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Matching of Asymptotic Expansions for an eigenvalue problem with two cavities linked by a thin hole

Bendali, Tizaoui, Tordeux

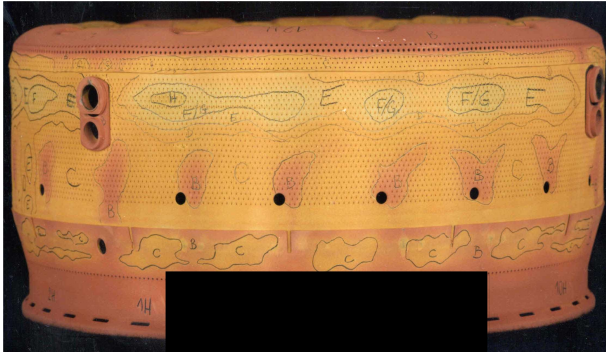
Institut de Mathématiques de Toulouse – INSA Toulouse

WAVES, 15 june 2009



Motivation

- The casing of a combustion chamber of a Turbo engine of Turbo Meca (SAFRAN Group, - [Acknowledgment](#) -).



- Goal: Study the effects of small holes on resonance frequencies using **Matching of Asymptotic Expansions (MAE)**.

A non Exhaustive Bibliography

- **Small holes:** Rauch and Taylor (75), Tuck (75), Sanchez-Hubert and Sanchez-Palencia (82), Taflov (88), Bonnet-BenDhia, Drissi and Gmati (04), Mendez and Nicoud (08), **Gadyl'shin (92)**.
- **Dumbell problems:** Beale (73), Jimbo and Morita (92) Brown, Hislop and Martinez (95), Arrieta (95), Anné (95).
- **Matching of Asymptotic Expansions:** Van Dyke (75), Il'in (92), Joly and Tordeux (06,08)
- **quasi-mode and min-max:** Bamberger and Bonnet (90), Dauge, Djurdjevic, Faou, and Rössle (99), Bonnaillie-Noël and Dauge (06).

A Toy Problem

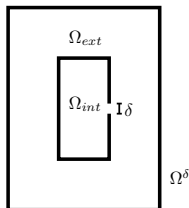
Problem: Find $u_n^\delta \in H_0^1(\Omega^\delta)$ and $\lambda_n^\delta \in \mathbb{R}$ satisfying

$$\begin{cases} -\nabla \cdot (a(x, y) \nabla u_n^\delta)(x, y) = \lambda_n^\delta b(x, y) u_n^\delta(x, y) \text{ in } \Omega^\delta, \\ u_n^\delta(x, y) = 0 \text{ on } \partial\Omega^\delta, \end{cases} \quad (1)$$

- with a and b two bounded positive regular functions with **two sides**

$$a_{int}(\mathbf{0}) \neq a_{ext}(\mathbf{0}) \text{ and } a_{int}(\mathbf{0}) \neq a_{ext}(\mathbf{0}) \quad (2)$$

- The small parameter $\delta > 0$ is the width of the hole in the domain Ω^δ .



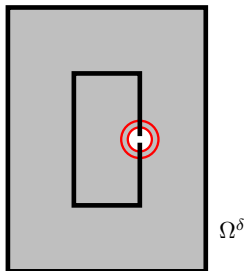
Hypothesis: The eigenvalues of Ω are simple.

Outline

- 1 Definition of the Asymptotic Expansion
- 2 Error Estimates
- 3 Numerical Simulations

The Matched Asymptotic Expansions Method

The MAE is based on a domain decomposition with overlapping



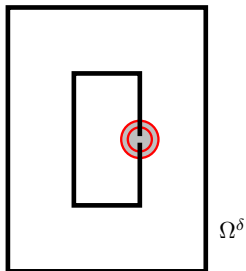
Far-field

The solution is described:

- with a **far**-field.
- with a **near**-field.

The Matched Asymptotic Expansions Method

The MAE is based on a domain decomposition with overlapping



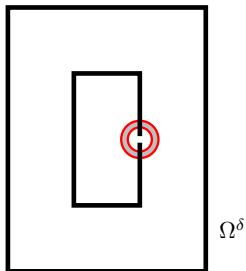
Near field

The solution is described:

- with a **far**-field.
- with a **near**-field.

The Matched Asymptotic Expansions Method

The MAE is based on a domain decomposition with overlapping



Matching zone

The solution is described:

- with a **far**-field.
- with a **near**-field.

The Asymptotic Expansions: The Eigenvalue Expansion

- The second order asymptotic expansion reads

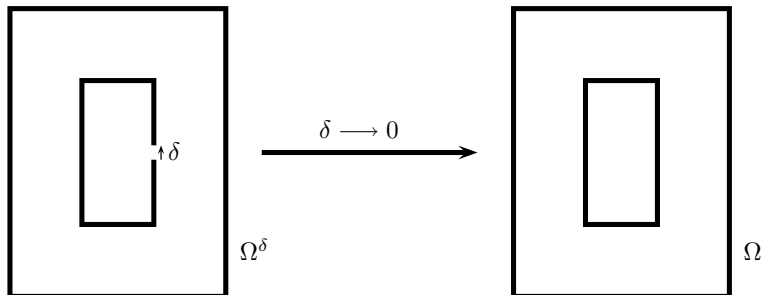
$$\lambda^\delta = \lambda^0 + \delta\lambda^1 + \delta^2\lambda^2 + \underset{\delta \rightarrow 0}{o}(\delta^2). \quad (3)$$

- polynomial gauge functions? not trivial: at fourth order

$$\lambda^\delta = \lambda^0 + \delta\lambda^1 + \delta^2\lambda^2 + \delta^3\lambda^3 + \delta^4\lambda^{4,0} + \delta^4 \ln \delta \lambda^{4,1} + \underset{\delta \rightarrow 0}{o}(\delta^4). \quad (4)$$

- The coefficients $\lambda^i \in \mathbb{R}$ and do not depend on δ
- Necessity of a proof!

The Asymptotic Expansions: The Far-field Expansion



Far-field (Asymptotic Expansion):

$$u^\delta = u^0 + \delta u^1 + \delta^2 u^2 + o_{\delta \rightarrow 0}(\delta^2)$$

The coefficients of the far-field asymptotic expansion u^i will be

- defined in the far-field domain Ω : The limit of Ω^δ when $\delta \rightarrow 0$,
- independent of δ .

Asymptotic Expansion: The Far-Field Expansion

They are solutions of the following problems

$$\left\{ \begin{array}{l} \text{Find } u^0 : \Omega \rightarrow \mathbb{R} \text{ and } \lambda^0 \in \mathbb{R} \text{ such that} \\ \nabla \cdot (a \nabla u^0) + \lambda^0 b u^0 = 0, \quad \text{in } \Omega, \\ u^0 = 0, \quad \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{array} \right. \quad (5)$$

$$\left\{ \begin{array}{l} \text{Find } u^1 : \Omega \rightarrow \mathbb{R} \text{ and } \lambda^1 \in \mathbb{R} \text{ such that} \\ \nabla \cdot (a \nabla u^1) + \lambda^0 b u^1 = -\lambda^1 b u^0, \quad \text{in } \Omega, \\ u^1 = 0, \quad \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{array} \right. \quad (6)$$

$$\left\{ \begin{array}{l} \text{Find } u^2 : \Omega \rightarrow \mathbb{R} \text{ and } \lambda^2 \in \mathbb{R} \text{ such that} \\ \nabla \cdot (a \nabla u^2) + \lambda^0 b u^2 = -\lambda^2 b u^0 - \lambda^1 b u^1, \quad \text{in } \Omega, \\ u^2 = 0, \quad \text{on } \partial\Omega \setminus \{\mathbf{0}\}. \end{array} \right. \quad (7)$$

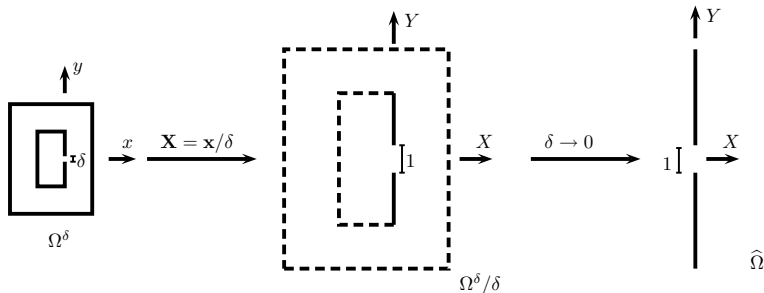
- The problem (5) can be interpreted in the following way:

u^0 is an eigenfunction in Ω associated to λ^0 .

- The coefficients u^1 and u^2 are possibly singular at $\mathbf{x} = 0$.
- The problems (6) and (7) do not uniquely define u^1 and u^2 .

Asymptotic Expansion: The Near-Field Expansion

Let $X = \frac{x}{\delta}$, $Y = \frac{y}{\delta}$, and we put $\Pi^\delta(X, Y) = u^\delta(\delta X, \delta Y)$.



Near-field (Asymptotic Expansion):

$$\Pi^\delta(X, Y) = \Pi^0(X, Y) + \delta \Pi^1(X, Y) + \delta^2 \Pi^2(X, Y) + o_{\delta \rightarrow 0}(\delta^2). \quad (8)$$

These functions will be

- defined on the near-field domain $\hat{\Omega}$.
- independent of δ .

Asymptotic Expansion: The near-field expansion

They are solutions of elliptic problems

- with coefficients of the principal symbol that are two sides constants
- on the near-field domain $\hat{\Omega}$

$$\hat{\Omega} := \mathbb{R}^2 \setminus \left(\{0\} \times \left(] - \infty, -\frac{1}{2}[U], \frac{1}{2}, +\infty[\right) \right). \quad (9)$$

- The problems do not uniquely define Π^0 , Π^1 , and Π^2 .

The Matched Asymptotic Expansions Method: Theoretical context

In order to ensure the uniqueness of u^1 , u^2 , Π^0 , Π^1 , and Π^2 we derive additional matching conditions. We use the following procedure to obtain these extra conditions.

- 1 We consider the far-field approximation of order m written with $\mathbf{x} = \delta \mathbf{X}$

$$\sum_{i=0}^m \delta^i u^i(\delta \mathbf{X}). \quad (10)$$

- 2 Then this sum is expanded up to $o(\delta^m)$. This defines the U_m^i in the \mathbf{X} coordinates

$$\sum_{i=0}^m \delta^i u^i(\delta \mathbf{X}) = \sum_{i=0}^m \delta^i U_m^i(\mathbf{X}) + o_{\delta \rightarrow 0}(\delta^m). \quad (11)$$

- 3 The matching conditions are the following

$$\Pi^i - U_m^i = o_{R \rightarrow +\infty} \left(\frac{1}{R^{m-i}} \right) \quad \forall i \in \llbracket 0, m \rrbracket. \quad (12)$$

The Limits

Using the **matching condition**, we get the problem defining u^0 , λ^0 , and Π^0 .

Far-field. The function u^0 is an eigenfunction of the Dirichlet Laplace:

$$\left\{ \begin{array}{ll} \text{Find } u^0 \in H^1(\Omega) \text{ such that} & \\ \nabla \cdot (a \nabla u^0) + \lambda^0 b u^0 = 0, & \text{in } \Omega, \\ u^0 = 0, & \text{on } \partial\Omega. \end{array} \right. \quad (13)$$

To simplify the presentation $u^0 \equiv 0$ in Ω_{ext}

The First Order Asymptotic Expansion

Using the **matching condition**, we get the problem defining u^1 , λ^1 , and Π^1 .

Far-field.

$$\begin{cases} \text{Find } u^1 \in H^1(\Omega) \text{ and } \lambda^1 \in \mathbb{R} \text{ such that} \\ \nabla \cdot (a \nabla u^1) + \lambda^0 b u^1 = -\lambda^1 u^0, & \text{in } \Omega, \\ u^1 = 0, & \text{on } \partial\Omega. \end{cases} \quad (14)$$

Note that u^1 is regular. Due to the Fredholm alternative, the second member has to be orthogonal to u^0

$$\lambda^1 \int_{\Omega} b(x, y) (u^0(x, y))^2 dx dy = 0. \quad (15)$$

We obtain $\lambda^1 = 0$.

The function u^1 is still defined up to its u^0 -component, which is classical for eigenvalue problems. Then

$$u^1 = \gamma u^0 \text{ in } \Omega_{int} \text{ and } u^1 = 0 \text{ in } \Omega_{ext}, \quad \text{with } \gamma \in \mathbb{R}. \quad (16)$$

We add the condition $\int_{\Omega} b(x, y) u^1(x, y) u^0(x, y) dx dy = 0 \Rightarrow u^1 \equiv 0$.

The Second Order Asymptotic Expansion

The **matching procedure** leads to the following problem

$$\left\{ \begin{array}{l} \text{Find } u^2 : \Omega \rightarrow \mathbb{R} \text{ and } \lambda^2 \in \mathbb{R} \text{ such that} \\ \nabla \cdot (a \nabla u^2) + \lambda^0 b u^2 = -\lambda^2 b u^0, \text{ in } \Omega, \\ u^2 = 0, \text{ on } \partial\Omega \setminus \{0\}. \\ u_{int}^2(\mathbf{x}) - \partial_x u_{int}^0(\mathbf{0}) \frac{1}{8} \frac{a_{int}(\mathbf{0})}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})} \frac{\sin(\theta)}{r} = O_{r \rightarrow 0}(1), \\ u_{ext}^2(\mathbf{x}) + \partial_x u_{int}^0(\mathbf{0}) \frac{1}{8} \frac{a_{ext}(\mathbf{0})}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})} \frac{\sin(\theta)}{r} = O_{r \rightarrow 0}(1), \end{array} \right. \quad (17)$$

- Note that u^2 is singular $u^2 \notin H^1(\Omega)$.
- Due to the singularity of u^2 , the Fredholm alternative theory can not be directly applied to obtain λ^2 .

Proposition: The problem (17) has solutions. Moreover if (u^2, λ^2) and (u_*^2, λ_*^2) are solutions, one has

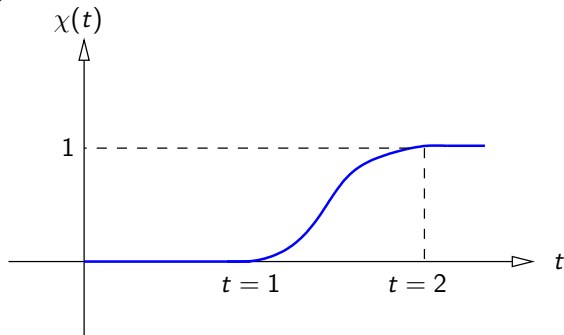
$$\lambda^2 = \lambda_*^2 = -\frac{\pi}{8} \frac{(a_{int}(\mathbf{0}))^2}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})} \frac{|\partial_x u_{int}^0(\mathbf{0})|^2}{\|u^0\|_0^2} \quad \text{and} \quad \exists \gamma \in \mathbb{R} : u_*^2 - u^2 = \gamma u^0.$$

The Second Order Asymptotic Expansion

Sketch of proof: We introduce the auxiliary function ω^2

$$\omega_{int,ext}^2 = u_{int,ext}^2 - \chi(r) \partial_x u_{int,ext}^0(\mathbf{0}) \alpha_{int,ext} \frac{\sin(\theta)}{r} \in H^1(\Omega_{int,ext}), \quad (18)$$

with χ the regular cut-off function:



Applying Fredholm alternative to ω^2 which is regular, we obtain λ^2 .

Asymptotic Expansion of the Eigenvalues

Theorem:

Let λ^0 be a simple eigenvalue of Ω . For all $\delta > 0$, there exists an eigenvalue λ^δ of the Dirichlet laplacian in Ω^δ , see (1), satisfying

$$\left| \lambda^\delta - (\lambda^0 + \delta^2 \lambda^2) \right| \leq C \delta^3 |\ln(\delta)| \quad (19)$$

with λ^2 given by

$$\begin{cases} \lambda^2 = -\frac{\pi}{8} \frac{(a_{int}(\mathbf{0}))^2}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})} \frac{|\partial_x u_{int}^0(\mathbf{0})|^2}{\int_{\Omega} b(u^0)^2}, & \text{if } u_{ext}^0 = 0, \\ \lambda^2 = -\frac{\pi}{8} \frac{(a_{ext}(\mathbf{0}))^2}{a_{int}(\mathbf{0}) + a_{ext}(\mathbf{0})} \frac{|\partial_x u_{ext}^0(\mathbf{0})|^2}{\int_{\Omega} b(u^0)^2}, & \text{if } u_{int}^0 = 0. \end{cases} \quad (20)$$

Generalization of the result of [Gadyl'shin \(92\)](#) that has looked to the case of constant coefficients.

Sketch of the Proof: A Quasi-Mode Approximation

- A mode $(u^\delta, \lambda^\delta) \in H_0^1(\Omega^\delta) \times \mathbb{R}$ satisfies $u^\delta \neq 0$ and

$$\int_{\Omega^\delta} \nabla u^\delta \nabla v - \lambda^\delta \int_{\Omega^\delta} u^\delta v = 0, \quad \forall v \in H_0^1(\Omega^\delta).$$

- An ε -quasi-mode $(w^\delta, \mu^\delta) \in H_0^1(\Omega^\delta) \times \mathbb{R}$ satisfies $w^\delta \neq 0$ and

$$\left| \int_{\Omega^\delta} \nabla w^\delta \nabla v - \mu^\delta \int_{\Omega^\delta} w^\delta v \right| \leq \varepsilon \|w^\delta\|_{L^2(\Omega^\delta)} \|v\|_{L^2(\Omega^\delta)}, \quad \forall v \in H_0^1(\Omega^\delta).$$

Lemma:

Let (w^δ, μ^δ) be an ε -quasi-mode. There exists a mode $(u^\delta, \lambda^\delta)$:

$$|\lambda^\delta - \mu^\delta| \leq \varepsilon$$

Sketch of the Proof: The Uniformly Valid Approximation

In this proof we consider the quasi-mode $(\tilde{w}^\delta, \lambda^0 + \delta^2 \lambda^2)$ defined by

$$\tilde{w}^\delta = \chi^\delta \left(u^0 + \delta u^1 + \delta^2 u^2 \right) + \Psi \left(\hat{\Pi}^0 + \delta \hat{\Pi}^1 + \delta^2 \hat{\Pi}^2 \right) - \chi^\delta \Psi \left(\hat{U}_2^0 + \delta \hat{U}_2^1 + \delta^2 \hat{U}_2^2 \right). \quad (21)$$

with

- the cut-off function χ^δ defined by $\chi^\delta(r, \theta) := \chi(r/\delta)$.
- the cut-off function Ψ defined by $\Psi(r, \theta) = 1 - \chi(r)$;
- $\hat{\Pi}^i$ the near-field terms expressed in the \mathbf{x} -coordinates, $\hat{\Pi}^i(\mathbf{x}) = \Pi^i(\mathbf{x}/\delta)$.
- \hat{U}^j the near-field behaviors expressed in the \mathbf{x} -coordinates, $\hat{U}^j(\mathbf{x}) = U^j(\mathbf{x}/\delta)$.

Sketch of the Proof: Final Estimate

We obtain

$$\left| \mathbf{a}(\tilde{w}^\delta, v) - (\lambda^0 + \delta^2 \lambda^2) (\tilde{w}^\delta, v)_0 \right| \leq C \delta^2 \|\tilde{w}^\delta\|_0 \|v\|_0, \quad \forall v \in H_0^1(\Omega^\delta).$$

⇒ Existence of an eigenvalue λ^δ of the Dirichlet laplacian in Ω^δ , satisfying

$$\lambda^\delta = \lambda^0 + \delta^2 \lambda^2 + \underset{\delta \rightarrow 0}{O}(\delta^2).$$

That is a **suboptimal** result.

To obtain an optimal result: use the **third order asymptotic expansion**

$$\lambda^\delta = \lambda^0 + \delta^2 \lambda^2 + \underset{\delta \rightarrow 0}{O}(\delta^3 \ln \delta)$$

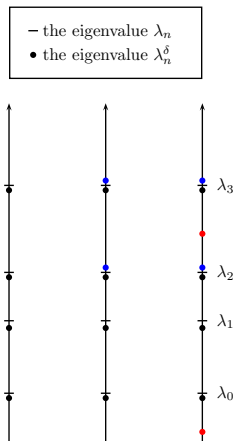
Do we have missed some eigenvalues?

The last result reveals:

- There exists a λ_n^δ in a small neighborhood of each λ_n
- $0 < \lambda_{\sigma(n)}^\delta \leq \lambda_n$ for δ small enough.

Some questions

- only one λ_n^δ in the neighborhood of λ_n ?
- other λ_n^δ ?



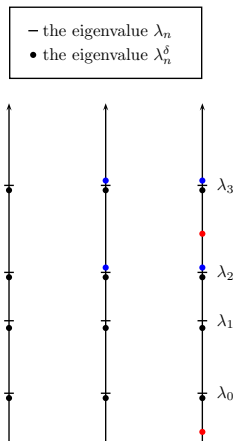
Do we have missed some eigenvalues?

The last result reveals:

- There exists a λ_n^δ in a small neighborhood of each λ_n
- $0 < \lambda_{\sigma(n)}^\delta \leq \lambda_n$ for δ small enough

Some answers (with **min-max**)

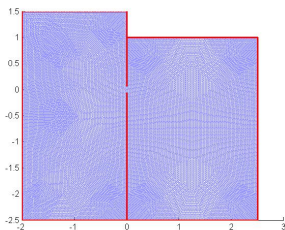
- only one λ_n^δ in the neighborhood of λ_n ? **no.**
- other λ_n^δ ? **no.**



Numerical Simulations: an Experiment

Let Ω^δ be the domain defined by the following figure with

$$\Omega_{int} =]-2, 0[\times]-2.5, 1.5[\quad \text{and} \quad \Omega_{ext} =]0, 2.5[\times]-1.5, 1[. \quad (22)$$

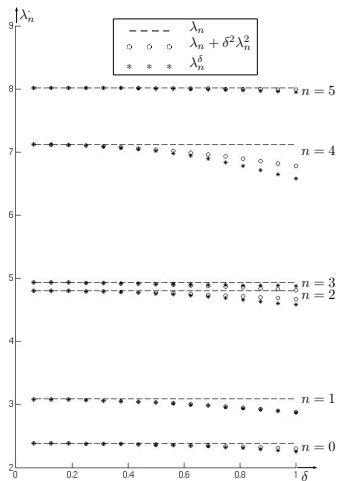


A computational mesh.

We recall that the eigenmodes of the limit problem in a domain $[0, a] \times [0, b]$ are

$$\lambda_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}, \quad U_{nm}(x, y) = \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right). \quad (23)$$

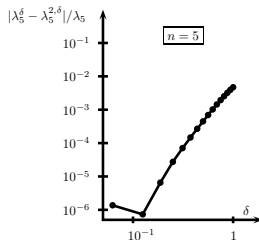
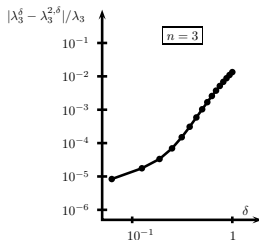
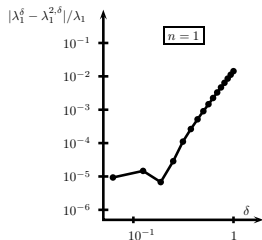
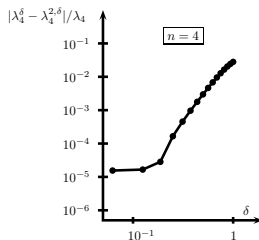
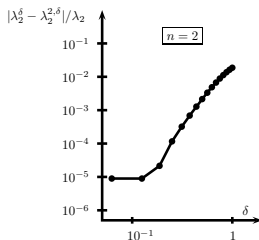
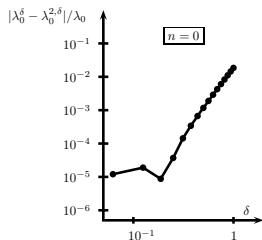
Numerical Simulations: First Experiment



- the λ_n^{δ} are numerically computed
- The λ_n and λ_n^2 are analytically computed

n	λ_n	λ_n^2
0	2.38	-0.087
1	3.08	-0.207
2	4.80	-0.135
3	4.93	-0.121
4	7.12	-0.347
5	8.02	-0.036

The Relative Errors



Conclusion

Main results: an approximation of eigenvalues with

- a theoretical background
- no mesh refinement required

Some publications:

- [Bendali, Tizaoui, Tordeux, Vila](#), SAM Research Report (2009)
- [Bendali, Huard, Tizaoui, Tordeux and Vila](#), CRAS (submitted)

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Thank you!

