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# On the Stock Estimation for a Harvested Fish Population

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## Abstract

We consider a stage-structured model of a harvested fish population and we are interested in the problem of estimating the unknown stock state for each class. The model used in this work to describe the dynamical evolution of the population is a discrete time system including a nonlinear recruitment relationship. To estimate the stock state, we build an observer for the considered fish model. This observer is an auxiliary dynamical system that uses the catch data over each time interval and gives a dynamical estimate of the stock state for each stage class. The observer works well even if the recruitment function in the considered model is not well known. The same problem for an age-structured model has been addressed in a previous work [26].

**Keywords:** Discrete-time system, Stage structured population models, Estimation, Harvested Fish Population, Observers.

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# 1 Introduction

Overfishing has negatively affected the stock of renewable marine resources. Analyzing the catch data during the last decade, it can be noticed that catch fishing has considerably decreased in many countries and some species are threatened with extinction. The collapse of catch fishing and the extinction of species can be explained in most cases by a gap in management policies due partly to the complex problem of controlling the dynamic evolution of marine resources. To improve the efficiency of exploiting stock and preventing the collapse of key fishery, it is essential to well understand the stock evolution mechanism and to have a good estimate of the population stock state. To overcome these types of problems in fishery management, mathematical models are very often used to describe the dynamic evolution of some fish population. With these models it becomes easier to observe and hence to regulate the dynamics of a given fish population.

In this paper, we are interested in the estimation problem of the unknown stock state of a harvested fish population whose dynamics are modeled by a discrete time system including a nonlinear stock recruitment relationship. As already mentioned in the abstract, the stock state estimation problem for a harvested fish population was first investigated [26] by considering an age-structured fish-population model. However determining the age for a fish population is a difficult task, it would be easier and more pertinent in practice to classify a fish population by other quantities that correlate with age such as size or weight. Moreover, most of the time the catch data that we have access to are generally evaluated in terms of size or weight. Hence a stage structured fish population model is more general and realistic than an age-structured one. The stage-structured fish model we consider here is the following (cf: [12], page 84 and 35):

$$\left\{ \begin{array}{l} x_1(k+1) = (1-p_1)s_1(k)x_1(k) + s_0f\left(\sum_{i=1}^n b_i x_i(k)\right) \\ x_2(k+1) = p_1s_1(k)x_1(k) + (1-p_2)s_2(k)x_2(k) \\ \quad \quad \quad \vdots \quad \quad \quad \vdots \\ x_{n-1}(k+1) = p_{n-2}s_{n-2}(k)x_{n-2}(k) + (1-p_{n-1})s_{n-1}(k)x_{n-1}(k) \\ x_n(k+1) = p_{n-1}s_{n-1}(k)x_{n-1}(k) + s_n(k)x_n(k) \end{array} \right. \quad (1)$$

where

- $n$  is the number of stage-classes.
- $x_i(k)$  is the number of individuals in stage class  $i$  at time  $k$ .
- $b_i$  is the fecundity rate in stage class  $i$ .
- $s_0$  is the individual survival rate of newborn.
- $s_i(k)$  is the individual survival rate in the stage class  $i$  at time  $k$ :  
 $s_i(k) = e^{-M_i - q_i E(k)\tau}$  (see Getz book: pp 221)

- $M_i$  is the natural mortality rate in stage class  $i$ .
- $q_i$  is the catchability coefficient of individuals in stage class  $i$ .
- $E(k)$  is the fishing effort at time  $k$ .
- $p_i$  is the proportion of individuals in stage class  $i$  that move to the next stage class  $i + 1$  after the time interval  $[k, k + 1[$ .
- $\tau$  is the length of harvesting season.
- $f$  is the stock-recruitment function.

The widely used recruitment functions [6, 12, 24, 30, 31] are:

$$\begin{aligned}
\text{Beverton and Holt} & f(x_0) = \alpha x_0 / (1 + \beta x_0) ; \\
\text{Ricker} & f(x_0) = \alpha x_0 e^{-\beta x_0} ; \\
\text{Power function} & f(x_0) = \alpha x_0^{1-\beta} ; \\
\text{Shepherd} & f(x_0) = \alpha x_0 / (1 + \beta x_0^c) , (c > 0). \\
\text{Deriso - Schnute} & f(x_0) = \alpha x_0 (1 - \beta x_0)^{1/\gamma} ; \\
\text{Saila - Lorda} & f(x_0) = \alpha x_0^\gamma e^{-\beta x_0}.
\end{aligned}$$

Where  $\alpha$ ,  $\beta$  and  $\gamma$  are positive parameters and  $x_0 = \sum_{i=1}^n b_i x_i$  is the number of newborns.

The recruitment function is very difficult to model because many factors such as water temperature, air speed and other natural phenomena can often perturb the growth of young individuals that are too vulnerable. One weakness of the model (1) is that most of the suggested mathematical expressions for the recruitment function are only valid for some restricted hypotheses. For example Beverton and Holt justified their recruitment function by making hypotheses on food competition whereas the Ricker recruitment function is based on eggs and juvenile predation. Hence the recruitment term in the first equation of model (1) is not well known and we can rewrite the model (1) by considering the recruitment function  $s_0 f\left(\sum_{i=1}^n b_i x_i(k)\right)$  as an unknown input  $r(x(k))$  to obtain the linear model with unknown single input  $r(x(k))$ :

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{pmatrix} = A(k) \begin{pmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{pmatrix} + \begin{pmatrix} r(x(k)) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (2)$$

where

$$A(k) = \begin{pmatrix} (1-p_1)s_1(k) & 0 & \dots & \dots & 0 \\ p_1s_1(k) & \ddots & \ddots & & \vdots \\ 0 & p_2s_2(k) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & (1-p_{n-1})s_{n-1}(k) & 0 \\ 0 & \dots & 0 & p_{n-1}s_{n-1}(k) & s_n(k) \end{pmatrix}$$

**Remark 1.** Note that if  $p_i = 1$ , for  $i = 1, \dots, n$  we obtain the age-structured model for which we have built an observer in [26].

While we focus on analyzing a fish-population model subjected to fishing action, quantifying the number of fish catch has great importance for the synthesis of the estimator stock states. Regarding the model we address above, the number of fish catch over the time period  $[k, k+1[$  in each stage class  $i$  is given by:

$$y_i(k) = \int_k^{k+\tau} q_i E(k) x_i(t) dt \quad (3)$$

(cf: Getz book 's, page 146).

The total fish catch number is then:

$$y(k) = \sum_{i=1}^n \int_k^{k+\tau} q_i E(k) x_i(t) dt \quad (4)$$

After few computations we can write

$$y(k) = \sum_{i=1}^n \frac{q_i E(k)}{q_i E(k) + M_i} \left( 1 - e^{(-M_i - q_i E(k))\tau} \right) x_i(k) \quad (5)$$

Let  $C_i(k) = \frac{q_i E(k)}{q_i E(k) + M_i} \left( 1 - e^{(-M_i - q_i E(k))\tau} \right)$ , then  $y(k)$  can be written as:

$$y(k) = \sum_{i=1}^n C_i(k) x_i(k) \quad (6)$$

To summarize, the model we consider in this work is:

$$\begin{cases} x(k+1) = A(k)x(k) + Dr(x) \\ y(k) = C(k)x(k) \end{cases} \quad (7)$$

where  $A(k)$  is defined above,  $C(k) = (C_1(k), \dots, C_n(k))$  and  $D = (1, 0, \dots, 0)^t$ .

Now we assume that the model with unknown single input (2), which can be written as (7), describes successfully the dynamics of the considered harvested fish population. To compute the state  $x(k)$  at time  $k$ , we need to know the value of the state  $x(k_0)$  at a given time  $k_0 < k$ . However the model does not give any information about the unknown state value  $x(k_0)$ . The model only provides information about relation between two consecutive states  $x(k)$  and  $x(k+1)$ . The model also gives measure of captures over time interval  $[k, k+1[$ . To make the model (2) useful, we have to find how to estimate the unknown value  $x(k)$ . This is the main concern we will address in this paper. To achieve this goal, we will use a tool from control theory called observer. We construct another dynamical system whose state  $\hat{x}$  will provide an estimate of the unknown state of the considered model and this will be true regardless of the initial condition of the observer. We briefly recall the definition of an *observer* in control theory. Let us consider the following dynamic system:

$$\begin{cases} x(k+1) = \mathcal{F}(x(k), u(k)) \\ y(k) = \mathcal{H}(x(k), u(k)). \end{cases} \quad (8)$$

where  $x(k) \in \mathbb{R}^n$  is the state of the system,  $u(k) \in U \subset \mathbb{R}^m$  is the input or the control and  $y(k) \in \mathbb{R}^q$  is the measurable output at time  $k$ .

**Definition 1.** *An observer for the dynamical system (8) is another dynamical system  $\Sigma$  whose inputs are the inputs and the outputs of system (8) and whose state  $\hat{x}(k)$  satisfies  $x(k) - \hat{x}(k)$  tends to zero as time  $k$  goes to infinity and must remain small if it starts small.*

$\Sigma$  is an exponential observer of (8) if there exists  $\rho < 1$  in such a way that for all  $k \geq 0$  and for all initial conditions  $(x(0), \hat{x}(0))$ , one has

$$| \hat{x}(k) - x(k) | \leq \rho^k | \hat{x}(0) - x(0) | .$$

Usually, a candidate observer  $\Sigma$  for system (8) is given by

$$\begin{cases} z(k+1) = \mathcal{G}(x(k), u(k), y(k)), \\ \hat{x}(k) = \mathcal{J}(z(k), y(k)). \end{cases} \quad (9)$$

More details on observer notions can be found in [16, 34]. Observer theory for linear discrete-time system with known inputs is well established. For linear invariant-time systems, the synthesis of the observer is completely given by the Luenberger method [23]. For linear time varying systems, several robust algorithms have been developed, one can cite for instance [1, 3, 5, 8, 20, 21]. In the special case of uncertain systems the synthesis of observers is still a very difficult task. The Kalman filter [2] is the most classical approach used to build observers for uncertain systems but the optimal filter require restricted hypothesis. Another approach which is based on dynamical interval analysis exists in the literature [9, 29], the idea consist in computing an interval known as "interval observer" which contains the state variable we need to estimate. Several algorithms on the synthesis of "interval observer" has been proposed in the literature for continuous systems [22, 28] and for discrete-time system [17, 32]. Some algorithms have also been developed to solve

the estimation problem for some classes of system with unknown inputs [7, 10, 13, 14, 15, 18, 19, 27, 33, 35]. Some of these observers carry out only the system state estimation, while other observers allow the estimation of the states as well as the unknown inputs of the system. One of the techniques used for the synthesis of the observer that estimate the states and the unknown inputs of a system consists in regrouping the state vector and the unknown inputs of the system in an augmented state-vector and in building an estimator for this augmented system [15, 19, 37]. To construct an observer for system with unknown inputs, one can use an approach that consists in eliminating the unknowns inputs in the equations of the system [14]. However, most of the available results concern linear time invariant systems. Here we are interested in the design of a global observer for the fishery system (2) which is a linear time varying system with known input  $E(k)$  and unknown input  $r(x(k))$ .

## 2 Observer for linear time varying systems with unknown inputs

We consider the following time-varying system:

$$\begin{cases} x(k+1) &= \Gamma(k)x(k) + \Lambda\zeta, \\ y(k) &= \theta(k)x(k), \end{cases} \quad (10)$$

where  $x(k)$  and  $y(k)$  are respectively the state and the measurable output at time  $k$ ,  $\Gamma(k)$  and  $\theta(k)$  are respectively time varying matrix,  $\Lambda$  is a column vector and  $\zeta$  is a scalar unknown input. In [13] M. Hou and P.C. Muller have developed a technique to synthesize observers for linear continuous time-invariant systems with unknown inputs. We adapt this technique for linear discrete time-varying systems with unknown inputs of the form (10).

Let  $y(k)$  be the measurable output of the system (10), we have

$$\begin{aligned} y(k+1) &= \theta(k+1)x(k+1) \\ &= \theta(k+1)(\Gamma(k)x(k) + \Lambda\zeta) \\ &= \theta(k+1)\Gamma(k)x(k) + \theta(k+1)\Lambda\zeta \end{aligned}$$

Assume that  $\theta(k+1)\Lambda$  is invertible  $\forall k \geq 0$ . Thus we obtain:

$$\zeta = (\theta(k+1)\Lambda)^{-1}(y(k+1) - \theta(k+1)\Gamma(k)x(k))$$

Then we have:

$$\begin{aligned} x(k+1) &= \Gamma(k)x(k) + \Lambda\zeta \\ &= \Gamma(k)x(k) + \Lambda\left((\theta(k+1)\Lambda)^{-1}(y(k+1) - \theta(k+1)\Gamma(k)x(k))\right) \\ &= \Gamma(k)x(k) + \Lambda(\theta(k+1)\Lambda)^{-1}y(k+1) - \Lambda(\theta(k+1)\Lambda)^{-1}\theta(k+1)\Gamma(k)x(k) \\ &= \left(\Gamma(k) - \Lambda(\theta(k+1)\Lambda)^{-1}\theta(k+1)\Gamma(k)\right)x(k) + \Lambda(\theta(k+1)\Lambda)^{-1}y(k+1) \\ &= \left(I_{n \times n} - \Lambda(\theta(k+1)\Lambda)^{-1}\theta(k+1)\right)\Gamma(k)x(k) + \Lambda(\theta(k+1)\Lambda)^{-1}y(k+1). \end{aligned}$$

We define:

$$\bar{\Gamma}(k) = \left( I_{n \times n} - \Lambda(\theta(k+1)\Lambda)^{-1}\theta(k+1) \right) \Gamma(k).$$

Thus we obtain:

$$x(k+1) = \bar{\Gamma}(k)x(k) + \Lambda(\theta(k+1)\Lambda)^{-1}y(k+1)$$

A candidate observer for system (10) is then given by:

$$\begin{cases} z(k+1) &= \left( \bar{\Gamma}(k) - L(k)\theta(k) \right) z(k) + \left( L(k) + \left( \bar{\Gamma}(k) - L(k)\theta(k) \right) \Lambda(\theta(k)\Lambda)^{-1} \right) y(k). \\ \hat{x}(k) &= z(k) + \Lambda(\theta(k)\Lambda)^{-1}y(k). \end{cases} \quad (11)$$

Let  $e(k) = x(k) - \hat{x}(k)$  the estimation error, the error dynamics are described by:

$$\begin{aligned} e(k+1) &= x(k+1) - \hat{x}(k+1) \\ &= \bar{\Gamma}(k)x(k) + \Lambda(\theta(k+1)\Lambda)^{-1}y(k+1) - z(k+1) - \Lambda(\theta(k+1)\Lambda)^{-1}y(k+1) \\ &= \bar{\Gamma}(k)x(k) - \left( \bar{\Gamma}(k) - L(k)\theta(k) \right) z(k) - \left( L(k) + \left( \bar{\Gamma}(k) - L(k)\theta(k) \right) \Lambda(\theta(k)\Lambda)^{-1} \right) y(k) \\ &= \left( \bar{\Gamma}(k) - L(k)\theta(k) \right) x(k) - \left( \bar{\Gamma}(k) - L(k)\theta(k) \right) z(k) \\ &\quad - \left( \bar{\Gamma}(k) - L(k)\theta(k) \right) \Lambda(\theta(k)\Lambda)^{-1}y(k) \\ &= \left( \bar{\Gamma}(k) - L(k)\theta(k) \right) (x(k) - \hat{x}(k)) \\ e(k+1) &= \left( \bar{\Gamma}(k) - L(k)\theta(k) \right) e(k) \end{aligned}$$

To summarize, we have the following result

**Proposition 1.** *Assume that the matrix  $\theta(k)\Lambda$  is invertible for all  $k$ . If there exists a matrix  $L(k)$  in such a way that the matrix  $\bar{\Gamma}(k) - L(k)\theta(k)$  satisfies  $\|\bar{\Gamma}(k) - L(k)\theta(k)\| \leq \delta$  for some  $\delta < 1$  then system (11) is a global exponential observer for system (10), i.e., the estimation error  $e(k)$  tends exponentially fast towards zero as  $k$  tends to  $+\infty$ .*

A sufficient condition for the existence of such a matrix  $L(k)$  is that the pair  $[\bar{\Gamma}(k), \theta(k)]$  is uniformly observable. We recall here what does uniform observability mean. Consider the system  $x(k+1) = \bar{\Gamma}(k)x(k)$  with measurable output  $y(k) = \theta(k)x(k)$ . Let  $\Psi_{i,k}$  for  $i \geq k$  be the transition matrix for the system  $x(k+1) = \bar{\Gamma}(k)x(k)$ , i.e.

$$\Psi_{i,i} = I_{n \times n} \text{ and } \Psi_{i,k} = \bar{\Gamma}(i-1)\bar{\Gamma}(i-2) \dots \bar{\Gamma}(k) \text{ for } i > k.$$

The observability Gramian for the system is defined as

$$G_{k+t,k} = \sum_{i=k}^{k+t} \Psi_{i,k}^T \theta(i)^T \theta(i) \Psi_{i,k} \text{ (where } A^T \text{ is the transpose matrix of } A\text{)}.$$

The pair  $[\bar{\Gamma}(k), \theta(k)]$  is uniformly observable if there exist a positive real  $\beta > 0$  and a positive integer  $t > 0$  such that one has for all  $k \geq 0$

$$G_{k+t,k} \geq \beta I_{n \times n}. \quad (12)$$

If the pair  $[\bar{\Gamma}(k), \theta(k)]$  is uniformly observable and the matrices  $\bar{\Gamma}(k), \theta(k)$  are bounded above then a possible choice for the gain matrix  $L(k)$  that makes the error equation  $e(k+1) = (\bar{\Gamma}(k) - L(k)\theta(k))e(k)$  globally exponentially stable is given by the following formula (see [25] for instance):

$$L(k) = \bar{\Gamma}(k) \Psi_{k,k-t-1} G_{k,k-t-1}^{-1} \Psi_{k,k-t-1}^T \theta(k)^T \quad (13)$$

### 3 Stock state estimation using an observer

In this section we shall apply the method developed in Section 2 in order to construct an observer that will provide an estimate of the state of system (2) using only the measurable output given by equation (5).

System (7) has the same form as system (10) with:

$$\Gamma(k) = A(k), \quad \Lambda = D, \quad \zeta = r(x(k)) \quad \text{and} \quad \theta(k) = C(k)$$

We assume that  $q_1 \neq 0$ . Therefore  $C(k)D = C_1(k) = \frac{q_1 E(k)}{q_1 E(k) + M_1} \left(1 - e^{(-M_1 - q_1 E(k))\tau}\right) \neq 0$  for all  $k > 0$ , so that  $C(k)D$  is invertible for all  $k > 0$ .

As in Section 2 we define:

$$\bar{A}(k) = \left(I_{n \times n} - D (C(k+1)D)^{-1} C(k+1)\right) A(k) = \left(I_{n \times n} - \frac{1}{C_1(k+1)} D C(k+1)\right) A(k).$$

Therefore a candidate observer for system (2) (or system (7)) is given by:

$$\begin{cases} z(k+1) &= \left(\bar{A}(k) - L(k)C(k)\right) z(k) + \left(L(k) + \left(\bar{A}(k) - L(k)C(k)\right) D (C(k)D)^{-1}\right) y(k) \\ \hat{x}(k) &= z(k) + D (C(k)D)^{-1} y(k) \end{cases} \quad (14)$$

To compute the appropriate matrix gain  $L(k)$ , one can use the formula (13). However, one has first to check the uniform observability of the pair  $[\bar{A}(k), C(k)]$  which is not an easy task due to the large number of parameters and when the condition (12) is satisfied, the formula (13) gives a huge expression for the matrix  $L(k)$ . These computations can be done when the parameter numerical values as well as the expression of the fishing effort are available. But here we prefer to give a simple expression for the observer, so, instead of computing  $L(k)$ , we just choose  $L(k) = 0$  and impose a weak condition on the fishing effort in such a way that the estimation error equation  $e(k+1) = \bar{A}(k) e(k)$  is exponentially stable. Therefore the candidate observer (14) becomes:



**Remark 4.** When the natural mortality rates are all equal, it is possible to give a weaker condition on the minimal value of the fishing effort that ensures the convergence of the observer.

**Proposition 3.** Assume that  $q_1 \leq q_i \forall i = 2, \dots, n$ , and moreover that the natural mortality coefficient is the same for all stages, that is,  $M_1 = M_2 = \dots = M_n = M$ . Then, the system (16) is a global exponential observer for the system (2) if

$$E_{min} > \frac{1}{q_1} \left( \frac{1}{\tau} \text{Log} \left[ \frac{q_{max}}{q_1} + 1 \right] - M \right).$$

**Remark 5.** The available catch data usually give the seasonal biomass yield, i.e., the total weight of the fishes caught during a season. If we denote  $w_i$  the mean weight of individuals of stage  $i$ ,  $m_i(k)$  the biomass of stage  $i$  at time  $k$  and  $F_i(k) = q_i E(k)$  the fishing mortality rate for stage  $i$ , then the measurable output (seasonal biomass yield) can be expressed as follows:

$$\begin{aligned} Y(k) &= \sum_{i=1}^n \frac{F_i(k)}{M_i + F_i(k)} (1 - e^{-(M_i - F_i(k))\tau}) w_i x_i(k) \\ &= \sum_{i=1}^n \frac{F_i(k)}{M_i + F_i(k)} (1 - e^{-(M_i - F_i(k))\tau}) m_i(k) \\ &= \sum_{i=1}^n C_i(k) m_i(k) \end{aligned} \tag{17}$$

The equation (16) of the observer can be modified in order to estimate the biomass  $m_i(k) = w_i x_i(k)$  for each stage instead of the number of individuals in each stage by replacing in (16)  $y(k)$  by  $Y(k)$  and  $\hat{x}_i(k)$  by  $\hat{m}_i(k)$ .

## 4 Numerical example

We now give some simulations to illustrate the efficiency of the observer (16). We suppose that the dynamics of a harvested population can be modeled by a three-stages model with a Ricker stock-recruitment function:

$$\begin{cases} x_1(k+1) &= (1 - p_1) s_1(k) x_1(k) + \alpha \sum_{i=1}^3 b_i x_i(k) e^{-\beta(\sum_{i=1}^3 b_i x_i(k))}, \\ x_2(k+1) &= p_1 s_1(k) x_1(k) + (1 - p_2) s_2(k) x_2(k), \\ x_3(k+1) &= p_2 s_2(k) x_2(k) + s_3(k) x_3(k). \end{cases} \tag{18}$$

We assume that the biological parameter values as well as the fishing strategy are known and are given by:

Parameters of the Ricker function	$\alpha = 1, \beta = 0.0002,$
Fecundity parameters	$b = [2 \ 5 \ 7],$
Catchability coefficients	$q = [0.24 \ 0.36 \ 0.42],$
Natural mortality rates	$M = [0.2 \ 0.2 \ 0.2],$
Transfer coefficients	$p_1 = 0.1, p_2 = 0.2$
Length of harvesting season	$\tau = 1/2,$
Fishing effort	$E(k) = 3 + \sin(k)$

Recall that since the value of  $(x_1(0), x_2(0), x_3(0))$  is not known, one can not use system (18) to evaluate  $(x_1(k), x_2(k), x_3(k))$  for a given positive time  $k$ . It is reasonable to assume that the value of the total catch is available for each fishing season, that is, the value of the quantity  $y(k) = \sum_{i=1}^3 \frac{q_i E(k)}{q_i E(k) + M_i} \left(1 - e^{(-M_i - q_i E(k))\tau}\right) x_i(k) = \sum_{i=1}^3 C_i(k) x_i(k)$  is known for each  $k$ . Therefore the observer (16) which uses the value of the total catch can be used to give a dynamical estimation of the stock state because we have proved in Section 3 that its solutions will converge (quite rapidly) towards the solutions of the model (18) regardless of the value of the initial conditions. Moreover, this observer can be implemented without using the expression of the recruitment function. To summarize, if the value of the total catch is available for each season then a good estimate  $(\hat{x}_1(k), \hat{x}_2(k), \hat{x}_3(k))$  of the stock state  $(x_1(k), x_2(k), x_3(k))$  can be obtained by choosing an arbitrary initial condition  $(z_1(0), z_2(0), z_3(0))$  and computing the solutions of the following observer:

$$\left\{ \begin{array}{l} z_1(k+1) = -\frac{C_2(k+1)p_1 s_1(k)}{C_1(k+1)} z_1(k) \\ \quad - \left( p_2 s_2(k) \frac{C_3(k+1)}{C_1(k+1)} + (1-p_2) s_2(k) \frac{C_2(k+1)}{C_1(k+1)} \right) z_2(k) \\ \quad - \frac{C_3(k+1) s_3(k)}{C_1(k+1)} z_3(k) - \frac{C_2(k+1)p_1 s_1(k)}{C_1(k+1)C_1(k)} y(k) \\ z_2(k+1) = p_1 s_1(k) z_1(k) + (1-p_2) s_2(k) z_2(k) + \frac{p_1 s_1(k)}{C_1(k)} y(k), \\ z_3(k+1) = p_2 s_2(k) z_2(k) + s_3(k) z_3(k), \\ \hat{x}_1(k) = z_1(k) + \frac{1}{C_1(k)} y(k), \\ \hat{x}_i(k) = z_i(k) \text{ for } i = 2, 3. \end{array} \right. \quad (19)$$

We have plotted the solutions of systems (18) and (19) using arbitrary initial conditions. The Figures (1), (2) and (3) confirm the fact that the  $\hat{x}_i(k)$  (dashed curves) delivered by the observer (19) are good estimates of the states  $x_i(k)$  of system (18). It can be noticed that the convergence of the state estimates  $\hat{x}_i(k)$  towards the real states  $x_i(k)$  is quite fast.

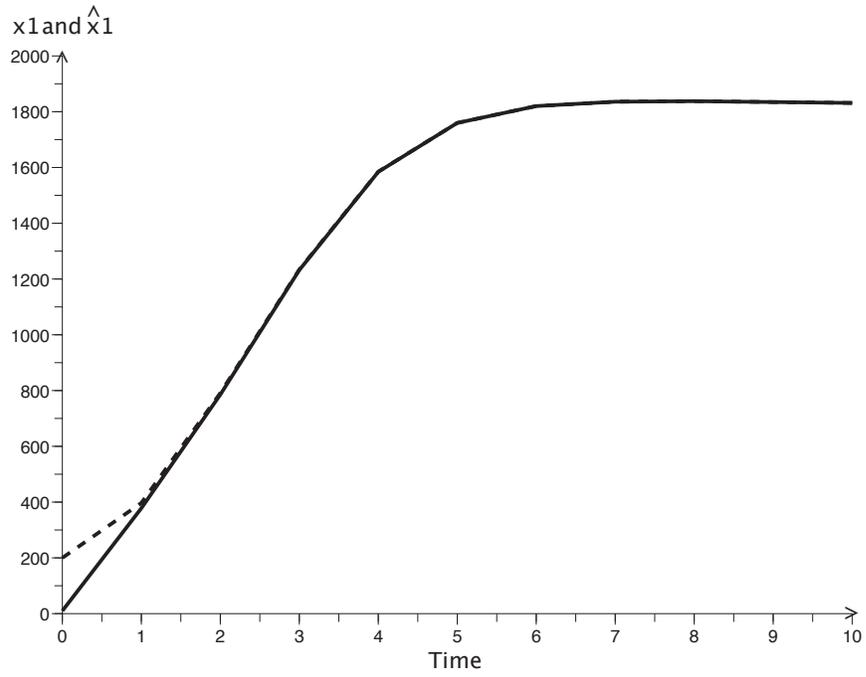


Figure 1: State  $x_1$  (solid line) and the corresponding estimate  $\hat{x}_1$  (dashed line) delivered by the observer (19)

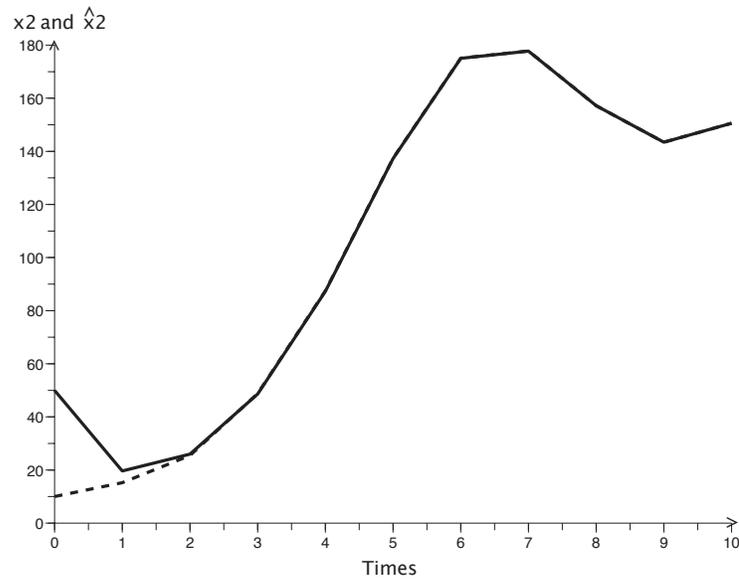


Figure 2: State  $x_2$  (solid line) and the corresponding estimate  $\hat{x}_2$  (dashed line) delivered by the observer (19)

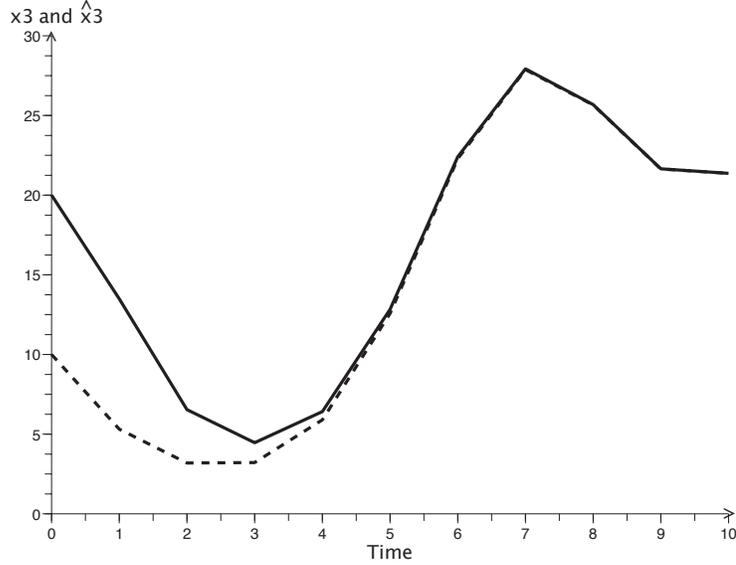


Figure 3: State  $x_3$  (solid line) and the corresponding estimate  $\hat{x}_3$  (dashed line) delivered by the observer (19)

## 5 Applications

### 5.1 Application using data from ICES

Now we shall illustrate how to use the the observer (16) with real data assessed by the Advisory Committee on Fisheries Management (ACFM) of the International Council for the Exploration of the Sea (ICES). One of the major task of the ICES is to collect scientific information in order to give advice to decision makers about the question of fisheries conservation and protection of the marine environment. In this context, the Advisory Committee on Fisheries Management of ICES has published a report on the evolution of the stock of the Arctic cod in the Atlantic North East [4] (see also [36]) between 1946 and 2000. The annual stock for immature and mature of the considered fish population between 1946 and 2000 are estimated in this report.

This report (Table 1) gives for each year:

- total catch of immature individuals ( $Y_1(k)$ );
- total catch of mature individuals ( $Y_2(k)$ );
- fishing mortality rate for immature ( $F_1(k)$ ) and for mature individuals ( $F_2(k)$ );
- the ICES annual stock estimates  $m_1(k)$  for immature and  $m_2(k)$  for mature.

In order to give an application of the proposed observer with these data we will consider the observer (16) with two classes ( $n = 2$ ): the class of immature and the class of mature individuals.

Using Table 1, we compute for each year  $k$  the measurable output  $y(k)$  which is the total catch of mature and immature individuals at year  $k$ , i.e.,  $Y(k) = Y_1(k) + Y_2(k)$ . Table 1 also gives the values of the fishing mortality rates  $F_1(k)$  and  $F_2(k)$ . The ICES assumes a natural mortality rate  $M_1 = M_2 = M = 0.2$  and  $\tau = 1$  (the length of harvesting season). The proportion  $p_1$  of immature individuals that move into the mature stage class at time  $k + 1$  has been chosen equal to  $p_1 = 0.1$ . Therefore, for all  $k$ , Table 1 allows to compute  $Y(k)$ ,  $C_i(k) = \frac{F_i(k)}{M_i + F_i(k)}(1 - e^{-(M_i + F_i(k))})$ , and  $s_i(k) = e^{-(M_i + F_i(k))}$ . Hence with all these data, it is possible to run the following observer (20) that will deliver stock estimates  $\hat{x}_1(k)$  and  $\hat{x}_2(k)$  for immature and mature individuals:

$$\left\{ \begin{array}{l} z_1(k+1) = -\frac{C_2(k+1)p_1s_1(k)}{C_1(k+1)}z_1(k) - \frac{C_2(k+1)}{C_1(k+1)}s_2(k)z_2(k) - \frac{C_2(k+1)p_1s_1(k)}{C_1(k+1)C_1(k)}Y(k), \\ z_2(k+1) = p_1s_1(k)z_1(k) + s_2(k)z_2(k) + \frac{p_1s_1(k)}{C_1(k)}Y(k), \\ \hat{m}_1(k) = z_1(k) + \frac{1}{C_1(k)}Y(k), \\ \hat{m}_2(k) = z_2(k). \end{array} \right. \quad (20)$$

To compare our estimations with those of ICES Advisory Committee on fisheries Management, we represent in the same graphs 4 and 5 the two estimations. These graphs show that our estimations have the same behavior as those of the ICES Advisory Committee on fisheries Management. Our observer slightly overestimates the biomass state for immature class.

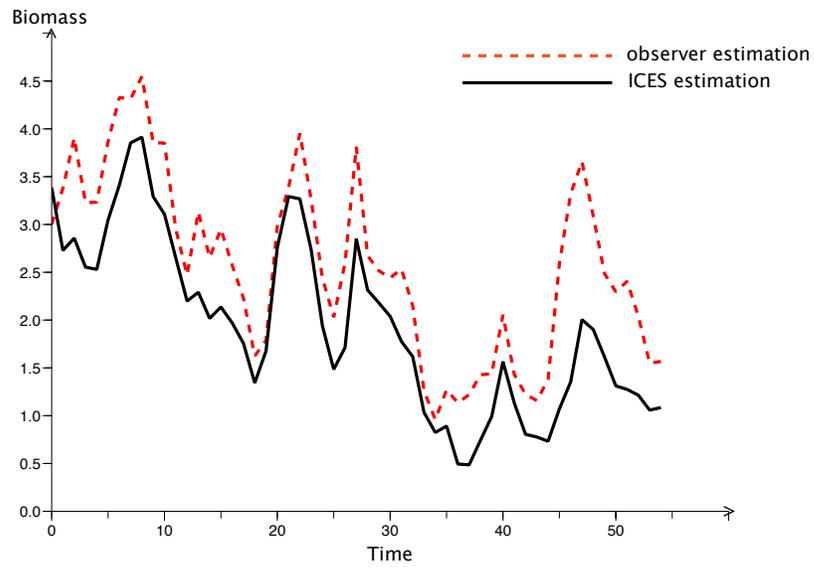


Figure 4: ICES biomass estimates and observer biomass estimates for the immature class.

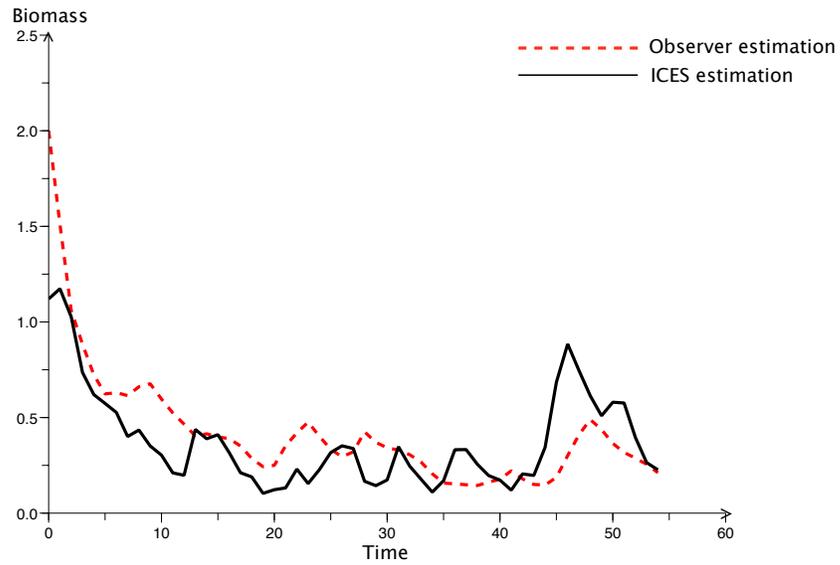


Figure 5: ICES biomass estimates and observer biomass estimates for the mature class.



age-classes 1, 2, 3, 4, 5, 6, 7<sup>+</sup>. The natural mortality rates  $M_i$  are assumed to be all equal to 0.5, the fishing mortality rates  $F_i(k)$  for each age-class are given in Table 3.

Using the data contained in the Tables 2 and 3 one can implement the observer (22) and compare the estimates delivered by this observer with the FAO estimates given by Table 4. It should be noticed that we only need the value of the total catch which is easier to measure than the catch for each age-class. The observer is implemented with two arbitrary different initial conditions to show that this initial condition didn't impact the convergence of the observer toward the "real states" of the stock. In Figures 6, 7, 8, 9, 10, 11 and 12 we compare the FAO stock estimates and the corresponding estimates computed by the observer (22) for each age-class.

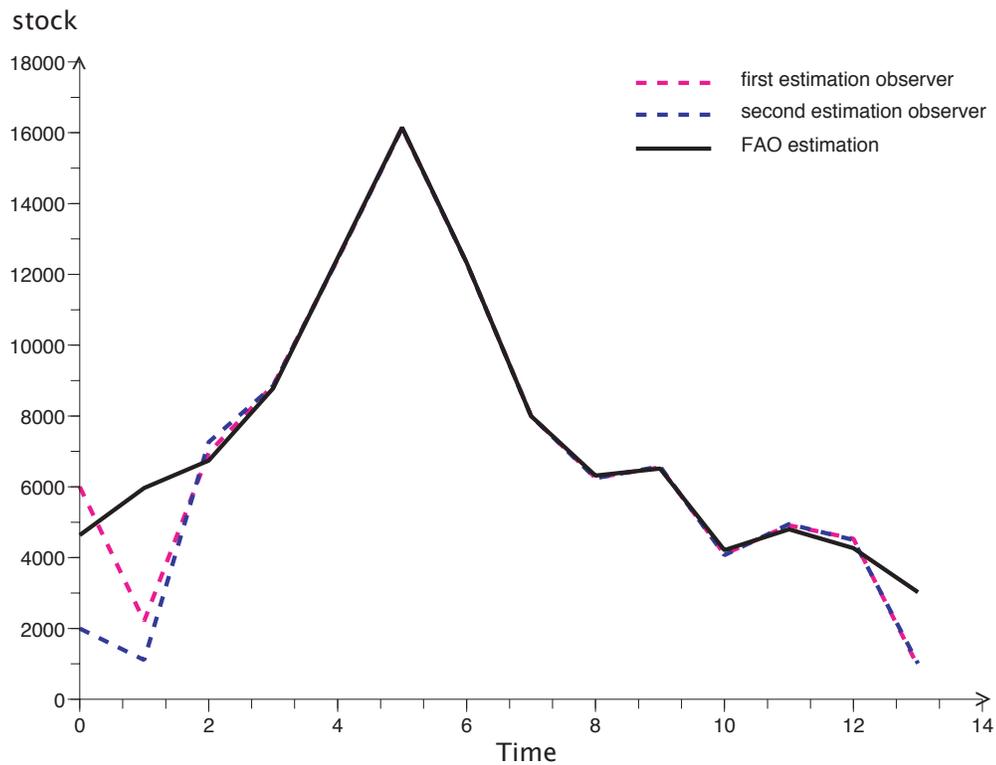


Figure 6: FAO estimates and the observer estimates for age class 1

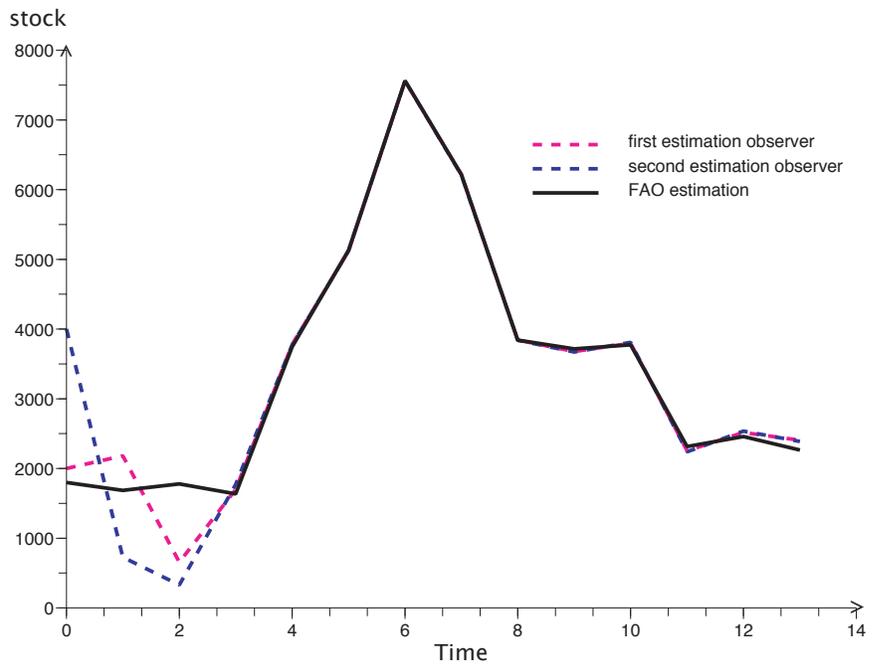


Figure 7: FAO estimates and the observer estimates for age class 2

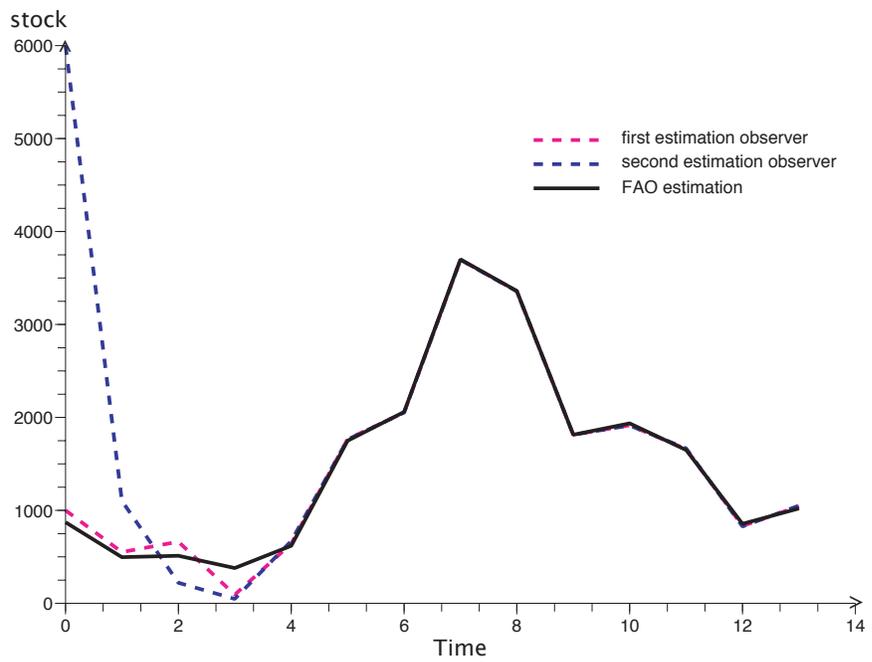


Figure 8: FAO estimates and the observer estimates for age class 3

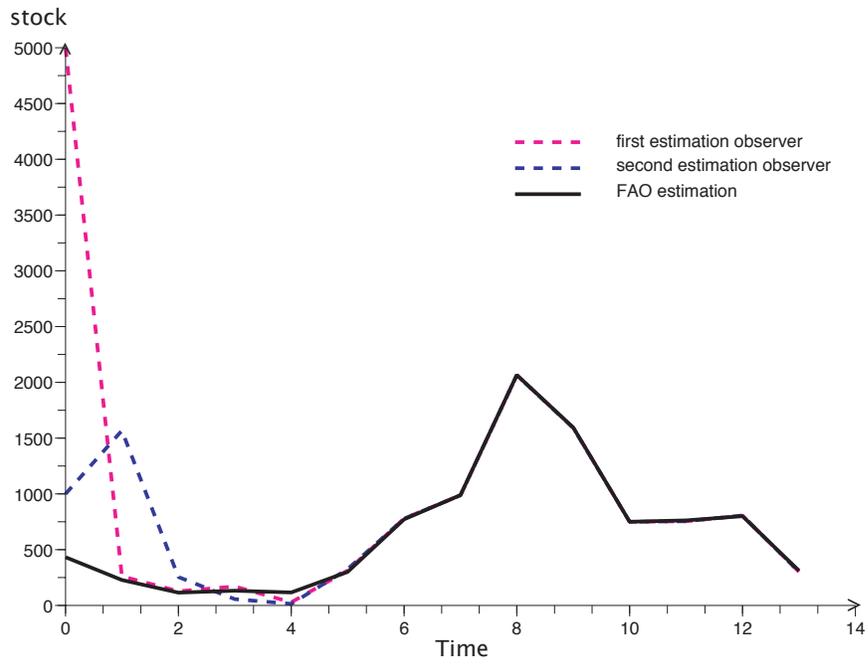


Figure 9: FAO estimates and the observer estimates for age class 4

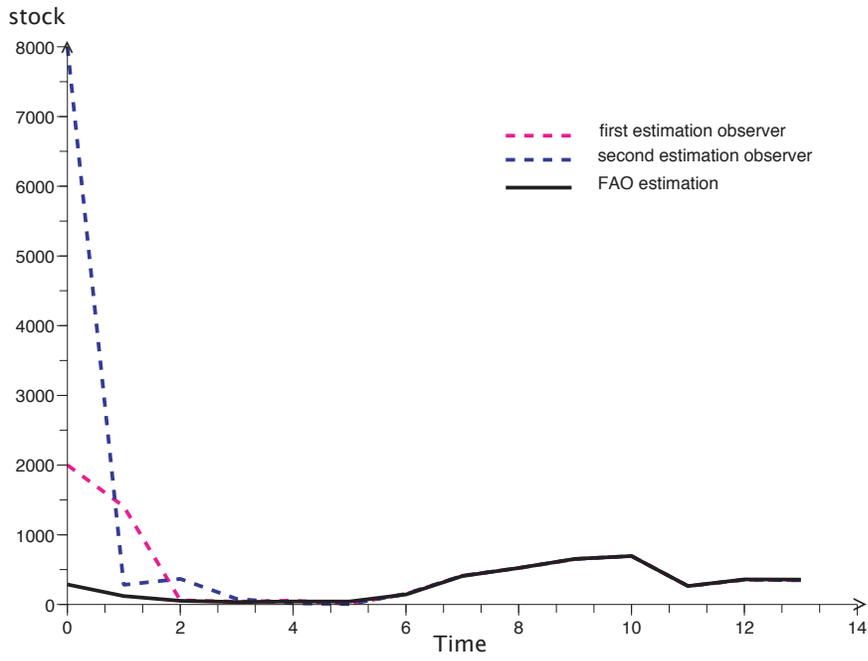


Figure 10: FAO estimates and the observer estimates for age class 5

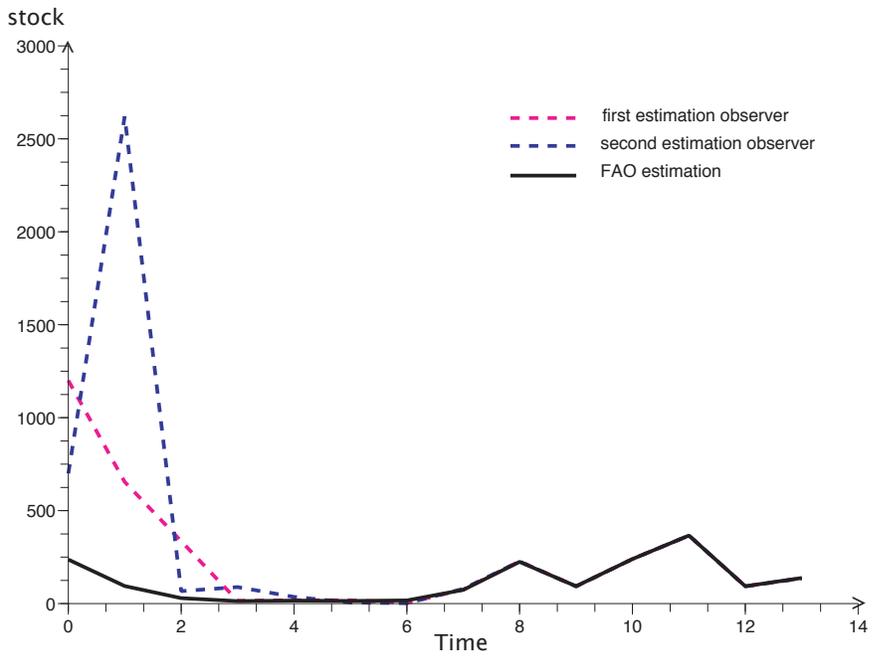


Figure 11: FAO estimates and the observer estimates for age class 6

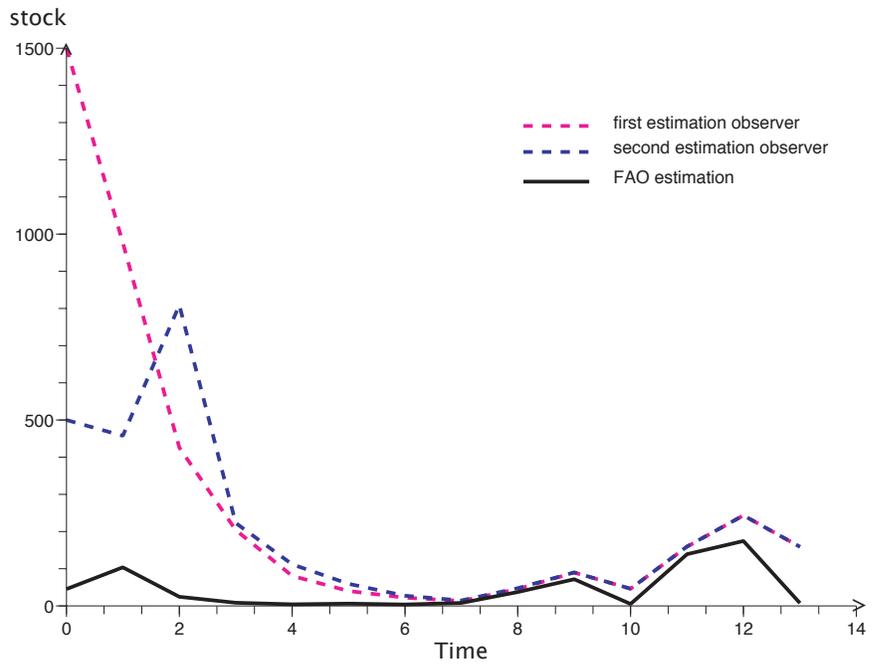


Figure 12: FAO estimates and the observer estimates for age class 7

## 6 Discussion

We have applied the observer (16 or 20) given by the stage-class structured model to a fish population structured in immature and mature classes. The observer is implemented with an arbitrary initial condition and delivers stock estimates for immature and mature classes for each year. These estimates are comparable to those of the ICES. However the observer (20) slightly overestimates the biomass of the immature stage. This mild difference may be due to the difference between our estimation method and the ICES estimation method. It could also be due to the value we have used for the parameter  $p$  (the proportion of individuals in immature class that moves to the mature class after one year) which is not given in the ICES report, we have used  $p = 0.1$ . In spite of this, the observer gives a good information about the fluctuation of the stock for the two stages during the considered period.

We have also applied the observer (22) to estimate the stock state for a harvested population structured in seven age-classes. The data corresponding to this population are given in a FAO report. We point out with simulations that even in choosing an initial condition of the observer very distant from the initial condition of the FAO estimates, the curve of the observer converges (quite rapidly) towards the curve of the FAO estimates. We also note that if we consider two different initial conditions of the observer, the two curves of the observer converge simultaneously toward the the graph of FAO estimates. Hence, in the absence of the FAO stock estimation, the states of the stock fish population are reached as soon as a curve of the observer reaches another curve of the observer with a different initial condition.

It must be noticed that for the applications with real data, we only need the value of the total catch for the whole population: the observer implementation does not require the knowledge of the captures for each stage.

When the analytic expression of the fishing effort as well as the the parameter numerical values are available it is possible to check the uniform observability condition and hence to compute the matrix gain  $L(k)$  in order to increase the convergence of the estimation error towards zero. We think that when the computation of such  $L(k)$  is possible, the convergence of the observer will be guaranteed without the condition of Proposition 2 concerning the minimal fishing effort value. A work in this direction is in progress.

## 7 Conclusion

We have built an observer for a stage-structured model of a harvested fish population. The model we have considered is a dynamical discrete-time system including a nonlinear recruitment relationship. We have shown how to use the catch data in order to give an estimate of the stock of each stage-class for the harvested fish population under consideration. The observer estimator has the particularity of not using the recruitment function which is very poorly known. It may be noticed that the observer we have built is easy to implement and it has been successfully tested by some real data from FAO and ICES Advisory Committee on fisheries Management. Historically, the tools and techniques of automatic control theory have been developed for artificial engineering systems but nowadays they are more and more applied to "natural systems". The work we address

in this paper is an example showing how observer theory and computer sciences can be helpful for good policies in management fishery.

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## Appendix A: Proof of proposition 2

We prove that system (15) is a global exponential observer for model (7). Let  $e(k) = x(k) - \hat{x}(k)$  be the error estimation at time  $k$ . It satisfies the equation:

$$e(k+1) = \bar{A}(k)e(k)$$

It's sufficient to prove that there exists  $\delta$  such that  $\|\bar{A}(k)\| \leq \delta < 1$ , for some matrix norm. Here we shall use the norm 1 defined by  $\|\bar{A}(k)\|_1 = \max_j \sum_{i=1}^n |\bar{a}_{ij}(k)|$ . Using the construction of the matrix  $\bar{A}(k)$ , we can write

$$\begin{aligned} \|\bar{A}(k)\|_1 &= \left\| \left( I_{n \times n} - D (C(k+1)D)^{-1} C(k+1) \right) A(k) \right\|_1 = \left\| I_{n \times n} - \frac{1}{C_1(k+1)} D C(k+1) A(k) \right\|_1 \\ \|\bar{A}(k)\|_1 &\leq \left\| I_{n \times n} - \frac{1}{C_1(k+1)} D C(k+1) \right\|_1 \|A(k)\|_1 \end{aligned}$$

We have:

$$\left\| I_{n \times n} - \frac{1}{C_1(k+1)} D C(k+1) \right\|_1 = \max_j \left( \frac{C_j(k+1)}{C_1(k+1)} + 1 \right), \text{ and } \|A(k)\|_1 = \max_j s_j(k).$$

Hence:

$$\|\bar{A}(k)\|_1 \leq \max_j \left( \frac{C_j(k+1)}{C_1(k+1)} + 1 \right) \max_j s_j(k).$$

The remaining part of the proof is quite similar to the proof of the main result of [26]. We give it here in order to be self-contained.

For  $j = 1 \dots n$ , Let:  $\gamma_j(k) = \frac{C_j(k+1)}{C_1(k+1)}$ .

We have:

$$\begin{aligned}\gamma_j(k) &= \frac{q_j E(k+1)(1 - e^{(-M_j - q_j E(k+1))\tau})}{M_j + q_j E(k+1)} \frac{M_1 + q_1 E(k+1)}{q_1 E(k+1)(1 - e^{-(M_1 - q_1 E(k+1))\tau})} \\ \gamma_j(k) &= \frac{M_1 q_j (1 - e^{(-M_j - q_j E(k+1))\tau})}{q_1 (M_j + q_j E(k+1)) (1 - e^{-(M_1 - q_1 E(k+1))\tau})} \\ &\quad + \frac{q_j q_1 E(k+1) (1 - e^{(-M_j - q_j E(k+1))\tau})}{q_1 (M_j + q_j E(k+1)) (1 - e^{-(M_1 - q_1 E(k+1))\tau})}\end{aligned}$$

Since  $\frac{q_j q_1 E(k+1)}{q_1 (M_j + q_j E(k+1))} < 1$ , we can write

$$\gamma_j(k) \leq \frac{M_1 q_j (1 - e^{(-M_j - q_j E(k+1))\tau})}{q_1 (M_j + q_j E(k+1)) (1 - e^{-(M_1 - q_1 E(k+1))\tau})} + \frac{1 - e^{(-M_j - q_j E(k+1))\tau}}{1 - e^{-(M_1 - q_1 E(k+1))\tau}}$$

The map  $x \mapsto \frac{1 - e^{-x\tau}}{x}$  is decreasing, hence:

$$\frac{1 - e^{(-M_j - q_j E(k+1))\tau}}{M_j + q_j E(k+1)} \leq \frac{1 - e^{(-M_{min} - q_{min} E_{min})\tau}}{M_{min} + q_{min} E_{min}}$$

So we obtain:

$$\gamma_j(k) \leq \frac{M_{max} q_{max}}{q_1} \frac{(1 - e^{(-M_{min} - q_{min} E_{min})\tau})}{M_{min} + q_{min} E_{min}} \frac{1}{1 - e^{-(M_1 - q_1 E(k+1))\tau}} + \frac{1 - e^{(-M_j - q_j E(k+1))\tau}}{1 - e^{-(M_1 - q_1 E(k+1))\tau}}$$

Since  $0 < e^{(-M_j - q_j E(k+1))\tau} \leq 1$ , we have  $1 - e^{(-M_j - q_j E(k+1))\tau} \leq 1$ . Therefore, we have for  $j = 1 \dots n - 1$ ,

$$\gamma_j(k) \leq \frac{M_{max} q_{max}}{q_1} \frac{(1 - e^{(-M_{min} - q_{min} E_{min})\tau})}{M_{min} + q_{min} E_{min}} \frac{1}{1 - e^{-(M_1 - q_1 E(k+1))\tau}} + \frac{1}{1 - e^{-(M_1 - q_1 E(k+1))\tau}}.$$

The map  $x \mapsto \frac{1}{1 - e^{-x\tau}}$  is decreasing; it follows that:

$$\frac{1}{1 - e^{-(M_1 - q_1 E(k+1))\tau}} \leq \frac{1}{1 - e^{(-M_{min} - q_{min} E_{min})\tau}}.$$

Therefore,

$$\gamma_j(k) \leq \frac{M_{max}q_{max}}{q_1} \frac{1}{M_{min} + q_{min}E_{min}} + \frac{1}{1 - e^{(-M_{min}-q_{min}E_{min})\tau}}, \quad \forall j = 1, \dots, n.$$

Otherwise  $s_i(k) \leq e^{-M_{min}-q_{min}\tau E_{min}} \leq e^{(-M_{min}-q_{min}E_{min})\tau}$ ,  $\forall j = 1, \dots, n$ .

Hence:

$$\|\bar{A}(k)\|_1 \leq \left( \frac{M_{max}q_{max}}{q_1} \frac{1}{M_{min} + q_{min}E_{min}} + \frac{1}{1 - e^{(-M_{min}-q_{min}E_{min})\tau}} + 1 \right) e^{(-M_{min}-q_{min}E_{min})\tau}.$$

We denote:

$$\delta(E_{min}) = \left( \frac{M_{max}q_{max}}{q_1} \frac{1}{M_{min} + q_{min}E_{min}} + \frac{1}{1 - e^{(-M_{min}-q_{min}E_{min})\tau}} + 1 \right) e^{(-M_{min}-q_{min}E_{min})\tau}.$$

We will show now that there exists  $\eta > 0$  in such a way that  $E_{min} > \eta$  implies  $\delta(E_{min}) < 1$ .

Let  $X = e^{(-M_{min}-q_{min}E_{min})\tau}$  and consider

$$P(X) = -\frac{\tau M_{max}q_{max} X}{q_1 \text{Log}(X)} + \frac{X}{1 - X} + X - 1.$$

One can remark that  $P(X) = \delta(E_{min}) - 1$ .

We have  $\lim_{X \rightarrow 0} P(X) = -1$  and  $\lim_{X \rightarrow 1^-} P(X) = +\infty$ . So there exists  $x_1 \in ]0, 1[$  such that  $P(x_1) = 0$ .

Let  $x^* = \inf\{x \in ]0, 1[ \mid P(x) = 0\}$  then we have  $P(X) < 0$  for all  $X$  satisfying  $0 < X < x^*$ . Now,

$0 < X < x^*$  is equivalent to  $E_{min} > \frac{-\text{Log}(x^*)}{q_{min}\tau} - \frac{M_{min}}{q_{min}}$  since  $X = e^{(-M_{min}-q_{min}E_{min})\tau}$ . It is then

sufficient to choose  $\eta = \frac{-\text{Log}(x^*)}{q_{min}\tau} - \frac{M_{min}}{q_{min}}$  and this completes the proof of Proposition 2.  $\blacksquare$

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## ICES data

Table 1: Table: Biomass and catch in million tonnes of the Arctic cod in the North-East between 1946 and 2000. These data are from the "ICES Advisory Committee on Fisheries Management" (Anon, 2001), table 3.7 (catch), table 3.19 (biomass) and table 3.23 (maturity).

ICES Biomass estimates (million tonnes)			Catch (million tonnes)		Fishing mortality rate	
Year (k)	Immature ( $m_1(k)$ )	Mature ( $m_2(k)$ )	Immature ( $y_1(k)$ )	Mature ( $y_2(k)$ )	Immature ( $F_1(k)$ )	Mature ( $F_2(k)$ )
1946	3.3866	1.1206	0.4222	0.2633	0.1247	0.2350
1947	2.7283	1.1740	0.4959	0.4688	0.1818	0.3993
1948	2.8562	1.0264	0.5176	0.3510	0.1812	0.3420
1949	2.5528	0.7360	0.5503	0.2563	0.2156	0.3482
1950	2.5292	0.6206	0.4284	0.2444	0.1694	0.3938
1951	3.0474	0.5741	0.5263	0.1940	0.1727	0.3379
1952	3.4114	0.5271	0.6951	0.2428	0.2038	0.4606
1953	3.8533	0.4010	0.5209	0.1425	0.1352	0.3554
1954	3.9139	0.4350	0.7196	0.1691	0.1839	0.3887
1955	3.2905	0.3512	0.9268	0.1526	0.2817	0.4345
1956	3.1074	0.3034	1.1425	0.1421	0.3677	0.4684
1957	2.6487	0.2103	0.7023	0.0900	0.2651	0.4280
1958	2.1979	0.1978	0.6019	0.0830	0.2739	0.4196
1959	2.2912	0.4382	0.6197	0.1806	0.2705	0.4121
1960	2.0169	0.3890	0.4343	0.1629	0.2153	0.4188

1961	2.1368	0.4104	0.5113	0.2026	0.2393	0.4937
1962	1.9697	0.3167	0.5785	0.1574	0.2937	0.4970
1963	1.7560	0.2118	0.6335	0.1257	0.3608	0.5935
1964	1.3421	0.1892	0.3270	0.0989	0.2436	0.5227
1965	1.6769	0.1036	0.2991	0.0458	0.1784	0.4421
1966	2.7595	0.1224	0.3425	0.0499	0.1241	0.4077
1967	3.2925	0.1318	0.4620	0.0628	0.1403	0.4765
1968	3.2669	0.2306	0.9045	0.0914	0.2769	0.3964
1969	2.7351	0.1546	1.0495	0.0886	0.3837	0.5731
1970	1.9394	0.2281	0.7180	0.1175	0.3702	0.5151
1971	1.4845	0.3167	0.4008	0.1546	0.2700	0.4882
1972	1.7130	0.3523	0.2886	0.1895	0.1685	0.5379
1973	2.8488	0.3380	0.4363	0.1733	0.1532	0.5127
1974	2.3150	0.1669	0.7231	0.0839	0.3124	0.5027
1975	2.1793	0.1440	0.6522	0.0678	0.2993	0.4708
1976	2.0396	0.1738	0.6082	0.0755	0.2982	0.4344
1977	1.7767	0.3473	0.6734	0.1740	0.3790	0.5010
1978	1.6184	0.2460	0.4947	0.1469	0.3057	0.5972
1979	1.0343	0.1778	0.2649	0.0980	0.2561	0.5512
1980	0.8233	0.1102	0.2400	0.0590	0.2915	0.5354
1981	0.8910	0.1699	0.2437	0.0942	0.2735	0.5544
1982	0.4938	0.3315	0.1283	0.1622	0.2598	0.4893
1983	0.4865	0.3328	0.1323	0.1916	0.2719	0.5757
1984	0.7402	0.2556	0.1484	0.1444	0.2005	0.5649
1985	0.9909	0.1966	0.2047	0.0977	0.2066	0.4969
1986	1.5661	0.1731	0.3319	0.0914	0.2119	0.5280
1987	1.1331	0.1203	0.4392	0.0724	0.3876	0.6018
1988	0.8044	0.2053	0.3171	0.1178	0.3942	0.5738
1989	0.7770	0.1970	0.2461	0.0904	0.3167	0.4589
1990	0.7320	0.3446	0.1112	0.0985	0.1519	0.2858

1991	1.0702	0.6847	0.1425	0.1927	0.1332	0.2814
1992	1.3533	0.8847	0.2037	0.2960	0.1505	0.3346
1993	2.0052	0.7461	0.2974	0.2769	0.1483	0.3711
1994	1.9007	0.6137	0.4394	0.3248	0.2312	0.5292
1995	1.6134	0.5093	0.4738	0.2640	0.2937	0.5184
1996	1.3109	0.5804	0.4360	0.2856	0.3326	0.4921
1997	1.2144	0.5763	0.4373	0.3248	0.3431	0.5636
1998	1.2146	0.3961	0.3558	0.2326	0.2929	0.5872
1999	1.0585	0.2645	0.3388	0.1477	0.3201	0.5584
2000	1.0854	0.2264	0.2836	0.1285	0.2613	0.5676

FAO data (from <http://www.fao.org/DOCREP/003/S4882F/S4882F09.htm>).

Table 2: *Trachurus* spp.: Catch in 10<sup>6</sup> of individuals by age classes.

Years	age							Total catch (1000 t)	Total catch 10 <sup>6</sup> individuals
	1	2	3	4	5	6	7+		
1972	1492.0	798.1	403.9	188.0	106.4	52.0	18.5	390.123	3058.9
1973	2457.0	680.9	250.9	114.5	60.3	43.2	48.5	383.052	3625.3
1974	3273.2	947.3	240.4	44.9	26.2	12.3	11.8	414.63	4562.8
1975	2078.1	493.6	152.3	48.9	7.7	4.8	2.6	302.317	2803.9
1976	3202.3	685.6	97.4	35.6	16.3	4.6	1.1	394.145	4060.3
1977	2927.2	1391.2	380.3	52.1	12.0	6.3	1.4	392.572	4787.6
1978	1650.4	1162.8	338.7	78.2	16.5	3.0	0.5	294.151	3268.1
1979	1323.3	534.7	231.0	99.0	29.6	10.0	0.8	223.165	2246.1
1980	150.2	676.6	582.1	799.6	306.6	86.0	11.4	503.081	2619.6
1981	230.4	415.2	462.2	357.9	207.2	72.6	14.1	357.935	1764.0
1982	310.5	840.4	542.4	254.1	71.5	8.1	1.1	310.464	2045.5
1983	591.1	727.3	262.9	135.9	88.4	62.5	31.8	280.000	1886.6
1984	424.6	621.4	274.9	167.1	105.7	68.4	41.9	300.000	1680.6
1985	145.6	800.1	360.3	109.8	126.4	48.7	2.5	320.000	1609.4

Table 3: Fishing mortality rates. (M= natural mortality rate=0.5).

Years	age-classes						
	1	2	3	4	5	6	7+
1972	0.511725	0.788882	0.841723	0.767820	0.616278	0.322343	0.705246
1973	0.710480	0.691405	0.962460	0.953076	0.933614	0.833445	0.850099
1974	0.911205	1.462620	0.858620	0.660721	0.921245	0.740458	0.879356
1975	0.351619	0.472105	0.686220	0.617037	0.312602	0.620453	0.488036
1976	0.387599	0.261425	0.221046	0.483193	0.632487	0.445506	0.397193
1977	0.258610	0.413621	0.318220	0.245832	0.424763	0.816728	0.332598
1978	0.184893	0.215220	0.232136	0.136975	0.167873	0.248101	0.185781
1979	0.233446	0.115179	0.082385	0.135158	0.096253	0.187148	0.132838
1980	0.030647	0.250393	0.245816	0.651817	1.229370	0.641243	0.481579
1981	0.045911	0.152152	0.383773	0.330540	0.502452	2.385422	0.283060
1982	0.098016	0.327470	0.430687	0.546851	0.139337	0.043822	0.308936
1983	0.168887	0.496638	0.224000	0.253589	0.540531	0.241525	0.337494
1984	0.134262	0.380006	0.511330	0.341390	0.457392	0.010983	0.357427
1985	0.063021	0.577001	0.577000	0.577000	0.577000	0.577000	0.530840

Table 4: Abundance (FAO estimates) in millions individuals per age class and total biomass in thousand tons.

Years	age-classes								Total Biomass (1000 t)
	0+	1	2	3	4	5	6	7+	
1972	9836.6	4635.1	1800	871.7	432	286.6	236.6	45.1	1191.9
1973	11053.1	5966.2	1685.3	496	227.9	121.6	93.9	104	1066.5
1974	15718.7	674.1	1779	512	114.9	53.3	29	24.7	1080.9
1975	21772	8779.6	1634.8	379	131.6	36	12.9	8.4	1273.9
1976	30610.6	12463.9	3746.5	618.4	115.7	43.1	16	4.2	1929.9
1977	20373.3	16140.9	5130.7	1749.6	300.7	43.3	13.9	6.2	2360.9
1978	13264.6	12329.1	7563.3	2057.8	771.9	142.6	17.2	3.7	2441.4
1979	10411.7	7999.2	6215.7	3699.1	989.5	408.2	73.9	8.1	2455.6
1980	10737.7	6315	3841.6	3359.8	2066.2	524.3	224.9	37.2	2488.7
1981	6939.1	6512.7	3714.7	1813.9	1593.7	653	93	71.8	2024.3
1982	7913.9	4208.8	3772.9	1935	749.6	694.5	239.7	5.2	1738.6
1983	7041.2	4800	2314.4	1649.3	763	263.1	366.5	139.1	1533.4
1984	6000	4270.7	2458.9	854.5	799.6	359.1	93	174.6	1290.9
1985	7300	3021.4	2264.9	1019.9	310.8	357.8	137.9	7.5	1037.3