

## Lecture 2. Numerical methods for time integration of NonSmooth Mechanical systems

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Time-stepping schemes

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# Objectives of the lecture

- ▶ Principles and Design of Event-tracking (Event-Driven) schemes.  
Pros and cons.
- ▶ Principles and Design of Event-capturing (Time-stepping) schemes.  
Pros and cons.
- ▶ Comparison
- ▶ Toward higher order schemes and adaptive time-step strategies

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## Definition (Nonsmooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ v^+ = \dot{q}^+ \end{cases} \quad (1)$$

where  $di$  is the reaction measure and  $dt$  is the Lebesgue measure.

## Decomposition of measure

$$\begin{cases} dv = \gamma dt + (v^+ - v^-) d\nu + dv_s \\ di = f dt + p d\nu + di_s \end{cases} \quad (2)$$

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Substituting the decomposition of measures into the nonsmooth Lagrangian Dynamics, one obtains

## Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \quad (3)$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \quad (4)$$

## Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t, q, v)dt = fdt \quad (5)$$

or

$$M(q)\gamma^+ + F(t, q, v^+) = f^+ [dt - a.e.] \quad (6)$$

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## The impact equations

The impact equations can be written at the time,  $t_i$  of discontinuities:

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \quad (7)$$

This equation will be solved at the time of impact together with an impact law. That is for an Newton impact law

$$\left\{ \begin{array}{l} M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \\ U_N^+(t_i) = H^T(q(t_i))v^+(t_i) \\ U_N^-(t_i) = H^T(q(t_i))v^-(t_i) \\ p_i = H(q(t_i))P_{N,i} \\ 0 \leq U_N^+(t_i) + eU_N^-(t_i) \perp P_{N,i} \geq 0 \end{array} \right. \quad (8)$$

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## The impact equations reduced on the local unknowns

One obtains the following LCP at time  $t_i$  of discontinuities of  $v$ :

$$\begin{cases} U_N^+(t_i) = H(q(t_i))(M(q(t_i)))^{-1}H(q(t_i))P_{N,i} + U_N^-(t_i) \\ 0 \leq U_N^+(t_i) + eU_N^-(t_i) \perp P_{N,i} \geq 0 \end{cases} \quad (9)$$

if the matrix  $M(q(t_i))$  is assumed to be invertible.

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## The smooth dynamics

The following smooth system are then to be solved ( $dt - a.e.$ ) :

$$\begin{cases} M(q(t))\gamma^+(t) + F(t, q, v^+) = f^+(t) \\ g = g(q(t)) \\ f^+ = H(q)F^+(t) \\ 0 \leq g \perp F^+(t) \geq 0 \end{cases} \quad (10)$$

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# Reformulations of the unilateral constraints on Different kinematics levels

## Differentiation of the constraints w.r.t time

The constraints  $g = g(q(t))$  can be differentiated with respect to time as follows in the Lagrangian setting:

$$\begin{cases} \dot{g}(q(t^+)) = U_N^+(t) = \nabla g^T(q(t))v^+(t) \\ \ddot{g}(q(t^+)) = \dot{U}_N^+(t) = \Gamma_N(t^+) = \nabla g^T(q(t))\gamma^+(t) + \frac{d}{dt}(\nabla g^T(q(t)))v^+(t) \end{cases} \quad (11)$$

## Comments. Index reduction techniques.

Solving the smooth dynamics requires that the complementarity condition  $0 \leq g \perp F^+(t) \geq 0$  must be written now at different kinematic level, i.e. in terms of right velocity  $U_N^+$  and in terms of accelerations  $\Gamma_N^+$ .

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# Reformulations of the unilateral constraints on Different kinematics levels

## At the velocity level

Assuming that  $U_N^+$  is right-continuous by definition of the right limit of a B.V. function, the complementarity condition implies, in terms of velocity, the following relation,

$$-F^+ \in \begin{cases} 0 & \text{if } g > 0 \\ 0 & \text{if } g = 0, U_N^+ > 0 \\ ] -\infty, 0] & \text{if } g = 0, U_N^+ = 0 \end{cases} . \quad (12)$$

A rigorous proof of this assertion can be found in (Glocker, 2001).

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# Reformulations of the unilateral constraints on Different kinematics levels

## Equivalent formulations

- Inclusion into  $N_{\mathbb{R}^+}(U_N^+)$

$$-F^+ \in \begin{cases} 0 & \text{if } g > 0 \\ N_{\mathbb{R}^+}(U_N^+) & \text{if } g = 0 \end{cases} \quad (12)$$

- Inclusion into  $N_{T_{\mathbb{R}^+(g)}}(U_N^+)$

$$-F^+ \in N_{T_{\mathbb{R}^+(g)}}(U_N^+) \quad (13)$$

- In a complementarity formalism

$$\begin{aligned} \text{if } g = 0 & \quad 0 \leq U_N^+ \perp F^+ \geq 0 \\ \text{if } g > 0 & \quad F^+ = 0 \end{aligned} \quad (14)$$

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## At the acceleration level

In the same way, the complementarity condition can be written at the acceleration level as follows.

$$-F^+ \in \begin{cases} 0 & \text{if } g > 0 \\ 0 & \text{if } g = 0, U_N^+ > 0 \\ 0 & \text{if } g = 0, U_N^+ = 0, \Gamma_N^+ > 0 \\ ] -\infty, 0] & \text{if } g = 0, U_N^+ = 0, \Gamma_N^+ = 0 \end{cases} \quad (15)$$

A rigorous proof of this assertion can be found in (Glocker, 2001).

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## Equivalent formulations

- ▶ Inclusion into a cone  $N_{\mathbb{R}^+}(\Gamma_N^+)$

$$-F^+ \in \begin{cases} 0 & \text{if } g > 0 \\ 0 & \text{if } g = 0, U_N^+ > 0 \\ N_{\mathbb{R}^+}(\Gamma_N^+) & \end{cases} \quad (15)$$

- ▶ Inclusion into  $N_{T_{\mathbb{R}^+}(\mathcal{g})}(U_N^+)(\Gamma_n^+)$

$$-F^+ \in N_{T_{\mathbb{R}^+}(\mathcal{g})}(U_N^+)(\Gamma_n^+) \quad (16)$$

- ▶ In the complementarity formalism,

$$\begin{array}{ll} \text{if } g = 0, U_N^+ = 0 & 0 \leq \Gamma_N^+ \perp F^+ \geq 0 \\ \text{otherwise} & F^+ = 0 \end{array} \quad (17)$$

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## Trivial inclusions

$$N_K(\mathbf{g}(\mathbf{q})) \supset N_{T_{\mathbb{R}^+}(\mathbf{g}(\mathbf{q}))}(U_N^+) \supset N_{T_{T_{\mathbb{R}^+}(\mathbf{g}(\mathbf{q}))}(U_N^+)}(\Gamma_n^+) \quad (18)$$

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# Reformulations of the smooth dynamics at acceleration level.

## The smooth dynamics as an inclusion

$$\left\{ \begin{array}{l} M(q(t))\gamma^+(t) + F(t, q, v^+) = f^+(t) \\ \Gamma_N = \nabla_q^T g(q)\gamma^+ + \frac{d}{dt}(\nabla_q^T g(q))v^+ \\ f^+ = \nabla_q g(q(t))F^+ \\ -F^+ \in N_{T_{T_{\mathbb{R}^+}(g)}(U_N^+)}(\Gamma_n) \end{array} \right. \quad (19)$$

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# Reformulations of the smooth dynamics at acceleration level.

## The smooth dynamics as a LCP

When the condition,  $g = 0$ ,  $U_N^+ = 0$  is satisfied, we obtain the following LCP

$$\begin{cases} M(q(t))\gamma^+(t) + F(t, q, v^+) = \nabla_q g(q(t))F^+(t) \\ \Gamma_N^+ = \nabla_q g^T(q)\gamma^+ + \frac{d}{dt}(\nabla_q g^T(q))v^+ \\ 0 \leq \Gamma_N^+ \perp F^+ \geq 0 \end{cases} \quad (20)$$

which can be reduced on variable  $\Gamma_N^+$  and  $F^+$ , if  $M(q(t))$  is invertible,

$$\begin{cases} \Gamma_N^+ = \nabla_q g^T(q)M^{-1}(q(t))(-F(t, q, v^+)) + \frac{d}{dt}(\nabla_q g^T(q))v^+ \\ \quad + \nabla_q g(q)M^{-1}\nabla_q g(q(t))F^+(t) \\ 0 \leq \Gamma_N^+ \perp F^+ \geq 0 \end{cases} \quad (21)$$

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# The case of a single contact.

## Two modes for the nonsmooth dynamics

1. *The constraint is not active.*  $F^+ = 0$

$$M(q)\gamma^+ + F(\cdot, q, v) = 0 \quad (22)$$

In this case, we associate to this step an integer,  $status_k = 0$ .

2. *The constraint is active.* Bilateral constraint  $\Gamma_N^+ = 0$ ,

$$\begin{bmatrix} M(q) & -\nabla_q g(q) \\ \nabla_q g^T(q) & 0 \end{bmatrix} \begin{bmatrix} \gamma^+ \\ F^+ \end{bmatrix} = \begin{bmatrix} -F(\cdot, q, v) \\ \nabla_q g^T(q)v^+ \end{bmatrix} \quad (23)$$

In this case, we associate to this step an integer,  $status_k = 1$ .

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[Case 1]  $status_k = 0$ .

Integrate the system (22) on the time interval  $[t_k, t_{k+1}]$

Case 1.1  $g_{k+1} > 0$ .

The constraint is still not active

$status_{k+1} \leftarrow 0$

Case 1.2  $g_{k+1} = 0, U_{N,k+1} < 0$

An impact occurs

Solve the impact equation (9) with  $U^- \leftarrow U_{N,k+1} < 0$

$U_{N,k+1} \leftarrow U^+$ .

Two cases are then possible:

Case 1.2.1  $U_+ > 0$ .

The constraint ceases to be active

$status_{k+1} \leftarrow 0$ .

Case 1.2.2  $U_+ = 0$ .

The relative post-impact velocity vanishes

Solve the LCP (20) to obtain the new status.

Three cases are then possible:

Case 1.2.2.1  $\Gamma_{N,k+1} > 0, F_{k+1} = 0$

The constraint is still not active

$status_{k+1} \leftarrow 0$ .

Case 1.2.2.2  $\Gamma_{N,k+1} = 0, F_{k+1} > 0$

The constraint has to be activated

$status_{k+1} \leftarrow 1$ .

Case 1.2.2.3  $\Gamma_{N,k+1} = 0, F_{k+1} = 0$

This case is undetermined.

We need to know the value of  $\hat{\Gamma}_N^+$ .

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[Case 1]  $status_k = 0$ .

Integrate the system (22) on the time interval  $[t_k, t_{k+1}]$

Case 1.3  $g_{k+1} = 0, U_{N,k+1} = 0$  we have grazing constraint

Solve the LCP (20) to obtain the new status assuming that

$$U^+ = U^- = U_{N,k+1}.$$

Three cases are then possible:

Case 1.3.1  $\Gamma_{N,k+1} > 0, F_{k+1} = 0$  The constraint is still not active

$$status_{k+1} \leftarrow 0.$$

Case 1.3.2  $\Gamma_{N,k+1} = 0, F_{k+1} > 0$  The constraint has to be activated

$$status_{k+1} \leftarrow 1.$$

Case 1.3.3  $\Gamma_{N,k+1} = 0, F_{k+1} = 0$  This case is undetermined.

We need to know the value of  $\dot{\Gamma}_N^+$ .

Case 1.4  $g_{k+1} = 0, U_{N,k+1} > 0$  Activation of constraints not detected.

Seek for the first time  $t_*$  such that  $g(q(t_*)) = 0$ .

$$t_{k+1} \leftarrow t_*.$$

Perform all of this procedure keeping with  $status_k \leftarrow 0$ .

Case 1.5  $g_{k+1} < 0$  Activation of constraints not detected.

Seek for the first time  $t_*$  such that  $g(q(t_*)) = 0$ .

$$t_{k+1} \leftarrow t_*.$$

Perform all of this procedure keeping with  $status_k \leftarrow 0$ .

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## [Case 2] $status_k = 1$

Integrate the system (23) on the time interval  $[t_k, t_{k+1}]$

**Case 2.1**  $g_{k+1} \neq 0$  or  $U_{N,k+1} = 0$

Something is wrong in the time integration or the drift from the constraints is too huge.

**Case 2.2**  $g_{k+1} = 0, U_{N,k+1} = 0$

In this case, we assume that  $U^+ = U^- = U_{N,k+1}$  and we compute  $\Gamma_{N,k+1}, F_{k+1}$  thanks to the LCP (20) assuming that  $U^+ = U^- = U_{N,k+1}$ . Three cases are then possible

**Case 2.2.1**  $\Gamma_{N,k+1} = 0, F_{k+1} > 0$

The constraint is still active. We set  $status_{k+1} = 1$ .

**Case 2.2.2**  $\Gamma_{N,k+1} > 0, F_{k+1} = 0$

The bilateral constraint is no longer valid. We seek for the time  $t_*$  such that  $F^+ = 0$ . We set  $t_{k+1} = t_*$  and we perform the integration up to this instant. We perform all of these procedure at this new time  $t_{k+1}$

**Case 2.2.3**  $\Gamma_{N,k+1} = 0, F_{k+1} = 0$

This case is undetermined. We need to know the value of  $\dot{\Gamma}_N^+$ .

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## Comments

- ▶ The Delassus example.  
In the one-contact case, a naive approach consists in to suppressing the constraint if  $F_{k+1} < 0$  after a integration with a bilateral constraints.  
→ Work only for the one contact case.
- ▶ The role of the “ $\varepsilon$ ”  
In practical situation, all of the test are made up to an accuracy threshold. All statements of the type  $g = 0$  are replaced by  $|g| < \varepsilon$ . The role of these epsilons can be very important and they are quite difficult to size.

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## Comments

- ▶ If the ODE solvers is able to perform the root finding of the function  $g = 0$  for  $status_k = 0$  and  $F^+ = 0$  for  $status_k = 1$   
→ the case 1.4, 1.5 and the case 2.2.2 can be suppressed.
- ▶ If the drift from the constraints is also controlled into the ODE solver by a error computation,  
→ the case 2.1 can also be suppressed
- ▶ Most of the case can be resumed into the following step
  - ▶ Continue with the same status
  - ▶ Compute  $U_{N,k+1}, P_{k+1}$  thanks to the LCP (9)(impact equations).
  - ▶ Compute  $\Gamma_{N,k+1}, F_{k+1}$  thanks to the LCP (20) (Smooth dynamics)

→ Rearranging the cases, we obtain the following algorithm.

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## The case of a single contact. An algorithm

**Require:**  $(g_k, U_{N,k}, status_k)$

**Ensure:**  $(g_{k+1}, U_{N,k+1}, status_{k+1})$

Time-integration of the system on  $[t_k, t_{k+1}]$ (22) if  $status_k = 0$  or of the system (23) if  $status_k = 1$  up to an event.

**if**  $g_{k+1} > 0$  **then**

$status_{k+1} = 0$  //The constraint is still not active. (case 1.1)

**end if**

**if**  $g_{k+1} = 0, U_{N,k+1} < 0$  **then**

//The constraint is active  $g_{k+1} = 0$  and an impact occur  $U_{N,k+1} < 0$  (case 1.2)

Solve the LCP (9) for  $U_N^- = U_{N,k+1}; U_{N,k+1} = U_N^+$

**if**  $U_{N,k+1} > 0$  **then**  $status_{k+1} = 0$

**end if**

**if**  $g_{k+1} = 0, U_{N,k+1} = 0$  **then**

//The constraint is active  $g_{k+1} = 0$  without impact (case 1.2.2, case 1.3, case 2.2)

solve the LCP (21)

**if**  $\Gamma_{N,k+1} = 0, F_{k+1} > 0$  **then**

$status_{k+1} = 1$

**else if**  $\Gamma_{N,k+1} > 0, F_{k+1} = 0$  **then**

$status_{k+1} = 0$

**else if**  $\Gamma_{N,k+1} = 0, F_{k+1} = 0$  **then**

//Undetermined case.

**end if**

**end if**

Go to the next time step

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## Index sets

The index set  $I$  is the set of all unilateral constraints in the system

$$I = \{1 \dots \nu\} \subset \mathbb{N} \quad (24)$$

The index-set  $I_c$  is the set of all active constraints of the system,

$$I_c = \{\alpha \in I, g^\alpha = 0\} \subset I \quad (25)$$

and the index-set  $I_s$  is the set of all active constraints of the system with a relative velocity equal to zero,

$$I_s = \{\alpha \in I_c, U_N^\alpha = 0\} \subset I_c \quad (26)$$

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## Impact equations

$$\left\{ \begin{array}{l} M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \\ U_N^+(t_i) = \nabla_q g^T(q(t_i))v^+(t_i) \\ U_N^-(t_i) = \nabla_q g^T(q(t_i))v^-(t_i) \\ p_i = \nabla_q g(q(t_i))P_{N,i} \\ \\ P_{N,i}^\alpha = 0; U_N^{\alpha,+}(t_i) = U_N^{\alpha,-}(t_i), \quad \forall \alpha \in I \setminus I_c \\ \\ 0 \leq U_N^{+,\alpha}(t_i) + eU_N^{-,\alpha}(t_i) \perp P_{N,i}^\alpha \geq 0, \quad \forall \alpha \in I_c \end{array} \right. \quad (27)$$

Using the fact that  $P_{N,i}^\alpha = 0$  for  $\alpha \in I \setminus I_c$ , this problem can be reduced on the local unknowns  $U_N^+(t_i), P_{N,i} \quad \forall \alpha \in I_c$ .

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Reformulations of the smooth dynamics at acceleration level.

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## Modes for the smooth Dynamics

- ▶ The smooth unilateral dynamics as a LCP

$$\left\{ \begin{array}{l} M(q)\gamma^+ + F_{int}(\cdot, q, v) = F_{ext} + \nabla_q g(q)F^+ \\ \Gamma_N^+ = \nabla_q g^T(q)\gamma^+ + \frac{d}{dt}(\nabla_q g^T(q))v^+ \\ F^{+, \alpha} = 0, \quad \forall \alpha \in I \setminus I_s \\ 0 \leq \Gamma_N^{+, \alpha} \perp F^{+, \alpha} \geq 0 \quad \forall \alpha \in I_s \end{array} \right. \quad (28)$$

- ▶ The smooth bilateral dynamics

$$\left\{ \begin{array}{l} M(q)\gamma^+ + F_{int}(\cdot, q, v) = F_{ext} + \nabla_q g(q)F^+ \\ \Gamma_N^+ = \nabla_q g^T(q)\gamma^+ + \frac{d}{dt}(\nabla_q g^T(q))v^+ \\ F^{+, \alpha} = 0, \quad \forall \alpha \in I \setminus I_s \\ \Gamma_N^{+, \alpha} = 0 \quad \forall \alpha \in I_s \end{array} \right. \quad (29)$$

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# The multi-contact case and the index-sets. an algorithm

**Require:**  $(g_k, U_{N,k}, I_{c,k}, I_{s,k}),$

**Ensure:**  $(g_{k+1}, U_{N,k+1}, I_{c,k+1}, I_{s,k+1})$

Time-integration on  $[t_k, t_{k+1}]$  of the system (29) according to  $I_{c,k}$  and  $I_{s,k}$  up to an event.

Compute the temporary index-sets  $I_{c,k+1}$  and  $I_{s,k+1}$ .

**if**  $I_{c,k+1} \setminus I_{s,k+1} \neq \emptyset$  **then**

*//Impacts occur.*

Solve the LCP (27).

Update the index-set  $I_{c,k+1}$  and temporary  $I_{s,k+1}$

Check that  $I_{c,k+1} \setminus I_{s,k+1} = \emptyset$

**end if**

**if**  $I_{s,k+1} \neq \emptyset$  **then**

Solve the LCP (28)

**for**  $\alpha \in I_{s,k+1}$  **do**

**if**  $\Gamma_{N,\alpha,k+1} > 0, F_{\alpha,k+1} = 0$  **then**

remove  $\alpha$  from  $I_{s,k+1}$  and  $I_{c,k+1}$

**else if**  $\Gamma_{N,\alpha,k+1} = 0, F_{\alpha,k+1} = 0$  **then**

*//Undetermined case.*

**end if**

**end for**

**end if**

*// Go to the next time step*

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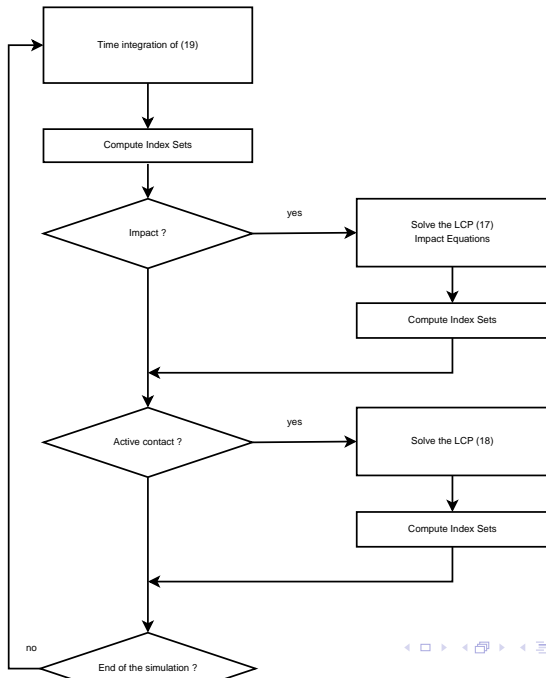
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## Extensions to Coulomb's friction

The set  $I_r$  is the set of sticking or rolling contact:

$$I_r = \{\alpha \in I_s, U_N^\alpha = 0, \|U_T\| = 0\} \subset I_s, \quad (30)$$

is the set of sticking or rolling contact, and

$$I_t = \{\alpha \in I_s, U_N^\alpha = 0, \|U_T\| > 0\} \subset I_s, \quad (31)$$

is the set of slipping or sliding contact.

## Remarks

In the 3D case, checking the events and the transition sticking/sliding and sliding/sticking is not a easy task.

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## Advantages and Weaknesses and the Event Driven schemes

### ▶ Advantages :

- ▶ Low cost implementation of time integration solvers (re-use of existing ODE solvers).
- ▶ Higher-order accuracy on free motion.
- ▶ Pseudo-localization of the time of events with finite time-step.

### ▶ Weaknesses

- ▶ Numerous events in short time.
- ▶ Accumulation of impacts.
- ▶ No convergence proof
- ▶ Robustness with the respect to thresholds " $\varepsilon$ ". Tuning codes is difficult.

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For sake of simplicity, the linear time invariant case is only considered.

$$\begin{cases} Mdv + (Kq + Cv^+) dt = F_{\text{ext}} dt + di. \\ v^+ = \dot{q}^+ \end{cases} \quad (32)$$

Integrating both sides of this equation over a time step  $]t_k, t_{k+1}]$  of length  $h$ ,

$$\begin{cases} \int_{]t_k, t_{k+1}] } Mdv + \int_{t_k}^{t_{k+1}} Cv^+ + Kq dt = \int_{t_k}^{t_{k+1}} F_{\text{ext}} dt + \int_{]t_k, t_{k+1}] } di, \\ q(t_{k+1}) = q(t_k) + \int_{t_k}^{t_{k+1}} v^+ dt. \end{cases} \quad (33)$$

By definition of the differential measure  $dv$ ,

$$\int_{]t_k, t_{k+1}] } M dv = M \int_{]t_k, t_{k+1}] } dv = M(v^+(t_{k+1}) - v^+(t_k)). \quad (34)$$

Note that the right velocities are involved in this formulation.

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The equation of the nonsmooth motion can be written under an integral form as:

$$\begin{cases} M(v(t_{k+1}) - v(t_k)) = \int_{t_k}^{t_{k+1}} -Cv^+ - Kq + F_{ext} dt + \int_{]t_k, t_{k+1}[} di, \\ q(t_{k+1}) = q(t_k) + \int_{t_k}^{t_{k+1}} v^+ dt. \end{cases} \quad (35)$$

The following notations will be used:

- ▶  $q_k \approx q(t_k)$  and  $q_{k+1} \approx q(t_{k+1})$ ,
- ▶  $v_k \approx v^+(t_k)$  and  $v_{k+1} \approx v^+(t_{k+1})$ ,

## Impulse as primary unknown

The impulse  $\int_{]t_k, t_{k+1}[} di$  of the reaction on the time interval  $]t_k, t_{k+1}[$  emerges as a natural unknown. we denote

$$p_{k+1} \approx \int_{]t_k, t_{k+1}[} di$$

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## Interpretation

The measure  $di$  may be decomposed as follows :

$$di = f dt + pd\nu$$

where

- ▶  $f dt$  is the abs. continuous part of the measure  $di$ , and
- ▶  $pd\nu$  the atomic part.

Two particular cases:

- ▶ Impact at  $t_* \in ]t_k, t_{k+1}]$  : If  $f = 0$  and  $pd\nu = p\delta_{t_{k+1}}$  then

$$p_{k+1} = p$$

- ▶ Continuous force over  $]t_k, t_{k+1}]$  : If  $di = fdt$  and  $p = 0$  then

$$p_{k+1} = \int_{t_k}^{t_{k+1}} f(t) dt$$

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## Remark

- ▶ A pointwise evaluation of a (Dirac) measure is a non sense. It practice using the value

$$f_{k+1} \approx f(t_{k+1})$$

yield severe numerical inconsistencies, since

$$\lim_{h \rightarrow 0} f_{k+1} = +\infty$$

- ▶ Since discontinuities of the derivative  $v$  are to be expected if some shocks are occurring, i.e.  $di$  has some Dirac atoms within the interval  $]t_k, t_{k+1}]$ , it is not relevant to use high order approximations integration schemes for  $di$ . It may be shown on some examples that, on the contrary, such high order schemes may generate artefact numerical oscillations.

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## Discretization of smooth terms

$\theta$ -method is used for the term supposed to be sufficiently smooth,

$$\int_{t_k}^{t_{k+1}} C v + K q dt \approx h [\theta (C v_{k+1} + K q_{k+1}) + (1 - \theta)(C v_k + K q_k)]$$

$$\int_{t_k}^{t_{k+1}} F_{ext}(t) dt \approx h [\theta (F_{ext})_{k+1} + (1 - \theta)(F_{ext})_k]$$

The displacement, assumed to be absolutely continuous is approximated by:

$$q_{k+1} = q_k + h [\theta v_{k+1} + (1 - \theta)v_k] .$$

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Finally, introducing the expression of  $q_{k+1}$  in the first equation of (34), one obtains:

$$\begin{aligned} [M + h\theta C + h^2\theta^2 K] (v_{k+1} - v_k) = & -hCv_k - hKq_k - h^2\theta K v_k \\ & + h[\theta(F_{ext})_{k+1}) + (1 - \theta)(F_{ext})_k] + p_{k+1}, \end{aligned} \quad (36)$$

which can be written :

$$v_{k+1} = v_{free} + \widehat{M}^{-1} p_{k+1} \quad (37)$$

where,

- ▶ the matrix  $\widehat{M} = [M + h\theta C + h^2\theta^2 K]$  is usually called the iteration matrix and,
- ▶ The vector

$$v_{free} = v_k + \widehat{M}^{-1} \left[ -hCv_k - hKq_k - h^2\theta K v_k + h[\theta(F_{ext})_{k+1}) + (1 - \theta)(F_{ext})_k] \right]$$

is the so-called “free” velocity, i.e. the velocity of the system when reaction forces are null.

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According to the implicit mind, the discretization of kinematic laws is proposed as follows.

For a constraint  $\alpha$ ,

$$U_{k+1}^\alpha = H^\alpha T(q_{k+1}) v_{k+1},$$

$$p_{k+1}^\alpha = H^\alpha(q_{k+1}) P_{k+1}^\alpha, \quad p_{k+1} = \sum_{\alpha} p_{k+1}^\alpha,$$

where

$$P_{k+1}^\alpha \approx \int_{]t_k, t_{k+1}] } d\lambda^\alpha.$$

For the unilateral constraints, it is proposed

$$g_{k+1}^\alpha = g_k^\alpha + h \left[ \theta U_{k+1}^\alpha + (1 - \theta) U_k^\alpha \right].$$

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# Discretization of the unilateral constraints

Recall that the unilateral constraint is expressed in terms of velocity as

$$-di \in N_{T_C(q)}(v^+) \quad (38)$$

or in local coordinates as

$$-d\lambda^\alpha \in N_{T_{\mathbb{R}^+}(g(q))}(U^{\alpha,+}) \quad (39)$$

The time discretization is performed by

$$-P_{k+1}^\alpha \in N_{T_{\mathbb{R}^+}(g^\alpha(\tilde{q}_{k+1}))}(U_{k+1}^\alpha) \quad (40)$$

where  $\tilde{q}_{k+1}$  is a forecast of the position for the activation of the constraints, for instance,

$$\tilde{q}_{k+1} = q_k + \frac{h}{2} v_k$$

In the complementarity formalism, we obtain

$$\text{if } g^\alpha(\tilde{q}_{k+1}) \leq 0, \text{ then } 0 \leq U_{k+1}^\alpha \perp P_{k+1}^\alpha \geq 0$$

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# Summary of the time discretized equations

One step linear problem	$\begin{cases} v_{k+1} = v_{free} + \widehat{M}^{-1} p_{k+1} \\ q_{k+1} = q_k + h [\theta v_{k+1} + (1 - \theta) v_k] \end{cases}$
Relations	$\begin{cases} U_{k+1}^\alpha = H^\alpha T(q_{k+1}) v_{k+1} \\ p_{k+1}^\alpha = H^\alpha(q_{k+1}) P_{k+1}^\alpha \end{cases}$
Nonsmooth Law	$\begin{cases} \text{if } g^\alpha(\tilde{q}) \leq 0, \text{ then} \\ 0 \leq U_{k+1}^\alpha \perp P_{k+1}^\alpha \geq 0 \end{cases}$

## One step Quasi-LCP

$$U_{k+1} = H^T(q_{k+1}) v_{free} + H^T(q_{k+1}) \widehat{M}^{-1} H(q_{k+1}) P_{k+1}$$

$$\text{if } g_p^\alpha \leq 0, \text{ then } 0 \leq U_{k+1}^\alpha \perp P_{k+1}^\alpha \geq 0$$

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## Principle

$$\left\{ \begin{array}{l} M(q_{k+\theta})(v_{k+1} - v_k) - h\tilde{F}_{k+\theta} = H(q_{k+\theta})P_{k+1}, \end{array} \right. \quad (41a)$$

$$q_{k+1} = q_k + hv_{k+\theta}, \quad (41b)$$

$$U_{k+1} = H^T(q_{k+\theta})v_{k+1} \quad (41c)$$

$$-P_{k+1} \in \partial\psi_{T_{\mathbb{R}^m_+}}(\tilde{y}_{k+\gamma})(U_{k+1} + eU_k), \quad (41d)$$

$$\tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma \in [0, 1]. \quad (41e)$$

with  $\theta \in [0, 1]$ ,  $\gamma \geq 0$  and  $x_{k+\alpha} = (1 - \alpha)x_{k+1} + \alpha x_k$  and  $\tilde{y}_{k+\gamma}$  is a prediction of the constraints.

## Properties

- ▶ Convergence results for one constraints
- ▶ Convergence results for multiple constraints problems with acute kinetic angles
- ▶ No theoretical proofs of order

## Principle

$$\left\{ \begin{array}{l} M(q_k + 1)(q_{k+1} - 2q_k + q_{k-1}) - h^2 F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta}) = p_{k+1} \quad (42a) \\ v_{k+1} = \frac{q_{k+1} - q_{k-1}}{2h}, \quad (42b) \\ -p_{k+1} \in N_K \left( \frac{q_{k+1} + eq_{k-1}}{1+e} \right), \quad (42c) \end{array} \right.$$

where  $N_K$  defined the normal cone to  $K$ .

For  $K = \{q \in \mathbb{R}^n, y = g(q) \geq 0\}$

$$0 \leq g \left( \frac{q_{k+1} + eq_{k-1}}{1+e} \right) \perp \nabla g \left( \frac{q_{k+1} + eq_{k-1}}{1+e} \right) P_{k+1} \geq 0 \quad (43)$$

## Properties

- ▶ Convergence results for one constraints
- ▶ Convergence results for multiple constraints problems with acute kinetic angles
- ▶ No theoretical proof of order

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## The bouncing Ball and the linear impacting oscillator

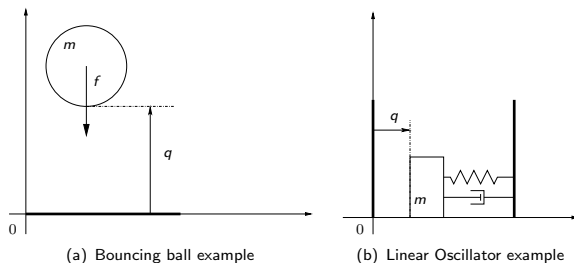


Figure: Academic test examples with analytical solutions

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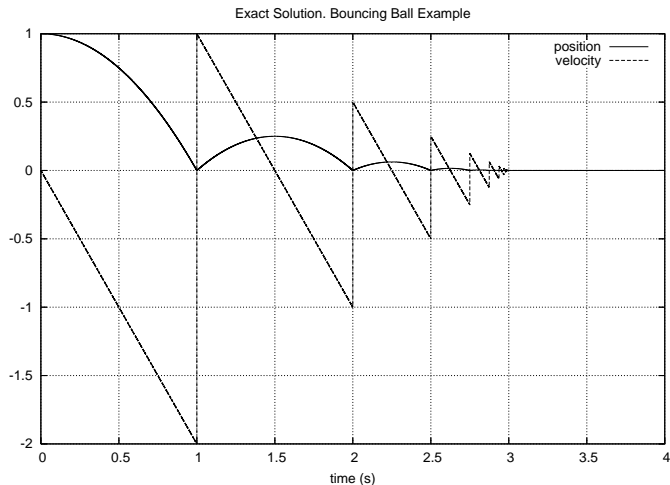


Figure: Analytical solution. Bouncing ball example

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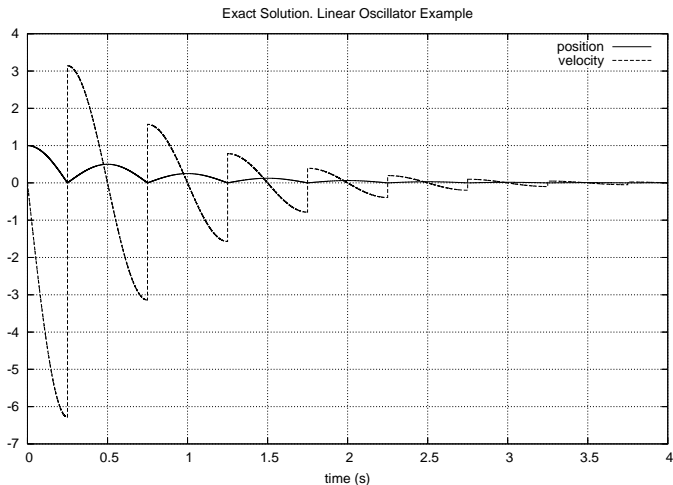


Figure: Analytical solution. Linear Oscillator

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## Convergence in the sense of filled-in graph (Moreau (1978))

$$gr^*(f) = \{(t, x) \in [0, T] \times \mathbb{R}^n, 0 \leq t \leq T \text{ and } x \in [f(t^-), f(t^+)]\}. \quad (44)$$

Such graphs are closed bounded subsets of  $[0, T] \times \mathbb{R}^n$ , hence, we can use the Hausdorff distance between two such sets with a suitable metric:

$$d((t, x), (s, y)) = \max\{|t - s|, \|x - y\|\}. \quad (45)$$

Defining the excess of separation between two graphs by

$$e(gr^*(f), gr^*(g)) = \sup_{(t,x) \in gr^*(f)} \inf_{(s,y) \in gr^*(g)} d((t, x), (s, y)), \quad (46)$$

the Hausdorff distance between two filled-in graphs  $h^*$  is defined by

$$h^*(gr^*(f), gr^*(g)) = \max\{e(gr^*(f), gr^*(g)), e(gr^*(g), gr^*(f))\}. \quad (47)$$

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An equivalent grid-function norm to the function norm in  $\mathcal{L}_1$

$$\|e\|_1 = h \sum_{i=0}^N |f_i - f(t_i)| \quad (48)$$

In the same way, the  $p$  - norm can be defined by

$$\|e\|_p = \left( h \sum_{i=0}^N |f_i - f(t_i)|^p \right)^{1/p} \quad (49)$$

The computation of this two last norm is easier to implement for piecewise continuous analytical function than the Hausdorff distance.

## Global order of convergence.

### Definition

A one-step time-integration scheme is of order  $q$  for a given norm  $\|\cdot\|$  if there exists a constant  $C$  such that

$$\|e\| = Ch^q + \mathcal{O}(h^{q+1}) \quad (50)$$

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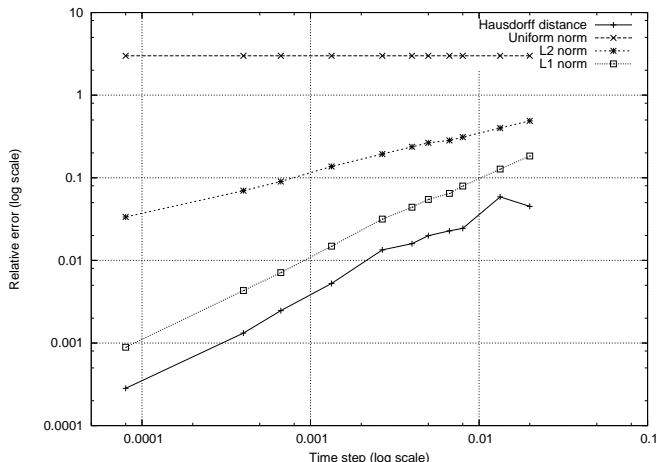
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# Empirical order of convergence. Moreau's time-stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

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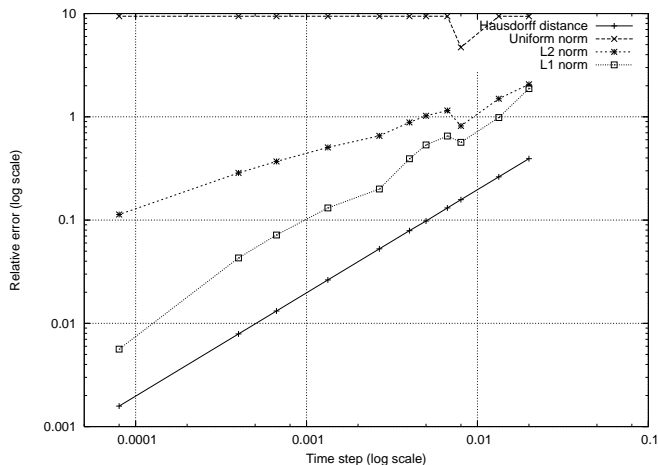
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# Empirical order of convergence. Moreau's time-stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

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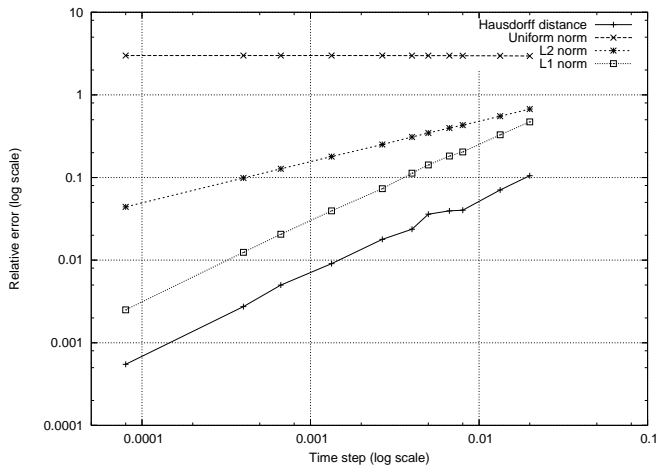
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# Empirical order of convergence. Schatzman–Paoli's time-stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Schatzman–Paoli's time-stepping scheme.

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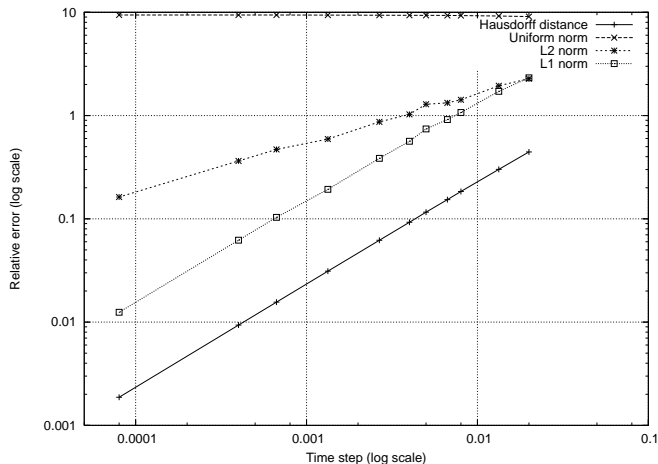
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# Empirical order of convergence. Schatzman–Paoli's time-stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Schatzman–Paoli's time-stepping scheme.

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Numerical time-integration methods for Nonsmooth Multibody systems (NSMBS):

## Nonsmooth event capturing methods (Time-stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

## Nonsmooth event tracking methods (Event-driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

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## Adaptive time-step strategies for time-stepping schemes

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## One-step numerical solvers for ODEs

Let us consider a ODE

$$\dot{x} = f(x, t), \quad (51)$$

where  $f$  is a mapping with sufficient regularity.

The one-step time-stepping method over the time-step  $[t_k, t_{k+1} = t_k + h]$  is generically denoted by

$$x_{k+1} = x_k + h\Phi(t_k, h, x_k). \quad (52)$$

## Order of consistency

The one-step time-stepping method is said to be consistent if  $\Phi(t, 0, x, x) = f(x, t)$  and has a consistency order  $p$  if there exists a constant  $C$  such that

$$e_{k+1} = x(t_{k+1}) - x_{k+1} = Ch^{p+1} + \mathcal{O}(h^{p+2}), \quad (53)$$

assuming that  $x_k = x(t_k)$ .

If the time-stepping method has an order of consistency  $p$  and converges, then the global order of convergence is  $p$ ,

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## Basic practical error evaluation

1. Two “small” time steps of size  $h/2 \implies x_{1/2}$ .
2. One “big” time-step  $h \implies x_1$ .

$$\begin{aligned} e_1 &= x(t_0 + h) - x_1 = C h^{p+1} + \mathcal{O}(h^{p+2}), \\ e_{1/2} &= x(t_0 + h) - x_{1/2} = 2C (h/2)^{p+1} + \mathcal{O}(h^{p+2}). \end{aligned} \quad (54)$$

This procedure permits us to evaluate the constant  $C$  and to obtain and a local error estimate such that:

$$e_2 = x(t_0 + h) - x_2 = \frac{x_{1/2} - x_1}{2^p - 1} + \mathcal{O}(h^{p+2}). \quad (55)$$

## Enhanced practical error evaluation

- ▶ Runge–Kutta Embedded pairs (Dormand-Price, Fehlberg)
- ▶ Milne’s device
- ▶ Nordsieck’s method

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## Automatic control of the time-step

$$\|e_k\| \leq etol = atol + rtol \circ \max(x_0, x_k) \quad (56)$$

The measure of the error is given by

$$\text{error} = \|e_k \circ invtol\| \quad (57)$$

with  $invtol = [1/etol_i, i = 1 \dots n]$ . The optima step size is then obtained by

$$h_{opt} = h \left( \frac{1}{\text{error}} \right)^{1/(p+1)} \quad (58)$$

Usually, the step size is not allowed to decrease or to increase too fast, thanks to the following heuristic rule

$$h_{new} = h \min(\alpha_{max}, \max(\alpha_{min}, \alpha \left( \frac{1}{\text{error}} \right)^{1/(p+1)})) \quad (59)$$

where  $\alpha$ ,  $\alpha_{min}$  and  $\alpha_{max}$  are some user parameters.

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# Local error estimates for the Moreau's time-stepping

Lecture 2. Numerical methods for time integration of NonSmooth Mechanical systems

Vincent Acary  
INRIA Rhône-Alpes,  
Grenoble.

## Notation

$$e = x(t_k + h) - x_{k+1} = \begin{bmatrix} e_v \\ e_q \end{bmatrix} = \begin{bmatrix} v^+(t_k + h) - v_{k+1} \\ q(t_k + h) - q_{k+1} \end{bmatrix} \quad (60)$$

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## Assumption 1 : Existence and uniqueness

A unique global solution over  $[0, T]$  for Moreau's sweeping process is assumed such that  $q(\cdot)$  is absolutely continuous and admits a right velocity  $v^+(\cdot)$  at every instant  $t$  of  $[0, T]$  and such that the function  $v^+ \in LBV([0, T], \mathbb{R}^n)$ .

→ Assumption 2 is ensured in the framework introduced by Ballard (Ballard, 2000) who proves the existence and uniqueness of a solution in a general framework mainly based on the analyticity of data.

## Assumption 2 : Smoothness of data

The following smoothness on the data will be assumed: a) the inertia operator  $M(q)$  is assumed to be of class  $C^p$  and definite positive, b) the force mapping  $F(t, q, v)$  is assumed to be of class  $C^p$ , c) the constraint functions  $g(q)$  are assumed to be of class  $C^{p+1}$  and d) the Jacobian matrix  $G(q) = \nabla_q^T g(q)$  is assumed to have full-row rank.

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# Local error estimates for the Moreau's time-stepping

## Lemma

Let  $I = [t_k, t_{k+1}]$ . Let us assume that the function  $f \in BV(I, \mathbb{R}^n)$ . Then we have the following inequality for the  $\theta$ -method,  $\theta \in [0, 1]$ ,

$$\left\| \int_{t_k}^{t_{k+1}} f(s) ds - h(\theta f(t_{k+1}) + (1 - \theta)f(t_k)) \right\| \leq C(\theta)(t_{k+1} - t_k) \text{var}(f, I), \quad (61)$$

where  $\text{var}(f, I) \in \mathbb{R}$  is the variation of  $f$  on  $I$  and  $C(\theta) = \theta$  if  $\theta \geq 1/2$  and  $C(\theta) = 1 - \theta$  otherwise. Furthermore, the value of  $C(\theta)$  yields a sharp bound in (61).

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# Local error estimates for the Moreau's time-stepping

## Proposition

*Under Assumptions 1 and 2, the local order of consistency of the Moreau time-stepping scheme for the generalized coordinates is*

$$e_q = \mathcal{O}(h)$$

*and at least for the velocities*

$$e_v = \mathcal{O}(1)$$

## Comments

The bounds are reached if an impact is located within the time-step and the activation of the constraint is not correct.

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One impact at time  $t_* \in (t_k, t_{k+1})$ 

## Assumption

$$di = p\delta_{t_*}, \quad \text{or equivalently} \quad dl = P\delta_{t_*}, \quad \text{with } P = G(t_*)p. \quad (62)$$

## Notation

$$\mathcal{I} = \{\alpha, \alpha \in \{1..m\}\} \quad (63)$$

$$\mathcal{I}_* = \{\alpha \in \mathcal{I}, P^\alpha \geq 0, U^{\alpha,+}(t_*) - U^{\alpha,-}(t_*) = -(1+e)U^{\alpha,-}(t_*)\} \quad (64)$$

$$\mathcal{I}_p = \{\alpha \in \mathcal{I}, P_{k+1}^\alpha \geq 0, U_{k+1}^\alpha - U_k^\alpha = -(1+e)U_k^\alpha\} \quad (65)$$

## Lemma

Let us assume that we have only one elastic impact at time  $t_* \in (t_k, t_{k+1})$  i.e. ,  $di = p\delta_{t_*} + r(t)dt$ .

1. If  $\mathcal{I}_* = \mathcal{I}_p$ , then the local order of consistency of the scheme is given by

$$e_v = K_v h + \mathcal{O}(h^2) \quad (66)$$

2. If  $\mathcal{I}_* \neq \mathcal{I}_p$ , then the local order of consistency of the scheme is given by

$$e_v = K_v + \mathcal{O}(h) \quad (67)$$

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## Example (The bouncing ball)

$$\begin{cases} \dot{v}(t) = f(t) + \lambda(t), & \dot{q}(t) = v(t), \\ 0 \leq q(t) \perp \lambda(t) \geq 0, & v^+(t) = -ev^-(t), \text{ if } q(t) = 0, \end{cases} \quad (68)$$

With chosen parameters as  $f = -2$ ,  $e = 1/2$  and the initial data as  $t_0 = 0$ ,  $q_0 = 1$  and  $v_0 = 0$ . The analytical solution reads as

- ▶ for  $t \in [0, 1)$ ,

$$\begin{cases} q(t) = -t^2 + 1, \\ v(t) = -2t, \end{cases} \quad (69)$$

- ▶ for  $t \in \left[3 - \frac{1}{2^{n-1}}, 3 - \frac{1}{2^n}\right)$ ,

$$\begin{cases} q(t) = -(t-3)^2 - \frac{3}{2^n}(t-1) + \frac{1}{2^{n-1}} \left(3 - \frac{1}{2^n}\right), \\ v(t) = -2(t-3) - \frac{3}{2^n}, \end{cases} \quad (70)$$

- ▶ and for  $t \in [3, +\infty)$

$$\begin{cases} q(t) = 0, \\ v(t) = 0, \end{cases} \quad (71)$$

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## Example (The bouncing ball (continued))

Let us consider a time interval such that the impacting time  $t_*$  belongs to  $(t_k, t_{k+1}]$ . The error is given by

$$\text{if } p_{k+1} = 0$$

$$\begin{cases} e_v = -(1 + e)[v_k + hf\sigma] \\ e_q = -q_k - h(e(1 - \sigma + 1))v_k - fh^2[e(1 - \sigma)\sigma + \frac{1}{2}(1 - \sigma)^2 + \theta] \end{cases}$$

$$\text{if } p_{k+1} > 0$$

$$\begin{cases} e_v = -hf[1 - \sigma - e\sigma] \\ e_q = -q_k - h((1 + e)(1 - \theta) - e\sigma)v_k - fh^2(e(1 - \sigma)\sigma + \frac{1}{2}(1 - \sigma)^2) \end{cases} \quad (72)$$

where  $\sigma = (t_k - t_*)/h \in (0, 1]$ .

The approximate solution of the Moreau scheme depends on the forecast of the active constraints, *i.e.*  $\bar{g}_{k+1} = q_k + \gamma hv_k$ .

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## Example (The bouncing ball (continued))

Using the fact that  $q(t_*) = q_k + v_k \sigma h + \frac{1}{2}(\sigma h)^2 = 0$ , we obtain that

$$q_k = -\sigma v_k h - \frac{1}{2}f(\sigma h)^2 \text{ and}$$

if  $p_{k+1} = 0$ ,

$$\begin{cases} e_v = -(1+e)[v_k + hf\sigma] \\ e_q = -h(e(1-\sigma+1) - \sigma)v_k - fh^2[e(1-\sigma)\sigma + \frac{1}{2}(1-\sigma)^2 - \frac{1}{2}(\sigma)^2] \end{cases}$$

i.e.  $e_v = \mathcal{O}(1)$  and  $e_q = \mathcal{O}(h)$

if  $p_{k+1} > 0$ ,

$$\begin{cases} e_v = -hf[1 - \sigma - e\sigma] \\ e_q = -h((1+e)(1-\theta - \sigma))v_k - fh^2(e(1-\sigma)\sigma + \frac{1}{2}(1-\sigma)^2 - \frac{1}{2}(\sigma)^2) \end{cases}$$

i.e.  $e_v = \mathcal{O}(h)$  and  $e_q = \mathcal{O}(h)$

(72)

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## Example (The bouncing ball (continued))

Near the finite accumulation of impact at time  $t = 3$ .

Let us consider a time step such that  $[t_k, t_{k+1}] = [3 - h, 3 + h]$  and  $n_0$  such that  $h \in [1/2^{n_0}, 1/2^{n_0-1}]$ . The local error in velocity is given if the impact is detected  $p_{k+1} > 0$  by

$$e_v = v(3 + h) - v_{k+1} = -2h - \frac{3}{2^{n_0}}. \quad (72)$$

As  $h \rightarrow 0$ , we have  $n_0 \rightarrow \infty$ , and  $\frac{1}{2^{n_0}} = \mathcal{O}(h)$  and then  $e_v = \mathcal{O}(h)$ .

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## To summarize

- ▶ In any case, we have  $\mathcal{O}(h)$  in the error in coordinates and it cannot be improved if a jump occurs.
- ▶ The local error in velocity is at least  $e_v = \mathcal{O}(1)$  if the impact is not well-forecast. In practice, this situation is usual. It illustrates the possible convergence problem that we can have in uniform norm
- ▶ *Finite accumulation* The order of the time-integration should be at least 0. Idea of the proof : use the fact that the velocity vanishes and is of bounded variations

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## Order “0” case

Standard error estimates do not apply for Order 0.

We propose to extend it to the order 0 of consistency by assuming that the the local error estimate is given by

$$e_{1/2} = 2(x_{1/2} - x_1) + \mathcal{O}(h^2) \quad (73)$$

where  $x_1$  is the result of the time integration with one time-step of length  $h$  and  $x_{1/2}$  with two time-steps of length  $h/2$ .

The adaptive time-step control used for smooth ODE is then apply directly Hairer et al. (1993).

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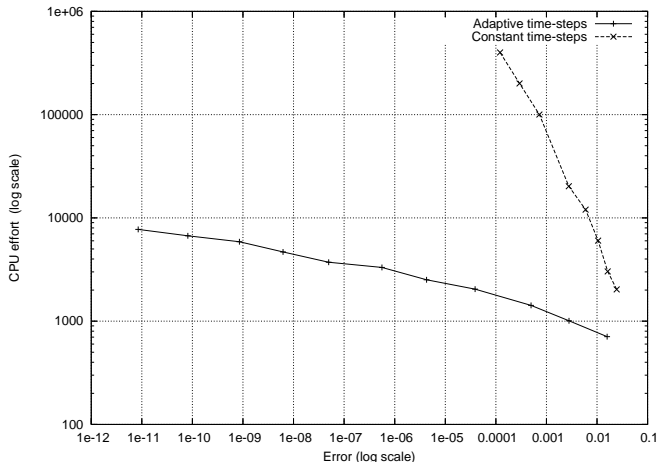
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# Order "0" time-step adjustment for the Moreau's time-stepping



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0

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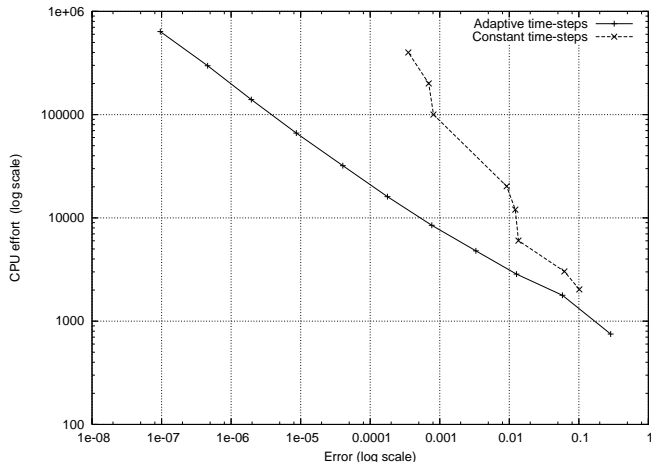
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# Order “0” time-step adjustment for the Moreau’s time-stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 0

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# Order “1” time-step adjustment for the Moreau’s time-stepping

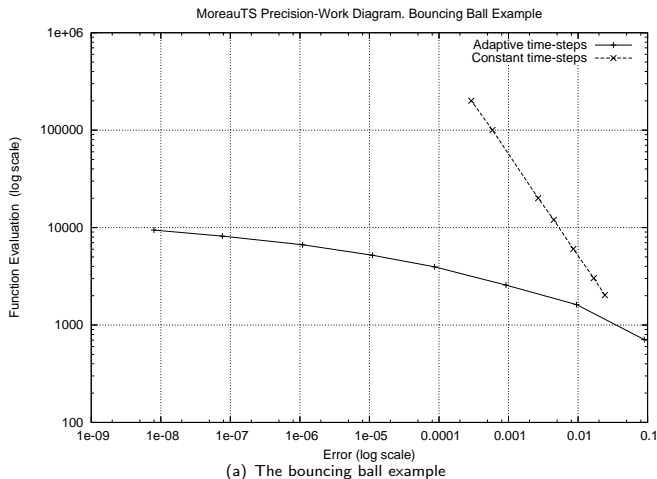


Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 1

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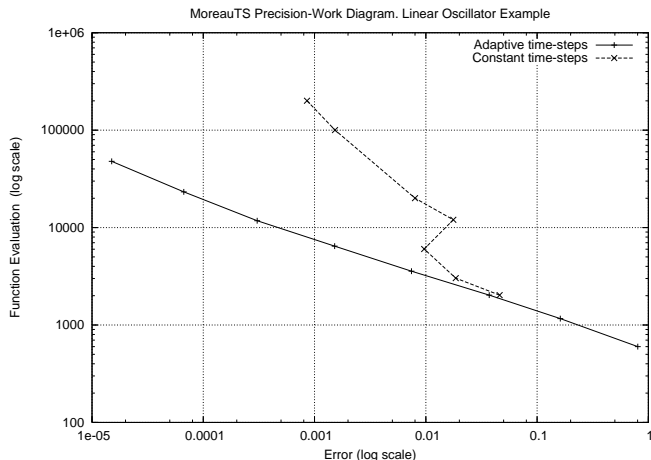
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# Order “1” time-step adjustment for the Moreau’s time-stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 1

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# Order “2” time-step adjustment for the Moreau’s time-stepping

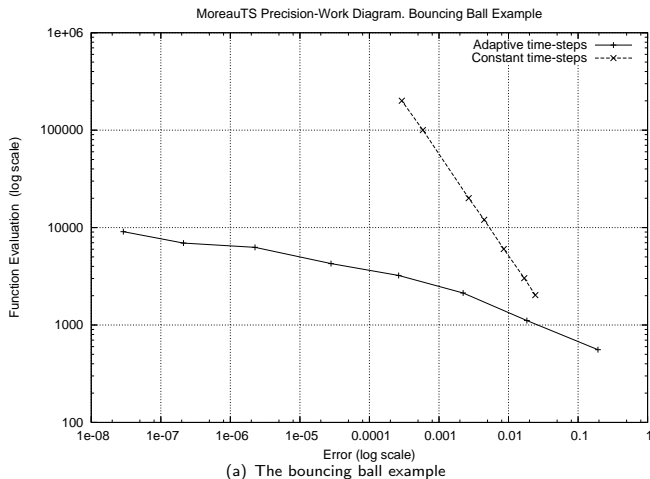


Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 2

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# Order "2" time-step adjustment for the Moreau's time-stepping

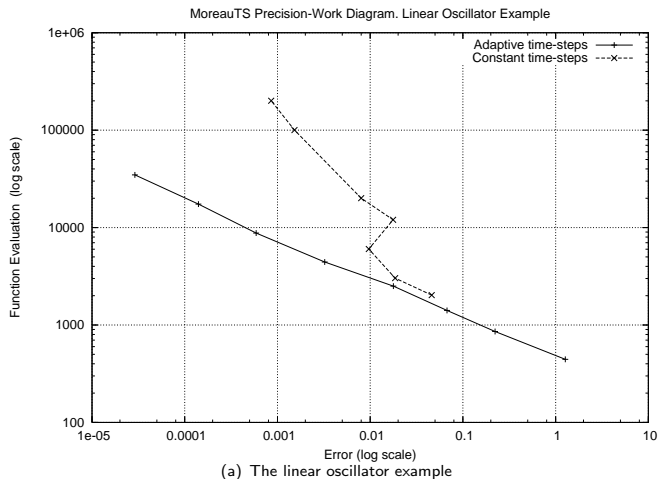


Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 2

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# Sizing the error in the violation of constraints

The violation of constraints is sized by the following rule:

$$e_{\text{violation}} = \|\min(0, g(q)) \circ \text{invtol}\|_{\infty} \quad (74)$$

Assuming that the scheme is of order 1 almost everywhere in smooth phase and may be controlled by  $e_{\text{violation}}$  when a nonsmooth event occurs, the step size adjustment is implemented by the means of the following error estimation

$$\text{error} = \max(e_{\text{violation}}, \|e_k \circ \text{invtol}\|_{\infty}) \quad (75)$$

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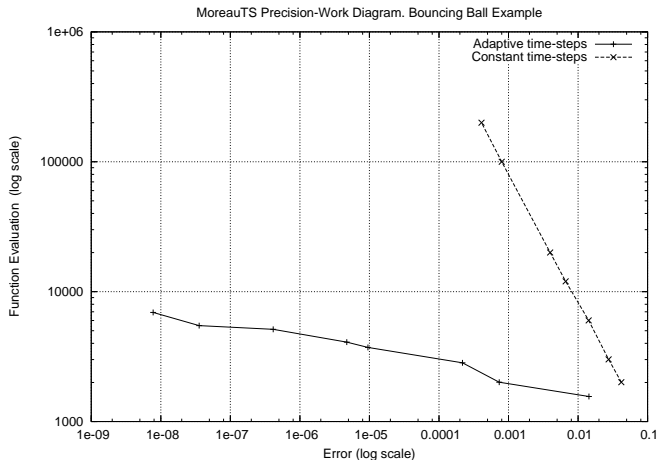
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# Results on two academic test examples



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

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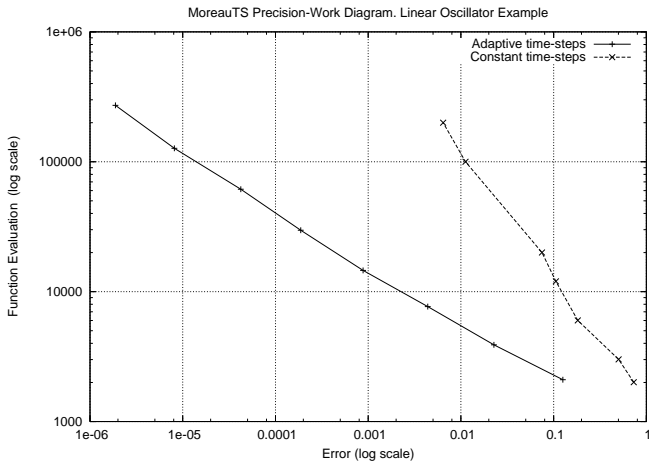
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(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

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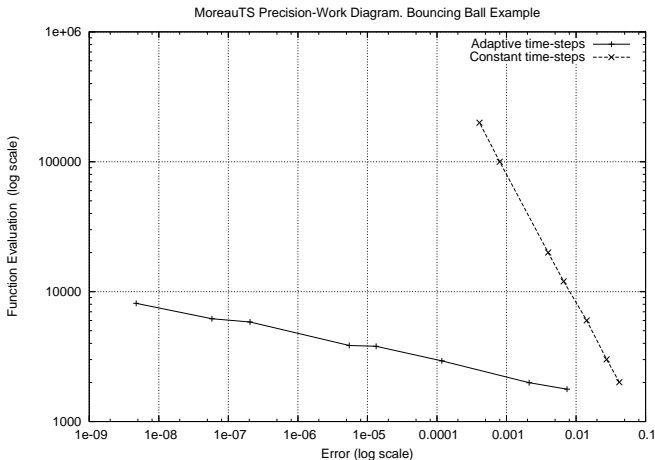
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(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

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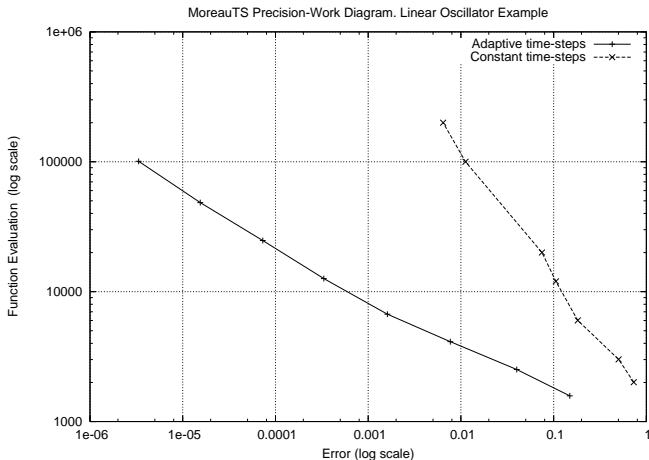
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(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

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# Variable order approach. Principle

Guess the order of consistency of the integration at each step.  
Adapt the practical error estimation

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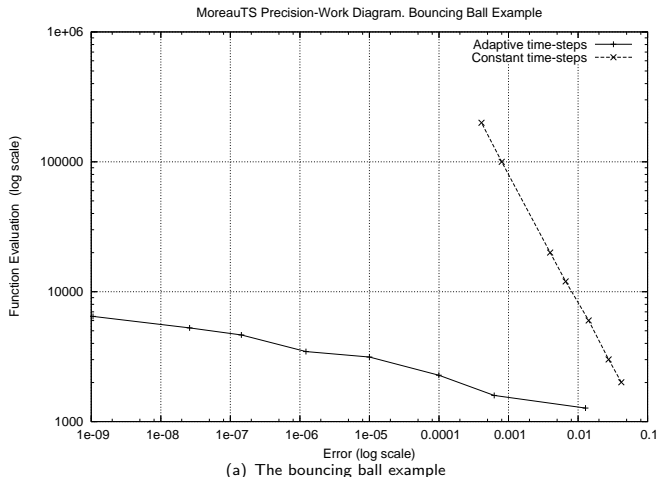
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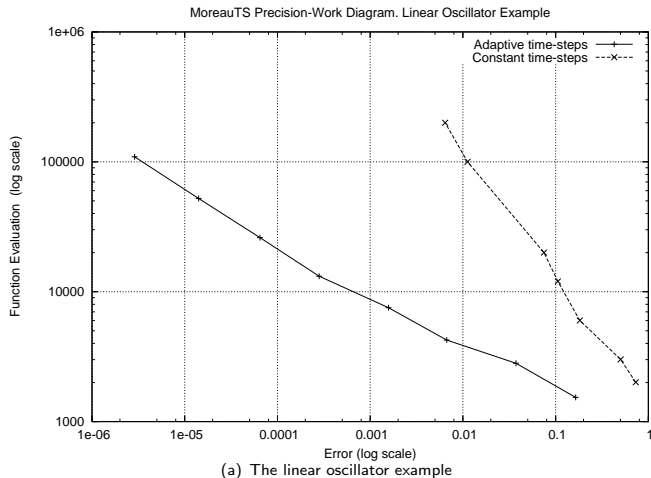
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Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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Figure: Precision Work diagram for the Moreau's time-stepping scheme.

## Time-stepping schemes of any order

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# First attempt

In Studer et al. (2008) ; Studer (2009) the first attempt to increase the efficiency of Moreau's scheme by an extrapolation method has been published.

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## Background

Work of Mannshardt (1978) on time-integration schemes of any order for ODE/DAEs with discontinuities (with transversality assumption)

## Principle

- ▶ Let us assume only one event per time-step at instants  $t_*$ .
- ▶ Choose any ODE/DAE solvers of order  $p$
- ▶ Perform a rough location of the event inside the time step of length  $h$   
Find an interval  $[t_a, t_b]$  such that

$$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \quad (76)$$

Dichotomy, Newton, Local Interpolants, Dense output,...

- ▶ Perform an integration on  $[t_k, t_a]$  with the ODE solver of order  $p$
- ▶ Perform an integration on  $[t_a, t_b]$  with Moreau's time-stepping scheme
- ▶ Perform an integration on  $[t_b, t_{k+1}]$  with the ODE solver of order  $p$

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Mainly for the sake of simplicity, the numerical integration over a smooth period is made with a Runge–Kutta (RK) method on the following index-1 DAE,

$$\begin{cases} M(q(t))\dot{v}(t) = F(t, q(t), v(t)) + G(q)\lambda(t), \\ \dot{q}(t) = v(t), \\ \gamma(t) = G(q(t))\dot{v}(t) = 0. \end{cases} \quad (77)$$

In practice, the time–integration is performed for the following system

$$\begin{cases} M(q(t))\dot{v}(t) = F(t, q(t), v(t)) + G(q)\lambda(t), \\ \dot{q}(t) = v(t), \\ 0 \leq \gamma(t) = G(q(t))\dot{v}(t) \perp \lambda(t) \geq 0 \end{cases} \quad (78)$$

on the time–interval  $I$  where the index set  $\mathcal{I}(t)$  of active constraints is assumed to be constant on  $I$  and  $\lambda(t) > 0$  for all  $t \in I$ .

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Using the standard notation for the RK methods (see Hairer et al. (1993) for details), the complementarity problem that we have to solve at each time-step reads

$$\left\{ \begin{array}{l} t_{ki} = t_k + c_i h, \\ v_{k+1} = v_k + h \sum_{i=1}^s b_i V'_{ki}, \\ q_{k+1} = q_k + h \sum_{i=1}^s b_i V_{ki}, \\ V'_{ki} = M^{-1}(Q_{ki}) [F(t_{ki}, Q_{ki}, V_{ki}) + G(Q_{ki}) \lambda_{ki}], \\ V_{ki} = v_k + h \sum_{j=1}^s a_{ij} V'_{nj}, \\ Q_{ki} = q_k + h \sum_{j=1}^s a_{ij} V_{nj}, \\ 0 \leq \gamma_{ki} = G(Q_{ki}) V'_{ki} \perp \lambda_{ki} \geq 0. \end{array} \right. \quad (79)$$

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### Assumption 3

Let  $I$  a smooth period time-interval (see Definition ??). We assume that

1. the local order of the RK method (79) is  $p$  that is

$$e_q = e_v = \mathcal{O}(h^{p+1}) \quad (80)$$

2. starting from inconsistent initial value  $\tilde{q}_k$  such that  $\tilde{q}_k - q_k = \mathcal{O}(h^{p+1})$ , the error made by the RK method (79) is

$$\tilde{q}_{k+1} - q_{k+1} = \mathcal{O}(h^{p+1}) \quad (81)$$

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## Theorem

*Let us assume that Assumptions 1, 2 and 3 hold. The local error of consistency of the scheme is of order  $p$  in the generalized coordinates that is*

$$e_q = \mathcal{O}(h^{p+1}). \quad (82)$$

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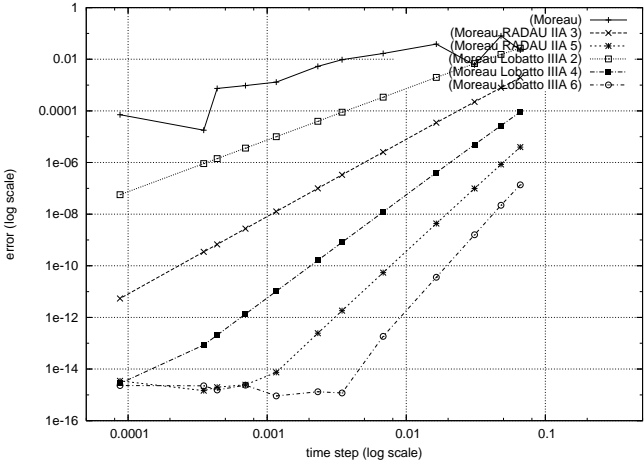
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# Results on the linear oscillator

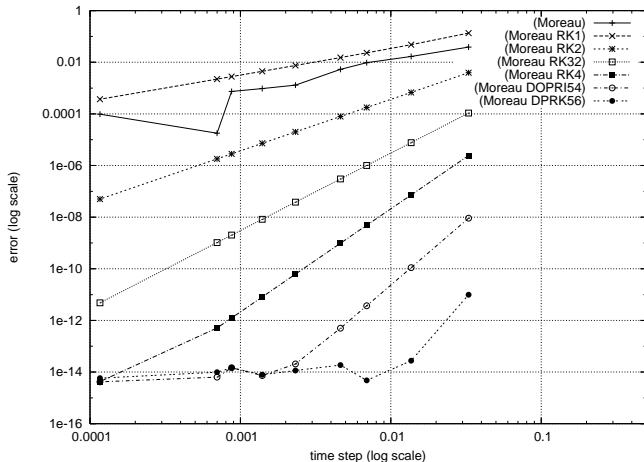


(a) The linear oscillator example with implicit Runge Kutta Method

Figure: Precision Work diagram for the Moreau's time-stepping scheme coupled with Runge-Kutta method.

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# Results on the linear oscillator



(a) The linear oscillator example with half explicit Runge Kutta Method

**Figure:** Precision Work diagram for the Moreau's time-stepping scheme coupled with Runge-Kutta method.

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## Finite accumulation

- ▶ Repeat the whole process on the remaining part of the interval  $[t_b, t_k]$
- ▶ By induction, repeat this process up to the end of the original time step.

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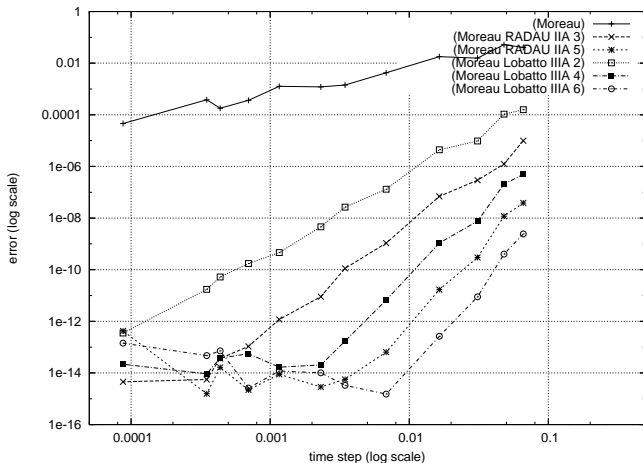
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# Results on the Bouncing Ball



(a) The Bouncing Ball example with implicit Runge Kutta Method

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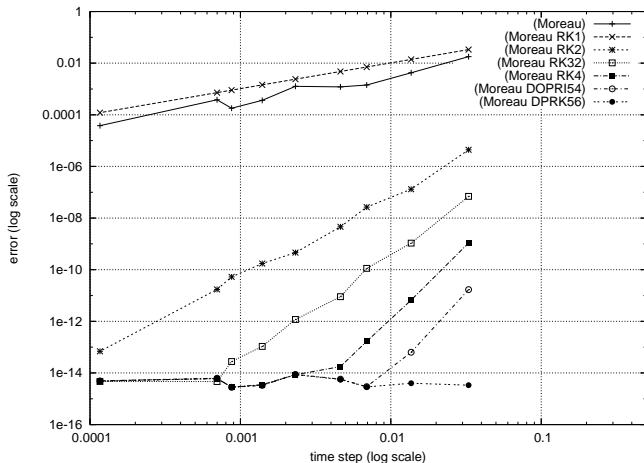
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Figure: Precision Work diagram for the Moreau's time-stepping scheme.

# Results on the Bouncing Ball



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Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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# Splitting-based methods.

## Principle for smooth ODEs

Let us consider a smooth ODE which can be written as

$$\dot{x}(t) = f(x, t) + g(x, t) \quad (83)$$

An example of splitting-based method is given by the following procedure

1. Perform the integration of  $f$  on  $[t_k, t_{k+1}]$  to obtain  $\tilde{x}(t_{k+1})$  that is

$$\tilde{x}(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f(x, t) dt \quad (84)$$

2. Perform the integration of  $g$  on  $[t_k, t_{k+1}]$  with initial value  $\tilde{x}(t_{k+1})$  to obtain  $\hat{x}(t_{k+1})$  that is

$$\hat{x}(t_{k+1}) = \tilde{x}(t_{k+1}) + \int_{t_k}^{t_{k+1}} g(x, t) dt \quad (85)$$

## Properties

- ▶  $x(t_k + 1) \neq \hat{x}(t_{k+1})$  is the general case. (except special linear case, constant dynamics, ...)
- ▶  $\hat{x}(t_{k+1}) \rightarrow x(t_{k+1})$  when  $t_{k+1} \rightarrow t_k$

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## Splitting-based for Moreau scheme without continuous contact forces

- ▶ The first part is

$$\begin{cases} M(q)\dot{v} = F(t, q, v), \\ \dot{q} = v, \\ q(t_k) = q_k, \quad v(t_k) = v_k \end{cases} \quad (86)$$

yielding to the approximations  $q_1 = q(t_{k+1})$  and  $v_1 = v(t_{k+1})$  which can be integrated by any smooth ODE solvers.

- ▶ The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}^+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \quad (87)$$

and leads to the approximation  $q_{k+1} = q(t_{k+1})$  and  $v_{k+1} = v(t_{k+1})$ .

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# Splitting-based methods with constants time-step.

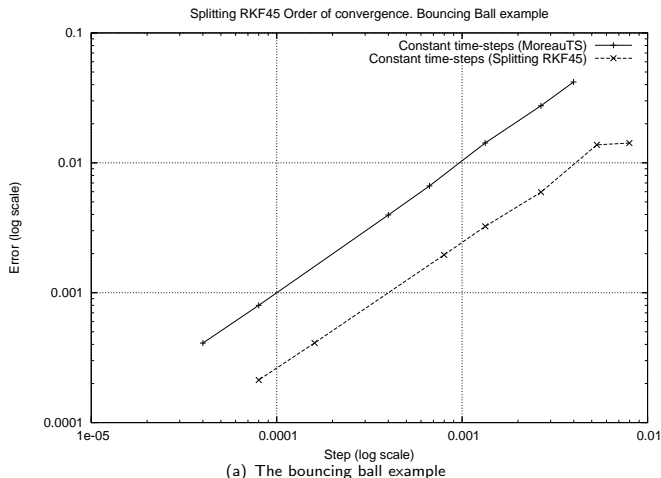


Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

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# Splitting-based methods with constants time-step.

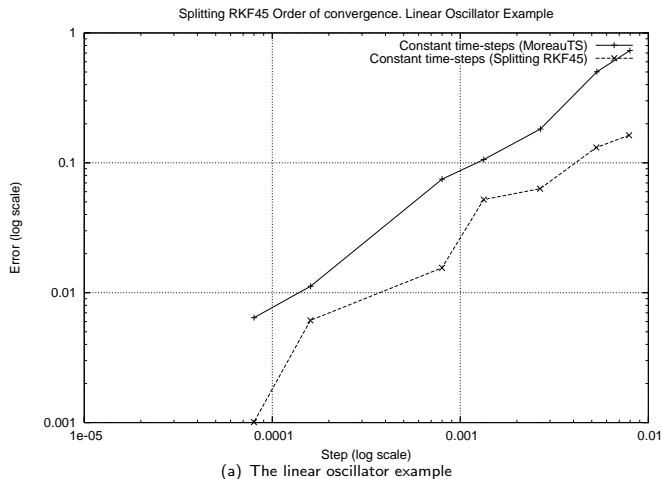


Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

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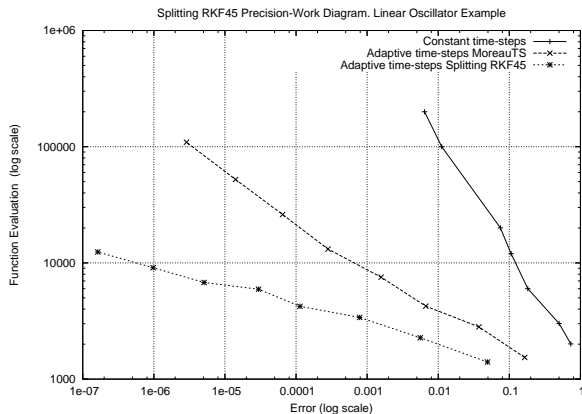
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# Splitting-based methods with adaptive time-step.



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

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## Splitting-based for Moreau scheme with continuous contact forces

- ▶ The first part is

$$\begin{cases} M(q)\dot{v} = F(t, q, v) + r(t), \\ \dot{q} = v, \\ y = g(q) \\ -r(t) \in \partial\psi_{T_{\mathbb{R}^+}(y)}(\dot{y}(t)) \\ q(t_k) = q_k, \quad v(t_k) = v_k \end{cases} \quad (88)$$

yielding to the approximations  $q_1 = q(t_{k+1})$  and  $v_1 = v(t_{k+1})$  which can be integrated by any smooth ODE solvers.

- ▶ The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}^+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \quad (89)$$

and leads to the approximation  $q_{k+1} = q(t_{k+1})$  and  $v_{k+1} = v(t_{k+1})$ .

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## Objectives

- The smooth dynamics and the impact equations
- Reformulations of the unilateral constraints on Different kinematics levels
- Reformulations of the smooth dynamics at acceleration level.
- The case of a single contact.
- The multi-contact case and the index-sets
- Comments and extensions

## Event-tracking schemes

- Time Discretization of the nonsmooth dynamics
- Time Discretization of the kinematics relations
- Discretization of the unilateral constraints
- Summary
- Schatzman–Paoli's scheme
- Empirical order

## Time-stepping schemes

### Comparison

- Smooth ODE time integration
- Local error estimates for the Moreau's Time-stepping scheme
- Adaptive time-step strategies
- A control based on violation
- Variable order approach

### Adaptive time-step strategies for time-stepping schemes

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### Time-stepping schemes of any order

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### Splitting based Time-stepping schemes

Thank you for your attention.

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