

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes, Grenoble.

Nonsmooth contact mechanics. Spring school. Aussois 2010

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Objectives of the lecture

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Formulation and numerical algorithms

Focus on discrete frictional contact 3D problem.

- ▶ Linear Complementarity Problem (LCP)
- ▶ Nonlinear Complementarity Problem (NCP)
- ▶ Generalized, Nonsmooth and semi-smooth equations,
- ▶ Variational Inequalities (VI) and Complementarity problems

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Objectives

The time-discretized problems

- The linear time-discretized problem
- The Index Set of Forecast Active Constraints
- Some further notation

LCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- The frictional contact problem as an LCP
- Comments

NCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- Comments

Nonsmooth Equations. Formulations and solving

- Principle
- Alart & Curnier's Formulation
- Variants and line-search procedures.

Formulation and Resolution as VI/CP

- VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Summary of the linear time discretized equations

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Frictionless case

One step linear problem

$$\begin{cases} v_{k+1} = v_{free} + \widehat{M}^{-1} p_{k+1} \\ q_{k+1} = q_k + h [\theta v_{k+1} + (1 - \theta) v_k] \end{cases}$$

Relations

$$\begin{cases} U_{k+1}^\alpha = H^\alpha T(q_{k+1}) v_{k+1} \\ p_{k+1}^\alpha = H^\alpha(q_{k+1}) P_{k+1}^\alpha \end{cases}$$

Nonsmooth Law

$$\begin{cases} \text{if } g^\alpha(\tilde{q}_{k+1}) \leq 0, \text{ then} \\ \quad 0 \leq U_{k+1}^\alpha \perp P_{k+1}^\alpha \geq 0 \\ \text{if } g^\alpha(\tilde{q}_{k+1}) > 0 \text{ then } P_{k+1}^\alpha = 0 \end{cases}$$

Objectives

The time-discretized problems

The linear time-discretized problem

The Index Set of Forecast Active Constraints
Some further notation

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Summary of the linear time discretized equations

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
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Grenoble.

Frictional case

One step linear problem

$$\begin{cases} v_{k+1} = v_{free} + \widehat{M}^{-1} p_{k+1} \\ q_{k+1} = q_k + h [\theta v_{k+1} + (1 - \theta) v_k] \end{cases}$$

Relations

$$\begin{cases} U_{k+1}^\alpha = H^\alpha T(q_{k+1}) v_{k+1} \\ p_{k+1}^\alpha = H^\alpha(q_{k+1}) P_{k+1}^\alpha \end{cases}$$

Nonsmooth Law

$$\begin{cases} \widehat{U}_{k+1}^\alpha = \begin{bmatrix} U_{N,k+1}^\alpha + e U_{N,k}^\alpha + \mu^\alpha \|U_{T,k+1}^\alpha\| \\ U_{T,k+1}^\alpha \end{bmatrix} \\ \text{if } g^\alpha(\check{q}_{k+1}) \leq 0 \text{ then} \\ \quad \mathbf{C}^{\alpha,*} \ni \widehat{U}_{k+1}^\alpha \perp P_{k+1}^\alpha \in \mathbf{C}^\alpha \\ \text{if } g^\alpha(\check{q}_{k+1}) > 0 \text{ then } P_{k+1}^\alpha = 0 \end{cases}$$

Objectives

The time-discretized problems

The linear time-discretized problem

The Index Set of Forecast Active Constraints
Some further notation

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Index Set of Forecast Active Constraints

The index set I of all unilateral constraints in the system is denoted by:

$$I = \{1 \dots \nu\} \subset \mathbb{N} \quad (1)$$

The index-set I_a is the set of all *forecast* active constraints of the system and it is denoted by

$$I_a(\tilde{\mathbf{q}}_{k+1}) = \{\alpha \in I \mid \mathbf{g}^\alpha(\tilde{\mathbf{q}}_{k+1}) \leq 0\} \subseteq I \quad (2)$$

where $\tilde{\mathbf{q}}_{k+1}$ is an explicit forecast of the position.

Objectives

The time-discretized problems

The linear time-discretized problem

The Index Set of Forecast Active Constraints

Some further notation

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Cone complementarity reformulation for one contact α

$$\mathbf{C}^{\alpha,*} \ni \hat{U}^\alpha \perp P_{k+1}^\alpha \in \mathbf{C}^\alpha, \quad \forall \alpha \in I_a(\tilde{q}_{k+1}) \quad (3)$$

assuming implicitly that $P^\alpha = 0$ for all $\alpha \in I \setminus I_a(\tilde{q}_{k+1})$ and introducing the modified local velocity

$$\hat{U}_{k+1}^\alpha = \left[U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha + \mu^\alpha \|U_{T,k+1}^\alpha\|, U_{T,k+1}^\alpha \right]^T \quad (4)$$

Objectives

The time-discretized problems

The linear time-discretized problem

The Index Set of Forecast Active Constraints

Some further notation

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The time-discretized problems

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

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Grenoble.

The time-discretized linear OSNSP (\mathcal{P}_L)

The time-discretized linear OSNSP, denoted by (\mathcal{P}_L) is given by

$$(\mathcal{P}_L) \left\{ \begin{array}{l} U_{k+1} = \widehat{W}P_{k+1} + U_{\text{free}} \\ \forall \alpha \in I_a(\tilde{\mathbf{q}}_{k+1}), \\ \widehat{U}_{k+1}^\alpha = [U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha + \mu^\alpha \|U_{T,k+1}^\alpha\|, U_{T,k+1}^\alpha]^T \\ \mathbf{C}^{\alpha,*} \ni \widehat{U}_{k+1}^\alpha \perp P_{k+1}^\alpha \in \mathbf{C}^\alpha \end{array} \right.$$

Objectives

The time-discretized problems

The linear time-discretized problem

The Index Set of Forecast Active Constraints

Some further notation

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The time-discretized problems

The time-discretized mixed linear OSNSP (\mathcal{P}_{ML})

$$(\mathcal{P}_{ML}) \left\{ \begin{array}{l} \widehat{M}(v_{k+1} - v_{\text{free}}) = p_{k+1} = \sum_{\alpha \in I_a(\tilde{q}_{k+1})} p_{k+1}^\alpha \\ \forall \alpha \in I_a(\tilde{q}_{k+1}), \\ \quad \left\{ \begin{array}{l} U_{k+1}^\alpha = H^{\alpha, T} v_{k+1} \\ p_{k+1}^\alpha = H^\alpha P_{k+1}^\alpha \\ \widehat{U}_{k+1}^\alpha = [U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha + \mu^\alpha \|U_{T,k+1}^\alpha\|, U_{T,k+1}^\alpha]^T \\ \mathbf{C}^{\alpha,*} \ni \widehat{U}_{k+1}^\alpha \perp P_{k+1}^\alpha \in \mathbf{C}^\alpha \end{array} \right. \end{array} \right.$$

Objectives

The time-discretized problems

The linear time-discretized problem

The Index Set of Forecast Active Constraints

Some further notation

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The time-discretized problems

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

The time-discretized mixed nonlinear OSNSP (\mathcal{P}_{MNL})

$$(\mathcal{P}_{MNL}) \left\{ \begin{array}{l} \mathcal{R}(v_{k+1}) = p_{k+1} = \sum_{\alpha \in I_a(\tilde{q}_{k+1})} p_{k+1}^\alpha \\ \forall \alpha \in I_a(\tilde{q}_{k+1}) \\ \left\{ \begin{array}{l} U_{k+1}^\alpha = H^{\alpha, T}(q_k + 1) v_{k+1} \\ p_{k+1}^\alpha = H^\alpha(q_k + 1) P_{k+1}^\alpha \\ \hat{U}_{k+1}^\alpha = [U_{N, k+1}^\alpha + e^\alpha U_{N, k}^\alpha + \mu^\alpha \|U_{T, k+1}^\alpha\|, U_{T, k+1}^\alpha]^T \\ \mathbf{C}^{\alpha, *} \ni \hat{U}_{k+1}^\alpha \perp P_{k+1}^\alpha \in \mathbf{C}^\alpha \end{array} \right. \end{array} \right.$$

Objectives

The time-discretized problems

The linear time-discretized problem

The Index Set of Forecast Active Constraints

Some further notation

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The time-discretized problems

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Delassus' Matrix notation

$$\left\{ \begin{array}{l} \widehat{W} = H^{a,T} \widehat{M}^{-1} H \\ \widehat{W}_{NT} = H_N^T \widehat{M}^{-1} H_T \\ \widehat{W}_{NN} = H_N^T \widehat{M}^{-1} H_N \\ \widehat{W}_{TT} = H_T^T \widehat{M}^{-1} H_T \end{array} \right. \quad (5)$$

Objectives

The time-discretized problems

The linear time-discretized problem

The Index Set of Forecast Active Constraints

Some further notation

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Objectives

The time-discretized problems

- The linear time-discretized problem
- The Index Set of Forecast Active Constraints
- Some further notation

LCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- The frictional contact problem as an LCP
- Comments

NCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- Comments

Nonsmooth Equations. Formulations and solving

- Principle
- Alart & Curnier's Formulation
- Variants and line-search procedures.

Formulation and Resolution as VI/CP

- VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Objectives

The time-discretized problems

- The linear time-discretized problem
- The Index Set of Forecast Active Constraints
- Some further notation

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Linear Complementarity Problem (LCP)

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Definition (Linear Complementarity Problem (LCP))

Given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the Linear Complementarity Problem, is to find a vector $z \in \mathbb{R}^n$, denoted by $\text{LCP}(M, q)$ such that

$$0 \leq z \perp Mz + q \geq 0 \quad (6)$$

The inequalities have to be understood component-wise and the relation $x \perp y$ means $x^T y = 0$.

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Linear Complementarity Problem (LCP)

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Definition (P -matrix)

A matrix, $M \in \mathbb{R}^{n \times n}$ is said to be a P -matrix if all its principal minors are positive.

Theorem

Let $M \in \mathbb{R}^{n \times n}$. The following statements are equivalent:

- (a) M is a P -matrix
- (b) M reverses the sign of no nonzero vector¹, i.e.
 $x \circ Mx \leq 0, \implies x = 0$ This property can be written equivalently,
$$\forall x \neq 0, \exists i \text{ such that } x_i (Mx)_i > 0. \quad (7)$$
- (c) All real eigenvalues of M and its principal submatrices are positive.

¹A matrix $A \in \mathbb{R}^{n \times n}$ reverses the sign of a vector $x \in \mathbb{R}^n$ if $x_i (Ax)_i \leq 0, \forall i \in \{1, \dots, n\}$.
The Hadamard product $x \circ y$ is the vector with coordinates $x_i y_i$.

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Linear Complementarity Problem (LCP)

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Theorem

A matrix $M \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if $\text{LCP}(M, q)$ has a unique solution for all vectors $q \in \mathbb{R}^n$.

Other properties

- ▶ In the worst case, the problem is N-P hard .i.e. there is no polynomial-time algorithm to solve it.
- ▶ In practice, this "P-matrix" assumption is difficult to ensure via numerical computation, but a definite positive matrix (not necessarily symmetric), which is a P-matrix is often encountered.

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Linear Complementarity Problem (LCP)

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Definition (Mixed Linear Complementarity Problem (MLCP))

Given the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$, $C \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{m \times n}$, and the vectors $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, the Mixed Linear Complementarity Problem denoted by $\text{MLCP}(A, B, C, D, a, b)$ consists in finding two vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that

$$\begin{cases} Au + Cv + a = 0 \\ 0 \leq v \perp Du + bv + b \geq 0 \end{cases} \quad (8)$$

Comments

The MLCP is a mixture between a LCP and a system of linear equations. Clearly, if the matrix A is non singular, we may solve the embedded linear system to obtain u and then reduced the MCLP to a LCP with $q = b - DA^{-1}a$, $M = b - DA^{-1}C$.

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Linear Complementarity Problem (LCP)

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Algorithms for LCP

- ▶ Splitting based methods
- ▶ Generalized Newton methods
- ▶ Interior point method
- ▶ Pivoting based method
- ▶ QP methods for a SDP matrix.

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictionless Case with Newton's Impact

The frictionless contact problem (\mathcal{P}_{LWF})

Remind that the frictionless contact problem in the form (\mathcal{P}_L) can be written as (\mathcal{P}_{LWF}):

$$(\mathcal{P}_{LWF}) \left\{ \begin{array}{l} U_{N,k+1} = \widehat{W}_{NN} P_{N,k+1} + U_{N,\text{free}} \\ \widehat{U}_{N,k+1}^\alpha = U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha \\ 0 \leq \widehat{U}_{N,k+1}^\alpha \perp P_{N,k+1}^\alpha \geq 0 \end{array} \right\} \forall \alpha \in I_a(\tilde{q}_{k+1})$$

LCP formulation

The formulation in terms of LCP is straightforward,

$$\left\{ \begin{array}{l} U_{N,k+1} = \widehat{W}_{NN} P_{N,k+1} + U_{N,\text{free}} \\ \widehat{U}_{N,k+1} = U_{N,k+1} + e \circ U_{N,k} \\ 0 \leq P_{N,k+1} \perp \widehat{U}_{N,k+1} \geq 0 \end{array} \right. \quad (9)$$

where the vector e collects the coefficients of restitution for $\alpha \in I_a(\tilde{q}_{k+1})$, and $x \circ y$ is the Hadamard product of the vectors x and y .

The Frictionless Case with Newton's Impact

LCP formulation

To obtain a proper LCP formulation it suffices to write

$$\begin{cases} \hat{U}_{N,k+1} = \widehat{W}_{NN} P_{N,k+1} + U_{N,\text{free}} - e \circ U_{N,k} \\ 0 \leq \hat{U}_{N,k+1} \perp P_{N,k+1} \geq 0 \end{cases} \quad (10)$$

and we can conclude that $(\hat{U}_{N,k+1}, P_{N,k+1})$ solves the following LCP

$$\text{LCP}(\widehat{W}_{NN}, U_{N,\text{free}} - e \circ U_{N,k}) \quad (11)$$

LCP Resolution

Almost all of the methods can be applied to solve (11).

- ▶ The matrix \widehat{W}_{NN} is a symmetric PSD matrix, provided that \widehat{M} is PD
- ▶ The fact that \widehat{W}_{NN} is PSD and not necessarily PD can cause troubles in numerical applications. This is due to the rank deficiency of H and can be interpreted in terms of redundant constraints.
- ▶ In practice, it may happen that the splitting-based algorithms have difficulties to converge.

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictionless Case with Newton's Impact and Linear Perfect Bilateral Constraints

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

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Grenoble.

The mixed linear OSNSP

Remind that the mixed linear OSNSP with linear perfect bilateral constraints $G^T q + b = 0$ is given by

$$(\mathcal{P}_{ML_b}) \left\{ \begin{array}{l} \widehat{M}(v_{k+1} - v_{\text{free}}) = p_{k+1} + GP_{\mu,k+1} \\ G^T v_{k+1} = 0 \\ U_{N,k+1}^\alpha = H_N^{\alpha,T} v_{k+1} \\ p_{k+1}^\alpha = H_N^\alpha P_{N,k+1}^\alpha \\ \widehat{U}_{N,k+1}^\alpha = U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha \\ 0 \leq \widehat{U}_{N,k+1}^\alpha \perp P_{N,k+1}^\alpha \geq 0 \end{array} \right\} \forall \alpha \in I_a(\tilde{q}_{k+1})$$

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictionless Case with Newton's Impact and Linear Perfect Bilateral Constraints

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

MCLP reformulation

Let us substitute the generalized velocities $\dot{q}_{k+1} = v_{k+1}$ thanks to

$$v_{k+1} = v_{\text{free}} + \widehat{M}^{-1} (G P_{\mu, k+1} + H_N P_{N, k+1}) = 0 \quad (12)$$

in order to obtain the following MLCP in the form,

$$\begin{cases} G^T v_{\text{free}} + G^T \widehat{M}^{-1} G P_{\mu, k+1} + G^T \widehat{M}^{-1} H_N P_{k+1} = 0 \\ \widehat{U}_{N, k+1}^\alpha = \left[H_N^T v_{\text{free}} + e \circ U_{N, k} + G^T \widehat{M}^{-1} G P_{\mu, k+1} + G^T \widehat{M}^{-1} H_N P_{k+1} \right] \\ 0 \leq \widehat{U}_{N, k+1}^\alpha \perp P_{N, k+1}^\alpha \geq 0 \end{cases} \quad (13)$$

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The 3D frictional contact problem as an LCP

The nonlinear nature of the friction cone

The second order cone \mathbf{C} which cannot be written as a polyhedral cone due to the nonlinear nature of the section of the friction cone, i.e., the disk $\mathbf{D}(\mu R_N)$ defined by

$$\mathbf{D}(\mu R_N) = \{R_T \mid \sigma(R_T) = \mu R_N - \|R_T\| \geq 0\} \quad (14)$$

Faceting of the Coulomb's cone.

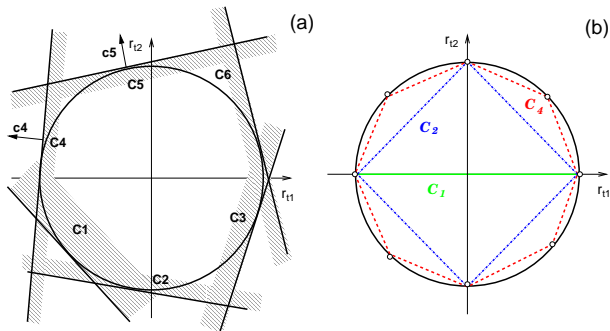


Figure: Approximation of the base of the Coulomb cone by an outer approximation (a) and by an interior 2ω -gon (b).

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP
Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The 3D frictional contact problem as an LCP

Outer Faceting

The friction disk \mathbf{D} can be approximated by an outer polygon (Klarbring, 1986 ; Klarbring and Björkman, 1988):

$$\mathbf{D}_{outer}(\mu R_N) = \bigcap_{i=1}^{\omega} \mathbf{D}_i(\mu R_N)$$

with (15)

$$\mathbf{D}_i(\mu R_N) = \{ R_T, \sigma_i(R_T) = \mu R_N - c_i^T R_T \geq 0 \}$$

We now assume that the contact law is of the form

$$- U_T \in N_{\mathbf{D}_{outer}(\mu R_N)}(R_T) \quad (16)$$

From (Rockafellar, 1970), the normal cone to $\mathbf{D}_{outer}(\mu R_N)$ is given by :

$$N_{\mathbf{D}_{outer}(\mu R_N)}(R_T) = \sum_{i=1}^{\omega} N_{\mathbf{D}_i(\mu R_N)}(R_T) \quad (17)$$

and the inclusion can be stated as:

$$- U_T = \sum_{i=1}^{\omega} \kappa_i \partial \sigma_i(R_T), \quad 0 \leq \sigma_i(R_T) \perp \kappa_i \geq 0 \quad (18)$$

Since $\sigma_i(R_T)$ is linear with respect to R_T , we obtain the following LCP :

$$- U_T = \sum_{i=1}^{\omega} \kappa_i c_i, \quad 0 \leq \sigma_i(R_T) \perp \kappa_i \geq 0 \quad (19)$$

The 3D frictional contact problem as an LCP

The time-discretized linear OSNSP, (\mathcal{P}_L)

$$(\mathcal{P}_L) \left\{ \begin{array}{l} U_{k+1} = \widehat{W}P_{k+1} + U_{\text{free}} \\ \forall \alpha \in I_a(\tilde{\mathbf{q}}_{k+1}), \\ \quad -U_{T,k+1}^\alpha = \sum_{i=1}^{\omega} \kappa_i^\alpha c_i \\ \quad \sigma_i(P_{T,k+1}^\alpha) = \mu P_{N,k+1}^\alpha - c_i^T P_{T,k+1}^\alpha \\ \quad 0 \leq U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha \perp P_{N,k+1}^\alpha \geq 0 \\ \quad 0 \leq \sigma_i^\alpha(P_{T,k+1}^\alpha) \perp \kappa_i^\alpha \geq 0 \end{array} \right. \quad (20)$$

LCP Formulation

Generally, the MLCP (20) can be reduced in an LCP in standard form assuming that at least one pair of vectors c_i is linearly independent. (see Glocker (2001)).

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The 3D frictional contact problem as an LCP

Inner Faceting

Another approach is based on an inner approximation as exposed in (Al-Fahed et al., 1991) and (Stewart and Trinkle, 1996),

$$\mathbf{D}_{inner}(\mu R_N) = \left\{ R_T = D\beta \mid \beta \geq 0, \mu R_N \geq \mathbb{1}^T \beta \right\} \quad (21)$$

where

- ▶ $\beta \in \mathbb{R}^2$,
- ▶ $\mathbb{1} = [1, \dots, 1]^T \in \mathbb{R}^\omega$,
- ▶ $D \in \mathbb{R}^{2 \times \omega}$ whose columns are the directions vectors d_j which are the coordinates of the vertices of the polygon.

MLCP formulation

$$\begin{cases} R_T = D\beta \\ 0 \leq \beta \perp \lambda \mathbb{1} + D^T U_T \geq 0 \\ 0 \leq \lambda \perp \mu R_N - \mathbb{1}^T \beta \geq 0 \end{cases} \quad (22)$$

where $\lambda \in \mathbb{R}$.

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The 3D frictional contact problem as an LCP

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

The time-discretized linear OSNSP, (\mathcal{P}_L)

$$\left\{ \begin{array}{l} U_{k+1} = \widehat{W}P_{k+1} + U_{\text{free}} \\ P_{T,k+1}^\alpha = D^\alpha \beta_{k+1}^\alpha \\ 0 \leq U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha \perp P_{N,k+1}^\alpha \geq 0 \\ 0 \leq \beta_{k+1}^\alpha \perp \lambda_{k+1}^\alpha \mathbb{1}^\alpha + D^{\alpha,T} U_{T,k+1}^\alpha \geq 0 \\ 0 \leq \lambda \perp \mu P_{N,k+1}^\alpha - \mathbb{1}^{T,\alpha} \beta_{k+1}^\alpha \geq 0 \end{array} \right\} \forall \alpha \in I_a(\tilde{q}_{k+1}) \quad (23)$$

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The 3D frictional contact problem as an LCP

LCP form for the inner faceting

$$\left\{ \begin{array}{l} \begin{bmatrix} U_{N,k+1} + eU_{N,k} \\ \kappa_{k+1} \\ \sigma_{k+1} \end{bmatrix} = M \begin{bmatrix} P_{N,k+1} \\ \beta_{k+1} \\ \lambda_{k+1} \end{bmatrix} + q \\ 0 \leq \begin{bmatrix} U_{N,k+1} + eU_{N,k}^\alpha \\ \kappa_{k+1} \\ \sigma_{k+1} \end{bmatrix} \perp \begin{bmatrix} P_{N,k+1} \\ \beta_{k+1} \\ \lambda_{k+1} \end{bmatrix} \geq 0 \end{array} \right. \quad (24)$$

where

$$M = \begin{bmatrix} \widehat{W}_{NN} & \widehat{W}_{NT}D & 0 \\ D^T \widehat{W}_{TN} & D^T \widehat{W}_{TT}D & \mathbb{1} \\ \mu & -\mathbb{1}^T & 0 \end{bmatrix} \quad (25)$$

and

$$q = \begin{bmatrix} U_{N,\text{free}} + eU_{N,\text{free}} \\ D^T(U_{T,\text{free}}) \\ 0 \end{bmatrix} \quad (26)$$

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The 3D frictional contact problem as an LCP

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

LCP form for the inner faceting

The variables $\kappa_{k+1} \in \mathbb{R}^\omega$ and $\sigma_{k+1} \in \mathbb{R}$ are given by the following equations:

$$\kappa_{k+1} = \lambda_{k+1} + D^T U_{T,k+1}, \quad \sigma_{k+1} = \mu P_{N,k+1} - \mathbb{1}^T \beta_{k+1} \quad (24)$$

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The 3D frictional contact problem as an LCP

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Proposition ((Stewart and Trinkle, 1996))

Let \widehat{W} be a PD matrix. The LCP defined by (24), (25) and (26) possesses solutions, which can be computed by Lemke's algorithm provided precautions are taken against cycling due to degeneracy.

Sketch of the proof

- ▶ M is co-positive

- ▶
$$z^T M z \geq 0 \text{ ; , for all } z \geq 0, \quad (25)$$

is satisfied

- ▶ Theorem 3.8.6 in (Cottle et al., 1992, page 179)

M only PSD

A condition has to be added on U_{free} to retrieve a similar result. This condition is related to the existence of a solution (Acary and Cadoux, 2009 ; Cadoux, 2009)

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The 3D frictional contact problem as an LCP

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Advantages and weaknesses of LCP formulation

- ⊕ Existence results and convergence for pivoting algorithms
- ⊖ Anisotropy in the frictional behaviour
- ⊖ Scaling issues. Number of variables and pivoting algorithms.

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Faceting process and induced anisotropy

Let us consider a ball of mass m lying on a horizontal plane under gravity g . A cycling external force defined by

$$F(t) = \begin{cases} \mu mg(\cos \frac{\pi}{3} \mathbf{i} + \sin \frac{\pi}{3} \mathbf{j}), & t \in [15k, 5 + 15k) \\ -\mu mg \mathbf{i}, & t \in [5 + 15k, 10 + 15k) \\ \mu mg(\cos \frac{\pi}{3} \mathbf{i} + -\sin \frac{\pi}{3} \mathbf{j}), & t \in [10 + 15k, 15(k + 1)) \end{cases}, \quad k \in \mathbb{N} \quad (26)$$

is applied to the ball.

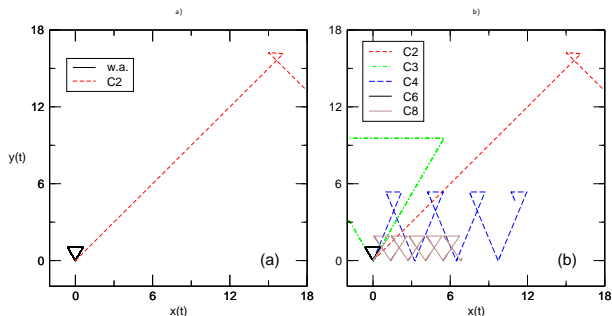


Figure: Ball trajectory under cycling loading.

Objectives

The time-discretized problems

- The linear time-discretized problem
- The Index Set of Forecast Active Constraints
- Some further notation

LCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- The frictional contact problem as an LCP
- Comments

NCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- Comments

Nonsmooth Equations. Formulations and solving

- Principle
- Alart & Curnier's Formulation
- Variants and line-search procedures.

Formulation and Resolution as VI/CP

- VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

The frictional contact problem as an LCP

Comments

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Nonlinear Complementarity Problem (NCP)

Definition (NCP)

Given a mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the NCP denoted by $\text{NCP}(F)$ is to find a vector $z \in \mathbb{R}^n$ such that

$$0 \leq z \perp F(z) \geq 0 \quad (27)$$

A vector z is called feasible (respectively strictly feasible) for the $\text{NCP}(F)$ if $z \geq 0$ and $F(z) \geq 0$ (respectively $z > 0$ and $F(z) > 0$).

Definition

A given mapping $F : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be

(a) monotone on X if

$$(x - y)^T (F(x) - F(y)) \geq 0, \text{ for all } x, y \in X \quad (28)$$

(b) strictly monotone on X if

$$(x - y)^T (F(x) - F(y)) > 0, \text{ for all } x, y \in X, x \neq y \quad (29)$$

(c) strongly monotone on X if there exists $\mu > 0$ such that

$$(x - y)^T (F(x) - F(y)) \geq \mu \|x - y\|^2, \text{ for all } x, y \in X \quad (30)$$

Nonlinear Complementarity Problem (NCP)

Theorem (Characterization of monotone mapping)

Given a continuously differentiable mapping $F : \mathcal{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ on the open convex set \mathcal{D} , the following statements are valid,

- (a) $F(\cdot)$ is monotone on \mathcal{D} if and only if $\nabla^T F(x)$ is PSD for all $x \in \mathcal{D}$
- (b) $F(\cdot)$ is strictly monotone on \mathcal{D} if $\nabla^T F(x)$ is PD for all $x \in \mathcal{D}$
- (c) $F(\cdot)$ is strongly monotone on \mathcal{D} if and only if $\nabla^T F(x)$ is uniformly PD for all $x \in \mathcal{D}$, i.e.

$$\exists \mu > 0, \quad z^T \nabla^T F(x) z^T \geq \mu \|z\|^2, \quad \forall x \in \mathcal{D} \quad (31)$$

Theorem

Given a continuous mapping $F : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, the following statements hold,

- (a) If $(\cdot)F$ is monotone on $X = \mathbb{R}_+^n$, the NCP(F) has a convex (possibly empty) solution set. Furthermore, if there exists a strictly feasible point, the NCP(F) has a non empty and compact solution set.
- (b) If $F(\cdot)$ is strictly monotone on $X = \mathbb{R}_+^n$, the NCP(F) has at most one solution.
- (c) If $F(\cdot)$ is strongly monotone on $X = \mathbb{R}_+^n$, the NCP(F) has a unique solution.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition

The Frictionless Case with Newton's Impact
Comments

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Nonlinear Complementarity Problem (NCP)

Definition

A given mapping $F : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be

(a) a P -function on X if

$$\max_{i=1\dots n} (x_i - y_i)(F_i(x) - F_i(y)) > 0, \quad \forall x, y \in X, x \neq y \quad (32)$$

(b) a uniform P -function if

$$\exists \mu > 0, \quad \max_{i=1\dots n} (x_i - y_i)(F_i(x) - F_i(y)) \geq \mu \|x - y\|^2, \quad \forall x, y \in X, x \neq y \quad (33)$$

Theorem

Given a continuous mapping $F : X \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, the following statements hold,

- (a) If $F(\cdot)$ is a P -function on X , then the $\text{NCP}(F)$ has at most one solution
- (b) If $F(\cdot)$ is a uniform P -function on X , then the $\text{NCP}(F)$ has a unique solution.

The proof can be found in (Moré, 1974).

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition

The Frictionless Case with Newton's Impact
Comments

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Nonlinear Complementarity Problem (NCP)

Newton–Josephy's and Linearization Methods

The standard Newton method to linearize $F(\cdot)$ is used, the following LCP

$$0 \leq z \perp F(z_k) + \nabla F(z_k)(x - x_k) \geq 0 \quad (34)$$

has to be solved to obtain z_{k+1} .

Newton–Robinson's

For theoretical considerations, Robinson (1988, 1992) proposed to use a linearization of the so-called normal map (see Section ?? for a general definition for VIs)

$$F^{nor}(y) = F(y^+) + (y - y^+) \quad (35)$$

where $y^+ = \max(0, y)$ stands for the positive part of y . Equivalence with the $NCP(F)$, is as follows: y is a zero of the normal map if and only if y^+ solves $NCP(F)$. The Newton–Robinson method uses a piecewise linear approximation of the normal map, namely

$$L_k(y) = F(y_k^+) + \nabla F(y_k^+)(y^+ - y_k^+) + y - y^+ \quad (36)$$

The Newton iterate y_{k+1} is a zero of $L_k(\cdot)$. The same y_{k+1} would be obtained by Newton–Josephy's method if z^k were set to y_k^+ in (34).

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

Comments

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The PATH solver

- ▶ The PATH solver (Dirkse and Ferris, 1995) is an efficient implementation of Newton–Robinson's method together with the path-search scheme.
- ▶ A “path-search” (as opposed to line-search) is then performed using the merit function $\|F_+(y)\|$. Standard theory of damped Newton's method can be extended to prove standard local and global convergence results (Ralph, 1994 ; Dirkse and Ferris, 1995).
- ▶ The construction of the piecewise linear path p_k is based on the use of pivoting methods. Each pivot corresponds to a kink in the path. In (Dirkse and Ferris, 1995), a modification of Lemke's algorithm is proposed to construct the path.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

Comments

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Nonlinear Complementarity Problem (NCP)

Generalized or Semismooth Newton's Methods

The principle of the generalized or semismooth Newton's method for LCPs is based on a reformulation in terms of possibly nonsmooth equations using the so-called C-function also called NCP-function.

Definition

A function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called a C-function (for complementarity) if

$$0 \leq w \perp z \geq 0 \iff \phi(w, z) = 0 \quad (37)$$

Well-known examples of C-function are

$$\phi(w, z) = \min(w, z) \quad (38a)$$

$$\phi(w, z) = \max(0, w - \rho z) - w, \rho > 0 \quad (38b)$$

$$\phi(w, z) = \max(0, z - \rho w) - z, \rho > 0 \quad (38c)$$

$$\phi(w, z) = \sqrt{w^2 + z^2} - z - w \quad (38d)$$

$$\phi(w, z) = \lambda(\sqrt{w^2 + z^2} - z - w) - (1 - \lambda)w_+z_+, \lambda \in (0, 1) \quad (38e)$$

$$\phi(w, z) = -wz + \frac{1}{2}\min^2(0, w + z) \quad (38f)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

Comments

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Nonlinear Complementarity Problem (NCP)

Then defining the following function associated with $\text{NCP}(F)$:

$$\Phi(z) = \begin{bmatrix} \phi(F_1(z), z_1) \\ \vdots \\ \phi(F_i(z), z_i) \\ \vdots \\ \phi(F_n(z), z_n) \end{bmatrix} \quad (39)$$

we obtain as an immediate consequence of the definitions of $\phi(\cdot)$ and $\Phi(\cdot)$ the following equivalence.

Lemma

Let $\phi(\cdot)$ be a C -function and the corresponding operator $\Phi(\cdot)$ defined by (39). A vector \bar{z} is a solution of $\text{NCP}(F)$ if and only if \bar{z} solves the nonlinear system of equations $\Phi(z) = 0$.

Numerical method

The standard Newton method is generalized to the nonsmooth case by the following scheme

$$z_{k+1} = z_k - H_k^{-1}\Phi(z_k), \quad H_k \in \partial\Phi(z_k) \quad (40)$$

Because the set $\partial\Phi(z_k)$ may not be a singleton (if z_k is a point of discontinuity of $\Phi(\cdot)$), we have to select an arbitrary element for H_k .

Nonlinear Complementarity Problem (NCP)

Comparison of the following implementation of solvers

- ▶ MILES (Rutherford, 1993) which is an implementation of the classical Newton–Joseph method,
- ▶ PATH
- ▶ NE/SQP (Gabriel and Pang, 1992 ; Pang and Gabriel, 1993) which is a generalized Newton's method based on the minimum function (38a); the search direction is computed by solving a convex QP at each iteration,
- ▶ QPCOMP (Billups and Ferris, 1995) which is an enhancement of the NE/SQP algorithm to allow iterates to escape from local minima,
- ▶ SMOOTH (Chen and Mangasarian, 1996) which is based on solving a sequence of smooth approximations of the NCP,
- ▶ PROXI (Billups, 1995) which is a variant of the QPCOMP algorithm using a nonsmooth Newton solver rather than a QP solver,
- ▶ SEMISMOOTH (DeLuca et al., 1996) which is an implementation of a semismooth Newton method using the Fischer–Bursmeister function,
- ▶ SEMICOMP (Billups, 1995) which is an enhancement of SEMISMOOTH based on the same strategy as QPCOMP.

All of these comparisons, which have been made in the framework of the MCP show that the PROXI, PATH and SMOOTH are superior on a large sample of test problems.

The Frictionless Case with Newton's Impact as an NCP

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

This problem can be stated in the following NCP for v_{k+1} and $P_{N,k+1}^\alpha$, $\alpha \in I_a(\tilde{q}_{k+1})$

$$\begin{cases} \mathcal{R}(v_{k+1}) = \sum_{\alpha \in I_a(\tilde{q}_{k+1})} H_N^\alpha(q_k + 1) P_{N,k+1}^\alpha \\ 0 \leq \widehat{H}_N^{\alpha, T}(q_k + 1) v_{k+1} + e^\alpha U_{N,k}^\alpha \perp P_{N,k+1}^\alpha \geq 0, \quad \forall \alpha \in I_a(\tilde{q}_{k+1}). \end{cases} \quad (41)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition

The Frictionless Case with Newton's Impact
Comments

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

A first NCP formulation

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

The Coulomb friction model (??) can be easily reformulated into the following CP:

$$\begin{cases} R_T \|U_T\| + \|R_T\| U_T = 0 \\ 0 \leq \|U_T\| \perp \mu R_N - \|R_T\| \geq 0 \end{cases} \quad (42)$$

Let us denote $\kappa = \|U_T\|$ the norm of U_T . The following CP can be stated

$$\begin{cases} \kappa = \|U_T\| \\ R_T \kappa + \|R_T\| U_T = 0 \\ 0 \leq \kappa \perp \mu R_N - \|R_T\| \geq 0 \end{cases} \quad (43)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

Comments

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

$$\left\{ \begin{array}{l} U_{k+1} = \widehat{W}P_{k+1} + U_{\text{free}} \\ \kappa_{k+1}^\alpha = \|U_{T,k+1}^\alpha\| \\ \tilde{U}_{N,k+1}^\alpha = U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha \\ P_{T,k+1}^\alpha \kappa_{k+1}^\alpha + \|P_{N,k+1}^\alpha\| U_{T,k+1}^\alpha = 0 \\ 0 \leq \kappa_{k+1}^\alpha \perp \mu^\alpha P_{N,k+1}^\alpha - \|P_{T,k+1}^\alpha\| \geq 0 \\ 0 \leq \tilde{U}_{N,k+1}^\alpha \perp P_{N,k+1}^\alpha \geq 0 \end{array} \right\} \forall \alpha \in I_a(\tilde{q}_{k+1}) \quad (44)$$

which can be casted into the MCP form (??) with $u = [U_{T,k+1}^\alpha, P_{N,k+1}^\alpha, P_{T,k+1}^\alpha]^T$ and $v = [\kappa_{k+1}^\alpha, U_N]^\top$.

Weaknesses of the formulation

Besides the difficulty to directly deal with a MCP in general form, the main drawback of this formulation is the lack of differentiability in the mapping involved in the CP.

A clever NCP formulation due to Glocker (1999)

Glocker (1999) adds three inequalities

$$\sigma_i(R_T) = \mu R_N - e_i^T R_T \geq 0, \quad i = 1, 2, 3 \quad (45)$$

where e_1, e_2, e_3 are three unit outwards vector defined by

$$e_i = [\cos \alpha_i, \sin \alpha_i], \quad \alpha_i = \frac{(4i - 3)\pi}{6} \quad (46)$$

We can remark that

$$\mathbf{D}(\mu R_N) = \bigcap_{i=1}^3 \mathbf{D}_i \cap \mathbf{D}(\mu r_N) \quad (47)$$

thus the Coulomb's frictional law remains as in (??). This normal cone condition leads to

$$-U_T \in \sum_{i=1}^3 e_i \kappa_i + \partial \sigma_D(R_T) \kappa_D \quad (48)$$

where $\sigma_D(R_T) = \mu^2 R_N^2 - \|R_T\|^2$ is a nonlinear friction saturation associated with the second-order cone.

The trick introduced by Glocker lies into the reformulation of this inclusion into an equation of the form

$$-U_T = \sum_{i=1}^3 e_i \kappa_i + 2R_T \kappa_D, \quad 0 \leq \kappa_j \perp \sigma_j \geq 0, \quad j = 1, 2, 3, D \quad (49)$$

A clever NCP formulation due to Glocker (1999)

The previous CP formulation yields

$$\left\{ \begin{array}{l} U_{k+1} = \widehat{W}P_{k+1} + U_{\text{free}} \\ -U_{T,k+1}^{\alpha} = \sum_{i=1}^3 e_i \kappa_{i,k+1}^{\alpha} + 2P_{T,k+1}^{\alpha} \kappa_{C,k+1}^{\alpha} \\ \sigma_i^{\alpha}(P_{T,k+1}^{\alpha}) = \mu^{\alpha} P_{N,k+1}^{\alpha} - e_i^T P_{T,k+1}^{\alpha}, i = 1, 2, 3 \\ \sigma_D^{\alpha}(P_{T,k+1}^{\alpha}) = (\mu^{\alpha} P_{N,k+1}^{\alpha})^2 - \|P_{T,k+1}^{\alpha}\|^2 \\ 0 \leq U_{N,k+1}^{\alpha} + e^{\alpha} U_{N,k}^{\alpha} \perp P_{N,k+1}^{\alpha} \geq 0 \\ 0 \leq \kappa_{j,k+1}^{\alpha} \perp \sigma_{j,k+1}^{\alpha} \geq 0, j = 1, 2, 3, D \end{array} \right\} \forall \alpha \in I_a(\tilde{q}_{k+1}) \quad (50)$$

Advantages

The mappings involved in the NCP formulation are differentiable, so standard NCP solvers should work

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

Comments

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

It is difficult to say something on the abilities of the NCP formulation to provide good numerical solvers, mainly due to the fact there is quite no attempt to use it in the literature.

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition

The Frictionless Case with Newton's Impact

Comments

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Objectives

The time-discretized problems

- The linear time-discretized problem
- The Index Set of Forecast Active Constraints
- Some further notation

LCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- The frictional contact problem as an LCP
- Comments

NCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- Comments

Nonsmooth Equations. Formulations and solving

- Principle
- Alart & Curnier's Formulation
- Variants and line-search procedures.

Formulation and Resolution as VI/CP

- VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Definition
The Frictionless Case with Newton's Impact
Comments

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictional contact 3D problem as a nonsmooth equation

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Principle

$$(\mathcal{P}_L) \left\{ \begin{array}{l} U_{k+1} = \widehat{W}P_{k+1} + U_{\text{free}} \\ \forall \alpha \in I_a(\tilde{q}_{k+1}), \\ \widehat{U}_{k+1}^\alpha = [U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha + \mu^\alpha \|U_{T,k+1}^\alpha\|, U_{T,k+1}^\alpha]^T \\ \mathbf{c}^{\alpha,*} \ni \widehat{U}_{k+1}^\alpha \perp P_{k+1}^\alpha \in \mathbf{C}^\alpha \end{array} \right.$$

is equivalent to

$$\Phi(U, R) = 0 \quad (51)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Principle

Alart & Curnier's Formulation

Variants and line-search procedures.

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Alart & Curnier's formulation

Curnier and Alart (1988) ; Alart and Curnier (1991) The previous linear problem can be written as

$$\left\{ \begin{array}{l} U_{k+1} = \widehat{W}P_{k+1} + U_{\text{free}} \\ P_{N,k+1}^\alpha = \text{proj}_{\mathbb{R}_+}(P_{N,k+1}^\alpha - \rho_N^\alpha (U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha)) \\ P_{T,k+1}^\alpha = \text{proj}_{\widehat{D}^\alpha(P_{N,k+1}^\alpha, U_{N,k+1}^\alpha)}(P_{T,k+1}^\alpha - \rho_T^\alpha \circ U_{T,k+1}^\alpha) \end{array} \right\} \forall \alpha \in I_a(\tilde{q}_{k+1}) \quad (52)$$

where $\rho_N^\alpha > 0$, $\rho_T^\alpha \in \mathbb{R}_+^2 \setminus \{0\}$ for all $\alpha \in I_a(\tilde{q}_{k+1})$ and the modified friction disk is

$$\widehat{D}^\alpha(P_{N,k+1}^\alpha, U_{N,k+1}^\alpha) = \mathbf{D}(\mu(\text{proj}_{\mathbb{R}_+}(P_{N,k+1}^\alpha - \rho_N^\alpha (U_{N,k+1}^\alpha + e^\alpha U_{N,k}^\alpha))) \quad (53)$$

for all $\alpha \in I_a(\tilde{q}_{k+1})$.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Principle

Alart & Curnier's Formulation

Variants and line-search procedures.

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictional contact 3D problem as a nonsmooth equation

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Alart& Curnier's formulation

The use of the projection operators $\text{proj}(\cdot)$, or more generally the natural and normal map, allows one to restate a CP or a VI into a system of nonlinear nonsmooth equations,

$$\Phi(U_{k+1}, P_{k+1}) = \begin{bmatrix} -U_{k+1} + \widehat{W}P_{k+1} + U_{\text{free}} \\ P_{N,k+1} - \text{proj}_{\mathbb{R}_+^a}(P_{N,k+1} - \rho_N \circ (U_{N,k+1} + e \circ U_{N,k})) \\ P_{T,k+1} - \text{proj}_{\widehat{D}(P_{N,k+1}, U_{N,k+1})}(P_{T,k+1} - \rho_T \circ U_{T,k+1}) \end{bmatrix} = 0 \quad (54)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Principle
Alart & Curnier's
Formulation

Variants and line-search
procedures.

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

The Frictional contact 3D problem as a nonsmooth equation

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Nonsmooth Newton's method

The standard Newton method is generalized to the nonsmooth case by the following scheme

$$z_{k+1} = z_k - H_k^{-1}\Phi(z_k), \quad H_k \in \partial\Phi(z_k) \quad (55)$$

Because the set $\partial\Phi(z_k)$ may not be a singleton (if z_k is a point of discontinuity of $\Phi(\cdot)$), we have to select an arbitrary element for H_k .

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Principle

Alart & Curnier's Formulation

Variants and line-search procedures.

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictional contact 3D problem as a nonsmooth equation

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Computation of a subgradient for the Alart–Curnier formulation

$H(U, P) \in \partial\Phi(U, P)$ which has the structure

$$H(U, P) = \begin{bmatrix} -I & 0 & \widehat{W}_{NN} & \widehat{W}_{NT} \\ 0 & -I & \widehat{W}_{TN} & \widehat{W}_{TT} \\ \partial_{U_N} \Phi_2(U, P) & 0 & \partial_{P_N} \Phi_2(U, P) & 0 \\ 0 & \partial_{U_T} \Phi_3(U, P) & \partial_{P_N} \Phi_3(U, P) & \partial_{P_T} \Phi_3(U, P) \end{bmatrix} \quad (56)$$

where the components of Φ are defined by

$$\begin{aligned} \Phi_1(U, P) &= -U_{k+1} + \widehat{W}P_{k+1} + U_{\text{free}} \\ \Phi_2(U, P) &= P_N - \text{proj}_{\mathbb{R}_+^a}(P_N - \rho_N \circ (U_N + e \circ U_{N,k})) \\ \Phi_3(U, P) &= P_T - \text{proj}_{\widehat{D}(P_N, U_N)}(P_{T,k+1} - \rho_T \circ U_T) \end{aligned} \quad (57)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Principle

**Alart & Curnier's
Formulation**

Variants and line-search
procedures.

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

The Frictional contact 3D problem as a nonsmooth equation

Computation of a subgradient for the Alart–Curnier formulation

$$\partial_{U_N} \Phi_2(U, P) = \begin{cases} \rho_N & \text{if } P_N - \rho_N U_N > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\partial_{P_N} \Phi_2(U, P) = \begin{cases} 0 & \text{if } P_N - \rho_N U_N > 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\partial_{U_T} \Phi_3(U, P) = \begin{cases} \rho_T I_{2 \times 2} & \text{if } P_T - \rho_T U_T \in \widehat{\mathbf{D}}(P_N, U_N) \\ \rho_T \mu P_N \Gamma(P_T - \rho_T U_T) & \text{otherwise} \end{cases}$$

$$\partial_{P_N} \Phi_3(U, P) = \begin{cases} 0 & \text{if } P_T - \rho_T U_T \in \widehat{\mathbf{D}}(P_N, U_N) \\ 0 & \text{if } P_T - \rho_T U_T \notin \widehat{\mathbf{D}}(P_N, U_N) \text{ and } P_N - \rho_N U_N \leq 0 \\ -\mu \frac{P_T - \rho_T U_T}{\|P_T - \rho_T U_T\|} & \text{otherwise} \end{cases}$$

$$\partial_{P_T} \Phi_3(U, P) = \begin{cases} 0 & \text{if } P_T - \rho_T U_T \in \widehat{\mathbf{D}}(P_N, U_N) \\ I_2 & \text{if } P_T - \rho_T U_T \notin \widehat{\mathbf{D}}(P_N, U_N) \text{ and } P_N - \rho_N U_N > 0 \\ I_2 - \mu(P_N - \rho_N U_N) \Gamma(P_T - \rho_T U_T) & \text{otherwise} \end{cases}$$

(58)

where the function $\Gamma(\cdot)$ is defined by $\Gamma(x) = \frac{I_{2 \times 2}}{\|x\|} - \frac{x x^T}{\|x\|^3}$

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Principle

Alart & Curnier's Formulation

Variants and line-search procedures.

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Software development and advice

References

The Frictional contact 3D problem as a nonsmooth equation

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Line-search procedure.

- ▶ (Christensen et al., 1998 ; Christensen and Pang, 1998 ; Christensen, 2000), the semi-smoothness is shown and a line-search procedure is based on a merit function such as

$$\Psi(U, P) = \frac{1}{2} \Phi(U, P)^T \Phi(U, P). \quad (59)$$

Let $\alpha^i = \rho^{m^i}$, where $\rho \in (0, 1)$ and m^i is the smallest non-negative integer m for which the following decrease criterion holds:

$$\Psi(z^i + \rho^m d^i) \geq (1 - 2\sigma\rho^m)\Psi(z^i) \quad (60)$$

where $\sigma \in (0, \frac{1}{2})$ is a given parameter.

- ▶ There is no clear study of the influence of the line-search procedure.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Principle

Alart & Curnier's Formulation

Variants and line-search procedures.

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictional contact 3D problem as a nonsmooth equation

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Variants

- ▶ (Park and Kwak, 1994 ; Leung et al., 1998) write the 2D problem as

$$\text{If } g(q) \leq 0, \quad \Psi_1(U_N, R_N) = \min(U_N, R_N) = 0 \quad (61)$$

$$\begin{aligned} \Psi_2(U_T, R_T) &= U_T + \min(0, \mu \max(0, R_N - U_N) + R_T - U_T) \\ &+ \max(0, -\mu \max(0, R_N - U_N) + R_T - U_T) = 0 \end{aligned} \quad (62)$$

and the 3D case, they introduce the sliding angle as variable.

- ▶ citepXuewen.ea2000, another direct equation-based reformulation is presented

$$\begin{cases} \Psi_1(U, R) = \min(U_N, R_N) = 0 \\ \Psi_2(U, R) = \min(\|U_T\|, \mu R_N - \|R_T\|) \\ \Psi_3(U, R) = |U_{T1}R_{T2} - U_{T2}R_{T1}| + \max(0, U_{T1}R_{T1}) \end{cases} \quad (63)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Principle

Alart & Curnier's
Formulation

Variants and line-search
procedures.

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Objectives

The time-discretized problems

- The linear time-discretized problem
- The Index Set of Forecast Active Constraints
- Some further notation

LCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- The frictional contact problem as an LCP
- Comments

NCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- Comments

Nonsmooth Equations. Formulations and solving

- Principle
- Alart & Curnier's Formulation
- Variants and line-search procedures.

Formulation and Resolution as VI/CP

- VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Principle

Alart & Curnier's
Formulation

Variants and line-search
procedures.

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

The Frictional contact 3D problem as a VI/CP problem

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

With the following definitions for the Cartesian product of Coulomb cones,

$$\mathbf{C} = \prod_{\alpha \in I_a(\tilde{q}_{k+1})} \mathbf{C}^\alpha, \quad \mathbf{C}^* = \prod_{\alpha \in I_a(\tilde{q}_{k+1})} \mathbf{C}^{\alpha,*}, \quad (64)$$

the following CP over cones can be written

$$\begin{cases} \widehat{U}_{k+1} = \widehat{W}P_{k+1} + U_{\text{free}} - G(P_{k+1}) \\ \mathbf{C}^* \ni \widehat{U}_{k+1} \perp P_{k+1} \in \mathbf{C}. \end{cases} \quad (65)$$

The function $G : \mathbb{R}^{3a} \rightarrow \mathbb{R}^{3a}$ defined by

$$G(P) = \left[\left[\mu^\alpha \|[WP + U_{\text{free}}]_{\text{T}}^\alpha\| + e^\alpha U_{\text{N},k}^\alpha, 0 \right], \alpha \in I_a(\tilde{q}_{k+1}) \right]^T \quad (66)$$

is a nonlinear and nonsmooth function of P .

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

VI/CP formulation

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

The Frictional contact 3D problem as a VI/CP problem

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

The formulation in terms of VI is straightforward due to the equivalence between VIs and CPs. The linear problem (\mathcal{P}_L) is equivalent to the following VI

$$(\widehat{W}P_{k+1} + U_{\text{free}} - G(P_{k+1}))^T(P^* - P_{k+1}) \geq 0, \quad \text{for all } P^* \in \mathbf{C}. \quad (67)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictional contact 3D problem as a VI/CP problem

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Projection-type methods for VI

We recall that the most basic projection-type method for the VI

$$F(P)^T(P^* - P) \geq 0, \quad \text{for all } P^* \in \mathbf{C} \quad (68)$$

is a fixed point method such that

$$P^{i+1} = \text{proj}_{\mathbf{C}}(P^i - \rho F(P^i)), \quad \rho > 0. \quad (69)$$

Variants: Extra-gradient, hyperplane method

Projection-type methods for VI

For the three-dimensional frictional contact, i.e.

$F(P) = \widehat{W}P + U_{\text{free}} - G(P)$, it yields

$$P^{i+1} = \text{proj}_{\mathbf{C}} \left[(I - \rho \widehat{W})P^i - \rho G(P^i) + \rho U_{\text{free}} \right] \quad \rho > 0. \quad (70)$$

This method has been initiated by De Saxcé and Feng (1990) and extensively tested in (Feng, 1991, 1995 ; De Saxcé and Feng, 1998). The authors term this method an Uzawa's method for solving the variational inequality.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictional contact 3D problem as a VI/CP problem

Projection/splitting methods. The NSGS method (Jean and Moreau, 1992)

1. The Delassus operator W is usually sparse block structured in multibody dynamics (see the formulation (5) and (??)). The splitting is chosen to take advantage of this structure.
2. Each subproblem of frictionless/frictional contact for a single contact α can be either analytically solved or easily approximated.

Principle

$$\left\{ \begin{array}{l} U_{k+1}^{\alpha,i+1} - \widehat{W}^{\alpha\alpha} P_{k+1}^{\alpha,i+1} = U_{\text{free}}^{\alpha} + \sum_{\beta < \alpha} \widehat{W}^{\alpha\beta} P_{k+1}^{\beta,i+1} + \sum_{\beta > \alpha} \widehat{W}^{\alpha\beta} P_{k+1}^{\beta,i} \\ \widehat{U}_{k+1}^{\alpha,i+1} = \left[U_N^{\alpha,i+1} + e^{\alpha} U_{N,k}^{\alpha} + \mu^{\alpha} \|U_{T,k+1}^{\alpha,i+1}\|, U_{T,k+1}^{\alpha,i+1} \right]^T \\ \mathbf{C}^{\alpha,*} \ni \widehat{U}_{k+1}^{\alpha,i+1} \perp P_{k+1}^{\alpha,i+1} \in \mathbf{C}^{\alpha} \end{array} \right. \quad (71)$$

for all $\alpha, \beta \in I_a(\tilde{\mathbf{q}}_{k+1})$.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

The Frictional contact 3D problem as a VI/CP problem

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Possible local solvers :

- ▶ Analytical solutions for the frictionless and the 2D frictional case
- ▶ Projection onto the friction cone or the friction disk
- ▶ Local newton solver on one contact.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

VI/CP formulation

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

The Frictional contact 3D problem as a VI/CP problem

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Newton's method for VI

the following nonsmooth equations holds

$$\mathbf{F}_C^{nat}(P) = P - \text{proj}_C((I - \widehat{W})P + U_{\text{free}} - G(P)) = 0 \quad (72)$$

or equivalently

$$\Phi(P) = P - \text{proj}_C((I - \rho\widehat{W})P + \rho U_{\text{free}} - \rho G(P)) = 0, \quad \rho > 0. \quad (73)$$

Use the Clarke generalized subgradient (see Section ??) to generate iterates such that

$$P^{i+1} = P^i - H^{i-1}\Phi(P^i), \text{ with } H^{i-1} \in \partial\Phi(P^i). \quad (74)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Objectives

The time-discretized problems

- The linear time-discretized problem
- The Index Set of Forecast Active Constraints
- Some further notation

LCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- The frictional contact problem as an LCP
- Comments

NCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- Comments

Nonsmooth Equations. Formulations and solving

- Principle
- Alart & Curnier's Formulation
- Variants and line-search procedures.

Formulation and Resolution as VI/CP

- VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

VI/CP formulation

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Optimization-based algorithms.

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Principle

Try to rely the solvers for the 3D frictional contact problem on (convex) optimization problems.

Why ?

- ▶ Reliable existing solvers may be available
- ▶ Convergence may be shown on stability arguments (decreasing an cost function)

Main approaches

Sequential convex QP on

- ▶ the Tresca's cylinder
- ▶ the second order cone

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Tresca's Friction

the so-called Tresca friction model can be invoked

$$\begin{cases} \text{If } U_T = 0 & \text{then } \|R_T\| \leq \theta \\ \text{If } U_T \neq 0 & \text{then } \|R_T(t)\| = \theta, \quad \text{and } \exists a \geq 0 \text{ such that } R_T(t) = -aU_T(t) \end{cases} \quad (75)$$

where θ is the friction threshold.

Equivalent formulation

$$-U \in \partial\psi_{\mathbf{T}_\theta}(R), \quad \text{or } R \in \partial\psi_{\mathbf{T}_\theta}^*(-U) \quad (76)$$

with $\mathbf{T}_\theta = \mathbb{R}_+ \times \mathbf{D}_\theta$ is the friction cylinder or the Tresca cylinder and \mathbf{D}_θ is a disk with radius θ .

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Optimization-based algorithms.

Inclusion into the normal cone for Tresca's friction.

Introducing the Cartesian product of the Tresca cylinders such that

$$\mathbf{T} = \prod_{\alpha \in I_a(\tilde{q}_{k+1})} \mathbf{T}_{\theta}^{\alpha}, \quad \mathbf{T}^* = \prod_{\alpha \in I_a(\tilde{q}_{k+1})} \mathbf{T}_{\theta}^{\alpha,*}, \quad (77)$$

the following inclusion can be written for the linear problem (\mathcal{P}_L) with Tresca's friction

$$-\left(\widehat{W}P_{k+1} + U_{\text{free}}^e\right) \in \partial\psi_{\mathbf{T}}(P_{k+1}) \quad (78)$$

where $U_{\text{free}}^e = U_{\text{free}} + \left[\left[e^{\alpha} \circ U_{N,k}^{\alpha}, 0 \right], \alpha \in I_a(\tilde{q}_{k+1}) \right]^T$.

Equivalent minimization problem

Under the assumption that \widehat{W} is a symmetric PSD matrix, we may consider the following minimization problem

$$\begin{aligned} &\text{minimize} && \frac{1}{2} P_{k+1}^T \widehat{W} P_{k+1} + P_{k+1}^T U_{\text{free}}^e \\ &\text{subject to} && P_{k+1} \in \mathbf{T} \end{aligned} \quad (79)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Optimization-based algorithms.

Require: $W, U_{\text{free}}^e, U_{\text{free}}, \mu$

Require: P_{k+1}^0

Ensure: U_{k+1}, P_{k+1} solution of the problem (\mathcal{P}_L)

$i \leftarrow 0$

while error $>$ tol **do**

$\theta^i \leftarrow \mu \circ P_{N,k+1}^i$

Solve (possibly inexactly) the minimization problem (79) that is

$$P_{k+1}^{i+1} \leftarrow \operatorname{argmin}_{P \in \mathcal{T}_{\theta^i}} \frac{1}{2} P^T \widehat{W} P + P^T U_{\text{free}}^e$$

$i \leftarrow i + 1$

Evaluate error.

end while

$U_{k+1} \leftarrow \widehat{W} P_{k+1} + U_{\text{free}}$

Fixed-point algorithm on the friction threshold

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

Vincent Acary
INRIA Rhône-Alpes,
Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Optimization-based algorithms.

PhD Thesis of Florent Cadoux. (co-supervised with C. Lemaréchal)

- ▶ Introduce **extra variable** s^i at each contact

$$s^i := \|U_T^i\| \quad (80)$$

- ▶ perform the **change of variables** (cf De Saxcé)

$$U \longrightarrow \hat{U} := U + \mu sn$$

Incremental problem

$$\begin{cases} Mv + f = H^T P \\ \hat{U} = Hv + w + \mu sn \\ C^* \ni \hat{U} \perp P \in C \end{cases} \quad (81)$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

Why should we do that ?

(81) are **KKT conditions** of two **convex** optimization problems
(SOCP: second order cone programs)

primal problem

$$\begin{cases} \min & J(v) := \frac{1}{2}v^\top Mv + f^\top v \\ & Hv + w + \alpha s \in C^* \end{cases} \quad (D_s)$$

dual problem

$$\begin{cases} \min & J_s(P) := \frac{1}{2}P^\top WP - b_s^\top P \\ & P \in C \end{cases} \quad (P_s)$$

with $W = HM^{-1}H^\top$ and $b_s = \alpha s + \beta$

Side note: when $\mu = 0$, incremental problem is a **QP** (Moreau)

Final reformulation

- ▶ Introducing

$$U(s) := \operatorname{argmin}_U(P_s) = \operatorname{argmin}_U(D_s)$$

practically **computable** by optimization software, and

$$F^i(s) := \|U_T^i(s)\|,$$

- ▶ the incremental problem becomes

fixed point problem

$$F(s) = s$$

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Assumption

Assumption

$$\exists v \in \mathbb{R}^m : Hv + w \in C^*$$

- **Interpretation:** it is kinematically possible to enforce

$$u \in C^*$$

at each contact

- **not only** the intuitive $u_N \geq 0$!

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Consequence

Using the assumption,

- ▶ the application $F : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ is **well-defined**
- ▶ it is **continuous**
- ▶ it is **bounded**
- ▶ apply Brouwer's theorem

Theorem

A fixed point exists

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Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Can this be used in practice ?

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

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Grenoble.

The fixed point equation $F(s) = s$ can be tackled by

- ▶ **fixed-point** iterations

$$s \leftarrow F(s)$$

- ▶ **Newton** iterations

$$s \leftarrow \text{Jac}[F](s) \setminus F(s)$$

- ▶ Variants possible (truncated resolution of inner problem...)

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Does it work ?

- ▶ fixed-point iterations:
 - ▶ expensive
 - ▶ not very robust
- ▶ Newton:
 - ▶ usually **very few** iterations...
 - ▶ ...but they are **expensive**
- ▶ **bottleneck**: SOCP solver
- ▶ practical interest is unclear yet
 - ▶ more robust ?
 - ▶ faster ?

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

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Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

Some comparisons and
advices

References

Objectives

The time-discretized problems

- The linear time-discretized problem
- The Index Set of Forecast Active Constraints
- Some further notation

LCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- The frictional contact problem as an LCP
- Comments

NCP formulations and solving

- Definition
- The Frictionless Case with Newton's Impact
- Comments

Nonsmooth Equations. Formulations and solving

- Principle
- Alart & Curnier's Formulation
- Variants and line-search procedures.

Formulation and Resolution as VI/CP

- VI/CP formulation

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

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Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving

Formulation and Resolution
as VI/CP

Optimization based
Algorithms

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Some comparisons and
advices

References

Comparisons between solvers

There is no thorough and systematic comparisons of the various methods

→ We need a standard library of benchmarks (as for MCPLib).

but

- ▶ For the 3D second-order cone, there is no results of convergence of solvers if a solution exists.
- ▶ When \widehat{W} has full rank, damped newton methods based on the Alart–Curnier formulation or the SOCP formulation provides quite good results.
- ▶ When \widehat{W} has not full rank, the projection–splitting based (projected gradient–like) methods provide a robust alternative but very slow convergence is achieved. Cycling can also be noted.
- ▶ The methods based on some optimization sub–problems can provide a good alternative to the previous methods, but their convergence are not shown. Moreover, they need specific developments of optimization solver that take into account the specific nature of the mechanical problem.

Thank you for your attention.

Objectives

The time-discretized
problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations.
Formulations and solving




Formulation and Resolution
as VI/CP

Optimization based
Algorithms

Theoretical interest

**Some comparisons and
advices**

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Lecture 3. Solvers for the time-discretized unilateral contact with Coulomb's friction problem.

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Grenoble.

Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

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Objectives

The time-discretized problems

LCP formulations and solving

NCP formulations and solving

Nonsmooth Equations. Formulations and solving

Formulation and Resolution as VI/CP

Optimization based Algorithms

Theoretical interest

Some comparisons and advices

References

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