2}}}, \quad U_{n m}(x, y)=\sin \left(\frac{n \pi}{a} x\right) \sin \left(\frac{m \pi}{b} y\right) . \tag{23}
\end{equation*}
$$

## Numerical Simulations: First Experiment



- the $\lambda_{n}^{2, \delta}$ are numerically computed
- The $\lambda_{n}$ and $\lambda_{n}^{2}$ are analatycally computed

| $n$ | $\lambda_{n}$ | $\lambda_{n}^{2}$ |
| :---: | :---: | :---: |
| 0 | 2.38 | -0.087 |
| 1 | 3.08 | -0.207 |
| 2 | 4.80 | -0.135 |
| 3 | 4.93 | -0.121 |
| 4 | 7.12 | -0.347 |
| 5 | 8.02 | -0.036 |

## The Relative Errors








## Conclusion

Main results: an approximation of eigenvalues with

- a theoretical background
- no mesh refinement required

Some publications:

- Bendali, Tizaoui, Tordeux, Vila, SAM Research Report (2009)
- Bendali, Huard, Tizaoui, Tordeux and Vila, CRAS (submitted)

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## Thank you!



